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Probability Theory and Related Fields

On Uniform Distribution of Subsequences

V. Losert[1] and R.F. Tichy[2]

[1] Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

[2] Institut für Analysis, Technische Mathematik und Versicherungsmathematik der TU Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria

1. Introduction and Results

In several papers [2, 8 and 10] G.M. Petersen and other authors have studied the summability of subsequences of a given sequence. These results on the summation of subsequences can be applied in the theory of uniform distribution (see [7, 9 and 10]) and one obtains that a sequence $\omega=(x_n)$ of real numbers is uniformly distributed modulo 1 if and only if almost all subsequences of ω are uniformly distributed. This result remains true for uniform distribution with respect to weighted means satisfying certain regularity conditions. The original proofs in [2, 8 and 10] are based on summability properties of the Rademacher functions.

In the present paper we consider in a more general situation r-dimensional sequences $\omega=(x_{n_1\dots n_r})$ (so called multi-sequences) and prove the above metric result by methods from probability theory. Furthermore it is shown that for certain weighted means P (not fulfilling the cited regularity conditions) there exists a sequence ω that is uniformly distributed with respect to P but almost no subsequences are uniformly distributed with respect to P. In the last part of the paper a sequence is constructed that is not uniformly distributed with respect to P but almost all subsequences are uniformly distributed with respect to P.

Notations. Let $r\in\mathbb{N}=\{1,2,3,\dots\}$ and $\mathbf{n}=(n_1,\dots,n_r)$, $\mathbf{N}=(N_1,\dots,N_r)$ etc. be vectors with r components. During the whole paper $\omega=(x_{\mathbf{n}})$ denotes an r-dimensional sequence with elements $x_{\mathbf{n}}$ in a compact Hausdorff space X with countable base of topology and μ a regular normed Borel measure on X that is not concentrated on a single point of X.

Furthermore $P=(p_1(n_1),\dots,p_r(n_r))$ denotes a positive r-dimensional weighted mean and $p(\mathbf{n})=\prod_{j=1}^{r} p_j(n_j)$, $P_j(n_j)=\sum_{k\leq n_j} p_j(k)$, $P(\mathbf{n})=\prod_{j=1}^{r} P_j(n_j)$.

In the following we introduce two concepts of uniform distribution of multi-dimensional sequences:

The multi-sequence $\omega = (x_{\mathbf{n}})$ is called P-uniformly distributed with respect to μ (in short (P, μ)-u.d., compare [6, 9 and 12]), if

$$\lim_{\mathbf{N} \to \infty} P(\mathbf{N})^{-1} \sum_{\mathbf{n} \leqq \mathbf{N}} p(\mathbf{n}) f(x_{\mathbf{n}}) = \int_X f(x)\, d\mu \tag{1.1.I}$$

for all continuous real-valued functions f on X. $\mathbf{n} \leqq \mathbf{N}$ denotes the usual product order and $\lim_{\mathbf{N} \to \infty} a(\mathbf{N}) = a$ means that for every $\varepsilon > 0$ there exists $\mathbf{N}(\varepsilon)$ such that for all $\mathbf{N} \geqq \mathbf{N}(\varepsilon)\, |a(\mathbf{N}) - a| < \varepsilon$.

The r-dimensional sequence $\omega = (x_{\mathbf{n}})$ is called P-weakly-uniformly distributed with respect to μ (in short (P, μ)-w.u.d.), iff

$$\lim_{N \to \infty} P(N, \ldots, N)^{-1} \sum_{\mathbf{n} \leqq (N, \ldots, N)} p(\mathbf{n}) f(x_{\mathbf{n}}) = \int_X f(x)\, d\mu \tag{1.1.II}$$

for all continuous real-valued functions f on X.

Obviously every (P, μ)-u.d. multi-sequence is also (P, μ)-w.u.d.; in the case $r = 1$ both concepts are equivalent.

If $\mathbf{t} = (t_1, \ldots, t_r) \in T = (\{0, 1\}^{\mathbb{N}})^r$, we denote by $\tau(\omega, \mathbf{t})$ the r-dimensional subsequence obtained from $\omega = (x_{\mathbf{n}})$ by deleting all $x_{\mathbf{k}}$ for which $t_j(k_j) = 0$ for some j, i.e. we consider only the indices belonging to the subset of $\mathbb{N}^r$ whose characteristic function equals $\prod_{j=1}^{r} t_j$. More explicitely, let $\tau(n, t_j)$ be the n-th index k for which $t_j(k) = 1$ (i.e. the n-th index of the subsequence determined by t_j), and put $\tau(\mathbf{n}, \mathbf{t}) = (\tau(n_1, t_1), \ldots, \tau(n_r, t_r))$, then $\tau(\omega, \mathbf{t}) = (x_{\tau(\mathbf{n}, \mathbf{t})})_{\mathbf{n} \in \mathbb{N}^r}$. We equip T with the product measure λ whose components assign probability $\frac{1}{2}$ to zero and one. In this way it is possible to speak of the measure of a set of subsequences, in particular the notions almost all and almost no subsequences refer always to λ. One can also identify (up to a set of measure zero) T with $[0, 1]^r$ equipped with Lebesgue measure by assigning to $t'_j \in [0, 1]$ the point $(R_n(t'_j))_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ $(n = 1, \ldots, r)$, where (R_n) denotes the system of Rademacher functions (this approach was used in [2 and 8]).

In §2 of this paper some auxiliary results are formulated and in §3 the following results are established:

Theorem I. *Let $\omega = (x_{\mathbf{n}})$ be an r-dimensional sequence and $P = (p_1(n_1), \ldots, p_r(n_r))$ a weighted mean such that*

(i) $p_j(k+1) \geqq p_j(k) > 0$ *for all* $k = 1, 2, 3, \ldots$ *and* $j = 1, \ldots, r$

(ii) $\sum_{k=1}^{\infty} \left(\frac{p_j(k)}{P_j(k)} \right)^2$ *converges for* $j = 1, \ldots, r$

(iii) $\frac{p_j(k)\, P_j(l)}{p_j(l)\, P_j(k)} < L$ *for all* $k = 1, 2, 3, \ldots$ *and* $l = 1, 2, \ldots, 3k$, $j = 1, \ldots, r$.

If ω is (P, μ)-w.u.d. then almost all subsequences of ω are (P, μ)-w.u.d.

Theorem II. *Let $\omega = (x_{\mathbf{n}})$ be an r-dimensional sequence and $P = (p_1(n_1), \ldots, p_r(n_r))$ a weighted mean such that*

(i) $p_j(k+1) \geqq p_j(k) > 0$ *for all* $k = 1, 2, \ldots$ *and* $j = 1, \ldots, r$

(ii) *$p_j(k)/P_j(k)$ decreases to* 0 *with* $k\to\infty$ $(j=1,\dots,r)$

(iii) $\dfrac{P_j(n)\,p_j(k)}{p_j(n)\,P_j(k)}<L$ *for* $\frac{1}{3}n\leqq k\leqq 3n$ $(j=1,\dots,r)$.

If the set of all (P,μ)-w.u.d. subsequences of ω has positive measure then ω is (P,μ)-w.u.d.

Remark. The statement of Theorem II remains true for (P,μ)-u.d. multi-sequences (instead of (P,μ)-w.u.d. multi-sequences). Furthermore it is shown that an analogon of Theorem I for (P,μ)-u.d. multi-sequences does not hold. In §3 we establish the following converse result:

Theorem III. *Let X be the discrete space $X=\{-1,0,1\}$ and μ the probability measure on X defined by $\mu(0)=0$ and $\mu(-1)=\mu(1)=1/2$; $P=(1,1)$ denotes the two-dimensional arithmetic mean. Then there exists a (P,μ)-u.d. double sequence $\omega=(x_{nk})$ such that almost no subsequences of ω are (P,μ)-u.d.*

By an explicit construction similar to that of Baayen and Hedrlin [1] one can show that the following condition is necessary and sufficient for the existence of (P,μ)-u.d. sequences on X:

$$\lim_{n\to\infty}\frac{p_j(n)}{P_j(n)}=0 \qquad \text{for some } j. \tag{1.2}$$

Nevertheless, it is possible that the family of all (P,μ)-u.d. sequences is small from a measure theoretic point of view. As usual, one considers the product measure $\mu_\infty=\bigotimes_{\mathbf{j}\in\mathbb{N}^r}\mu_{\mathbf{j}}$ on $X^{\mathbb{N}^r}$ (where $\mu_{\mathbf{j}}$ is a copy of μ for all $\mathbf{j}\in\mathbb{N}^r$; cf. [9], p. 182). In the one-dimensional case, it was shown in [6] Satz 11 that Hill's condition (compare [5]) is sufficient in order that μ_∞-almost all sequences are (P,μ)-u.d. If $p(n+1)\geqq p(n)$ for each n, it has recently been shown by the authors that this is also necessary. In the r-dimensional case put $A_j(n)=P_j(n)^{-2}\sum_{k=1}^{n}p_j(k)^2$ (for $j=1,\dots,r$). Then the same proof as in [6], Satz 11 shows that μ_∞-almost all r-dimensional sequences are (P,μ)-u.d. if

$$\sum_{\mathbf{n}\geqq\mathbf{N}}\exp\left(-\delta\prod_{j=1}^{r}A_j(n_j)^{-1}\right)<\infty \qquad \text{for some } \mathbf{N}\in\mathbb{N}^r$$

holds for each $\delta>0$. This can be shown to be true iff there exists $j_0\in\{1,\dots,r\}$ such that

$$\sum_{n=1}^{\infty}\exp\left(-\frac{\delta}{A_j(n)}\right)<\infty \quad \text{and} \quad \begin{array}{l}\text{for some } \delta>0 \text{ if } j\neq j_0\\ \text{for all } \delta>0 \text{ if } j=j_0\end{array}. \tag{1.3}$$

(We will call this Hill's condition too.)

In the last two theorems we consider weights that are constant on intervals of the form $[2^{n-1},2^n]$. More explicitly we assume that

$$\begin{aligned} p_j(k)&=a_j(n) && \text{for } 2^{n-1}\leqq k<2^n,\ n\in\mathbb{N},\ j=1,\dots,r.\\ a_j(n+1)&\geqq a_j(n) && \text{for all } n\in\mathbb{N},\ j=1,\dots,r.\\ \lim_{n\to\infty}\frac{a_j(n+1)}{a_j(n)2^n}&=0 && \text{for some } j. \end{aligned} \tag{1.4}$$

It is not hard to see that the last condition corresponds to (1.2). Hill's condition (1.3) gets the form

$$\sum_{n=1}^{\infty} \exp\left(-\delta \frac{2^n a_j(n)}{a_j(n+1)}\right) < \infty.$$

Theorem IV. *Let P be a weighted mean of the type* (1.4). *If $2^{-\frac{n}{2}} a_j(n)^{-1} a_j(n+1)$ is unbounded for some j, then there exists a (P,μ)-u.d. r-dimensional sequence $\omega=(x_n)$ on X such that almost no subsequences of ω are (P,μ)-u.d. Consequently, if P is of type* (1.4) *and the conclusions of Theorem I hold, then*

$$\sum_{n=1}^{\infty} \left(\frac{p_j(n)}{P_j(n)}\right)^2 < \infty \quad \textit{for each } j.$$

For example, if $a_j(n)=2^{n^{2/3}}$, Hill's condition is satisfied, but there exist (P,μ)-u.d. sequences such that almost no subsequences are (P,μ)-u.d.

Theorem V. *There exists a weighted mean P of the type* (1.4) *satisfying Hill's condition* (1.3) *and an r-dimensional sequence ω such that ω is not (P,μ)-u.d., but almost all of its subsequences $\tau(\omega,\mathbf{t})$ are (P,μ)-u.d.*

§2. Auxiliary Results

In this chapter we present some auxiliary results from the theory of uniform distribution, from the theory of summation methods and from probability theory, most of them without proofs because they are well-known or simple consequences of well-known facts.

Proposition 2.1. *There exists a countable class $\mathfrak{H}=\{f_0,f_1,f_2,\dots\}$ of continuous real-valued functions f_j on X, $f_0\equiv 1$, $|f_j|\leqq 1$ for $j\geqq 1$ and $\int_X f_j\,d\mu=0$ $(j\geqq 1)$ such that* (1.1.J) *holds for all continuous functions if it holds for all $f_j\in\mathfrak{H}$* (J = I, II).

The proof runs as in [9], p. 175.

It is easy to see that the random variable $\tau(1,.)$ has a geometric distribution, i.e. $\lambda\{t:\tau(1,t)=k\}=2^{-k}$ (see e.g. [3, Vol. I], p. 47). The increments $\tau(n+1,.)-\tau(n,.)$ are mutually independent and have the same distribution as $\tau(1,.)$.

Let F be the normal distribution, i.e. $F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt.$

Proposition 2.2. *We have uniformly in $x\in\mathbb{R}$:*

$$\lim_{n\to\infty} \lambda\{t:\tau(n,t)<2n+x\sqrt{2n}\}=F(x).$$

In particular: $\lim_{n\to\infty} \frac{\tau(n,t)}{n}=2$ *almost everywhere.*

Proof. An easy computation shows that $\tau(1,.)$ has expectation $E(\tau(1,.))=2$ and variance $\operatorname{Var}(\tau(1,.))=2$. Therefore, the first statement follows from the central limit theorem and the second from the law of large numbers.

The following result gives sufficient conditions for the regularity of an r-dimensional Riesz summation method (compare to [4], p. 58):

Proposition 2.3. *Let* $P=(p_1(n_1),\dots,p_r(n_r))$, $Q=(q_1(n_1),\dots,q_r(n_r))$ *be two r-dimensional weighted means such that*

(i) $p_j(n+1)\geqq p_j(n)>0$, $q_j(n+1)\geqq q_j(n)>0$ $(j=1,\dots,r$ *and* $n\in\mathbb{N})$

(ii) $\dfrac{q_j(k)\,P_j(k)}{p_j(k)\,Q_j(k)}<H$ $(j=1,\dots,r$ *and* $k\in\mathbb{N})$

(iii) $\sum_{k=1}^{m}\left|\dfrac{q_j(k)}{p_j(k)}-\dfrac{q_j(k+1)}{p_j(k+1)}\right|\dfrac{P_j(k)}{Q_j(m)}<L$ $(j=1,\dots,r$ *and* $m\in\mathbb{N})$.

If the r-dimensional sequence $s(\mathbf{n})$ *is bounded and convergent to the limit* σ *with respect to the mean P then it is convergent to* σ *with respect to Q, too.*

Remark. Proposition 2.3 is valid for the weak notion of convergence (as defined in (1.1.II)) and for the (stronger) notion of convergence (as defined in (1.1.I)). We omit the proof since it is standard.

Proposition 2.4. *If* $\omega=(x_n)$ *is a (one-dimensional) sequence in X, then the set of all t for which* $\tau(\omega,t)$ *is* (P,μ)*-u.d. has either measure one or zero.*

Proof. The property of being uniformly distributed is clearly a tail event on $\tau(n,.)$. Therefore the result follows from the Hewitt-Savage zero-one law (see [3, Vol. II], p. 122; cf. he remarks preceeding Proposition 2.2).

In the following we give a short proof of a multi-dimensional strong law of large numbers; in the case of independent random variables see Smythe [11]. A system S of random variables is said to be multiplicative, iff

$$E(\xi_1^{g_1}\xi_2^{g_2}\xi_3^{g_3}\xi_4^{g_4})=E(\xi_1^{g_1})\,E(\xi_2^{g_2})\,E(\xi_3^{g_3})\,E(\xi_4^{g_4})$$
$$\text{for all } \xi_j\in S(\xi_i\neq\xi_j \text{ for } i\neq j) \quad\text{and}\quad g_j\in\{0,1,2\},$$

where $E(\xi)$ denotes the expectation of the random variable ξ.

Proposition 2.5. *Let* $P=(p_1(n_1),\dots,p_r(n_r))$ *be an r-dimensional weighted mean satisfying* $\lim_{k\to\infty}P_j(k)=\infty$, $0<p_j(k)\leqq p_j(k+1)$ $(k\in\mathbb{N})$ *and*

$$\sum_{k=1}^{\infty}\left(\frac{p_j(k)}{P_j(k)}\right)^2<\infty \qquad \textit{for } j=1,\dots,r.$$

Furthermore let $a(\mathbf{k},n)$ $(\mathbf{k}\in\mathbb{N}^r,\ n\in\mathbb{N})$ *be an* $(r+1)$*-dimensional sequence of real numbers bounded by* 1 *and let* $\xi_{\mathbf{k}}$ $(\mathbf{k}\in\mathbb{N}^r)$ *be an multiplicative system of random variables bounded by* 1 *and with expectations* $E(\xi_{\mathbf{k}})=0$ *for all* $\mathbf{k}\in\mathbb{N}^r$.

Then

$$\lim_{n\to\infty}\frac{1}{P(n,\dots,n)}\sum_{\mathbf{k}\leqq(n,\dots,n)}p(\mathbf{k})\,a(\mathbf{k},n)\,\xi_{\mathbf{k}}=0$$

with probability 1.

Proof. For $n=(n,\dots,n)$ and

$$X_n=P(\mathbf{n})^{-1}\sum_{\mathbf{k}\leqq\mathbf{n}}p(\mathbf{k})\,a(\mathbf{k},\mathbf{n})\,\xi_{\mathbf{k}}$$

we have

$$E(X_n^4) \leqq 3 \sum_{\mathbf{n}\in\mathbb{N}^r} P(\mathbf{n})^{-4} \Big(\sum_{\mathbf{k}\leqq\mathbf{n}} p(\mathbf{k})^2\Big)^2$$
$$\leqq 3 \sum_{n\in\mathbb{N}^r} \frac{(p(\mathbf{n})\,P(\mathbf{n}))^2}{P(\mathbf{n})^4} = 3 \sum_{n\in\mathbb{N}^r} \left(\frac{p(\mathbf{n})}{P(\mathbf{n})}\right)^2 < \infty,$$

since $\sum_{\mathbf{k}\leqq\mathbf{n}} p(\mathbf{k})^2 \leqq p(\mathbf{n}) P(\mathbf{n})$ for increasing weights p_j. By the multi-dimensional version of Beppo Levi's theorem the proof of Proposition 2.5 is complete.

§3. Proofs of Theorems I, II and III

Recall that $T = (\{0,1\}^{\mathbb{N}})^r$. We identify $\mathbf{t} = (t_1, \ldots, t_r) \in T$ with the function $\prod_{j=1}^{r} t_j \colon \mathbb{N}^r \to \{0,1\}$.

Proposition 3.1. *Let $\omega = (b_\mathbf{n})$ be an r-dimensional sequence bounded by 1 and $P = (p_1(n_1), \ldots, p_r(n_r))$ a weighted mean fulfilling the conditions (i), (ii), (iii) of Theorem I. If*

$$\lim_{N\to\infty} \frac{1}{P(N, \ldots, N)} \sum_{\mathbf{n}\leqq(N,\ldots,N)} p(\mathbf{n})\, b_\mathbf{n} = 0$$

then for almost all subsequences $(b_{\tau(\mathbf{n},\mathbf{t})})$ of ω

$$\lim_{N\to\infty} \frac{1}{P(N, \ldots, N)} \sum_{\mathbf{n}\leqq(N,\ldots,N)} p(\mathbf{n})\, b_{\tau(\mathbf{n},\mathbf{t})} = 0.$$

Proof. Put $t_j(n_j) - \frac{1}{2} = X_j(n_j)$. Then

$$\mathbf{t}(\mathbf{n}) = \prod_{j=1}^{r} (X_j(n_j) + \tfrac{1}{2}) = \sum_{\emptyset \neq M \subseteqq \{1,\ldots,r\}} 2^{|M|-r} X_M(\mathbf{n}_M) + 2^{-r}, \qquad (*)$$

where the sum runs through all non-empty subsets $M = \{j_1, \ldots, j_m\}$ with cardinality $m = |M|$; $X_M(\mathbf{n}_M)$ denotes the random variable

$$X_{j_1}(n_{j_1}) \cdot \ldots \cdot X_{j_m}(n_{j_m}) \quad \text{and} \quad \mathbf{n}_M = (n_{j_1}, \ldots, n_{j_m}).$$

Similarly to (*) we split up

$$P(\mathbf{N})^{-1} \sum_{\mathbf{n}\leqq\mathbf{N}} p(\mathbf{n})\, \mathbf{t}(\mathbf{n})\, b_\mathbf{n} \quad (\mathbf{N} = (N, \ldots, N))$$

into 2^r terms. Applying Proposition 2.5 to $2^r - 1$ of these terms and using the hypothesis yields

$$\lim_{N\to\infty} P(\mathbf{N})^{-1} \sum_{\mathbf{n}\leqq\mathbf{N}} p(\mathbf{n})\, \mathbf{t}(\mathbf{n})\, b_\mathbf{n} = 0$$

almost everywhere. Similarly we get

$$\lim_{N\to\infty} P(\mathbf{N})^{-1} \sum_{\mathbf{n}\leqq\mathbf{N}} p(\mathbf{n})\, \mathbf{t}(\mathbf{n}) = \frac{1}{2^r},$$

hence

$$\lim_{N\to\infty} \frac{\sum_{\mathbf{n}\leqq\mathbf{N}} p(\mathbf{n})\,\mathbf{t}(\mathbf{n})\, b_{\mathbf{n}}}{\sum_{\mathbf{n}\leqq\mathbf{N}} p(\mathbf{n})\,\mathbf{t}(\mathbf{n})}=0$$

for almost all $\mathbf{t}\in T$; consequently

$$\lim_{N\to\infty} \frac{1}{\sum_{\mathbf{n}\leqq\mathbf{N}} p(\tau(\mathbf{n},\mathbf{t}))} \sum_{\mathbf{n}\leqq\mathbf{N}} p(\tau(\mathbf{n},\mathbf{t}))\, b_{\tau(\mathbf{n},\mathbf{t})}=0$$

for almost all $\mathbf{t}\in T$. By [8], p. 36 and p. 39 and Proposition 2.2 the conditions (i), (ii) and (iii) of Proposition 2.3 are fulfilled for almost all $(p_1(\tau_1(n_1,t_1)), \ldots, p_r(\tau_r(n_r,t_r)))$. Therefore almost all $(b_{\tau(\mathbf{n},\mathbf{t})})$ are P-summable to the limit 0 by Proposition 2.3 and 3.2 is proved.

Proof of Theorem I. By Proposition 2.1 a countable system of functions $\mathfrak{H}=\{f_0, f_1, f_2, \ldots\}$ with $|f_j|\leqq 1$, $f_0\equiv 1$ and $\int_X f_j(x)\,d\mu=0$ for $j\geqq 1$ exists on the space X with countable base. If $\omega=(x_{\mathbf{n}})$ is (P,μ)-u.d. then $f_j(x_{\mathbf{n}})$ is P-summable to the limit 0. Let E_j be the family of all subsequences $(f_j(x_{\tau(\mathbf{n},t)})$ that are P-summable to 0. $\lambda(E_j)=1$ by Prop. 3.1, therefore $\lambda\left(\bigcap_{j=1}^{\infty} E_j\right)=1$. So almost all subsequences of ω are (P,μ)-w.u.d.

The following Proposition immediately yields Theorem II:

Proposition 3.2. *If $P=(p_1(n_1),\ldots,p_r(n_r))$ is a weighted mean fulfilling the conditions (i), (ii) and (iii) of Theorem II, then a bounded r-dimensional sequence $\omega=(b_{\mathbf{n}})$ that is not P-summable has almost no subsequences that are P-summable.*

The proof of 3.2 is a consequence of the proof of Theorem 5 in [8].

Proof of Theorem III. Let $R_n(t)$ denote the n-th Rademacher-function; we define a double-sequence $\omega=(x_{nk})$ by

$$x_{nk}=R_n(t_{mj}) \quad \text{if } 9^m<n\leqq 9^{m+1}, 9^m<k\leqq 9^{2^{9^{m+1}}+m} \; (m=0,1,2,\ldots)$$

$$\text{and} \quad \max_{9^m<N\leqq 9^{m+1}} \left|\sum_{n=9^m+1}^{N} R_n(t_{mj})\right|\leqq N^{2/3}$$

$$\text{with} \quad t_{mj}=(2j+1)\,2^{-9^{m+1}-1}$$

$$(\text{for } 9^{m+j}<k\leqq 9^{m+j+1}, 0\leqq j<2^{9^{m+1}});$$

$$x_{nk}=0 \quad \text{otherwise.}$$

For $9^m<N\leqq 9^{m+1}$ we obtain

$$\left|\sum_{n=1}^{N} x_{nk}\right| \ll 9^{2/3\cdot 1}+\ldots+9^{2/3\cdot m}+N^{2/3}\leqq N^{2/3}(9+1)=10N^{2/3}.$$

Hence ω is (1,1)-u.d. on $X=\{-1,0,1\}$ with respect to the measure μ defined by $\mu(0)=0$, $\mu(1)=\mu(-1)=1/2$. In the following we show that $(x_{nk}t_1(n)t_2(k))$ is not u.d. for almost all $(t_1,t_2)\in T=(\{0,1\}^{\mathbb{N}})^2$.

We write $t_1(n)=\frac{1+R_n(t)}{2}$ with $t\in[0,1[$ (t irrational); the strong law of large numbers yields for almost all $t\in[0,1[$ and almost all $t_2\in\{0,1\}^{\mathbb{N}}$:

$$\left|\frac{1}{N^{2/3}}\sum_{n=1}^{N} R_n(t)\right|<\varepsilon,\qquad \left|\frac{1}{N^{2/3}}\sum_{n=1}^{N} t_2(k)-\frac{1}{2}\right|<\varepsilon \tag{$*$}$$

for all $N=N_0=N_0(\varepsilon)$; we can take $\varepsilon=1/217$. In the following estimates we consider a fixed $m\geqq N_0$ and $t\in I_j$, where I_j denotes the uniquely determined interval

$$[j\cdot 2^{-9^{m+1}},(j+1)2^{-9^{m+1}}[\qquad (j=0,\dots,2^{9^{m+1}}-1)$$

such that $t_{mj}\in I_j$. The following estimates are valid for almost all $t\in I_j$ and almost all $t_2\in\{0,1\}^{\mathbb{N}}$. Since R_n is constant on I_j for $n\leqq 9^{m+1}$ we have

$$\sum_{9^m<n\leqq 9^{m+1}} R_n(t)x_{nk}=8\cdot 9^m \qquad \text{for } 9^{m+j}<k\leqq 9^{m+j+1}.$$

By ($*$) we obtain for almost all $(t_1,t_2)\in T$

$$\Big|\sum_{9^m<n\leqq 9^{m+1}} t_1(n)-4\cdot 9^m\Big|<\varepsilon\cdot 10\cdot 9^m,$$

$$\Big|\sum_{9^{m+j}<k\leqq 9^{m+j+1}} t_2(k)-4\cdot 9^{m+j}\Big|<\varepsilon\cdot 10\cdot 9^{m+j}.$$

Hence

$$\sum_{\substack{9^m<n\leqq 9^{m+1}\\ 9^{m+j}<k\leqq 9^{m+j+1}}} x_{nk}R_n(t)\,t_2(k)=8\cdot 9^m\sum_{9^{m+j}<k\leqq 9^{m+j+1}} t_2(k)$$
$$\geqq 8\cdot 9^m(4-10\varepsilon)\,9^{m+j}, \tag{$**$}$$

and

$$\sum_{\substack{9^m<n\leqq 9^{m+1}\\ 9^{m+j}<k\leqq 9^{m+j+1}}} x_{nk}\,t_2(k)=\sum_{\substack{9^m<n\leqq 9^{m+1}\\ 9^{m+j}<k\leqq 9^{m+j+1}}} R_n(t)\,t_2(k)=\sum(2t_1(n)-1)\,t_2(k)$$
$$\geqq -\varepsilon 20\cdot 9^m(4+10\varepsilon)\,9^{m+j}. \tag{$***$}$$

Combining ($**$) and ($***$) we obtain

$$\sum_{\substack{9^m<n\leqq 9^{m+1}\\ 9^{m+j}<k\leqq 9^{m+j+1}}} x_{nk}\,t_1(n)\,t_2(k)\geqq 9^{2m+j}(8(2-5\varepsilon)-10\varepsilon(4+10\varepsilon))$$
$$\geqq 9^{2m+j}(16-180\varepsilon).$$

Furthermore we have

$$\sum_{\substack{1\leqq n\leqq 9^m\\ 1\leqq k\leqq 9^{m+j+1}}} x_{nk}\,t_1(n)\,t_2(k)\geqq -\sum t_1(n)\,t_2(k)$$
$$\geqq -(\tfrac{1}{2}+\varepsilon)\,9^m(\tfrac{1}{2}+\varepsilon)\,9^{m+j+1}\geqq -9^{2m+j+1}(\tfrac{1}{4}+2\varepsilon),$$

and

$$\sum_{\substack{9^m < n \leq 9^{m+1} \\ 1 \leqq k \leqq 9^{m+j}}} x_{nk} t_1(n) t_2(k) \geqq -(4+10\varepsilon) 9^m (1/2+\varepsilon) 9^{m+j} \geqq -9^{2m+j}(2+19\varepsilon).$$

Hence it follows that

$$\sum_{\substack{1 \leqq n < 9^{m+1} \\ 1 \leqq k \leqq 9^{m+j+1}}} x_{nk} t_1(n) t_2(k) \geqq 9^{2m+j}(16-180\varepsilon - \tfrac{9}{4} - 18\varepsilon - 2 - 19\varepsilon) = 9^{2m+j}(\tfrac{47}{4} - 217\varepsilon) \geqq 10 \cdot 9^{2m+j+1},$$

and so $(x_{nk} t_1(n) t_2(k))$ is not u.d. for almost all $(t_1, t_2) \in T$.

§4. Proof of Theorem IV

We will consider only the one-dimensional case. Therefore we simply write $a(n)$, $p(n)$, x_n etc.

In the r-dimensional case one uses the same construction in a fixed coordinate j and gets the same estimates.

The following observation (which follows from the monotonicity of $a(n)$) will be used repeatedly: $2^{n-1} a(n) \leqq P(2^n - 1) \leqq 2^n a(n)$.

In the following, we fix a (P, μ)-u.d. sequence (x_n) (which exists, cf. §1). If X is uncountable, there exists a point $\bar{x} \in X$ with $\mu(\bar{x}) = 0$, if X is countable (and consists of more than one point), there exists an open (and closed) point $\bar{x} \in X$ with $\mu(\bar{x}) < 1$. We fix such a point $\bar{x}$ and denote by f_0 the characteristic function of $\{\bar{x}\}$. Put $\alpha_0 = \mu(\bar{x})$. Then any (P, μ)-u.d. sequence (y_n) should satisfy

$$\lim_{N \to \infty} P(N)^{-1} \sum_{k=1}^{N} p(k) f_0(y_k) = \alpha_0.$$

Put $y_k = \bar{x}$ for $2^n - 2^{n/2} \leqq k < 2^n$ and $y_k = x_k$ in all other cases. We claim that (y_k) is (P, μ)-u.d.

Let f be a continuous function on X, bounded by 1. If $2^{n-1} \leqq N < 2^n$, then

$$\left| \sum_{k=2^{n-1}}^{N} p(k)(f(x_k) - f(y_k)) \right| \leqq 2^{n/2} a(n).$$

If $N < 2^n - 2^{n/2}$, the left hand side equals zero. Therefore

$$\left| P(N)^{-1} \sum_{k=1}^{N} p(k)(f(x_k) - f(y_k)) \right| \leqq 2^{-n+1} a(n-1)^{-1} \sum_{m=1}^{n-1} 2^{m/2} a(m) + P(N)^{-1} 2^{n/2} a(n) \leqq n 2^{(1-n)/2} + P(N)^{-1} 2^{n/2} a(n).$$

If $N < 2^n - 2^{n/2}$, the second term is not needed and if $N \geqq 2^n - 2^{n/2}$, we have

$$P(N)^{-1} 2^{n/2} a(n) \leqq (N - 2^{n-1})^{-1} 2^{n/2} \leqq (2^{n-1} - 2^{n/2})^{-1} 2^{n/2} \leqq 2^{-n/2+2}$$

(for $n \geqq 4$). This shows that

$$\lim_{N\to\infty} P(N)^{-1} \sum_{k=1}^{N} p(k) f(y_k) = \lim_{N\to\infty} P(N)^{-1} \sum_{k=1}^{N} p(k) f(x_k)$$

and consequently, (y_k) is (P,μ)-u.d.

Now assume that $2^{-n/2} a(n-1)^{-1} a(n)$ is unbounded. Then we will show that $(y_{\tau(k,t)})$ is not (P,μ)-u.d. for almost all $t \in T$, thus proving the first part of Theorem IV.

Put $E_n = \{t \in T : 2^n - 2^{n/2} \leqq \tau(2^{n-1}, t), \tau(2^{n-1} + 2^{n/2-2}, t) < 2^n\}$. By independence we have

$$\lambda(E_n) \geqq \lambda\{t \in T : 2^n - 2^{n/2} \leqq \tau(2^{n-1}, t) < 2^n - 2^{n/2-1}\} \cdot \lambda\{t \in T : \tau(2^{n/2-2}, t) < 2^{n/2-1}\}.$$

It follows from Proposition 2.2 that

$$\liminf_{n\to\infty} \lambda(E_n) \geqq \lim_{n\to\infty} (F(-\tfrac{1}{2}) - F(-1)) F(0) > 0.$$

If $t \in E_n$, then $y_{\tau(k,t)} = \bar{x}$ for $2^{n-1} \leqq k < 2^{n-1} + 2^{n/2-2}$. Put $N(=N(n)) = 2^{n-1} + 2^{n/2-2} - 1$. Then

$$P(N)^{-1} \sum_{k=1}^{N} p(k) f_0(y_{\tau(k,t)}) \geqq (2^{n-1} a(n-1) + 2^{n/2-2} a(n))^{-1} 2^{n/2-2} a(n) = (2^{n/2+1} a(n-1) a(n)^{-1} + 1)^{-1}.$$

By assumption, there exists a subsequence M of $\mathbb{N}$ such that

$$\lim_{\substack{n\to\infty \\ n \in M}} 2^{n/2} a(n)^{-1} a(n-1) = 0.$$

Put $E = \bigcap_{n=1}^{\infty} \bigcup_{\substack{m \geqq n \\ m \in M}} E_m$. Then $\lambda(E) > 0$.

If $t \in E$, then $t \in E_m$ for infinitely many $m \in M$, therefore $(y_{\tau(k,t)})$ is not (P,μ)-u.d. Consequently, the set of all $t \in T$ for which $(y_{\tau(k,t)})$ is not (P,μ)-u.d. has positive measure. By Proposition 2.4, it has measure one.

The last statement in Theorem IV can be seen as follows: If the conclusions of Theorem I hold, then $2^{-n/2} a(n-1)^{-1} a(n)$ is bounded. If $2^{n-1} \leqq k < 2^n$, then

$$\frac{p(k)}{P(k)} \leqq \frac{a(n)}{2^{n-2} a(n-1) + (k - 2^{n-1}) a(n)}.$$

If $k < 2^{n-1}(a(n-1)a(n)^{-1} + 1)$ this is estimated by $\dfrac{a(n)}{2^{n-2} a(n-1)}$ and if $k \geqq 2^{n-1}(a(n-1)a(n)^{-1} + 1)$ by $(k - 2^{n-1})^{-1}$.

This gives:

$$\sum_{k=2^{n-1}}^{2^n-1} \left(\frac{p(k)}{P(k)}\right)^2 \leqq 2^{n-1} \frac{a(n-1)}{a(n)} \left(\frac{a(n)}{2^{n-2} a(n-1)}\right)^2 + \frac{1}{2} \sum_{k \geqq 2^{n-1} a(n-1) a(n)^{-1}} \frac{1}{k^2}$$
$$\leqq \frac{a(n)}{a(n-1) 2^{n-3}} + \frac{1}{2} \frac{a(n)}{a(n-1) 2^{n-2}} = \ll 2^{-n/2}$$

and the proof of Theorem IV is complete.

Remark. Using more careful estimates of similar type, one can show that in the one-dimensional case the following conditions for a weighted mean of type (1.4) are necessary and sufficient to get the conclusion of Theorem I:

$$\sum_{n=1}^{\infty} \exp\left(-\delta\left(\frac{a(n-1)2^{n/2}}{a(n)}\right)^2\right) < \infty \quad \text{for each } \delta > 0$$

$$\frac{a(n)^2}{a(n-1)a(n+1)} \leqq L \quad \text{for all } n.$$

§5. Proof of Theorem V

The construction will be done for $X = \{-1, 1\}$, $\mu(-1) = \mu(1) = \frac{1}{2}$ (cf. the Remark after the proof). Put

$$a(2n-1) = a(2n) = n! \quad (n \in \mathbb{N}).$$

These weights satisfy Hill's condition, and even $\sum_{n=1}^{\infty}\left(\frac{p(n)}{P(n)}\right)^2 < \infty$. Put $x_k = (-1)^k$.

We claim that almost all subsequences $\tau(x,t)$ are (P,μ)-u.d. By the strong law of large numbers, [3, Vol. II, p. 238] $\left(\text{observe that } \sum_{k=1}^{N}(x_{\tau(k,t)} + \frac{1}{3}x_{\tau(k-1,t)}) \text{ is a martingale}\right)$, we have for $\gamma > \frac{1}{2}$:

$$\lim_{N\to\infty} N^{-\gamma} \sum_{k=1}^{N} x_{\tau(k,t)} = 0 \quad \text{for almost all } t.$$

Fix such a t. Then there exists $c > 0$ such that $\left|\sum_{k=M+1}^{N} x_{\tau(k,t)}\right| \leqq c N^{\gamma}$ for all $N \geqq M$. But the sum can also be estimated by $N - M$. If $2^{n-1} \leqq N < 2^n$, it follows that

$$\left|\sum_{k=1}^{N} p(k) x_{\tau(k,t)}\right| \leqq c \sum_{l=1}^{n-1} a(l) 2^{\gamma l} + a(n)\min(cN^{\gamma}, N - 2^{n-1} + 1).$$

We have $P(N) \geqq 2^{n-2} a(n-1) + (N - 2^{n-1} + 1) a(n)$.

If $\varepsilon > 0$, $N - 2^{n-1} + 1 < 2^{n\gamma}\varepsilon^{-1}$, then

$$\begin{aligned}\left|P(N)^{-1}\sum_{k=1}^{N} p(k) x_{\tau(k,t)}\right| &\leqq c \sum_{l=1}^{n-1} 2^{\gamma l - n + 2} + 2^{n\gamma - n + 2}\varepsilon^{-1} a(n) a(n-1)^{-1} \\ &\leqq c\, 2^{(\gamma-1)n+1}(2^{\gamma}-1)^{-1} \\ &\quad + 2^{n(\gamma-1)+2}\varepsilon^{-1} a(n) a(n-1)^{-1}.\end{aligned}$$

This tends to zero if $\gamma < 1$.

If $N - 2^{n-1} + 1 \geqq 2^{n\gamma}\varepsilon^{-1}$, we use the other part of the minimum and get $cN^{\gamma} 2^{-n\gamma}\varepsilon \leqq c\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the sane conclusion holds.

Now put $y_k = 1$ for $2^{2n} \leqq k < 2^{2n}\left(1 + \frac{1}{n}\right)$, $y_k = x_k$ in all other cases. Then for $N = \left[2^{2n}\left(1 + \frac{1}{n}\right)\right]$

$$P(N)^{-1}\sum_{k=2^{2n}}^{N} p(k)y_k \geq (2^{2n}a(2n)+(2^{2n}n^{-1}+1)a(2n+1))^{-1}\cdot 2^{2n}n^{-1}a(2n+1) \sim \tfrac{1}{2} \quad \text{for } n\to\infty.$$

It follows easily that (y_k) is not (P,μ)-u.d.

By Proposition 2.2 $\lim_{k\to\infty}\frac{\tau(k,t)}{k}=2$ for almost all t. Fix such a t. Then for any $s>1$ there exists $n_0(s)$ such that $2^{2n}\leq\tau(k,t)<2^{2n}\left(1+\frac{1}{n}\right)$ implies $s^{-1}2^{2n-1}<k<s2^{2n-1}\left(1+\frac{1}{n}\right)$ for $n\geq n_0(s)$. It follows easily from this that

$$\lim_{N\to\infty} P(N)^{-1}\sum_{k=1}^{N} p(k)(x_{\tau(k,t)}-y_{\tau(k,t)})=0,$$

and therefore $(y_{\tau(k,t)})$ is (P,μ)-u.d. for almost all t.

Remark. The same result holds for arbitrary compact spaces X and arbitrary Radon probability measures μ (not concentrated in a point). The main point is to construct a sequence in X for which almost all subsequences are (P,μ)-u.d. This can be done by combinatorial arguments based on similar ideas as in the foregoing proof. (In general, this depends on the growth of the coefficients of P. The existence of a (P,μ)-u.d. sequence – i.e. (1.2) – is not sufficient to ensure the existence of a sequence as above).

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