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Probability Theory and Related Fields

Convolution Tails, Product Tails and Domains of Attraction

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Summary. A distribution function is said to have an exponential tail $\bar{F}(t) = F(t, \infty)$ if $e^{\alpha u} \bar{F}(t+u)$ is asymptotically equivalent to $\bar{F}(t)$, $t \to \infty$, for all u. In this case $\bar{F}(\ln t)$ is regularly varying. For two such distributions, F and G, the convolution $H = F * G$ also has an exponential tail. We investigate the relationship between $\bar{H}$ and its components $\bar{F}$ and $\bar{G}$, providing conditions for $\lim \bar{H}/\bar{F}$ to exist. In addition, we are able to describe the asymptotic nature of $\bar{H}$ when the limit is infinite, for many cases. This corresponds to determining both the domain of attraction and the norming constants for the *product* of independent variables whose distributions have regularly varying tails.

In addition, we compare the tails of $H = F * G$ with $H_1 = F_1 * G_1$ when $\bar{F}$ is asymptotically equivalent to $\bar{F}$ and $\bar{G}$ is equivalent to $\bar{G}_1$. Such a comparison corresponds to the "balancing" consideration for the product of independent variables in stable domains of attraction. We discover that there are several distinct comparisons possible.

1. Introduction

Assume that F and G are distributions on $[0, \infty]$ and let $H = F * G$ be their convolution. We use the common convention to denote distribution tails, namely $\bar{F}(t) = F(t, \infty)$. We will also write $\bar{F}_1 \sim k\bar{F}$ for asymptotic equivalence, that is, for

$$\lim_{t \to \infty} \frac{\bar{F}_1(t)}{\bar{F}(t)} = k \in (0, \infty).$$

Definition 1. A distribution function F has exponential tails with rate $\alpha > 0$ ($F \in \mathscr{L}_\alpha$) if

$$\lim_{t \to \infty} \frac{\bar{F}(t-u)}{\bar{F}(t)} = e^{\alpha u} \quad \text{for all real } u. \quad \#$$

Such a distribution has the representation

$$\bar{F}(t)=a(t)\exp\left[-\int_0^t \alpha(v)\,dv\right], \quad \text{where} \quad a(t)\to a>0, \alpha(t)\to\alpha, \quad \text{as} \quad t\to\infty. \quad (1.1)$$

The convergence is necessarily uniform for $u \leqq u_0$.

A complimentary definition is the following:

Definition 2. A measurable function U is regularly varying with exponent ρ ($U \in \mathscr{R}_\rho$) if

$$\lim_{t\to\infty} \frac{U(yt)}{U(t)} = y^\rho \quad \text{for all } y>0. \quad \#$$

Such a function has the (Karamata) representation

$$U(t)=a(t)\exp\left[\int_0^t \frac{\rho(v)}{v}\,dv\right], \quad \text{where} \quad a(t)\to a>0, \rho(t)\to\rho, \quad \text{as} \quad t\to\infty. \quad (1.2)$$

Clearly $F \in \mathscr{L}_\alpha$ if and only if $\bar{F}(\ln t) \in \mathscr{R}_{-\alpha}$. For excellent discussions of regularly varying functions see de Haan (1970) and Bingham, Goldie and Teugels (1983).

The class $\mathscr{L}_0$, of course, can be defined as in Definition 1 with $\alpha=0$. This class in fact has received more attention in the literature on convolution tails. Chistyakov (1964) and Teugels (1975) have studied a subclass of $\mathscr{L}_0$ known as the subexponential distributions, with a view toward applications in branching processes and renewal theory. In some sense, $\mathscr{L}_0$ is a wider class of distributions than is $\mathscr{L}_\alpha$ for $\alpha>0$. However, we are interested in applications which require $\bar{F}(\ln t)$ to be regularly varying with exponent $-\alpha<0$. We will therefore concentrate on, but not limit our attention to, these cases. The foundation of our work includes Chover, Wainger and Ney (1973) and Embrechts and Goldie (1980, 1982) which extend the work of Chistyakov and Teugels to the cases $\alpha>0$.

We have three main objectives in this paper. Our primary objective is to describe the asymptotic behavior of $\bar{H}(t)=\overline{F*G}(t)$ when F and G each have exponential tails. For example, we will provide conditions for which $\bar{H}\sim k\bar{F}$. In addition, $\lim \bar{H}(t)/\bar{F}(t)$ is infinite in many cases and for these it is often possible to describe the behavior of $\bar{H}$ in terms of the tails and the truncated transforms of F and G. Our second objective is to consider when it is possible to compare two convolution tails $\bar{H}=\overline{F*G}$ and $\bar{H}_1=\overline{F_1*G_1}$, assuming $\bar{F}\sim\bar{F}_1$ and $\bar{G}\sim\bar{G}_1$.

Finally, we want to apply these results to problems involving the weak convergence of sums (or extremes) of products of two independent variables. If X and Y are independent random variables whose distributions have regularly varying tails, then it is readily seen that, with $F(t)=P[X\leqq e^t]$ and $G(t)=P[Y\leqq e^t]$, we need to describe the asymptotic nature of $\bar{H}(t)=\overline{F*G}(t)$ in order to establish the attraction corresponding to XY and in order to calculate the necessary norming constants. In addition, if the problem involves the sum of variables like XY, then convergence to a stable distribution ($\alpha<2$) requires showing that the tails $P[XY<-t]$ and $P[XY>t]$ balance. Accomplishing our second objective enables us to determine when this is possible. Specific applications for partial sums of products may be found in Davis and Resnick (1984 a, b) and in Cline (1984). In these applications, an additional requirement is that $\lim P[XY>t]/P[X>t]$ either exists or is infinite.

One crucial consideration in these problems is the transform (moment generating function)

$$m_F(\gamma)=\int_0^\infty e^{\gamma u} F(du),$$

which has a singularity at $\gamma=\alpha$. In fact $m_F(\gamma)<\infty$ for all $\gamma<\alpha$ and $m_F(\alpha)$ may or may not be finite (cf. Embrechts and Goldie, 1982, Lemma 2.4 and remark following).

An easy solution occurs when F and G have exponential tails with different rates. We repeat a lemma first proved (in a different form) by Breiman (1965, Proposition 3).

Lemma 1 (Breiman, 1965). *If $F\in\mathscr{L}_\alpha$ and $m_G(\gamma)<\infty$ for some $\gamma>\alpha$, then*

$$\lim_{t\to\infty}\frac{\bar{H}(t)}{\bar{F}(t)}=m_G(\alpha). \quad \#$$

The proof is quite direct, using the representation (1.1) for F and dominated convergence.

Since the situation with different rates is easy, we will concentrate on the case where both $F\in\mathscr{L}_\alpha$ and $G\in\mathscr{L}_\alpha$. Embrechts and Goldie (1980, Theorem 3) have shown that $H\in\mathscr{L}_\alpha$ follows. Under certain conditions we can have $\bar{H}\sim k\bar{F}$ for some $k\in(0,\infty)$. This is the situation which has been most extensively studied in the literature. We will present the major results and will weaken known conditions in Sect. 2. This situation requires $m_F(\alpha)<\infty$, but as we show, it is not nearly sufficient.

In the third section we will investigate $\bar{H}$ without requiring $\bar{H}\sim k\bar{F}$. Instead we will impose some regularity (but not absolute continuity) on F and G. We find that the asymptotic nature of $\bar{H}$ is quite varied, depending on which conditions we impose. Some specific examples we will investigate include

$$\bar{F}(t)=b(t)\,e^{-\alpha t}, \quad b\in\mathscr{R}_\beta \qquad \text{(Theorem 4, Sect. 3)},$$

$$\bar{F}(t)=e^{-\chi(t)-\alpha t}, \quad \chi\in\mathscr{R}_\rho,\ 0<\rho<1 \qquad \text{(Theorem 3, Sect. 2)}$$

and

$$\bar{F}(t)=e^{\chi(t)-\alpha t}, \quad \chi\in\mathscr{R}_\rho,\ 0<\rho<1 \qquad \text{(Theorem 5, Sect. 3)}.$$

We return to our second objective in Sect. 4, where we are again obliged to consider separate cases. The final section briefly considers application to the product distribution given by $H(\ln t)$.

Before we continue to Sect. 2, we will prove a simplifying lemma.

Lemma 2. *If $F\in\mathscr{L}_\alpha$, $G\in\mathscr{L}_\alpha$, $\alpha>0$, then for any $s(t)\to\infty$, $t-s(t)\to\infty$,*

$$\bar{H}(t)\sim\int_0^{s(t)}\bar{F}(t-u)\,G(du)+\int_0^{t-s(t)}\bar{G}(t-u)\,F(du).$$

Proof. Since

$$\bar{H}(t)=\int_0^{s(t)}\bar{F}(t-u)\,G(du)+\int_0^{t-s(t)}\bar{G}(t-u)\,F(du)+\bar{F}(t-s(t))\,\bar{G}(s(t)),$$

then it suffices to show that the third term is asymptotically negligible when compared to $\bar{H}(t)$.

Assume first that G is absolutely continuous with the property

$$\lim_{t\to\infty}\frac{G'(t)}{\bar{G}(t)}=\alpha. \tag{1.3}$$

We note that (1.3) is sufficient for $G\in\mathscr{L}_\alpha$. Changing variables and then using Fatou's lemma,

$$\varliminf_{t\to\infty}\int_0^{s(t)}\frac{\bar{F}(t-u)\,G(du)}{\bar{F}(t-s(t))\,\bar{G}(s(t))}=\varliminf_{t\to\infty}\int_0^{s(t)}\left(\frac{\bar{F}(t-s(t)+u)}{\bar{F}(t-s(t))}\right)\left(\frac{G'(s(t)-u)}{\bar{G}(s(t))}\right)du$$

$$\geqq\int_0^{\infty}(e^{-\alpha u})(\alpha e^{\alpha u})\,du=\infty.$$

This shows that the first term dominates the last and hence

$$\lim_{t\to\infty}\frac{\bar{F}(t-s(t))\,\bar{G}(s(t))}{\bar{H}(t)}=0.$$

If G does not satisfy (1.3), then by the representation (1.1) for G, $\bar{G}(t)=a(t)\,\bar{G}_1(t)$ where $a(t)\to 1$ and G_1 does satisfy (1.3). In fact we can always choose $a(t)$ to satisfy

$$\frac{1}{2}\leqq\frac{a(t)}{a(0)}\leqq 2 \quad \text{for all } t.$$

Then it is easy to show that $\bar{H}(t)\geqq\frac{a(0)}{2}\bar{H}_1(t)$, where $H_1=F*G_1$. Therefore

$$\lim_{t\to\infty}\frac{\bar{F}(t-s(t))\,\bar{G}(s(t))}{\bar{H}(t)}\leqq 4\lim_{t\to\infty}\frac{\bar{F}(t-s(t))\,\bar{G}_1(s(t))}{a(0)\,\bar{H}_1(t)}=0.$$

And this verifies the lemma. #

We will ordinarily choose $s(t)=t-s(t)=t/2$.

When both F and G are absolutely continuous,

$$H'(t)=\int_0^{t/2}F'(t-u)\,G(du)+\int_0^{t/2}G'(t-u)\,F(du).$$

Thus a simple consequence of Lemma 2 is that H satisfies property (1.3) whenever both F and G satisfy it. We note that (1.3) is stronger than the statement $G\in\mathscr{L}_\alpha$ and G is absolutely continuous.

2. Convolution Equivalency

In this section we are interested in conditions for which $\bar{H} \sim k\bar{F}$. Chistyakov (1964) and Chover, Wainger and Ney (1973) introduced the following class.

Definition 3. A distribution $F \in \mathscr{L}_\alpha$, $\alpha \geqq 0$, is convolution equivalent ($F \in \mathscr{S}_\alpha$) if

$$\lim_{t \to \infty} \frac{\overline{F * F}(t)}{\bar{F}(t)} \quad \text{exists finite.} \quad \#$$

This class has also been examined in Teugels (1975) and in Embrechts and Goldie (1982). The class $\mathscr{S}_0$ is called the subexponential class. We summarize the primary result, which combines results from the second and fourth papers mentioned above.

Theorem 1 (Chover, Wainger and Ney, 1973; Embrechts and Goldie, 1982). *Suppose $F \in \mathscr{S}_\alpha$, then*

$$\lim_{t \to \infty} \frac{\overline{F * F}(t)}{\bar{F}(t)} = 2m_F(\alpha) < \infty. \tag{2.1}$$

Furthermore, if $k_i = \lim_{t \to \infty} \frac{\bar{G}_i(t)}{\bar{F}(t)}$ *exists finite for distributions G_1 and G_2, then*

$$\lim_{t \to \infty} \frac{\overline{G_1 * G_2}(t)}{\bar{F}(t)} = k_1 m_{G_2}(\alpha) + k_2 m_{G_1}(\alpha).$$

And if $k_i > 0$, then $G_i \in \mathscr{S}_\alpha$. #

The proof of the limit (2.1) provided by Chover, Wainger and Ney is quite involved and uses Banach algebra techniques. A real analytic proof may be found in Cline (1985). The second part of Theorem 1 is Lemma 2.7 in Embrechts and Goldie, when $k_i > 0$. The limit holds for $k_i = 0$, also, as can readily be seen by taking $F_i = (1-\varepsilon) G_i + \varepsilon F$, considering $\overline{F_1 * F_2}$ and letting ε tend to zero.

Returning now to our original question, we see that if $F \in \mathscr{S}_\alpha$ and $\bar{G} \sim k\bar{F}$, $0 \leqq k < \infty$, then $\bar{H} \sim (m_G(\alpha) + k m_F(\alpha)) \bar{F}$. However, Theorem 1 does not explain the asymptotic nature of $\bar{H}$ when both $F \in \mathscr{S}_\alpha$, $G \in \mathscr{S}_\alpha$ and $\lim_{t \to \infty} \frac{\bar{G}(t)}{\bar{F}(t)}$ does *not* exist.

Corollary 1. *Let $H = F * G$ where $F \in \mathscr{S}_\alpha$, $G \in \mathscr{L}_\alpha$ and $\sup_t \frac{\bar{G}(t)}{\bar{F}(t)} < \infty$. Then $\bar{H}(t) \sim m_G(\alpha) \bar{F}(t) + m_F(\alpha) \bar{G}(t)$ and $H \in \mathscr{S}_\alpha$.*

Proof. See Cline (1985, Corollary 3.2 and Theorem 3.4). An alternate proof which requires $\bar{G}/\bar{F}$ bounded away from zero is similar to that of Theorem 1 in Embrechts and Goldie (1980). #

We next provide conditions for F to be in $\mathscr{S}_\alpha$. This is a substantial sharpening of similar results in Chover, Wainger and Ney and in Teugels. Different sets of conditions appear in Theorems 3 and 4.

Theorem 2. *Suppose $\bar{F}(t)=a(t)e^{-\psi(t)}$ where $a(t)\to a>0$ and ψ is eventually concave.*

If there exists nondecreasing $v(u)\to\infty$, $v(u)\geqq 2u$, such that

i) $\dfrac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}$ *is integrable on* $[0,\infty]$

and ii) $\lim_{u\to\infty}\left|(v(u)-2u)\dfrac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}\right|=0,$

then $F\in\mathcal{S}_\alpha$ for some $\alpha\geqq 0$.

Conversely, if $F\in\mathcal{S}_\alpha$, $\alpha>0$, then $\lim_{u\to\infty}\left|u\dfrac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}\right|=0$, for any function $v(u)\geqq 2u$.

Proof. Because it is eventually concave, ψ is eventually absolutely continuous with bounded, nonincreasing density ψ'. We may modify $\psi(t)$ and $a(t)$ on a finite interval so that ψ' exists finite and is nonincreasing for all t. Let $\alpha=\lim_{t\to\infty}\psi'(t)$. Then $\alpha\geqq 0$ and

$$\bar{F}(t)=a(t)\exp\left[-\int_0^t\psi'(v)\,dv\right].$$

From (1.1) we see that this indicates $F\in\mathcal{L}_\alpha$. By the second part of Theorem 1, it suffices to prove this theorem with $a(t)=1$ for all t. Thus $F(t)=e^{-\psi(t)}$ and F has density $\psi'(t)\bar{F}(t)$.

Since $\psi'(t)\downarrow\alpha$,

$$\alpha u\leqq\psi(t)-\psi(t-u)\leqq\psi(s)-\psi(s-u),\qquad s\leqq t.$$

This translates as

$$e^{\alpha u}\bar{F}(u)\leqq\frac{\bar{F}(t-u)\bar{F}(u)}{\bar{F}(t)}\leqq\frac{\bar{F}(s-u)\bar{F}(u)}{\bar{F}(s)},\qquad s\leqq t. \tag{2.5}$$

Define

$$s(t)=\sup\{u: v(u)\leqq t\}.$$

Then $s(t)\geqq u$ if and only if $t\geqq v(u)$. From (2.5),

$$e^{\alpha u}\bar{F}(u)\leqq\frac{\bar{F}(t-u)\bar{F}(u)}{\bar{F}(t)}\leqq\frac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))},\qquad u\leqq s(t).$$

The rightmost quantity is integrable by assumption.

This says

$$m_F(\alpha)=\int_0^\infty e^{\alpha u}F(du)=\int_0^\infty\psi'(u)e^{\alpha u}\bar{F}(u)\,du$$

$$\leqq\psi'(0)\int_0^\infty\frac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}\,du<\infty.$$

We may also apply dominated convergence,

$$\lim_{t\to\infty}\int_0^{s(t)}\frac{\bar F(t-u)}{\bar F(t)}F(du)=\int_0^\infty\psi'(u)\lim_{t\to\infty}\left[\frac{\bar F(t-u)\bar F(u)}{\bar F(t)}1_{[0,s(t)]}(u)\right]du$$
$$=\int_0^\infty\psi'(u)e^{\alpha u}\bar F(u)\,du=m_F(\alpha). \tag{2.6}$$

Now for $s(t)\leqq u\leqq t-s(t)$, the concavity of ψ implies

$$\psi(t-u)+\psi(u)\geqq\psi(t-s(t))+\psi(s(t)).$$

That is,

$$\frac{\bar F(t-u)\bar F(u)}{\bar F(t)}\leqq\frac{\bar F(t-s(t))\bar F(s(t))}{\bar F(t)}.$$

Hence,

$$\int_{s(t)}^{t-s(t)}\frac{\bar F(t-u)}{\bar F(t)}F(du)\leqq\psi'(0)(t-2s(t))\frac{\bar F(t-s(t))\bar F(s(t))}{\bar F(t)}.$$

Changing variables, $t=v(u)$, $s(t)\geqq u$,

$$\overline{\lim_{t\to\infty}}\int_{s(t)}^{t-s}\frac{\bar F(t-u)}{\bar F(t)}F(du)\leqq\psi'(0)\overline{\lim_{t\to\infty}}(v(u)-2s(v(u)))\frac{\bar F(v(u)-s(v(u)))\bar F(s(v(u)))}{\bar F(v(u))}$$
$$\leqq\psi'(0)\overline{\lim_{t\to\infty}}(v(u)-2u)\frac{\bar F(v(u)-u)\bar F(u)}{\bar F(v(u))}=0. \tag{2.7}$$

Applying Lemma 2 in concert with (2.6) and (2.7) we finally have

$$\lim_{t\to\infty}\frac{\overline{F*F}(t)}{\bar F(t)}=2\lim_{t\to\infty}\int_0^{s(t)}\frac{\bar F(t-u)}{\bar F(t)}F(du)+\lim_{t\to\infty}\int_{s(t)}^{t-s}\frac{\bar F(t-u)}{\bar F(t)}F(du)$$
$$=2m_F(\alpha).$$

To prove the partial converse, assume $F\in\mathscr{S}_\alpha$, $\alpha>0$. Again, it suffices to assume $\psi(t)$ has bounded, nonincreasing density $\psi'(t)$ for all t and that $a(t)\equiv 1$. From Lemma 2, with $s(t)=t/4$,

$$\overline{F*F}(t)\sim 2\int_0^{t/4}\bar F(t-u)F(du)+\int_{t/4}^{3t/4}\bar F(t-u)F(du).$$

But by Fatou's lemma and the fact that $F\in\mathscr{L}_\alpha$,

$$\underline{\lim_{t\to\infty}}\int_0^{t/4}\frac{\bar F(t-u)}{\bar F(t)}F(du)\geqq\int_0^\infty e^{\alpha u}F(du)=m_F(\alpha).$$

From Theorem 1,

$$\lim_{t\to\infty}\frac{\overline{F*F}(t)}{\bar F(t)}=2m_F(\alpha).$$

Therefore, we must have

$$\overline{\lim_{t\to\infty}} \int_{t/4}^{3t/4} \frac{\bar{F}(t-u)}{\bar{F}(t)} F(du) \leqq 2m_F(\alpha) - 2m_F(\alpha) = 0. \tag{2.8}$$

Now since ψ is concave,

$$\psi(t-u)+\psi(u) \leqq 2\psi(t/2) \qquad \text{for } t/4 \leqq u \leqq 3t/4.$$

That is,

$$\frac{\bar{F}(t-u)\bar{F}(u)}{\bar{F}(t)} \geqq \frac{\bar{F}(t/2))^2}{\bar{F}(t)}, \qquad t/4 \leqq u \leqq 3t/4.$$

Using (2.8),

$$\begin{aligned}\overline{\lim_{t\to\infty}}\, t/2 \frac{\bar{F}(t/2))^2}{\bar{F}(t)} &= \overline{\lim_{t\to\infty}} \int_{t/4}^{3t/4} \frac{(\bar{F}(t/2))^2}{\bar{F}(t)} du \\ &\leqq \overline{\lim_{t\to\infty}} \frac{1}{\alpha} \int_{t/4}^{3t/4} \psi'(u) \frac{\bar{F}(t-u)\bar{F}(u)}{\bar{F}(t)} du \\ &= 0.\end{aligned}$$

Changing variables, $t=2u$, and using (2.5) with $v(u) \geqq 2u$

$$0 = \overline{\lim_{t\to\infty}}\, t/2 \frac{(\bar{F}(t/2))^2}{\bar{F}(t)} = \overline{\lim_{t\to\infty}}\, u \frac{(\bar{F}(u))^2}{\bar{F}(2u)} \geqq \overline{\lim_{t\to\infty}}\, u \frac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}. \quad \#$$

If $\alpha=0$ we may replace ii) in Theorem 2 with

$$\lim_{t\to\infty} \left[\psi'(u)(v(u)-2u) \frac{\bar{F}(v(u)-u)\bar{F}(u)}{\bar{F}(v(u))}\right] = 0,$$

which is weaker, since $\psi'(u) \to 0$. When $\alpha > 0$, it is simplest and generally sufficient to take $v(u)=2u$. The conditions in Theorem 2 can be further weakened to include cases where ψ is not concave.

Corollary 2. *Suppose $\bar{F}(t)=a(t)e^{-\psi(t)}$ where $a(t)\to a>0$ and $\psi(t)$ is eventually absolutely continuous with the property that $\lim_{t\to\infty} t|\psi'(t)-\psi_1'(t)|=0$ for some ψ_1 which is eventually concave. Then both statements in Theorem 2 hold for F.*

Proof. The proof proceeds by first showing $F \in \mathscr{L}_\alpha$ and then that the dominated convergence argument of Theorem 2 is valid. The details are straightforward so we do not present them here. #

Intuitively, one would suspect that if $\chi(t) \in \mathscr{R}_\rho$, $0<\rho<1$, then $F(t)=1-e^{-\alpha t-\chi(t)}$ would be in $\mathscr{S}_\alpha$. The following counterexample illustrates that this is not the case. Let $\chi(t)=(1-b(t))t^\rho$ where $0<\rho<1$ and

$$b(t) = \begin{cases} e^{-\rho n} & t_{n-1}+1 \leqq t \leqq t_n = e^n \\ e^{-\rho n}(1-(1-e^{-\rho})(t-t_n)), & t_n \leqq t \leqq t_n+1. \end{cases}$$

Note that $\chi(t)$ is both absolutely continuous and nondecreasing. In order to have $F\in\mathscr{L}_\alpha$, we must show

$$\lim_{t\to\infty}[\chi(t)-\chi(t-u)]=0, \qquad \text{for all } u.$$

But this is not case. With $t_n=e^n$,

$$\lim_{n\to\infty}(\chi(t_n+1)-\chi(t_n))=\lim_{n\to\infty}((t_n+1)^\rho-t_n^\rho)+\lim_{n\to\infty}(b(t_n)\,t_n^\rho-b(t_n+1)(t_n+1)^\rho)$$
$$=0+\lim_{n\to\infty}(1-e^{-\rho(n+1)}(e^n+1)^\rho)=1-e^{-\rho}>0.$$

Therefore $F\notin\mathscr{L}_\alpha$ and hence $F\notin\mathscr{S}_\alpha$.

We can, however, provide a subclass of $\mathscr{S}_\alpha$ which contains many examples with regularly varying $\chi(t)$.

Theorem 3. *Suppose* $\bar F(t)=a(t)\,e^{-\alpha t-\chi(t)}$, $a(t)\to a>0$, $\alpha\geqq 0$, $\chi(t)\uparrow\infty$. *Assume there exists* $\rho<1$ *and* $s\geqq 0$ *such that*

i) $\quad \dfrac{\chi(yt)}{\chi(t)}\leqq y^\rho \quad$ for all $y\geqq 1,\ t\geqq s$

and ii) $\quad e^{-(2-2\rho)\chi(t)} \quad$ is integrable on $[2,\infty]$.

Then $F\in\mathscr{S}_\alpha$.

Proof. As before, we may assume $a(t)\equiv 1$. First note that

$$\frac{\chi(t)}{t}\leqq\left(\frac{t}{s}\right)^{\rho-1}\frac{\chi(s)}{s}\leqq\frac{\chi(s)}{s} \qquad \text{for all } t\geqq s.$$

Thus $\overline{\lim_{t\to\infty}}\,\dfrac{\chi(t)}{t}\leqq\dfrac{\chi(s)}{s}\,\overline{\lim_{t\to\infty}}\left(\dfrac{t}{s}\right)^{\rho-1}=0.$

Now fix $u\leqq s$ and let $t\geqq s$.

$$0\leqq\chi(t)-\chi(t-u)\leqq(1-(1-u/t)^\rho)\,\chi(t)\leqq\frac{\chi(t)}{t}\,u\leqq\frac{\chi(s)}{s}\,u.$$

This shows that χ is absolutely continuous with bounded density $\chi'(t)$, $t\geqq s$. It also demonstrates

$$\overline{\lim_{t\to\infty}}|\chi(t)-\chi(t-u)|\leqq u\,\overline{\lim_{t\to\infty}}\,\frac{\chi(t)}{t}=0.$$

And this is equivalent to $F\in\mathscr{L}_\alpha$ since $\dfrac{\bar F(t-u)}{\bar F(t)}=e^{\alpha u+\chi(t)-\chi(t-u)}$. Furthermore, the convergence is uniform for $u\leqq s$. Hence

$$\lim_{t\to\infty}\int_0^s\frac{\bar F(t-u)}{\bar F(t)}\,F(du)=\int_0^s e^{\alpha u}\,F(du). \tag{2.9}$$

On the other hand, for $u \geqq s$, $u \leqq t/2$,

$$\begin{aligned}\chi(t)-\chi(t-u)-\chi(u) &\leqq (1-(1-u/t)^{\rho})\,\chi(t)-\chi(u)\\ &\leqq \left[(1-(1-u/t)^{\rho})\left(\frac{t}{u}\right)^{\rho}-1\right]\chi(u)\\ &\leqq -(2-2^{\rho})\,\chi(u).\end{aligned}$$

Since $e^{-(2-2^{\rho})\chi(u)}$ is integrable, dominated convergence assures

$$\begin{aligned}\lim_{t\to\infty}\int_s^{t/2}\frac{\bar F(t-u)}{\bar F(t)}F(du) &= \lim_{t\to\infty}\int_s^{t/2}(\alpha+\chi'(u))\,e^{\chi(t)-\chi(t-u)-\chi(u)}\,du\\ &=\int_s^{\infty}(\alpha+\chi'(u))\,e^{-\chi(u)}\,du\\ &=\int_s^{\infty}e^{\alpha u}F(du). \qquad (2.10)\end{aligned}$$

Limits (2.9) and (2.10) are sufficient to have $F\in\mathscr{S}_\alpha$. #

Although each of our sets of conditions have required $\chi(t)$ (or $\psi(t)$) to be absolutely continuous, it must be remembered that (1.1) requires $\bar F$ to be asymptotic to just such an example.

We now present a counterexample which helps to illustrate some of the problems encountered in attempting to show $F\in\mathscr{S}_\alpha$. Note that this is a new example for which $F\in\mathscr{L}_\alpha$, $m_F(\alpha)<\infty$, but $F\notin\mathscr{S}_\alpha$ (cf. Embrechts & Goldie, sec. 3, 1980). Let $\bar F(t)=e^{-\alpha t-\chi(t)}$, where $\alpha>0$ and

$$\chi(t)=t^{1/2+\delta\cos(\ln t)}, \qquad t\geqq 1,\ 0<\delta<1/2.$$

Again, χ is absolutely continuous, but not monotone

$$\chi'(t)=(1/2+\delta\cos(\ln t)-\delta\ln t\sin(\ln t))\,t^{-1/2+\delta\cos(\ln t)}.$$

Since $\lim_{t\to\infty}\chi'(t)=0$, $F\in\mathscr{L}_\alpha$. Furthermore, $m_F(\gamma)<\infty$ if and only if $\gamma\leqq\alpha$. However, setting $t_n=e^{2n\pi}$, it is not difficult to show that

$$\begin{aligned}\frac{2\chi(t_n/2)}{\chi(t_n)} &= \frac{2(1/2e^{2n\pi})^{1/2+\delta\cos(\ln 2)}}{e^{(1+2\delta)n\pi}}\\ &\leqq 2e^{-2\delta n\pi(1-\cos(\ln 2))}\to 0.\end{aligned}$$

Therefore

$$\overline{\lim_{t\to\infty}}\,\frac{\overline{F*F}(t)}{\bar F(t)}\geqq\overline{\lim_{t\to\infty}}\,\frac{(\bar F(t/2))^2}{\bar F(t)}=\overline{\lim_{t\to\infty}}\,e^{\chi(t_n)-2\chi(t_n/2)}=\infty.$$

3. More on Convolution Tails

In this section we will investigate the asymptotic nature of $\bar H$ without requiring $F\in\mathscr{S}_\alpha$. As far as we know, this has not been pursued before. Unfortunately, we are unable to present a general pattern. Indeed, the very results depend heavily on the conditions assigned to F and G. The conclusions obtained in Theorems 4 and 5 below are concise and provide simple formulations of the tails of H. But these conclusions are at variance with each other and with Corollary 1 and offer little suggestion for a unified theorem.

Theorem 4 is characterized by the assumption that $e^{\alpha t}\bar{F}(t)$ is regularly varying, while Theorem 5 assumes $\ln(e^{\alpha t}\bar{F}(t))$ is regularly varying. For the first result we need the coefficients

$$I_{\beta,\gamma}=\begin{cases}1, & \gamma\leqq -1\\ (1+\gamma)\,2^{1+\gamma}\int_0^{1/2}(1-y)^{\beta}y^{\gamma}\,dy, & \gamma>-1.\end{cases}$$

Lemma 4. *Assume that $F,\ G\in\mathscr{L}_{\alpha}$, $\alpha>0$, and that $b(t)=e^{\alpha t}\bar{F}(t)\in\mathscr{R}_{\beta}$ and $c(t)=e^{\alpha t}\bar{G}(t)\in\mathscr{R}_{\gamma}$, where β and γ are any real values. Then*

$$\bar{H}(t)\sim I_{\beta,\gamma}\bar{F}(t)\int_0^{t/2}e^{\alpha u}G(du)+I_{\gamma,\beta}\bar{G}(t)\int_0^{t/2}e^{\alpha u}F(du).$$

Proof. In light of Lemma 2, it suffices to show

$$\int_0^{t/2}\bar{F}(t-u)\,G(du)\sim I_{\beta,\gamma}\bar{F}(t)\int_0^{t/2}e^{\alpha u}G(du). \tag{3.1}$$

For large t, $u<t/2$, the Karamata representation (1.2) applied to b yields

$$(1-\varepsilon)(1-u/t)^{\beta}\leqq\frac{b(t-u)}{b(t)}\leqq(1+\varepsilon)(1-u/t)^{\beta}.$$

Therefore,

$$\begin{aligned}\int_0^{t/2}\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G(dv)&=\int_0^{t/2}\frac{b(t-u)}{b(t)}\,e^{\alpha u}G(du)\\ &\sim\int_0^{t/2}(1-u/t)^{\beta}e^{\alpha u}G(du).\end{aligned} \tag{3.2}$$

Of course, $(1-u/t)^{\beta}\leqq 2^{|\beta|}$. If $m_G(\alpha)<\infty$ (requiring $\gamma\leqq -1$) then dominated convergence proves

$$\lim_{t\to\infty}\int_0^{t/2}\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G(du)=\int_0^{\infty}e^{\alpha u}G(du)=m_G(\alpha).$$

This satisfies (3.1).

Assume instead that $m_G(\alpha)=\infty$, which requires $\gamma\geqq -1$. Then we first define

$$C(t)=\int_0^{t}e^{\alpha u}G(du).$$

Integrating by parts and recalling $c(t)=e^{\alpha t}\bar{G}(t)$,

$$C(t)=1-c(t)+\alpha\int_0^{t}c(u)\,du.$$

Since $\int_0^t c(u)\,du \in \mathscr{R}_{1+\gamma}$ (de Haan, 1970, p. 15) and

$$\lim_{t\to\infty} \frac{1}{tc(t)} \int_0^t c(u)\,du = \frac{1}{1+\gamma},$$

then it follows that $C(t)$ is also in $\mathscr{R}_{1+\gamma}$. Furthermore,

$$\begin{aligned}\int_0^{t/2} (1-u/t)^\beta e^{\alpha u} G(du) &= 2^{-\beta} C(t/2) + \frac{\beta}{t} \int_0^{t/2} (1-u/t)^{\beta-1} C(u)\,du \\ &= 2^{-\beta} C(t/2) + \beta \int_0^{1/2} (1-y)^{\beta-1} C(yt)\,dy.\end{aligned}$$

Since $C(t)$ is nondecreasing, then dominated convergence gives

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{C(t/2)} \int_0^{t/2} (1-u/t)^\beta e^{\alpha u} G(du) &= 2^{-\beta} + \lim_{t\to\infty} \beta \int_0^{1/2} (1-y)^{\beta-1} \frac{C(yt)}{C(t/2)}\,dy \\ &= 2^{-\beta} + \beta \int_0^{1/2} (1-y)^{\beta-1} (2y)^{1+\gamma}\,dy \\ &= (1+\gamma)\,2^{1+\gamma} \int_0^{1/2} (1-y)^\beta y^\gamma\,dy = I_{\beta,\gamma}. \qquad (3.3)\end{aligned}$$

The last equality is integration by parts.

Combining (3.3) with (3.2) we have

$$\lim_{t\to\infty} \frac{1}{\bar{F}(t)\,C(t/2)} \int_0^{t/2} \bar{F}(t-u)\,G(du) = I_{\beta,\gamma},$$

which is our result as stated in (3.1). #

As corollary to Lemma 4, we have Theorem 4 to distinguish cases according to the finiteness of the moment generating function.

Theorem 4. *Assume F and G are as in Lemma 4.*

i) *The result in Lemma 4 can be simplified according to*

$$I_{\beta,\gamma} \int_0^{t/2} e^{\alpha u} G(du) \sim \begin{cases} m_G(\alpha), & \text{if } m_G(\alpha) < \infty \\ \int_0^{t/2} e^{\alpha u} G(du), & \text{if } \gamma = -1 \text{ and } m_G(\alpha) = \infty,\ \alpha > 0 \\ \left(\int_0^{1/2} (1-y)^\beta y^\gamma\,dy\right) \alpha t e^{\alpha t} \bar{G}(t), & \beta > -1,\ \alpha > 0. \end{cases}$$

In particular,

ii) *If $m_F(\alpha) < \infty$, then $F \in \mathscr{S}_\alpha$*

iii) *If $m_F(\alpha) < \infty$ and $m_G(\alpha) < \infty$, then $H \in \mathscr{S}_\alpha$ and*

$$\bar{H}(t) \sim m_G(\alpha)\,\bar{F}(t) + m_F(\alpha)\,\bar{G}(t).$$

iv) *If* $m_G(\alpha)<\infty$, $m_F(\alpha)=\infty$, *and* $\beta>\gamma$, *then* $\bar{H}(t)\sim m_G(\alpha)\bar{F}(t)$. *This is not necessarily true when* $\beta=\gamma=-1$.

v) *If* $\beta>-1$ *and* $\gamma>-1$, *then* $\bar{H}(t)\sim\dfrac{\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(2+\beta+\gamma)}\alpha t e^{\alpha t}\bar{F}(t)\bar{G}(t)$.

Proof. i) The first two expressions, corresponding to $\gamma\leqq -1$, are obvious. When $\gamma>-1$, then as shown in the proof of the lemma,

$$C(t)=\int_0^t e^{\alpha u}G(du)\sim\frac{\alpha t}{1+\gamma}c(t)$$

and $C\in\mathscr{R}_{1+\gamma}$. Therefore

$$\begin{aligned}I_{\beta,\gamma}\int_0^{t/2} e^{\alpha u}G(du)&=I_{\beta,\gamma}C(t/2)\sim\frac{\alpha I_{\beta,\gamma}t}{2^{1+\gamma}(1+\gamma)}c(t)\\&=\left(\int_0^{1/2}(1-y)^{\beta}y^{\gamma}dy\right)\alpha t e^{\alpha t}\bar{G}(t).\end{aligned}$$

ii) With $F=G$, we have $\overline{F*F}\sim 2m_F(\alpha)\bar{F}$.

iii) Clearly, $\bar{H}\sim m_G(\alpha)\bar{F}+m_F(\alpha)\bar{G}$. By ii), $F*F$ and $G*G$ satisfy the conditions of Lemma 4 so that this applies to $H*H$ as well. Using the same argument as at the end of the proof of Corollary 1, it follows that $H\in\mathscr{S}_\alpha$.

iv) Define $B(t)=\int_0^t e^{\alpha u}F(du)$ and $C^*(t)=\int_t^\infty e^{\alpha u}G(du)$. Similarly to the remarks in i) and in the proof of Lemma 4, $B(t)\in\mathscr{R}_{1+\beta}$ and $B(t)\sim\dfrac{\alpha}{1+\beta}tb(t)$. In the same manner we can show $C^*\in\mathscr{R}_{1+\gamma}$ and $C^*(t)\sim\dfrac{-\alpha}{\gamma+1}tc(t)$. Note $\gamma\leqq -1\leqq\beta$. In case of equality, the asymptotic expressions above hold if treated in the obvious manner.

Now $\bar{H}\sim m_G(\alpha)\bar{F}$ if and only if

$$\begin{aligned}0&=\lim_{t\to\infty}\frac{\bar{G}(t)}{\bar{F}(t)}\int_0^{t/2}e^{\alpha u}F(du)=\lim_{t\to\infty}\frac{c(t)}{b(t)}B(t/2)\\&=(1/2)^{1+\beta}\lim_{t\to\infty}\frac{c(t)}{b(t)}B(t).\end{aligned}\tag{3.4}$$

When $\gamma<-1\leqq\beta$, $\frac{c}{b}B$ is in $\mathscr{R}_{1+\gamma}$ and thus (3.4) is automatically true. When $\gamma=-1<\beta$, then

$$\lim_{t\to\infty}\frac{c(t)}{b(t)}B(t)=\frac{-\gamma-1}{1+\beta}\lim_{t\to\infty}C^*(t)=0.$$

However, if $\gamma=\beta=-1$, we will show that (3.4) does not have to hold and that in fact even if

$$\lim_{t\to\infty}\frac{\bar{G}(t)}{\bar{F}(t)}=\lim_{t\to\infty}\frac{c(t)}{b(t)}=0,$$

the limit in (3.4) does not have to exist. To see this let $d(t)$ be any nonintegrable function in $\mathscr{R}_{-1}$ such that $td(t)\to 0$. Define

$$b(t)=d(t)\exp\left[\int_0^t d(v)\,dv\right]. \tag{3.5}$$

This implies $\int_0^t b(u)\,du=\exp\left[\int_0^t d(v)\,dv\right]\to\infty$ and $b\in\mathscr{R}_{-1}$, so (3.5) describes a class of functions b which satisfy our conditions. Since $B(t)=1-b(t)+\alpha\int_0^t b(u)\,du$ then

$$d(t)=\frac{b(t)}{\int_0^t b(u)\,du}\sim\frac{\alpha b(t)}{B(t)}.$$

On the other hand, we may choose $c(t)$ to be any integrable $\mathscr{R}_{-1}$ function since $m_G(\alpha)<\infty$ if and only if $c(t)\to 0$ and $c(t)$ is integrable. This means we must have

$$\varliminf_{t\to\infty}\frac{c(t)}{b(t)}B(t)=\varliminf_{t\to\infty}\frac{c(t)}{d(t)}=0.$$

But it is not necessary that $\varlimsup_{t\to\infty}\frac{c(t)}{d(t)}$ is even finite $\left(\text{e.g. } c(t)=c(t)=\frac{1}{t(\ln t)^2},\ d(t)=\frac{1+\left(1-\frac{1}{t}\right)\sin(\ln\ln t)}{t\ln t}\right)$. Hence (3.4) does not have to hold when $\beta=\gamma=-1$.

v) Define $C(t)$ and $B(t)$ as in the proofs for i) and iv), respectively. Then $B\in\mathscr{R}_{1+\beta}$, $C\in\mathscr{R}_{1+\gamma}$ and

$$B(t/2)=\int_0^{t/2} e^{\alpha u}F(du)\sim\frac{\alpha t b(t)}{(1+\beta)2^{1+\beta}}$$

$$C(t/2)=\int_0^{t/2} e^{\alpha u}G(du)\sim\frac{\alpha t c(t)}{(1+\gamma)2^{1+\gamma}}.$$

Therefore, by Lemma 4,

$$\begin{aligned}e^{\alpha t}\bar{H}(t)&\sim I_{\beta,\gamma}b(t)\,C(t/2)+I_{\gamma,\beta}c(t)\,B(t/2)\\&\sim\alpha t\left(\frac{I_{\beta,\gamma}}{(1+\beta)2^{1+\beta}}+\frac{I_{\gamma,\beta}}{(1+\gamma)2^{1+\gamma}}\right)b(t)\,c(t)\\&=\frac{\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(2+\beta+\gamma)}\alpha t b(t)\,c(t).\end{aligned}$$

Thus $\bar{H}(t)\sim\frac{\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(2+\beta+\gamma)}\alpha t e^{\alpha t}\bar{F}(t)\bar{G}(t)$. #

An intriguing consequence of part v) is, if $\bar{F}(t)=b(t)\,e^{-\alpha t}$, $b(t)\in\mathscr{R}_\beta$, $\beta>-1$, then

$$\bar{F}^{*n}(t)\sim\frac{(\alpha\Gamma(1+\beta))^n}{a\Gamma(n(1+\beta))}\,t^{n-1}\,b^n(t)\,e^{-\alpha t},\qquad n\geqq 1.$$

The formulation of the tail in Lemma 4 is not unpleasant and seems to agree reasonably well with Corollary 1. The difference is that moment generating functions are truncated and that additional coefficients are included. The value of the coefficients, however, depend on the specific regularity of F and G. Our next theorem will provide a different looking formulation for another class of distributions. For this result we need a lemma which is easy to prove, although we have not seen it elsewhere.

Lemma 5. *Suppose $\chi(t)$ has derivative $\chi'(t)$. Then for any real ρ,*

$$\lim_{t\to\infty}\frac{t\,\chi'(t)}{\chi(t)}=\rho \tag{3.6}$$

if and only if

$$\lim_{t\to\infty,\,\eta\to 0}\frac{\chi((1+\eta)\,t)-\chi(t)}{\eta\,\chi(t)}=\rho.$$

If $\chi''(t)$ exists, then

$$\lim_{t\to\infty}\frac{t^2\,\chi''(t)}{\chi(t)}=\rho(\rho-1) \tag{3.7}$$

if and only if

$$\lim_{t\to\infty,\,\eta\to 0}\frac{\chi((1+\eta)\,t)-2\chi(t)+\chi((1-\eta)\,t)}{\eta^2\,\chi(t)}=\rho(\rho-1).$$

Proof. Assume that (3.6) holds. Clearly, $\chi\in\mathscr{R}_\rho$. Then

$$\frac{\chi((1+\eta)\,t)-\chi(t)}{\chi(t)}=\int_1^{1+\eta}\frac{t\,v\,\chi'(v\,t)}{\chi(v\,t)}\,\frac{\chi(v\,t)}{v\,\chi(t)}\,dv.$$

The integrand converges uniformly in $|v|\leqq 1/2$ so that

$$\lim_{t\to\infty}\frac{\chi((1+\eta)\,t)-\chi(t))}{\eta\,\chi(t)}=\frac{1}{\eta}\int_1^{1+\eta}\rho\,v^{\rho-1}\,dv=\frac{(1+\eta)^\rho-1}{\eta}$$

uniformly for $|\eta|\leqq 1/2$.

Therefore, the double limit is the iterated limit,

$$\lim_{t\to\infty,\,\eta\to 0}\frac{\chi(t+\eta\,t)-\chi(t)}{\eta\,\chi(t)}=\lim_{\eta\to 0}\frac{(1+\eta)^\rho-1}{\eta}=\rho.$$

Conversely, suppose

$$\lim_{t\to\infty,\,\eta\to 0}\frac{\chi(t+\eta\,t)-\chi(t)}{\eta\,\chi(t)}=\rho.$$

Since $\lim_{\eta\to 0} \frac{\chi(t+\eta t)-\chi(t)}{\eta\chi(t)} = \frac{t\chi'(t)}{\chi(t)}$ exists for all t, then it follows that

$$\rho = \lim_{t\to\infty} \lim_{\eta\to 0} \frac{\chi(t+\eta t)-\chi(t)}{\eta\chi(t)} = \lim_{t\to\infty} \frac{t\chi'(t)}{\chi(t)}.$$

The proof for the second assertion is handled in much the same way. Assume (3.7) holds. Then

$$\begin{aligned} &\lim_{t\to\infty} \frac{\chi((1+\eta)t)-2\chi(t)+\chi((1-\eta)t)}{\eta^2\chi(t)} \\ &= \lim_{t\to\infty} \frac{1}{\eta^2} \int_1^{1+\eta} \int_{v-\eta}^{v} \frac{(\varepsilon t)^2\chi''(\varepsilon t)}{\chi(\varepsilon t)} \frac{\chi(\varepsilon t)}{\varepsilon^2\chi(t)} d\varepsilon dv \\ &= \frac{1}{\eta^2} \int_1^{1+\eta} \int_{v-\eta}^{v} \rho(\rho-1)\varepsilon^{\rho-2} d\varepsilon dv = \frac{(1+\eta)^\rho - 2 + (1-\eta)^\rho}{\eta^2}, \end{aligned} \tag{3.8}$$

uniformly for $|\eta| \leq 1/2$.

The converse is also as simple. #

The uniform convergence in (3.8) can be stated more strongly. Indeed, for $\eta = \eta(t)$ such that $\eta(t)\to 0$ and $t-|\eta(t)|t\to\infty$,

$$|\chi(t+\eta(t)t)-2\chi(t)+\chi(t-\eta(t)t)| \leq (\rho(1-\rho)-\varepsilon)\eta^2(t)\chi(t) \tag{3.9}$$

whenever t is large enough.

One important conclusion we may draw from Lemma 5 is, if $\rho>0$ and $\eta = \eta(t) = \frac{y}{t\chi'(t)}$, then

$$\lim_{t\to\infty} e^{\chi(t+y/\chi'(t))-\chi(t)} = e^y, \quad \text{all } y.$$

This places $e^{\chi(t)}$ in the class of Γ-varying functions with auxilary function $\frac{1}{\chi'(t)}$ (cf. de Haan, 1970, pp. 43–50). We exploit this fact in the next theorem.

Theorem 5. *Suppose* $\bar{F}(t) = a(t)e^{-\alpha t+\chi(t)}$ *where* $a(t)\to a>0$, $\alpha>0$ *and* $\chi(t)$ *is eventually twice differentiable and* $\lim_{t\to\infty} \frac{t\chi''(t)}{\chi'(t)} = \rho - 1$ *with* $0<\rho<1$. *Then*

$$\overline{F*F}(t) \sim \frac{\alpha}{2}\sqrt{\frac{\pi}{\rho(1-\rho)}}\, \frac{t}{\chi^{1/2}(t/2)} \bar{F}^2(t/2).$$

Proof. The given condition implies (3.6) and (3.7) hold. We may modify $a(t)$ and $\chi(t)$ so that χ is twice differentiable for all t and (by (3.6) and (3.7)) so that χ is both monotone and concave for all t.

Since $\chi(t)\geq 0$, then $m_F(\alpha)=\infty$. Let $\bar{F}_1(t) = e^{-\alpha t+\chi(t)}$. By Theorem 6, in the final section, $\overline{F*F}(t) \sim a^2\overline{F_1 * F_1}(t)$. It suffices then to assume $a(t)\equiv a$.

Note that

$$\lim_{t\to\infty} \chi'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} t\chi'(t) = \infty.$$

As remarked above, $b(t)=e^{\chi(t)}$ is Γ-varying with auxilary function $f(t)=\frac{1}{\chi'(t)}$. Consequences of this include (cf. de Haan, 1970)

$$\lim_{t\to\infty}\frac{\chi'(t+uf(t))}{\chi'(t)}=1, \qquad \text{all } u,$$

$$\lim_{t\to\infty}\frac{b(yt)}{b(t)}=\infty, \qquad \text{all } y>1,$$

and

$$\lim_{t\to\infty}\frac{\int_0^t b(u)\,du}{f(t)\,b(t)}=1.$$

Also, $\int_0^t b(u)\,du$ is Γ-varying with auxilary $f(t)$, as is

$$B(t)=\int_0^t e^{\alpha u}F(du)=1-b(t)+\alpha\int_0^t b(u)\,du.$$

Note that $B(t)\sim\alpha f(t)\,b(t)$. Thus for $y>1$

$$\lim_{t\to\infty}\frac{t\,b(yt)}{B(t)}=\lim_{t\to\infty}\left(\frac{t}{f(t)}\right)\left(\frac{f(t)\,b(t)}{B(t)}\right)\left(\frac{b(yt)}{b(t)}\right)=\infty.$$

Now, to apply this to our problem,

$$\begin{aligned}\frac{1}{2a^2}\,e^{\alpha t}\,\overline{F*F}(t)&\sim\frac{1}{a^2}\,e^{\alpha t}\int_0^{t/2}\bar F(t-u)\,F(du)\sim\int_0^{t/2}b(t-u)\,B(du)\\&=\int_0^{t/2}b(t-u)(\alpha-\chi'(u))\,b(u)\,du.\end{aligned} \tag{3.10}$$

Changing variables and applying Fatou's lemma,

$$\begin{aligned}\varliminf_{t\to\infty}\int_0^{t/2}\frac{b(t-u)}{b(t)}\,\frac{1}{B(f(t))}\,B(du)&\geqq\varliminf_{t\to\infty}\int_{f(t)}^{2f(t)}\frac{b(t-u)}{b(t)}\,\frac{1}{B(f(t))}\,B(du)\\&\geqq\int_1^2\varliminf_{t\to\infty}\left[\left(\frac{b(t-yf(t))}{b(t)}\right)\left(\frac{f(t)\,b(yf(t))}{B(f(t))}\right)(\alpha-\chi'(yf(t)))\right]dy\\&=\int_1^2(e^{-y})(\infty)(\alpha)\,dy=\infty.\end{aligned} \tag{3.11}$$

But

$$\int_0^{f(t)}b(t-u)\,B(du)\leqq b(t)\,B(f(t)). \tag{3.12}$$

Thus, using (3.10), (3.11) and (3.12),

$$\frac{1}{2a^2} e^{\alpha t} \overline{F*F}(t) \sim \int_0^{t/2} b(t-u)\, B(du) \sim \int_{f(t)}^{t/2} b(t-u)\, B(du)$$
$$= \int_{f(t)}^{t/2} b(t-u)\left(\alpha - \frac{1}{f(u)}\right) b(u)\, du$$
$$\sim \alpha \int_{f(t)}^{t/2} b(t-u)\, b(u)\, du.$$

The final equivalence occurs since $f(t) \to \infty$.

Define

$$d(t) = \int_0^{t-f(2t)} b(t+u)\, b(t-u)\, du,$$
$$= \int_{f(2t)}^{t} b(2t-u)\, b(u)\, du.$$

We have, then,

$$\overline{F*F}(t) \sim 2\alpha a^2 d(t/2)\, e^{-\alpha t}. \tag{3.13}$$

(Note that up to this point we have used only the facts that $b(t)$ is monotone and Γ-varying with auxilary $f(t)$ and that $f(t) \to \infty$.)

Let $\eta(t) = y\chi(t)^{-1/2}$, for fixed y, and $s(t) = t\chi(t)^{-1/2}$. By Lemma 5, since (3.7) holds and since $\eta(t) \to 0$,

$$\lim_{t\to\infty} [\chi(t+ys(t)) - 2\chi(t) + \chi(t-ys(t))]$$
$$= \lim_{t\to\infty} y^2 \left[\frac{\chi(t+\eta(t)\,t) - 2\chi(t) + \chi(t-\eta(t)\,t)}{\eta^2(t)\,\chi(t)}\right]$$
$$= -\rho(1-\rho)\, y^2. \tag{3.14}$$

Furthermore, (3.9) holds since if $y \leqq r(t) = \dfrac{t - f(2t)}{t}\chi^{1/2}(t)$, then $t - |\eta(t)|\, t \to \infty$. Hence

$$\frac{b(t+ys(t))\, b(t-ys(t))}{b^2(t)} = \exp[\chi(t+\eta(t)\,t) - 2\chi(t) + \chi(t-\eta(t)\,t)]$$
$$\leqq \exp\left[-(1-\rho-\varepsilon)\, t\chi'(t)\frac{y^2}{\chi(t)}\right]$$
$$\leqq \exp[-(\rho-\varepsilon)(1-\rho-\varepsilon)\, y^2], \qquad y \leqq r(t).$$

We can therefore use dominated convergence and (3.14) to obtain

$$\lim_{t\to\infty} \frac{\chi^{1/2}(t)\, d(t)}{t b^2(t)} = \lim_{t\to\infty} \frac{\chi^{1/2}(t)}{t} \int_0^{t-f(2t)} \frac{b(t+u)\, b(t-u)}{b^2(t)}\, du$$
$$= \int_0^\infty \lim_{t\to\infty} \left[\frac{b(t+ys(t))\, b(t-ys(t))}{b^2(t)}\, 1_{[0,r,(t)]}(y)\right] dy$$
$$= \int_0^\infty e^{-\rho(1-\rho)y^2}\, dy = \frac{1}{2}\sqrt{\frac{\pi}{\rho(1-\rho)}}. \tag{3.15}$$

From (3.13) and (3.15),

$$\overline{F*F}(t) \sim \alpha a^2 \sqrt{\frac{\pi}{\rho(1-\rho)}}\,\frac{t/2}{\chi^{1/2}(t/2)}\, b^2(t/2)\, e^{-\alpha t}$$

$$\sim \frac{\alpha}{2}\sqrt{\frac{\pi}{\rho(1-\rho)}}\,\frac{t}{\chi^{1/2}(t/2)}\,\bar{F}^2(t/2). \quad \#$$

Corollary 3. *Properly interpreted, the conclusion of Theorem 5 also holds if*

i) $\rho=0,\ t\chi'(t)\to\infty$ and $\dfrac{t\chi''(t)}{\chi'(t)}\to -1$

or ii) $\rho=1,\ \chi'(t)\downarrow 0$ and $\dfrac{t\chi''(t)}{\chi'(t)}\to 0.$

Proof. i) $\rho=0$. Since $\dfrac{t\chi''(t)}{\chi'(t)}\to -1=\rho-1$, both (3.6) and (3.7) hold. In addition, $t\chi'(t)\to\infty$ implies $\chi(t)\to\infty$. However, we still need to check that $b(t)=e^{\chi(t)}$ is Γ-varying with auxiliary $f(t)=\dfrac{1}{\chi'(t)}$. Since $\chi'(t)\in\mathcal{R}_{-1}$ and $\dfrac{y}{t\chi'(t)}\to 0$,

$$\lim_{t\to\infty}\frac{\chi'(t+y/\chi'(t))}{\chi'(t)}=1, \qquad \text{locally uniformly in } y.$$

Thus

$$\lim_{t\to\infty}[\chi(t+y/\chi'(t))-\chi(t)]=\lim_{t\to\infty}\int_0^y \frac{\chi'(t+v/\chi'(t))}{\chi'(t)}\,dv=y,$$

which is sufficient. The proof then follows through, using Fatou's Lemma in (3.14) rather than dominated convergence. That is

$$\varliminf_{t\to\infty}\frac{\overline{F*F}(t)\,\chi^{1/2}(t/2)}{t e^{\alpha t}F^2(t/2)}=\infty.$$

ii) $\rho=1$. Again (3.6) and (3.7) hold. Also, $b(t)$ is Γ-varying. The only condition still required for the proof to go through is $\chi'(t)\downarrow 0$. Again Fatou's Lemma is used for (3.14). #

Corollary 4. *Suppose* $\bar{F}(t)=a_1(t)\,e^{-\alpha t+\chi_1(t)}$, $\bar{G}(t)=a_2(t)\,e^{-\alpha t+\chi_2(t)}$ *each satisfy the conditions of Theorem 5. Assume also that*

$$\lim_{t\to\infty}\left[\chi_2^{1/2}(t)\left(\frac{\chi_1'(t)}{\chi_2'(t)}-1\right)\right]=k, \qquad |k|<\infty. \tag{3.16}$$

Then $\overline{F*G}(t)\sim\dfrac{\alpha}{2}\sqrt{\dfrac{\pi}{\rho(1-\rho)}}\,\dfrac{t}{\chi_1^{1/2}(t/2)}\,\bar{F}(t/2)\,\bar{G}(t/2).$

Proof. Let $b(t)=e^{\chi_1(t)}$ and $c(t)=e^{\chi_2(t)}$. Also define

$$B(t)=1-b(t)+\alpha\int_0^t b(u)\,du, \qquad C(t)=1-c(t)+\alpha\int_0^t c(u)\,du.$$

Note that (3.16) implies $\chi_1' \sim \chi_2'$ and $\chi_1 \sim \chi_2$. Therefore b, c, B, C are all Γ-varying with the same auxilary function $f(t) = \frac{1}{\chi_2'(t)}$ (cf. de Haan, 1970, p. 45). Replacing b with c and B with C where appropriate, we follow the proof of Theorem 5 and find

$$e^{\alpha t} \int_0^{t/2} \bar{F}(t-u)\, G(du) \sim \alpha a_1 a_2 d_1(t/2),$$

where

$$d_1(t) = \int_0^{t-f(2t)} b(t+u)\, c(t-u)\, du.$$

From (3.14), applied to $c(t)$ in place of $b(t)$,

$$\lim_{t\to\infty} \frac{c(t+ys(t))\, c(t-ys(t))}{c^2(t)} = e^{-\rho(1-\rho)y^2}$$

and these sequences are bounded by $e^{-[\rho(1-\rho)-\varepsilon]y^2}$, $y \leqq r(t) = \frac{t-f(2t)}{t} \chi_2^{1/2}(t)$. We will show that

$$\lim_{t\to\infty} \frac{b(t+ys(t))}{b(t)} \frac{c(t)}{c(t+ys(t))} = e^{k\rho y} \tag{3.17}$$

and that the sequences are bounded by $e^{k_1 y}$, $k_1 < \infty$, $y \leqq r(t)$. We can therefore use dominated convergence as before to obtain

$$\lim_{t\to\infty} \frac{d_1(t)\chi_2^{1/2}(t)}{tb(t)\, c(t)} = \lim_{t\to\infty} \int_0^{r(t)} \frac{b(t+ys(t))\, c(t-ys(t))}{b(t)\, c(t)}\, dy$$
$$= \int_0^{\infty} (1)(e^{k\rho y})(e^{-\rho(1-\rho)y^2})\, dy.$$

Thus

$$\int_0^{t/2} \bar{F}(t-u)\, G(du) \sim \frac{\alpha}{2}\left(\int_0^{\infty} e^{k\rho y - \rho(1-\rho)y^2}\, dy\right) \frac{t}{\chi_2^{1/2}(t/2)} \bar{F}(t/2)\, \bar{G}(t/2).$$

Similarly,

$$\int_0^{t/2} \bar{G}(t-u)\, F(du) \sim \frac{\alpha}{2}\left(\int_0^{\infty} e^{-k\rho y - \rho(1-\rho)y^2}\, dy\right) \frac{t}{\chi_1^{1/2}(t/2)} \bar{F}(t/2)\, \bar{G}(t/2).$$

Finally, since $\chi_1 \sim \chi_2$

$$\overline{F * G}(t) \sim \int_0^{t/2} \bar{F}(t-u)\, G(du) + \int_0^{t/2} \bar{G}(t-u)\, F(du)$$
$$\sim \frac{\alpha}{2}\left(\int_0^{\infty} e^{k\rho y - \rho(1-\rho)y^2}\, dy + \int_0^{\infty} e^{-k\rho y - \rho(1-\rho)y^2}\, dy\right) \frac{t}{\chi_1^{1/2}(t/2)} \bar{F}(t/2)\, \bar{G}(t/2)$$
$$= \frac{\alpha}{2}\sqrt{\frac{\pi}{\rho(1-\rho)}}\, \frac{t}{\chi_1^{1/2}(t/2)} \bar{F}(t/2)\, \bar{G}(t/2).$$

Now to show that (3.17) holds, first note that (3.16) implies

$$\chi_1((1+\eta)t)-\chi_1(t)-\chi_2((1+\eta)t)+\chi_2(t)=\int_t^{(1+\eta)t}\left[\chi_2^{1/2}(v)\left(\frac{\chi_1'(v)}{\chi_2'(v)}-1\right)\right]\frac{\chi_2'(v)}{\chi_2^{1/2}(v)}\,dv$$
$$\leqq 2(k+\varepsilon)(\chi_2^{1/2}((1+\eta)t)-\chi_2^{1/2}(t)).$$

Using the alternate inequality and applying Lemma 5 to $\chi_2^{1/2}(t)$,

$$\lim_{t\to\infty,\eta\to 0}\frac{\chi_1((1+\eta)t)-\chi_1(t)-\chi_2((1+\eta)t)+\chi_2(t)}{\eta\chi_2^{1/2}(t)}$$
$$=2k\lim_{t\to\infty,\eta\to 0}\frac{\chi_2^{1/2}((1+\eta)t)-\chi_2^{1/2}(t)}{\eta\chi_2^{1/2}(t)}=k\rho.$$

Letting $\eta=\eta(t)=y\chi_2(t)^{-1/2}$, $s(t)=t\chi_2(t)^{-1/2}$, this is (3.17). Recalling the bounds in the proof of (3.6), we see we can take the bound k_1 to be

$$k_1=4|k|(2^{\rho/2}-1). \quad \#$$

4. Comparing Convolution Tails

In this section we return to the second objective stated in the introduction, namely, if $\bar{F}_1\sim b\bar{F}$ and $\bar{G}_1\sim c\bar{G}$ can we write $\overline{F_1*G_1}\sim k\overline{F*G}$ for some k? In fact the answer is no, in general, even if F and G are reasonably regular. As an example, suppose $F\in\mathcal{S}_\alpha$, $G\in\mathcal{S}_\alpha$, but

$$0<\varliminf_{t\to\infty}\frac{\bar{G}(t)}{\bar{F}(t)}<\varlimsup_{t\to\infty}\frac{\bar{G}(t)}{\bar{F}(t)}<\infty.$$

By Corollary 1, $\bar{H}(t)=\overline{F*G}(t)\sim m_G(\alpha)\bar{F}(t)+m_F(\alpha)\bar{G}(t)$. Now suppose also that $\bar{F}_1(t)\sim\bar{F}(t)$ and let $\bar{G}_1(t)=\bar{G}(t)$. Then $\bar{H}(t)=\overline{F_1*G}(t)\sim m_G(\alpha)\bar{F}(t)+m_{F_1}(\alpha)\bar{G}(t)$. Because $\bar{F}$ and $\bar{G}$ are not asymptotically equivalent to each other, it is clear that $\bar{H}_1$ and $\bar{H}$ will also not be, except in the special case $m_{F_1}(\alpha)=m_F(\alpha)$.

We can, however, provide three distinct situations in which the result does hold. It is of special interest that the value of $\lim\bar{H}_1/\bar{H}$ differs between the three cases.

Theorem 6. *Let F and G be distributions on $[0,\infty]$ such that $F\in\mathcal{L}_\alpha$. Let F_1 and G_1 be any distributions such that $\bar{F}_1\sim b\bar{F}$, $b>0$ and $\bar{G}_1\sim c\bar{G}$, $c\geqq 0$. Define $H=F*G$, $H_1=F_1*G_1$.*

i) *Suppose $F\in\mathcal{S}_\alpha$ and $\bar{G}\sim a\bar{F}$ for some $a\geqq 0$. Then*

$$\lim_{t\to\infty}\frac{\bar{H}_1(t)}{\bar{H}(t)}=\frac{bm_{G_1}(\alpha)+acm_{F_1}(\alpha)}{m_G(\alpha)+cm_F(\alpha)}.$$

ii) *Suppose* $m_G(\alpha)<\infty$ *and* $\bar{H}\sim m_G(\alpha)\bar{F}$. *Then*

$$\lim_{t\to\infty}\frac{\bar{H}_1(t)}{\bar{H}(t)}=\frac{b\,m_{G_1}(\alpha)}{m_G(\alpha)}.$$

iii) *Suppose* $G\in\mathscr{L}_\alpha$, $c>0$ *and* $m_F(\alpha)=m_G(\alpha)=\infty$. *Then*

$$\lim_{t\to\infty}\frac{\bar{H}_1(t)}{\bar{H}(t)}=bc.$$

Proof. i) This follows automatically from Theorem 1.

ii) We need only to show that $\bar{H}_1\sim b\,m_{G_1}(\alpha)\bar{F}$. We will first show that $\overline{F*G_1}\sim m_{G_1}(\alpha)\bar{F}$. Assume, without loss of generality, that $\bar{F}(0)=1$. For large enough s,

$$\begin{aligned}\int_s^t\bar{F}(t-u)\,G_1(du)+\bar{G}_1(t)&=\int_0^{t-s}\bar{G}_1(t-u)\,F(du)+\bar{G}_1(s)\,\bar{F}(t-s)\\&\leqq(c+\varepsilon)\left[\int_0^{t-s}\bar{G}(t-u)\,F(du)+\bar{G}(s)\,\bar{F}(t-s)\right]\\&=(c+\varepsilon)\left[\int_s^t\bar{F}(t-u)\,G(du)+\bar{G}(t)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\varlimsup_{t\to\infty}\left[\int_s^t\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G_1(du)+\frac{\bar{G}_1(t)}{\bar{F}(t)}\right]&\leqq(c+\varepsilon)\lim_{t\to\infty}\left[\int_s^t\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G(du)+\frac{\bar{G}(t)}{\bar{F}(t)}\right]\\&=(c+\varepsilon)\lim_{t\to\infty}\left[\frac{\bar{H}(t)}{\bar{F}(t)}-\int_0^s\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G(du)\right]\\&=(c+\varepsilon)\int_s^\infty e^{\alpha u}\,G(du).\end{aligned}$$

And this is arbitrarily small for s large enough.

On the other hand, by the uniform convergence,

$$\lim_{t\to\infty}\int_0^s\frac{\bar{F}(t-u)}{\bar{F}(t)}\,G_1(du)=\int_0^s e^{\alpha u}\,G_1(du)$$

and this is arbitrarily close to $m_{G_1}(\alpha)$, if s is large enough. Thus we must have $\overline{F*G_1}\sim m_{G_1}(\alpha)\bar{F}$. This allows us to use dominated convergence next, since we can choose $k>0$ such that $\bar{F}_1(t)\leqq k\bar{F}(t)$, all t. Therefore,

$$\lim_{t\to\infty}\frac{\bar{H}_1(t)}{\bar{F}(t)}=\int_0^\infty\lim_{t\to\infty}\frac{\bar{F}_1(t-u)}{\bar{F}(t)}\,G_1(du)=b\int_0^\infty e^{\alpha u}\,G_1(du)=b\,m_{G_1}(\alpha).$$

iii) Since $m_G(\alpha)=\infty$, then Fatou's lemma shows

$$\varliminf_{t\to\infty}\frac{\bar{H}(t)}{\bar{F}(t)}\geqq m_G(\alpha)=\infty.$$

As a consequence, for any $s>0$,

$$\overline{\lim_{t\to\infty}} \int_0^s \frac{\bar{F}(t-u)}{\bar{H}(t)} G(du) \leqq \overline{\lim_{t\to\infty}} \frac{\bar{F}(t-s)}{\bar{H}(t)} = e^{\alpha s} \overline{\lim_{t\to\infty}} \frac{\bar{F}(t)}{\bar{H}(t)} = 0.$$

Similarly, $\overline{\lim_{t\to\infty}} \int_0^s \frac{\bar{G}(t-u)}{\bar{H}(t)} F(du) = 0.$

Therefore, if $t \geqq 2s$,

$$\begin{aligned}\bar{H}(t) &= \int_0^s \bar{F}(t-u)\, G(du) + \int_0^s \bar{G}(t-u)\, F(du) + \int_s^{t-s} \bar{F}(t-u)\, G(du) + \bar{F}(s)\, \bar{G}(t-s)\\ &\sim \int_s^{t-s} \bar{F}(t-u)\, G(du) + \bar{F}(s)\, \bar{G}(t-s), \qquad \text{all } s>0. \end{aligned} \tag{4.5 a}$$

Likewise,

$$\bar{H}_1(t) \sim \int_s^{t-s} \bar{F}_1(t-u)\, G_1(du) + \bar{F}_1(s)\, \bar{G}_1(t-s), \qquad \text{all } s>0. \tag{4.5 b}$$

But for s large enough, $t \geqq 2s$,

$$\begin{aligned} &\int_s^{t-s} \bar{F}_1(t-u)\, G_1(du) + \bar{F}_1(s)\, \bar{G}_1(t-s)\\ &\quad \leqq (1+\varepsilon)\, b \left[\int_s^{t-s} \bar{F}(t-u)\, G_1(du) + \bar{F}(s)\, \bar{G}_1(t-s)\right]\\ &\quad = (1+\varepsilon)\, b \left[\int_s^{t-s} \bar{G}_1(t-u)\, F(du) + \bar{G}_1(s)\, \bar{F}(t-s)\right]\\ &\quad \leqq (1+\varepsilon)^2\, b c \left[\int_s^{t-s} \bar{G}(t-u)\, F(du) + \bar{G}(s)\, \bar{F}(t-s)\right]\\ &\quad = (1+\varepsilon)^2\, b c \left[\int_s^{t-s} \bar{F}(t-u)\, G(du) + \bar{F}(s)\, \bar{G}(t-s)\right]. \end{aligned} \tag{4.6}$$

Combining (4.6) with (4.5 a, b),

$$\overline{\lim_{t\to\infty}} \frac{\bar{H}_1(t)}{\bar{H}(t)} \leqq (1+\varepsilon)^2\, b c.$$

In the same manner of proof,

$$\underline{\lim_{t\to\infty}} \frac{\bar{H}_1(t)}{\bar{H}(t)} \geqq (1-\varepsilon)^2\, b c.$$

Since ε can be chosen arbitrarily, this proves the result. #

5. Product Tails and Domains of Attraction

Suppose X and Y are independent nonnegative random variables such that $P[X>t] \in \mathscr{R}_{-\alpha}$, $P[Y>t] \in \mathscr{R}_{-\alpha}$, $\alpha>0$. Define

$$\bar{F}(t) = P[X>e^t], \qquad \bar{G}(t) = P[Y>e^t] \quad \text{and} \quad \bar{H}(t) = \overline{F*G}(t) = P[XY>e^t].$$

Clearly, $F, G \in \mathscr{L}_\alpha$. From Embrechts and Goldie (1980, Theorem 3) we know that $H \in \mathscr{L}_\alpha$ and hence XY has distribution with a regularly varying tail. This results in a simple proposition, based on standard results.

Proposition I. *Suppose* (X_i, Y_i), $i = 1, 2, 3, \ldots$, *are distributed independently like* (X, Y) *above. Then there exists* a_n, b_n *such that*

$$\text{i)} \lim_{n \to \infty} P[\max_{i \leq n} (X_i Y_i) \leqq a_n t] = \exp(-t^{-\alpha}), \qquad t \geqq 0$$

and

$$\text{ii)} \lim_{n \to \infty} P\left[\sum_{i=1}^{n} (X_i Y_i) \leqq a_n t + n b_n\right] = S(t),$$

where S *is the fully asymmetric stable* (α) *law if* $\alpha < 2$ *and the normal law if* $\alpha \geqq 2$. #

The proposition itself is simple. The real problem is in calculating the norming and centering constants, a_n and b_n, if one does not already have an explicit expression for the distribution of XY. For example, a_n should be chosen so that

$$\frac{1}{n} \sim P[XY > a_n] = \bar{H}(\ln a_n), \qquad \text{as } n \to \infty. \tag{5.1}$$

Our previous work has shown that it is often possible to approximate $\bar{H}(t)$ with only the tails and truncated moment generating function of F and G (equivalently, the tails and truncated α-moments of X and Y). This eliminates calculation of a convolution. While we cannot promise an easy solution, it will generally be easier to perform the inversion required in (5.1) with the approximation than with the actual convolution. For example, if $\bar{H} \sim m_G(\alpha) \bar{F} + m_F(\alpha) \bar{G}$, one instead may solve

$$\frac{1}{n} \sim m_G(\alpha) \bar{F}(\ln a_n) + m_F(\alpha) \bar{G}(\ln a_n), \qquad n \to \infty.$$

Cline (1984) considers the joint convergence of $\left(\sum_{i=1}^{n} X_i Y_i, \sum_{i=1}^{n} X_i^2\right)$. When $\alpha < 2$, this requires that $\bar{H} \sim k\bar{F}$, where k is possibly infinite. Clearly, if k is finite, the correct normalization is $a_n \sim k^{1/\alpha} a_n'$, where a_n' is the normalization for $\sum_{i=1}^{n} X_i$ $\left(\text{i.e., } \frac{1}{n} \sim \bar{F}(\ln a_n')\right)$. As another example, which includes cases where k is infinite, suppose both F and G satisfy the conditions of Lemma 4. It is well known that if a_n' satisfies

$$\frac{1}{n} \sim \bar{F}(\ln a_n'),$$

then $a_n' \in \mathscr{R}_{1/\alpha}$. In fact, since $b(t) = e^{\alpha t} \bar{F}(t) \in \mathscr{R}_\beta$, we can further say

$$a_n' \sim [b(\ln a_n') n]^{1/\alpha} \sim [\alpha^{-\beta} b(\ln n) n]^{1/\alpha}.$$

Thus we have

Corollary 5. *Suppose* $b(t)=e^{\alpha t}\bar{F}(t)\in\mathscr{R}_\beta$ *and* $c(t)=e^{\alpha t}\bar{G}(t)\in\mathscr{R}_\gamma$. *Let* a_n *satisfy* (5.1). *Then*

i) if $m_F(\alpha)<\infty$ and $m_G(\alpha)<\infty$,

$$a_n\sim([\alpha^{-\beta}m_G(\alpha)\,b(\ln n)+\alpha^{-\gamma}m_F(\alpha)\,c(\ln n)]\,n)^{1/\alpha}.$$

ii) if $m_G(\alpha)<\infty$ and $m_F(\alpha)=\infty$, $\beta>\gamma$,

$$a_n\sim[\alpha^{-\beta}m_G(\alpha)\,b(\ln n)\,n]^{1/\alpha}.$$

iii) if $m_F(\alpha)=m_G(\alpha)=\infty$, $\beta>-1$, $\gamma>-1$,

$$a_n\sim[m(\ln n)\,b(\ln n)\,c(\ln n)\,n]^{1/\alpha},$$

where $m=\dfrac{\Gamma(1+\beta)\,\Gamma(1+\gamma)}{\Gamma(2+\beta+\gamma)}\,\alpha^{-\beta-\gamma}$.

Proof. These are direct applications of the remarks above and Theorem 4 iii), iv) and v). #

Another method for inverting distribution tails is given by the following.

Lemma 6. *Suppose* $\bar{F}(t)=e^{\alpha\chi(t)-\alpha t}$, *where* $\chi(t)\geqq 0$ *and satisfies* (3.6) *with* $\rho<1$. *Choose* $j>\dfrac{\rho}{1-\rho}$. *Define* $s_i(t)=t$, $s_i(t)=t+\chi(s_{i-1}(t))$, $i=2,\dots,j$. *If* a'_n *satisfies* $\dfrac{1}{n}\sim\bar{F}(\ln a'_n)$, *then*

$$a'_n\sim\exp\left[\chi\left(s_j\left(\frac{1}{\alpha}\ln n\right)\right)\right]n^{1/\alpha}.$$

Proof. Since $s-\chi(s)\to\infty$ there exists $s(t)$ such that $s(t)-\chi(s(t))=t$. Then

$$\lim_{t\to\infty}\frac{t}{s(t)}=\lim_{t\to\infty}\left(1-\frac{\chi(s(t))}{s(t)}\right)=1.$$

That is, $s(t)\sim t$ and $\chi(s(t))\sim\chi(t)$. It follows that for each $i\geqq 1$, $s_i(t)=t+\chi(s_{i-1}(t))\sim t$. Let $\eta_i(t)=\dfrac{s(t)-s_i(t)}{s_i(t)}$. Then by Lemma 5,

$$\begin{aligned}\chi(s(t))-\chi(s_i(t))&\sim\rho\,\eta_i(t)\,\chi(s_i(t))\\&\sim\rho\,\frac{\chi(t)}{t}(s(t)-s_i(t))\\&=\rho\,\frac{\chi(t)}{t}(\chi(s(t))-\chi(s_{i-1}(t))).\end{aligned}$$

Therefore, by induction,

$$\lim_{t\to\infty}[\chi(s(t))-\chi(s_j(t))]=\lim_{t\to\infty}\rho^j\frac{\chi^{j+1}(t)}{t^j}=0. \tag{5.2}$$

We now apply this to F. We can assume without loss of generality that

$$\frac{1}{n}=\bar{F}(\ln a_n')=e^{\alpha\chi(\ln a_n')}(a_n')^{-\alpha}.$$

Thus,

$$\ln a_n'=\frac{1}{\alpha}\ln n+\chi(\ln a_n')=s\left(\frac{1}{\alpha}\ln n\right).$$

Using (5.2) we obtain

$$a_n'=e^{\chi\left(s\left(\frac{1}{\alpha}\ln n\right)\right)}n^{1/\alpha}\sim e^{\chi\left(s_j\left(\frac{1}{\alpha}\ln n\right)\right)}n^{1/\alpha}. \quad \#$$

This lemma can be applied to H, for example, when F and G satisfy the conditions in Corollary 4. This ends our discussion on the norming constants a_n.

As for the centering constants b_n, we can take $b_n=0$ if $\alpha<1$ or $b_n=EXY$ if this is finite. The only real difficulty occurs when $\alpha=1$ and $EXY=\infty$. In this case one choice is

$$b_n=E[XY1_{XY\leqq a_n}]=\int_0^{\ln a_n} e^u H(du).$$

(Actually, if $EY<\infty$, $b_n=E[X1_{X\leqq a_n}]\,EY$ is satisfactory.)

However,

$$\int_0^t e^u H(du)=M_1 * M_2(t), \quad \text{where}$$

$$M_1(t)=\int_0^t e^u F(du)=E[X1_{X\leqq e^t}],$$

$$M_2(t)=\int_0^t e^u G(du)=E[Y1_{Y\leqq e^t}].$$

Since $\alpha=1$, $M_1(\ln t)$ and $M_2(\ln t)$ are slowly varying (in $\mathscr{R}_0$). As an example of how one might approximate $M_1 * M_2(t)$, we have the following lemma.

Lemma 7. *Suppose M_1 and M_2 are infinite measures on $[0,\infty)$.*

i) *$M_i(t)\in\mathscr{R}_{\beta_i}$, $\beta_i\geqq 0$, $i=1,2$, then*

$$M_1 * M_2(t)\sim\frac{\Gamma(1+\beta_1)\Gamma(1+\beta_2)}{\Gamma(1+\beta_1+\beta_2)}M_1(t)M_2(t).$$

ii) *If $M_i(t)=a_i(t)e^{\chi_i(t)}$ where $a_i(t)$ and $\chi_i(t)$ are as in Corollary 4, then*

$$M_2 * M_2(t)\sim\sqrt{\frac{\pi\rho}{1-\rho}}\,\chi_1^{1/2}(t/2)M_1(t/2)M_2(t/2).$$

Proof. The proofs of i) and ii) are similar to those of Lemma 4 and Corollary 4, respectively. #

Unfortunately, asymptotic equivalence is not a sufficient approximation. If $M_3(t)$ is to be an approximation for $M_1 * M_2(t)$ and we use the centering constants $b'_n = M_3(\ln a_n)$, these must satisfy

$$\lim_{n\to\infty} \frac{n}{a_n}(b_n - b'_n) \quad \text{exists finite.}$$

Equivalently,

$$\lim_{t\to\infty} \frac{M_1 * M_2(t) - M_3(t)}{e^t \bar{H}(t)} \quad \text{must exist finite.} \tag{5.3}$$

And this is a stronger condition than $M_1 * M_2(t) \sim M_3(t)$. We leave open the question of choosing M_3 so that (5.3) holds.

In Proposition I ii) we saw that the distribution of XY is in a stable domain of attraction when X and Y are nonnegative, independent and each in a stable domain of attraction. Unfortunately, this result does not generalize when the word "nonnegative" is removed. The problem is that the "balancing" condition does not necessarily hold. However, Theorem 6 allows us to list a number of conditions for which XY is in a stable domain of attraction.

Assume that X and Y are independent and satisfy

$$P[|X|>t] \in \mathscr{R}_{-\alpha}, \quad p_1 = \lim_{t\to\infty} \frac{P[X>t]}{P[|X|>t]} \quad \text{exists}$$

and

$$P[|Y|>t] \in \mathscr{R}_{-\alpha}, \quad p_2 = \lim_{t\to\infty} \frac{P[Y>t]}{P[|Y|>t]} \quad \text{exists, } 0<\alpha\leqq 2.$$

For $\alpha<2$, these conditions are necessary and sufficient for the distributions of X and Y to be in stable domains of attraction (Feller, 1971, p. 577). Define $X_+ = \max(0, X)$, $X_- = \max(0, -X)$ and similarly for Y_+, Y_-.

Proposition II. *Under each of the following conditions* $q = \lim_{t\to\infty} \frac{P[XY>t]}{P[|XY|>t]}$ *exists and hence the distribution of* XY *is in a stable* (α) *domain of attraction.*

i) $P[X>t] = p_1 P[|X|>t]$ *and either* $p_1 = 1/2$ *or* $P[Y>t] = p_2 P[|Y|>t]$. *In this case* $q = p_1 p_2 + (1-p_1)(1-p_2)$.

ii) $P[|X|>e^t]$ *and* $P[|Y|>e^t]$ *are both in* $\mathscr{S}_\alpha$ *and* $\lim_{t\to\infty} \frac{P[|Y|>t]}{P[|X|>t]} = k < \infty$. *In this case*

$$q = \frac{(p_1 EY_+^\alpha + (1-p_1) EY_-^\alpha) + k(p_2 EX_+^\alpha + (1-p_2) EX_-^\alpha)}{E|Y|^\alpha + kE|X|^\alpha}$$

iii) *Either* $\frac{P[|XY|>t]}{P[|X|>t]} \to E|Y|^\alpha < \infty$ *or* $\frac{P[XY>t]}{P[X>t]} \to EY_+^\alpha < \infty$, $p_1 > 0$. *In this case*

$$q = \frac{p_1 EY_+^\alpha + (1-p_1) EY_-^\alpha}{E|Y|^\alpha}.$$

iv) $E|X|^\alpha = E|Y|^\alpha = \infty$. *In this case* $q = p_1 p_2 + (1-p_1)(1-p_2)$.

Proof. Define $\bar{F}_+(t)=P[X>e^t]$, $\bar{F}_-(t)=P[X<-e^t]$, $\bar{F}(t)=P[|X|>e^t]$ and similarly for $\bar{G}_+$, $\bar{G}_-$, $\bar{G}$ in terms of Y's distribution. Then

$$\bar{H}_+(t)=P[XY>e^t]=\overline{F_+*G_+}(t)+\overline{F_-*G_-}(t)$$
$$\bar{H}_-(t)=P[XY<-e^t]=\overline{F_+*G_-}(t)+\overline{F_-*G_+}(t),$$

and $\bar{H}(t)=P[|XY|>e^t]=\bar{H}_+(t)+\bar{H}_-(t)$.

The proof of i) follows by direct calculation and ii), iii) and iv) are applications of Theorem 6 i), ii) and iii), respectively. #

If X and Y are in stable domains of attraction with different indices, then Lemma 1 can be applied and the product will be as in iii) above, with α equal to the smaller index (Breiman, 1965). When X and Y are independent and each in the domain of attraction of the normal distribution, then XY will always be in the domain of attraction of the normal (Maller, 1981).

The norming constants, a_n, that would be used as a result of Proposition II can be calculated from $\bar{H}(t)=P[|XY|>e^t]$, as suggested in the remarks following Proposition I. Again the primary difficulty in choosing the centering constants b_n occurs only when $\alpha=1$ and $E|X|=E|Y|=\infty$.

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References

Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. To be published by Cambridge University Press 1983

Breiman, L.: On some limit theorems similar to the arc-sine law. Theor. Probab. Appl. **10**, 323–331 (1965)

Chistykov, V.P.: A theorem on the sums of independent positive random variables and its applications to branching random processes. Theor. Probab. Appl. **9**, 640–648 (1964)

Chover, J., Ney, P., Wainger, S.: Functions of probability measures. J. Analyse Math. **26**, 255–302 (1973)

Cline, D.B.H.: Least squares regression with infinite variance. Tech. Report, Texas A & M University (1986)

Cline, D.B.H.: Convolutions of Distributions with Exponential and Subexponential Tails. Tech. Report Texas A & M University (1985)

Davis, R.A., Resnick, S.I.: Limit theory for the sample covariance and correlation functions of moving averages. Tech. Report 68, Dept. Stat., Univ. of North Carolina (1984)

Davis, R.A., Resnick, S.I.: More limit theory for the sample correlation function of moving averages. Tech. Report, Colorado St. Univ. (1984)

Embrechts, P., Goldie, C.M.: On closure and factorization properties of subexponential distributions. J. Aust. Math. Soc. (Ser. A) **29**, 243–256 (1980)

Embrechts, P., Goldie, C.M.: On convolution tails. Stochastic Processes Appl. **13**, 263–278 (1982)

Feller, W.: One sided analogues of Karamata's regular variation. Enseign. Math. **15**, 107–121 (1969)

Feller, W.: An introduction to probability theory and its applications, Vol. II, New York: Wiley 1971 de Haan, L.: On regular variation and its application to the weak convergence of sample extremes. Math. Cent. Tracts **32**, Mathematisch Centrum, Amsterdam (1970)

Maller, R.A.: A theorem on products of random variables, with application to regression. Aust. J. Stat. **23**, 177–185 (1981)

Pitman, E.J.G.: Subexponential distribution functions. J. Aust. Math. Soc. (Ser. A) **29**, 337–347 (1980)

Teugels, J.L.: The class of subexponential distributions. Ann. Probab. **3**, 1000–1011 (1975)

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