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Probability Theory and Related Fields

A Ratio Ergodic Theorem for Increasing Additive Functionals

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Summary. Let B be a 1-dimensional Brownian motion. In this paper ratios of the form $A^+(t)/A^-(t)$, where A^+ is the $(0, \infty)$-occupation time functional of B and A^- is a local time integral of an infinite (but locally finite) measure m with support in $(-\infty, 0]$, are studied. Conditions on m are given which ensure that such a ratio will be unbounded a.s. (or go to zero a.s.) as $t \to \infty$.

1. Introduction and Statement of Main Results

Let $\{B(t),\ t \geqq 0\}$ be a standard Brownian motion on $\mathbb{R}$ with $B(0)=0$ and let $L(t, x)$ be the local time functional

$$L(t, x) = \lim_{\varepsilon \to 0+} meas.\{s;\ s \leqq t,\ x \leqq B(s) \leqq x+\varepsilon\}/\varepsilon.$$

Let m be a measure *concentrated on* $(-\infty, 0]$ and which satisfies

(1.1) $$m(-\infty, 0] = \infty, \qquad m\{I\} < \infty \qquad \text{for bounded } I.$$

For $t \geqq 0$ put

(1.2) $$A^-(t) = \int_{-\infty}^{0+} L(t, x)\, m\{dx\},$$
$$A^+(t) = \int_0^t I_{(0,\infty)}(B_s)\, ds = \int_0^\infty L(t, x)\, dx.$$

(Note that $A^-(t) = \int_0^t m^\bullet(B_s)\, ds$ in the case that $m\{dx\} = m^\bullet(x)\, dx$.) The purpose of this paper is to determine when the ratio

$$K(t) = A^+(t)/A^-(t)$$

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is bounded or unbounded, a.s., as $t \to \infty$. More specifically let us write

$$k^* = \limsup_{t \to \infty} K(t) \quad \text{and} \quad k_* = \liminf_{t \to \infty} K(t).$$

By the 0–1 law k^* and k_* are constants a.s. which, as we will see, independently of each other (but for $k_* \leqq k^*$), must be 0 or ∞ and our goal is to find criteria expressed as directly as possible in terms of m for deciding which of the two possibilities prevails. Note that if m were a finite measure, then we would immediately obtain from the ratio ergodic theorem, [5], p. 228, that $k^* = k_* = \infty$, so our results could be viewed as extensions of that theorem.

Our motivation for studying these limits stems in part from a recent (1982) paper by London, McKean, Rogers and Williams [7]. Let $A = A^+ - A^-$. A is a continuous additive functional which decreases when $B(t) \in \mathrm{supp}(m)$ and increases (linearly) when $B(t) > 0$. Put $A^{-1}(t) = \inf\{s\colon A(s) \geqq t\}$, $\inf \phi = \infty$, and $Y(t) = B(A^{-1}(t))$, $B(\infty) =$ cemetary point δ. Y is a Feller Brownian motion: a strong Markov process with state space $[0, \infty) \cup \delta$ which behaves like Brownian motion on $(0, \infty)$ (i.e. its local generator is $(1/2) d^2/dy^2$ there). The relationship of the measure m to the behavior of Y at the origin is the subject of [7]. The results of our paper pertain directly to the finiteness of the lifetime $\eta = \inf\{t\colon Y(t) = \delta\}$. As one may easily show

$$\eta = \infty \qquad \text{if and only if} \qquad a^* = \limsup_{t \to \infty} A(t) = +\infty,$$

and, under (1.1), $a^* = +\infty$ or $-\infty$ according as $k^* = \infty$ or 0. Note that in terms of the characteristics $(p_1, p_2, 0, p_4)$ of Y, $\eta = \infty$ if and only if p_1, the killing rate at the origin, is 0. See [7], p. 44. We will not make any further reference to the process Y in the remainder of the paper.

Statement of Main Results. For any α put

$$D_\alpha = \inf\{t\colon B(t) = \alpha\}$$

and let

$$W = A^-(D_1) = \int_{-\infty}^{0+} L(D_1, x)\, m\{dx\}. \tag{1.3}$$

We will occasionally write f_t for $f(t)$ for functions on $[0, \infty)$.

Theorem 1. (i) $k^* = 0$ *or* ∞ *according as*

$$\int_1^\infty \left(\int_0^x P(W > t)\, dt\right)^{-1} \frac{dx}{\sqrt{x}} < \infty \quad \text{or} \quad = \infty. \tag{1.4i}$$

(ii) $k_* = \infty$ *or* 0 *according as*

$$\int_1^\infty P(W > t) \frac{dt}{\sqrt{t}} < \infty \quad \text{or} \quad = \infty. \tag{1.4ii}$$

(Note that the convergence of the integral (1.4ii) is equivalent to the finiteness of the moment $E\sqrt{W}$.)

For any positive decreasing function h on $(0,\infty)$, define

(1.5)
$$J^*(h)=\int_1^\infty \left(\int_0^x h(t)\,dt\right)^{-1}\frac{dx}{\sqrt{x}}$$
$$J_*(h)=\int_1^\infty h(x)\frac{dx}{\sqrt{x}}.$$

With m as in (1.1) put, for $x\geqq 0$,

$$m_+\{dx\}=m\{-dx\},\qquad m_+(x)=m_+[0,x]=m\{[-x,0]\},$$

and define $\alpha(t)$ and $\beta(t)$ by

(1.6)
$$\alpha(t)=\inf\{a\colon\, a\,m_+(a)\geqq t\}\vee 1,$$

(1.7)
$$\beta(t)=\sup\left\{b\colon\, \frac{1}{b}\int_0^{b+} x^2 m_+\{dx\}\leqq t\right\}\vee 1.$$

It is useful to note that under (1.1)

(1.8)
$$\alpha(t)\uparrow\infty,\qquad \beta(t)\uparrow\infty,\qquad \alpha(t)\leqq\beta(t),\qquad \alpha(t)=o(t),\qquad \text{as } t\to\infty.$$

We leave the easy verification to the reader. Note that α is also continuous.

Theorem 2. *We have the following implications:*

(i) $J^*(1/\alpha)=\infty\Rightarrow k^*=\infty$.
(ii) $J^*(1/\beta)<\infty\Rightarrow k^*=k_*=0$.
(iii) $J_*(1/\alpha)<\infty\Rightarrow k_*=k^*=\infty$.
(iv) $J_*(1/\beta)=\infty\Rightarrow k_*=0$.

Remark. It may be helpful to note that (1.8) implies $J^*(1/\alpha)\leqq J^*(1/\beta)$, and $J_*(1/\beta)\leqq J_*(1/\alpha)$, and that for any $0<h\in\downarrow$ at most one of $J^*(h)$, $J_*(h)$ can be finite. This latter may be proved as in [2], p. 376.

Theorem 3. *If in addition to* (1.1) *we also have*

(1.9)
$$\limsup m_+(x)/m_+(2x)<1,$$

then (i) $k^*=\infty\Leftrightarrow J^*(1/\alpha)=\infty$; (ii) $k_*=0\Leftrightarrow J_*(1/\alpha)=\infty$. (For a generalization see 5(b).)

Example. Interesting cases occur when m is near Lebesgue measure. Suppose for example $m\{dx\}=(\log^+|x|)^r\,dx$ for $x\leqq 0$. Then by Theorems 1 and 3

$$\limsup_{t\to\infty}\int_0^t I_{(0,\infty)}(B_s)\,ds\Big/\int_0^t (\log^+|B_u\wedge 0|)^r\,du=\infty\ \text{ or } 0$$

according as $r\leqq 2$ or $r>2$, and the $\liminf_{t\to\infty}(\bullet)=0$ or ∞ according as $r\geqq -2$ or $r<-2$.

We prove Theorem 1 by a method similar, initially, to the proof of the ergodic theorem in [5]. By sampling A^+ and A_- at the successive passage times (up and down) across a fixed interval, we find that $K(t)$ can be replaced by a ratio of two independent sums of positive independent random variables with infinite means. An application of some random walk methods in Erickson [2] and Kesten [6] completes the proof. The proof of Theorems 2 and 3 requires an asymptotic evaluation of $P(W>t)$. This is accomplished by conditioning the integral at (1.3) on $b=\min\{B_s;\ s\leqq D_1\}$, applying Ray's representation of $L(D_1, x)$ in terms of BES(4) and estimating the integrals which develop. In the special case that m is regularly varying, application of a Tauberian theorem gives a more precise evaluation of $P(W>t)$, see § 5(a).

2. Proof of Theorem 1

Step 1. Let A_0^+ be occupation time of $[1, \infty)$: $A_0^+(t)=\int_0^t I_{[1,\infty)}(B_s)\,ds$. Then

$$k^*(k_*)=\limsup_{t\to\infty}(\inf)\, A_0^+(t)/A^-(t).$$

Proof. This follows immediately from

$$\lim_{t\to\infty}\int_0^t I_{(0,1)}(B_s)\,ds/C(t)=o \quad \text{a.s.}$$

where C stands for any one of the functionals $A+, A^-, A_0^+$. See [5], p. 228.

Step 2. Define stopping times $T_0, T_1, T_2, \ldots$, by $T_0=0$, $T_1=D_1$, and

$$T_j=\begin{cases}\min\{t\geqq T_{j-1};\ B(t)=0\} & \text{for } j \text{ even},\\ \min\{t\geqq T_{j-1};\ B(t)=1\} & \text{for } j \text{ odd},\end{cases}$$

and for $n\geqq 1$, let

$$\begin{aligned} W_n&=\int_{-\infty}^{0+}[L(T_{2n-1},x)-L(T_{2n-2},x)]\,m\{dx\}=A^-(T_{2n-1})-A^-(T_{2n-2}),\\ V_n&=\int_1^{\infty}[L(T_{2n},x)-L(T_{2n-1},x)]\,dx=A_0^+(T_{2n})-A_0^+(T_{2n-1}).\end{aligned} \tag{2.1}$$

Because A_0^+, A^-, increases only on intervals $[T_j, T_{j+1}]$ with j odd, j even, respectively, and is otherwise constant, we see immediately that for $n=1,2,\ldots$,

$$\frac{V_1+\ldots+V_{n-1}}{W_1+\ldots+W_n}\leqq\frac{A_0^+(t)}{A^-(t)}\leqq\frac{V_1+\ldots+V_r}{W_1+\ldots+W_r}, \qquad T_{2n-2}\leqq t\leqq T_{2n}, \tag{2.2}$$

where $r=n-1$ when $T_{2n-2}\leqq t<T_{2n-1}$ and $r=n$ when $T_{2n-1}\leqq t\leqq T_{2n}$. Continuity of the sample paths of B and the strong Markov property imply

(2.3) $\{V_n\}$ and $\{W_n\}$ are each sequences of independent, identically distributed, positive random variables and the two sequences are independent of each other.

In particular, the two sequences of partial sums $\{V_1+\ldots+V_{n-1}, n\geqq 2\}$ and $\{V_2+\ldots+V_n, n\geqq 2\}$ are identical in law and are independent of the sequence $\{W_n\}$. (From the point of view of the V's the sequence $\{W_1+\ldots+W_n, n\geqq 1\}$ may be regarded simply as a sequence of constants $\uparrow\infty$.) From these facts, (2.2), and Step 1, we conclude

$$k^*(k_*)=\limsup(\inf)\frac{V_1+\ldots+V_n}{W_1+\ldots+W_n} \quad \text{a.s.} \tag{2.4}$$

Now, by the method of problem 1, p. 230, in [5], it follows that $E[L(T_{2n-1},x)-L(T_{2n-2},x)]=EL(D_1,x)=2$, for $x\leqq 0$, and $E[L(T_{2n},x)-L(T_{2n-1},x)]=E[L(T_2,x)-L(D_1,x)]=2$, for $x\geqq 1$. Hence, see (2.1),

$$EV_1=\infty \quad \text{and} \quad EW_1=\infty. \tag{2.5}$$

Step 3. For positive random variables V and W define

$$J(V,W)=\int_1^\infty \left(x\Big/\int_0^x P(W>t)\,dt\right) P\{V\in dx\}.$$

Lemma A. *Let $\{V_n\}$ and $\{W_n\}$ be any two sequences of r.v.'s on the same probability space which satisfy* (2.3) *and* (2.5). *Then $J(V_1,W_1)+J(W_1,V_1)=\infty$ and the following implications hold*

(i) $$J(V_1,W_1)=\infty \Rightarrow \limsup\frac{V_n}{W_1+\ldots+W_n}=\infty \quad \text{a.s.}$$

(ii) $$J(V_1,W_1)<\infty \Rightarrow \limsup\frac{V_1+\ldots+V_n}{W_1+\ldots+W_n}=0 \quad \text{a.s.}$$

Proof. That at most one of $J(V_1,W_1)$, $J(W_1,V_1)$ is finite can be proved with slight modification as in [2], pp. 375–6. From the estimates in [2], pp. 377–8 for any fixed $\varepsilon>0$

$$\sum_{n=1}^\infty P(W_1+\ldots+W_n\leqq\varepsilon x)\asymp x\Big/\int_0^x P(W_1>t)\,dt, \qquad x\to\infty, \tag{2.6}$$

($\asymp$ means the ratio of both sides is bounded away from 0 and ∞). It follows that $J(V_1,W_1)=\infty$ implies $\sum P(W_1+\ldots+W_n\leqq\varepsilon V_n)=\infty$ and then, by Lemma 2 in [6], p. 1192, that for every $\varepsilon>0$ $P(W_1+\ldots+W_n\leqq\varepsilon V_n \text{ i.o.})=1$. Now suppose $J(V_1,W_1)<\infty$. Then by (2.6) and the Borel-Cantelli Lemma we have (*) $P(W_1+\ldots+W_n\leqq V_n \text{ i.o.})=0$. Suppose, contrary to the conclusion of (ii), $P(V_1+\ldots+V_n\geqq\varepsilon(W_1+\ldots+W_n) \text{ i.o.})>0$ for some $\varepsilon>0$. This probability must be 1 and, as in [6], p. 1191, we get $P(V_n\geqq\varepsilon\min_{j\geqq n}(W_1+\ldots+W_j)/j \text{ i.o.})=1$ and then by Lemmas 3 and 4 of [6] we get $P(V_n\geqq c(W_1+\ldots+W_n) \text{ i.o.})=1$ for any $c>0$ which contradicts (*).

Step 4. It is now clear from Lemma A and (2.4) that $k^*=0$ or ∞ according as $J(V_1,W_1)$ is finite or infinite, and, by interchanging the V's and W's in Lemma A, we also get $k^*=\infty$ or 0 according as $J(W_1,V_1)$ is finite or infinite. To complete the proof we need to show that the integrals at (1.4i) and (1.4ii) are equivalent to $J(V_1, W_1)$ and $J(W_1, V_1)$, respectively, and to do this it suffices to show that

$$P\{V_1 \in dx\} \asymp x^{-3/2}\, dx \quad \text{as } x \to \infty. \tag{2.7}$$

This must be well known but we lack a ready reference so here is a quick proof. Let E_x denote expectation for paths starting at x and w_s^+ the shifted path: $B_t(w_s^+) = B_{t+s}(w)$. Then for any path w, $V_1 = A_0^+(s + D_0(w_s^+); w) - A_0^+(s; w) = A_0^+(D_0(w_s^+); w_s^+)$, $s = D_1(w)$, as everyone knows, hence, for any $\lambda \geqq 0$,

$$E_0 \exp(-\lambda V_1) = E_1 \exp(-\lambda A_0^+(D_0)).$$

If $A_\varepsilon^+(t) = \int_{-\infty}^{\infty} L(t,x)\, \delta_\varepsilon(x)\, dx$ where $\delta_\varepsilon(x) = 1$ for $x \geqq 1$, $\delta_\varepsilon(x) = \varepsilon$ for $x < 1$, then $A_\varepsilon^+(D_0)$ is the first passage time to 0 for the diffusion process on natural scale whose speed measure is $2\delta_\varepsilon(x)\, dx$. It follows that $g(x) = E_x \exp(-\lambda A_\varepsilon^+(D_0))$ satisfies $g(0) = 1$, g is bounded, g and g' are continuous, and $g''(x)/2 = \lambda \delta_\varepsilon(x)\, g(x)$, $x \neq 1$. Solving for g and setting $x = 1$, we obtain

$$E \exp(-\lambda V_1) = \lim_{\varepsilon \to 0+} g(1) = \frac{1}{1 + \sqrt{2\lambda}}, \qquad \lambda \geqq 0. \tag{2.8}$$

The function $x \mapsto 2\int_0^\infty e^{-t} P\{B(x) > t\}\, dt$ is a distribution function which also has the Laplace transform $(1 + \sqrt{2\lambda})^{-1}$. (To see this note that it is the distribution function of $T(\eta)$ where $\{T(t),\, t \geqq 0\}$ is a stable process of index 1/2 and rate $\sqrt{2}$ and η is an independent Exp(1)-distributed random variable.) It follows that

$$\begin{aligned} \frac{d}{dx} P(V_1 \leqq x) &= \frac{d}{dx} 2\int_0^\infty e^{-t} P\{B(x) > t\}\, dt \\ &= x^{-3/2} \int_0^\infty t\, e^{-t} e^{-t^2/2x}\, dt / \sqrt{2\pi}. \end{aligned}$$

This easily yields (2.7) and concludes the proof of Theorem 1.

3. Proof of Theorem 2

By Ray's Theorem, see Williams [9], p. 873, or Ray [8], for any fixed $b < 0$

$$\begin{aligned} &Law[\{L(D_1, x)\colon\ b \leqq x \leqq 0\} \,|\, \min_{s \leqq D_1} B_s = b,\ B_0 = 0] \\ &= Law\left[\left\{(1-x)^2 Z\left(\frac{1}{1-x} - \frac{1}{1-b}\right)^2;\ b \leqq x \leqq 0\right\}\right], \end{aligned}$$

where Z is the radial part of a 4-dimensional Brownian motion starting at 0. (Note that our local time is twice the Ito-McKean local time.) Also

$$P[\min_{s \leqq D_1} B_s \leqq b] = \frac{1}{1-b}, \qquad b \leqq 0.$$

Hence, making the change of variable $x\to -x$, $b\to -b$,

$$P(W>t)=\int_0^\infty P\left[\int_0^{b+}(1+x)^2 Z\left(\frac{1}{1+x}-\frac{1}{1+b}\right)^2 m_+\{dx\}>t\right]\frac{db}{(1+b)^2}.$$

Lemma B. *With α as at* (1.6), *there is a constant c_1, $0<c_1<\infty$, such that*

$$P(W>t)\leqq c_1/\alpha(t) \quad \text{for } t\geqq 0. \tag{3.2}$$

Proof. Let B^4 be a 4-dimensional Brownian motion starting at the origin and put $H(s)=B^4(s)-sB^4(1)$, then the process

$$x\mapsto\sqrt{1+b}\,H\left(\frac{1+x}{1+b}\right), \quad -1\leqq x\leqq b, \tag{3.3}$$

and the process

$$x\mapsto(1+x)\,B^4\left(\frac{1}{1+x}-\frac{1}{1+b}\right), \quad -1\leqq x\leqq b, \tag{3.4}$$

are law equivalent. To see this note that they are both 0 mean Gaussian processes in $\mathbb{R}^4$ with covariance $EU_i(x_1)\,U_j(x_2)=(1+x_1)(b-x_2)(1+b)^{-1}\delta_{ij}$, $x_1<x_2$, where U is either of the processes (3.3) or (3.4), δ_{ij} is the Kronecker delta, $i,j=1,\dots,4$. From this equivalence we get

$$P(W>t)\leqq\int_0^\infty P\left[N>\frac{t}{(1+b)\,m_+(b)}\right]\frac{db}{(1+b)^2}, \tag{3.5}$$

where

$$\begin{aligned}N&=\max\left\{\left\|H\left(\frac{1+x}{1+b}\right)\right\|^2;\ -1\leqq x\leqq b\right\}\\&=\max\{\|H(s)\|^2;\ 0\leqq s\leqq 1\}.\end{aligned}$$

According to Fernique [4], for some $\varepsilon>0$ we have $E\exp(\varepsilon N)=c_2<\infty$. Hence

$$P[N>u]\leqq c_2\,e^{-\varepsilon u}, \quad u\geqq 0. \tag{3.6}$$

Put $c_3=\varepsilon/(1+2m_+(1))$. Then

$$P\left[N\geqq\frac{t}{(1+b)\,m_+(b)}\right]\leqq\begin{cases}c_2\,e^{-c_3 t}, & 0\leqq b\leqq 1,\\ c_2\,e^{-\varepsilon t/2bm_+(b)}, & b\geqq 1.\end{cases}$$

If $\alpha(t)>2$ then for $1\leqq b\leqq\alpha(t)/2$,

$$\begin{aligned}\exp(-\varepsilon/2b\,m_+(b))&\leqq\exp(-\varepsilon\,t/2b\,m_+(\alpha(t)/2))\\&\leqq\exp(-\varepsilon\,\alpha(t)/4b).\end{aligned}$$

Going back to (3.5) with these bounds we get

$$\begin{aligned}P(W>t)&\leqq\int_0^1+\int_1^{\alpha/2}+\int_{\alpha/2}^\infty(\cdot)\\&=0(e^{-c_3 t})+0\left(\int_1^{\alpha/2}e^{-\varepsilon\alpha(t)/4b}\frac{db}{b^2}\right)+0\left(\frac{1}{\alpha(t)}\right)\\&=0(e^{-c_3 t})+0\left(\frac{1}{\alpha(t)}\right), \quad t\to\infty.\end{aligned} \tag{3.8}$$

From (1.8) we have $0(e^{-c_3 t})=0(1/t)=0(1/\alpha(t))$, as $t\to\infty$. With this (3.2) is established.

Lemma C. *With β as at* (1.7), *there is a constant $c_0>0$ such that*

$$P(W>t)\geqq c_0/\beta(t), \quad \textit{for } t\geqq 0. \tag{3.9}$$

Proof. Using Brownian scaling we have for $b\geqq 1$

$$\begin{aligned}
&\min\left\{Z\left(\frac{1}{1+x}-\frac{1}{1+b}\right)^2,\ 0\leqq x\leqq b/2\right\}\\
&\quad =_d \min\left\{\frac{1}{b}Z(br)^2;\ \frac{2}{2+b}-\frac{1}{1+b}\leqq r\leqq\frac{b}{1+b}\right\}\\
&\quad \geqq \inf\{Z(s)^2;\ \tfrac{1}{6}\leqq s<\infty\}/b=M/b
\end{aligned}$$

where $=_d$ means equality in distribution. Since $Z(0)=0$ and since $\lim_{u\to\infty} Z(u)=\infty$ a.s. (four-dimensional Brownian motion is transient), it is clear that $P(M>\alpha)>0$ for every $\alpha>0$. Using these facts in (3.1) we obtain

$$\begin{aligned}
P(W>t)&\geqq\frac{1}{4}\int_1^\infty P\left[\int_0^{b/2} x^2 Z\left(\frac{1}{1+x}-\frac{1}{1+b}\right)^2 m_+\{dx\}>t\right]\frac{db}{b^2}\\
&\geqq\frac{1}{4}\int_1^\infty P\left[M>\frac{bt}{\sigma(b/2)}\right]\frac{db}{b^2},
\end{aligned}$$

where $\sigma(x)=\int_0^x y^2 m_+\{dy\}$. With β as at (1.7) we have $(b/2)/\sigma(b/2)\leqq 1/t$ whenever $b/2\geqq\beta(t)>1$, hence

$$P\left[M>\frac{bt}{\sigma(b/2)}\right]\geqq P[M>2]>0,$$

for $b\geqq 2\beta(t)$. Consequently

$$P(W>t)>\tfrac{1}{4}\int_{2\beta(t)}^\infty(\bullet)\geqq c_0/\beta(t),$$

for all $t\geqq 0$ for some constant c_0, $0<c_0\leqq\frac{1}{16}P[M>2]$.

The reader may easily complete the proof of Theorem 2 by substituting the bounds at (3.2) and (3.9) into the integrals which occur in Theorem 1 and reading off the implications.

Remark 1. It might be thought that one could get $c/\alpha(t)$ as a lower bound for $P(W>t)$ by staying with the process (3.3). Unfortunately $H(1/(1+b))\to 0$ as $b\to\infty$ and $H(1)=0$, so there is trouble at both endpoints of $(-1,b)$ and this trouble neatly foils the attempt.

Remark 2. The assumption $m_+(\infty)=\infty$ is not necessary to get (3.9). It suffices to require only that $(1/b)\int_0^{b+} x^2 m_+\{dx\}\to\infty$ as $b\to\infty$.

Remark 3. Here is an example. Suppose

$$m\{dx\}=(|x|\log|x|)^{-1}\,dx, \qquad x\leqq -e.$$

Then $m_+(x)=\log\log x$, $\int_{-x}^{0} y^2 m\{dy\}=\frac{1}{2}(x^2/\log x)(1+o(1))$ for $x\to\infty$. So $\alpha(t)=(t/\log\log t)(1+o(1))$, $\beta(t)=(2t\log t)(1+o(1))$, and then

$$C_0\frac{1}{t\log t}\leqq P(W>t)\leqq C_1\frac{\log\log t}{t}, \qquad t\geqq 3.$$

It is not clear which bound is the best asymptotically, though one might suspect it is the lower one. See the last remarks in § 5(a).

4. Proof of Theorem 3

By (1.9) we can choose ε, $0<\varepsilon<1$ and $x_0>1$ and $x_0>1$ so that for $x\geqq x_0$, $m_+(x)/m_+(2x)\leqq 1-\varepsilon$. Then for $b\geqq 2x_0$

$$\begin{aligned}\varepsilon\frac{b^2}{4}m_+(b)&\leqq\frac{b^2}{4}\left[m_+(b)-m_+\left(\frac{b}{2}\right)\right]\\&\leqq\int_0^{b+}x^2 m_+\{dx\}\leqq b^2 m_+(b).\end{aligned}$$

From these inequalities and (1.6)–(1.7), it follows that $\alpha(t)\leqq\beta(t)\leqq\alpha(4t/\varepsilon)$ for all t sufficiently large. A simple scale change of variables now shows that the integrals $J^*(1/\beta)$ and $J_*(1/\beta)$ of Theorem 2 are equivalent to $J^*(1/\alpha)$ and $J_*(1/\alpha)$ respectively. (Note that the same argument shows that (1.9) can be replaced by $\limsup(m_+(x)/m_+(cx))<1$ for some $c>1$.)

Remark. One should note that (1.9) does not imply regular variation. For example, the measure $m\{dx\}=\exp(|x|)\,dx$, $x\leqq 0$, satisfies (1.9) but not (5.7) in the next section.

5. Miscellaneous Comments

(a) A more direct method is available for getting at the distribution of W in (1.3) which yields exact asymptotic estimates in special cases. Consider the process

$$\left\{y\mapsto A^-(D_y)=\int_{-\infty}^{0+}L(D_y;x)\,m\{dx\},\ y\geqq 0;\ P_0\right\}. \tag{5.1}$$

The strong Markov property of B and the fact that $B(D_y)=y$ shows that (5.1) is a process with independent (but generally not stationary) increments. For $\lambda>0$ write

$$u(y)=E_0\,e^{-\lambda A^-(D_y)}. \tag{5.2}$$

Then for $h>0$,

$$\begin{aligned}u(y+h)&=u(y)\,E_y\,e^{-\lambda A^-(D_{y+h})}\\&=u(y)\,[E_y(e^{-\lambda A^-(D_{y+h})};\,D_0>D_{y+h})+E_y(e^{-\lambda A^-(D_{y+h})};\,D_0\leqq D_{y+h})]\\&=u(y)\left[\frac{y}{y+h}+\frac{h}{y+h}\,u(y+h)\right],\end{aligned}$$

which, keeping in mind that (5.1) is continuous in probability, gives

$$u^+(y)=\lim_{h\to 0+}(u(y+h)-u(y))/h=\frac{1}{y}\,u(y)\,(u(y)-1), \tag{5.3}$$

for $y>0$ and $u(0+)=u(0)=1$. Solving (5.3) we arrive at the formula

$$E\,e^{-\lambda A^-(D_y)}=E\exp\left(-\lambda\int_{-\infty}^{0+}L(D_y,x)\,m\{dx\}\right)=(j(\lambda;m)\,y+1)^{-1} \tag{5.4}$$

where j does not depend on y and $j(0+;m)=0$. The Brownian scaling property states that the path transformation $B(\bullet)\mapsto s\,B(\bullet/s^2)$ is P_0-measure preserving for fixed $s>0$. This and the definition of local time show that for any fixed $s>0$, $A^-(D_y)$ has the same distribution, under P_0, as $s\int_{-\infty}^{0+}L(D_{y/s},x/s)\,m\{dx\}$. Consequently

$$E\,e^{-\lambda A^-(D_y)}=E\exp(-\lambda\,s\,m_+(s)\,A_s^-(D_{y/s})) \tag{5.5}$$

where $A_s^-(D_z)=\int_{-\infty}^{0+}L(D_z,x)\,m_s\{dx\}$ and $m_s\{dx\}=m\{s\,dx\}/m_+(s)$. Combining (5.4) and (5.5) gives

$$j(\lambda;m)=j(\lambda\,s\,m_+(s);\,m_s)/s. \tag{5.6}$$

Now let us assume that m is regularly varying in the sense that for every $x\geqq 0$, the

$$\lim_{s\to\infty}m_+(s\,x)/m_+(s)=\mu(x) \tag{5.7}$$

exists and is finite. Then, necessarily, $\mu(x)=x^q$ for some $q\geqq 0$ (q is the "exponent") and, supposing $q>0$,

$$\int_{-\infty}^{0+}f(x)\,m_s\{dx\}\to q\int_{-\infty}^{0}f(x)\,|x|^{q-1}\,dx,\qquad s\to\infty,$$

for every continuous, compact f. Hence, since $x\mapsto L(D_y,x)$ has compact support a.s., $A_s^-(D_y)\to A_\mu^-(D_y)$ a.s. as $s\to\infty$ and then

$$\lim_{s\to\infty}j(\lambda;m_s)=j(\lambda;\mu)$$

uniformly on bounded intervals of $\lambda\geqq 0$. Properties of regular variation, see, for example, Bojanic and Seneta (1971), imply that the function α defined at (1.6) is also regularly varying (with exponent $(q+1)^{-1}$) and that

$$\lambda s m_+(s) \to 1 \quad \text{as} \quad \lambda \to 0+, \; s = \alpha(1/\lambda).$$

Setting $s=\alpha(1/\lambda)$ in (5.6) we get

$$\begin{aligned}\int_0^\infty e^{-\lambda t} P(W>t)\,dt &= \frac{1-E e^{-\lambda W}}{\lambda} = \frac{j(\lambda; m)/\lambda}{j(\lambda; m)+1} \\ &= \frac{j(1+o(1), m_s)}{o(1)+1}\left(\frac{1}{\lambda\,\alpha(1/\lambda)}\right) \\ &= c(q)\,(\lambda\,\alpha(1/\lambda))^{-1}(1+o(1)),\end{aligned}$$

as $\lambda \to 0+$, where $c(q) = E_0 \exp\left(-q \int_{-\infty}^{0} L(D_1, x)\,|x|^{q-1}\,dx\right)$. Applying a Tauberian theorem, see Feller (1971), p. 446, now gives

$$P(W>t) = c(q)\,\Gamma\left(\frac{q}{q+1}\right)^{-1}\left(\frac{1}{\alpha(t)}\right)(1+o(1)) \tag{5.10}$$

as $t \to \infty$. If $q=0$ about the best one can do with this argument is to show that $P(W>t) = o(1/\alpha(t))$. We omit the details. It should be pointed out, however, that in the case $q=0$, even though (5.10) does not apply and even though Theorem 3 does not apply, it can still be shown, via Theorem 2 and properties of regularly varying functions, that $k^* = k_* = \infty$. This leads one to suspect that (1.9) in Theorem 3 is not necessary.

(b) If A^+, the occupation time functional of $(0, \infty)$, is replaced by a more general functional say

$$A^+(t) = \int_0^\infty L(t, x)\, m^+\{dx\}$$

(m^+ not to be confused with m_+ of § 1), then the methods of this paper will yield results on the boundedness or unboundedness of $A^+(t)/A^-(t)$ (A^- as before). Unfortunately the analogue of Theorem 2 is rather unwieldy. Under a regularity condition such as (1.9), however, matters improve. Here is a sample. Write m^- for m. We assume $m^\pm(I) < \infty$ for bounded I and that $m^+(0, \infty) = m^-(-\infty, 0] = \infty$. Suppose also that for some $0 < \varepsilon < 1$

$$m^+(0, x]/m^+(0, 2x] \leqq 1-\varepsilon$$
$$m^-[-x, 0]/m[-2x, 0] \leqq 1-\varepsilon$$

for all x sufficiently large. Let $\alpha^\pm$ be defined by $\alpha^- = \alpha$ at (1.6) and $\alpha^+(t) = \inf\{x\colon x m^+(0, x] \geqq t\}$. Put $s(x) = \int_0^x \alpha^-(t)^{-1}\,dt$.

Theorem 4. *Under these assumptions* $\limsup_{t\to\infty} A^+(t)/A^-(t) = \infty$ a.s. *if and only if* $\int^\infty [s(x) - x/\alpha^-(x)]\, s(x)^{-2}\, \alpha^+(x)^{-1}\,dx = \infty$.

We omit the proof. As noted before our motivation for studying the particular case $m^+ =$ Lebesgue measure on $(0, \infty)$ derives from the London et al. (1982) paper, but it also seems natural to compare an arbitrary additive functional with occupation time.

(c) A more interesting problem than the one discussed in (b) concerns the a.s. boundedness of $A_1(t)/A_2(t)$, $t \to \infty$, where A_i is now an arbitrary increasing continuous additive functional: $A_i(t) = \int_{-\infty}^{\infty} L(t, x)\, m_i\{dx\}$. I do not have a good conjecture but, as before, the ratio ergodic theorem does take care of the case in which at least one of m_1 or m_2 is a finite measure.

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