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Probability Theory and Related Fields


# Strong Laws for the $k$-th Order Statistic when $k \leqq c \log_2 n$

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**Summary.** Under general regularity assumptions, we characterize the upper and lower almost sure classes of $U_{k,n}$, where $U_{1,n} < \ldots < U_{n,n}$ are the order statistics of an i.i.d. sample of size $n$ from the uniform distribution on (0, 1), and where $k = k_n$ is a non-decreasing integer sequence such that $1 \leqq k = O(\log_2 n)$ as $n \to \infty$.

## 1. Introduction and Results

Let $U_1, U_2, \ldots$ be an i.i.d. sequence of uniformly distributed random variables on (0, 1), and denote for $n = 1, 2, \ldots$, by $U_{1,n} < U_{2,n} < \ldots < U_{n,n}$ the order statistics of $U_1, \ldots, U_n$.

Let $k = k_n$, $n = 1, 2, \ldots$ be a non-decreasing integer sequence such that, for $n = 1, 2, \ldots, 1 \leqq k_n \leqq n$. In this paper, we shall be concerned with the limiting strong behavior of $U_{k,n}$ as $n \to \infty$, with emphasis on the case where $k_n = O(\log_2 n)$ as $n \to \infty$.

Before stating our theorems, it is worthwhile to review the known results concerning strong limiting bounds for $U_{k,n}$. The case where $k$ is constant has received a complete treatment. In this case, we have:

(1) (Kiefer, 1972). If $c_n \downarrow$ then

$$P(nU_{k,n} \leqq c_n \text{ i.o.}) = 0 \text{ or } 1, \text{ according as } \sum_{n=1}^{\infty} \frac{1}{n} c_n^k < \infty \text{ or } = \infty.$$

(2) (Shorack and Wellner (1978)). If $n^{-1} c_n \downarrow$ and either $c_n \uparrow$ or $\liminf_{n\to\infty} c_n / \log_2 n \geqq 1$, where $\log_j$ stands for the $j$-th iterated logarithm, then

$$P(nU_{k,n} \geqq c_n \text{ i.o.}) = 0 \text{ or } 1, \text{ according as } \sum_{n=1}^{\infty} \frac{1}{n} c_n^k \exp(-c_n) < \infty \text{ or } = \infty.$$

It may be remarked here that (1) is due to Geffroy (1958, 1959) for $k = 1$, while (2) is due to Robbins and Siegmund (1972) for $k = 1$. Earlier, Barndorff-

Nielsen (1961) had given a variation of the same result, assuming that $n^{-1} c_n \downarrow$ and that $(1-n^{-1} c_n)^n \downarrow$, in which case, we have

(3) $P(nU_{1,n} \leqq c_n \text{ i.o.}) = 0$ or 1, according as $\sum_{n=1}^{\infty} \frac{\log_2 n}{n} (1-n^{-1} c_n)^n < \infty$ or $=\infty$.

A direct application of either of these criteria shows that, for any fixed $p \geqq 4$,

$$(4) \qquad \begin{aligned} P(nU_{k,n} &\leqq \{(\log n)(\log_2 n)\dots(\log_p n)^{1+\varepsilon}\}^{-1/k} \text{ i.o.}) \\ &= P(nU_{k,n} \geqq \log_2 n + (k+1)\log_3 n \\ &\quad + \log_4 n + \dots + (1+\varepsilon)\log_p n \text{ i.o.}) = 0 \text{ or } 1, \end{aligned}$$

according as $\varepsilon > 0$ or $\varepsilon \leqq 0$. These bounds are given in Deheuvels (1974).

When $k_n \uparrow \infty$, the situation is more complex, and, up to now, only first order terms are known, from Kiefer (1972). We state now his results, which we reformulate, using different notations.

(5) (Kiefer, 1972). Suppose that $k_n \uparrow \infty$, and that $n^{-1} k_n \sim p_n \downarrow 0$. Then:

(5.1) If $k_n/\log_2 n \to \infty$ $\left(\text{and } \frac{np_n}{\log_2 n} \uparrow\right)$, then $\limsup_{n\to\infty} \pm \frac{nU_{k,n} - k}{\sqrt{2k \log_2 n}} = 1$ a.s.;

(5.2) If $k_n/\log_2 n \to v \in (0, \infty)$, and if $-1 < \delta' < 0 < \delta''$ are the two roots (in $\delta$) of the equation $v^{-1} = \delta - \log(1+\delta)$, then

$$\liminf_{n\to\infty} \frac{nU_{k,n} - k}{k} = \delta' \quad \text{and} \quad \limsup_{n\to\infty} \frac{nU_{k,n} - k}{k} = \delta'' \quad \text{a.s.}$$

(5.3) If $k_n/\log_2 n \to 0$, then $\limsup_{n\to\infty} \frac{nU_{k,n} - k}{\log_2 n} = 1$ a.s.;

(5.4) If $k_n/\log_2 n \to 0$ $\left(\text{together with } np_n \uparrow \infty \text{ and } \frac{np_n}{\log_2 n} \downarrow 0\right)$ then

$$P(nU_{k,n} < k\{\log n\}^{-(1+\varepsilon)/k} \text{ i.o.}) = 0 \text{ or } 1, \text{ according as } \varepsilon > 0 \text{ or } \varepsilon < 0.$$

In the expressions above, we remark that the use of an auxiliary sequence $p_n$ to ensure regularity conditions on the rate of increase of $k_n$ is necessary, due to the fact that there is no non-ultimately constant non-decreasing sequence $k_n = o(n)$ such that $k_n/n$ is non-increasing.

In the sequel, we shall precise these bounds. Our main results are stated below.

**Theorem 1.** (upper-upper class). *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Let also $c_n$ be a sequence such that $\lim_{n\to\infty} k_n^{-1/2}(c_n - k_n) = +\infty$, and*

$$(6) \qquad \sum_{n}^{\infty} \frac{1}{n} k^{1/2} \left(\frac{e}{k} c_n\right)^k \exp(-c_n) < \infty.$$

*In addition, suppose that one of the conditions* (i), (ii) *or* (iii) *below is satisfied:*

(i) $n^{-1} c_n \downarrow$;

(ii) *For any* $n \geqq 1$ *such that* $k_{n+1} = k_n$, $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$; $\limsup_{n\to\infty} (c_{n+1}/c_n) < \infty$; $\liminf_{n\to\infty} k_n^{-1}(c_n - k_n) > 0$; $\lim_{n\to\infty} (\log k_{n+1})/\log k_n = 1$;

(iii) *For any* $n \geqq 1$ *such that* $k_{n+1} = k_n$, $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$;

$$0 < \liminf_{n\to\infty} \frac{c_n - k_n}{\log_2 n} \leqq \limsup_{n\to\infty} \frac{c_n}{\log_2 n} < \infty.$$

*Then* $P(n U_{k,n} > c_n \text{ i.o.}) = 0$.

**Theorem 2** (lower-lower class). *Let* $k = k_n \geqq 1$ *be a non-decreasing integer sequence such that* $\limsup_{n\to\infty} n^{-1} k_n < 1$. *Let also* $c_n$ *be a sequence such that:*

(a) *If* $\lim_{n\to\infty} k_n = \infty$, *then* $\lim_{n\to\infty} k_n^{-1/2}(c_n - k_n) = -\infty$;

(b) *If* $\lim_{n\to\infty} k_n < \infty$, *then* $\lim_{n\to\infty} c_n = 0$;

(c) $\sum_n \frac{1}{n} k^{1/2} \left(\frac{e}{k} c_n\right)^k \exp(-c_n) < \infty$.

*In addition, suppose that one of the conditions* (i) *or* (ii) *below is satisfied.*

(i) $n^{-1} c_n \downarrow$;

(ii) $k_n \to \infty$; $k_n = O(n^{1/2})$; $\lim_{n\to\infty} (k_n \log k_n)^{-1/2}(c_n - k_n) = -\infty$; *for any* $n \geqq 1$ *such that* $k_n = k_{n+1}$, $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$.

*Then* $P(n U_{k,n} \leqq c_n \text{ i.o.}) = 0$.

*Remark 1.* The change of variable $c_n = k_n(1 + \delta_n)$ transforms (6) into the equivalent form

$$\sum_{n=1}^{\infty} \frac{k^{1/2}}{n} \exp(-k(\delta - \log(1+\delta))) < \infty, \tag{7}$$

where $k = k_n$ and $\delta = \delta_n$. Observe that the same condition (7) is used in Theorems 1–2 but with $\delta_n$ ultimately $> 0$ in Theorem 1 and ultimately $< 0$ in Theorem 2.

It will be proved in the sequel that $P(n U_{k,n} > c_n) \to 0$ iff $k_n^{-1/2}(c_n - k_n) \to \infty$, and likewise $P(n U_{k,n} < c_n) \to 0$ iff (a) or (b) in Theorem 2 holds. It follows that these conditions have to be assumed in order to have $P(n U_{k,n} > c_n$ i.o.$) = 0$ or $P(n U_{k,n} < c_n$ i.o.$) = 0$ repectively.

If we assume in Theorem 1 that $n^{-1} k_n \sim p_n \downarrow 0$ and that $c_n - k_n \sim \gamma \log_2 n$ for some $0 < \gamma < \infty$, then the conditions in this theorem sum up to (7) and $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$ for $k_{n+1} = k_n$. Likewise we see that for $k_n = O(\log_2 n)$, the regularity conditions in Theorem 2 are satisfied for all sequences $c_n$ of interest whenever $n^{-1} k_n \sim p_n \downarrow 0$.

**Theorem 3** (lower-upper class, small $k$'s). *Let* $k = k_n \geqq 1$ *be a nondecreasing integer sequence such that* $k_n = o(\log_2 n)$ *as* $n \to \infty$, *and such that there exists a* *sequence* $p_n\downarrow$ *with* $n^{-1}k_n\sim p_n$ *as* $n\to\infty$. *Assume further that* $c_n$ *is a sequence such that:*

(a) $0<\liminf_{n\to\infty}\frac{c_n}{\log_2 n}\leqq\limsup_{n\to\infty}\frac{c_n}{\log_2 n}<\infty$;

(b) *If* $k_n=k_{n+1}$, *then* $c_n\leqq c_{n+1}$ *and* $n^{-1}c_n\geqq(n+1)^{-1}c_{n+1}$;

(c) $\sum_n\frac{1}{n}k^{1/2}\left(\frac{e}{k}c_n\right)^k\exp(-c_n)=\infty$.

*Then* $P(nU_{k,n}>c_n \text{ i.o.})=1$.

**Corollary 1.** *Let* $k=k_n\geqq 1$ *be a nondecreasing sequence such that* $k_n=o(\log_2 n)$ *as* $n\to\infty$ *and such that there exists a sequence* $p_n\downarrow$ *with* $n^{-1}k_n\sim p_n$ *as* $n\to\infty$. *Then*

$$\limsup_{n\to\infty}\frac{nU_{k,n}-\log_2 n}{(k+1)\log\left(\frac{\log_2 n}{k}\right)}=1 \quad a.s. \tag{8}$$

**Theorem 4** (lower-upper class, the limit case). *Let* $k=k_n$ *be a non-decreasing integer sequence such that* $k_n=O(\log_2 n)$ *as* $n\to\infty$, *and such that there exists a sequence* $p_n\downarrow$ *with* $n^{-1}k_n\sim p_n$ *as* $n\to\infty$. *Suppose that* $\liminf_{n\to\infty}k_n/\log_2 n>0$. *Let* $c_n$ *be a sequence such that* $c_n\sim\frac{A}{n}\log_2 n$ for $0<A<\infty$, *and such that, whenever* $k_n=k_{n+1}$, $c_{n+1}\geqq c_n$ *and* $(n+1)^{-1}c_{n+1}\leqq n^{-1}c_n$. *Then, if*

$$\sum_n\frac{1}{n}k^{1/2}\left(\frac{e}{k}c_n\right)^k\exp(-c_n)=\infty, \tag{7}$$

$P(nU_{k,n}>c_n \text{ i.o.})=1$.

**Corollary 2.** *Let* $k=k_n=[v\log_2 n]$, *where* $v\in(0,\infty)$ *is fixed. Let* $\delta''>0$ *be the positive root of the equation (in* $\delta$*)* $v^{-1}=\delta-\log(1+\delta)$. *Then, for any fixed* $p\geqq 5$, *we have*

$$P\Big(nU_{k,n}>v(1+\delta'')\log_2 n + \left(\frac{1+\delta''}{\delta''}\right)\left(\tfrac{3}{2}\log_3 n+\log_4 n+\ldots+(1+\varepsilon)\log_p n\right) \text{ i.o.}\Big)=0 \text{ or } 1, \tag{9}$$

*according as* $\varepsilon>0$ *or* $\varepsilon<0$.

*Remark* 2. Let $F(\cdot)$ be a distribution function, and let $G(u)=\inf\{x\colon 1-F(x)\leqq u\}$ for $0<u<1$. It is easily seen that $X_n=G(U_n)$, $n=1,2,\ldots$ defines an i.i.d. sequence of random variables with distribution function $F(\cdot)$, and such that $M_{k,n}=G(U_{k,n})$ is the $k$-th maximum of $X_1,\ldots,X_n$.

It is therefore easy to translate the preceding results in terms of $M_{k,n}$ in order to obtain strong limiting bounds for $M_{k,n}$. By Theorems 1–4, we see that, under suitable regularity conditions on $k=k_n$, $\mu_n$ and $F(\cdot)$, we have

$$(10) \qquad P(M_{k,n} < \mu_n \text{ i.o.}) = 0 \text{ or } 1,$$

according as

$$(11) \qquad \sum_n \frac{k^{1/2}}{n} \left\{ \frac{en}{k} (1 - F(\mu_n)) \right\}^k \exp(-n(1 - F(\mu_n))) < \infty \text{ or } = \infty.$$

*Remark 3*. For the lower-upper bound, we have only considered the case where $k = k_n = O(\log_2 n)$. It is worthwhile to notice that (6) gives also a sharp result when $k_n/\log_2 n \to \infty$. This can be seen from the equivalent form (7):

$$(7) \qquad \sum_n \frac{k^{1/2}}{n} \exp(-k(\delta - \log(1+\delta))) < \infty.$$

If we take in (7) $\delta = \pm(1+\varepsilon)\left(\frac{1}{k} \log_2 n\right)^{1/2}$, using the fact that $\delta - \log(1+\delta) \sim \frac{1}{2}\delta^2$, as $\delta \to 0$, we see that the series in (7) converges or diverges according as $\varepsilon > 0$ or $\varepsilon < 0$. The same can be said about the lower-lower bound. This last result corresponds to (5.1).

*Remark 4*. Note that, for $k = k_n$ constant and $c_n < k$, we have

$$(k^{1/2}) \left(\frac{1}{n} c_n^k\right) \leqq \frac{k^{1/2}}{n} \left(\frac{e}{k} c_n\right)^k \exp(-c_n) \leqq (k^{1/2} e^k) \left(\frac{1}{n} c_n^k\right).$$

In this case we have (6)$\Leftrightarrow$(1), which means that (6) is then necessary and sufficient for $P(nU_{k,n} < c_n \text{ i.o.}) = 0$.

*Remark 5*. The series (6) converges or diverges, according as the same happens for

$$(12) \qquad \sum_n \frac{k}{n} \left(\frac{1}{k!} c_n^k \exp(-c_n)\right).$$

*Remark 6*. Corollary 1 gives a result very similar to that obtained in Deheuvels and Devroye (1984), where it was proved that if $M_n^k$ denotes the maximal $k$-spacing generated by uniformly distributed random variables on (0, 1), then, if $k = k_n \to \infty$ together with $k_n = o(\log_2 n)$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \frac{nM_n^k - \log n}{k \log\left(\frac{\log n}{k}\right)} = 1 \quad \text{a.s.}$$

This hints that some kind of identification of $M_n^k$ and $U_{k,n}$ should be possible (after changing $\log n$ into $\log_2 n$) via a strong invariance principle.

We do not offer here a proof of the sharpness of the conditions in Theorem 2. This will be made in a forthcoming publication. The proofs concerning the upper-upper, lower-lower and lower-upper classes are given in the next sections.

## 2. The Upper-Upper Class

Throughout, we shall make use of the following Borel-Cantelli-type lemma, due to Barndorff-Nielsen (1961) (see also Devroye (1981)).

**Lemma 1.** *Let $E_n$, $n=1,2,\ldots$ be a sequence of events such that $P(E_n)\to 0$ as $n\to\infty$. Then, if $\sum_n P(E_n^c\cap E_{n+1})<\infty$ or $\sum_n P(E_n\cap E_{n+1}^c)<\infty$, we have $P(E_n$ i.o.$)=0$.*

In this section, we shall consider the events defined by $E_n=\{U_{k,n}>n^{-1}c_n\}$. First, we give conditions for which $P(E_n)\to 0$.

**Lemma 2.** *Let $k=k_n\geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1}k_n<1$. Then $P(E_n)=P(U_{k,n}>n^{-1}c_n)\to 0$ as $n\to\infty$ iff*

$$k_n^{-1/2}(c_n-k_n)\to\infty.$$

*Proof.* Assume in the first place that $k_n\uparrow k<\infty$. Then $nU_{k,n}$ converges weakly to a $\Gamma(k)$ distribution as $n\to\infty$. Hence $P(E_n)\to 0$ iff $c_n\to\infty\Leftrightarrow k_n^{-1/2}(c_n-k_n)\to\infty$.

Next, if $k_n\to\infty$ and $n-k_n\to\infty$, it is well known (see e.g. Balkema and De Haan (1978) Th. 2.2) that

$$\{k(n-k)n^{-3}\}^{-1/2}(U_{k,n}-n^{-1}k)\to N(0,1) \quad \text{in distribution.}$$

It follows that, when $\limsup_{n\to\infty} n^{-1}k_n<1$, the condition $k_n^{-1/2}(c_n-k_n)\to\infty$ is necessary and sufficient for $P(E_n)\to 0$.

Note here that if $c_n=k_n(1+\delta_n)$, this condition amounts to $k_n^{1/2}\delta_n\to\infty$ or equivalently to $k_n\delta_n^2\to\infty$.

**Lemma 3.** *Let $k=k_n\geqq 1$ be a non-decreasing integer sequence, and let $0<c_n<n$ for $n=1,2,\ldots$. Then*

$$P(E_n\cap E_{n+1}^c)\leqq P\left(\frac{c_n}{n}<U_{k-1,n}\leqq\frac{c_{n+1}}{n+1}\right)\frac{c_{n+1}}{n+1}$$
$$+P\left(U_{k-1,n}\leqq\frac{c_n}{n}<U_{k,n}\right)\frac{c_{n+1}}{n+1}1_{\{k_n=k_{n+1}\}}+P\left(\frac{c_n}{n}<U_{k,n}\leqq\frac{c_{n+1}}{n+1}\right).$$

*Proof.* 1°) Assume that $k=k_n=k_{n+1}$. Then

$$E_n\cap E_{n+1}^c=\left\{U_{k,n}>\frac{c_n}{n},\ U_{k,n+1}\leqq\frac{c_{n+1}}{n+1}\right\}$$
$$\subset\left(\left\{U_{k-1,n}\leqq\frac{c_{n+1}}{n+1}\right\}\cap\left\{U_{k,n}>\frac{c_n}{n}\right\}\cap\left\{U_{n+1}\leqq\frac{c_{n+1}}{n+1}\right\}\right)\cup\left\{\frac{c_n}{n}<U_{k,n}\leqq\frac{c_{n+1}}{n+1}\right\},$$

as requested.

2°) Let now $k_{n+1}>k=k_n$. Then $U_{k_{n+1},n+1}\geqq U_{k_n,n}$ and hence

$$E_n\cap E_{n+1}^c\subset\left\{\frac{c_n}{n}<U_{k,n}\leqq\frac{c_{n+1}}{n+1}\right\}.$$

This completes the proof of our lemma.

**Lemma 4.** *Let $1 \leqq k < n$ and $0 < p < 1$. Let also $Z$ denote a random variable with a binomial $B(n, p)$ distribution. Then we have the following identities:*

$$P(U_{k-1,n} \leqq p < U_{k,n}) = P(Z = k-1) = \binom{n}{k-1} p^{k-1}(1-p)^{n-k+1},$$

$$P(U_{k,n} > p) = P(Z \leqq k-1) = \sum_{j=0}^{k-1} \binom{n}{j} p^j (1-p)^{n-j},$$

$$P(U_{k,n} \leqq p) = P(Z \geqq k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j},$$

*where $U_{0,n} = 0$.*

*Proof.* Straightforward.

**Lemma 5.** *Let $k = k_n \geqq 1$ be a sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Let $c_n = k_n(1+\delta_n) = k(1+\delta) \in (0, n)$. Then, as $n \to \infty$, we have, uniformly in $\delta > 0$,*

$$P(U_{k-1,n} \leqq n^{-1} c_n < U_{k,n}) = O\left\{\frac{k^{-1/2}}{1+\delta} \exp(-k(\delta - \log(1+\delta)))\right\}.$$

*Proof.* By Lemma 4 and Stirling's formula,

$$P(U_{k-1,n} \leqq n^{-1} c_n < U_{k,n}) = O\left\{\left(k\left(1-\frac{k}{n}\right)\right)^{-1/2} (1+\delta)^{k-1} \left(1 - \frac{k\delta}{n-k}\right)^{n-k}\right\},$$

which gives the result, using the bound $(1-a)^r \leqq e^{-ra}$, $r > 0$, $0 \leqq a < 1$.

**Lemma 6.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Let $c_n = k_n(1+\delta_n) \geqq k_n$ be a sequence such that $n^{-1} c_n \downarrow$. Then $\sum_n P(E_n \cap E_{n+1}^c) < \infty$ whenever*

$$\sum_n \frac{1}{n} k_n^{1/2} \exp(-k_n(\delta_n - \log(1+\delta_n))) < \infty.$$

*Proof.* By Lemma 3, $P(E_n \cap E_{n+1}^c) \leqq n^{-1} c_n P(U_{k-1,n} \leqq n^{-1} c_n < U_{k,n})$, which, by Lemma 5, implies the result.

**Lemma 7.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Let $c_n = k_n(1+\delta_n) \in (k_n, n)$ be such that $k_n^{1/2} \delta_n \to \infty$. Suppose further that one of the following set of conditions is satisfied:*

(i) $n^{-1}(1+\delta_n) = c_n/k_n \downarrow$;

$$\sum_n k_n^{-1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n))) < \infty;$$

(ii) *For any $n \geqq 1$ such that $k_{n+1} = k_n$, we have $(n+1) c_{n+1} \leqq n^{-1} c_n$;*

$$\sum_n k_n^{1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n))) < \infty; \qquad \limsup_{n\to\infty} (c_{n+1}/c_n) < \infty.$$

*Then* $\sum_n \left\{P\left(\frac{c_n}{n} < U_{k,n} \leqq \frac{c_{n+1}}{n+1}\right) + P\left(\frac{c_n}{n} < U_{k-1,n} \leqq \frac{c_{n+1}}{n+1}\right) \frac{c_{n+1}}{n+1}\right\} < \infty.$

*Proof.* Let $P_n = P(n^{-1} c_n < U_{k,n} \leqq (n+1)^{-1} c_{n+1})$. Clearly, $P_n = 0$ when $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$. Therefore, it suffices to consider the case where $(n+1)^{-1} c_{n+1} > n^{-1} c_n$. When (i) holds, we shall use the bound

$$(n+1)^{-1} c_{n+1} - n^{-1} c_n \leqq (k_{n+1} - k_n)\left(\frac{1+\delta_n}{n}\right).$$

On the other hand, if (ii) holds, the bound $(n+1)^{-1} c_{n+1} - n^{-1} c_n \leqq (n+1)^{-1} c_{n+1}$ suffices for our needs.

By the assumption $k_n^{1/2} \delta_n \to \infty$, we have, for $n$ large enough,

$$n^{-1} c_n = n^{-1} k_n(1+\delta_n) > \frac{k_n - 1}{n-1} = n^{-1} k_n(1 + O(n^{-1}) + O(k_n^{-1})).$$

This, in turn, implies that

$$\begin{aligned} P_n &= \int_{n^{-1} c_n}^{(n+1)^{-1} c_{n+1}} \binom{n}{k} k x^{k-1} (1-x)^{n-k} \, dx \\ &\leqq \left\{\frac{c_{n+1}}{n+1} - \frac{c_n}{n}\right\} \binom{n}{k} \frac{n}{1+\delta} \left(\frac{k}{n}(1+\delta)\right)^k \left(1 - \frac{k}{n}(1+\delta)\right)^{n-k}. \end{aligned}$$

By the same arguments as in the proof of Lemma 5, we have in case (i):

$$P_n = O\{k_n^{-1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n)))\},$$

while in case (ii), we get

$$P_n = O\{k_n^{1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n)))\},$$

as desired.

Let $Q_n = P(n^{-1} c_n < U_{k-1,n} \leqq (n+1)^{-1} c_{n+1})$. We get likewise, for $n$ large enough,

$$Q_n \leqq \left\{\frac{c_{n+1}}{n+1} - \frac{c_n}{n}\right\} \binom{n}{k} \frac{n}{(1+\delta)^2} \left(\frac{k}{n}(1+\delta)\right)^k \left(1 - \frac{k}{n}(1+\delta)\right)^{n-k}.$$

In case (i), we obtain

$$Q_n \frac{c_{n+1}}{n+1} = O(Q_n(1+\delta_n)) = O\{k_n^{-1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n)))\}.$$

In case (ii), we get

$$Q_n(k_{n+1} - k_n) \frac{c_{n+1}}{n+1} \geqq Q_n \frac{c_{n+1}}{n+1} = O\left\{\frac{c_n^2}{n^2} \binom{n}{k} \frac{n}{(1+\delta)^2} \left(\frac{k}{n}(1+\delta)\right)^k \left(1 - \frac{k}{n}(1+\delta)\right)^{n-k}\right\}.$$

It suffices now to remark that $c_n^2 = k^2(1+\delta)^2 \leqq n k (1+\delta)^2$ to complete the proof.

**Lemma 8.** *Let $k_n \uparrow \infty$, $\delta_n > 0$, and assume that*

$$\limsup_{n\to\infty} \frac{\log k_{n+1}}{\log k_n} = 1 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{k_n \delta_n^2}{\log k_n} > 1 + 2r,$$

*where $r \geqq 0$ is fixed. Then*

$$\sum_n k_n^{r-1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n))) < \infty.$$

*Proof.* Note in the first place that $x - \log(1+x) \sim \frac{1}{2}x^2$ as $x \to 0$. It follows that there exists an $\varepsilon > 0$ and an $n_\varepsilon$ such that $n \geqq n_\varepsilon \Rightarrow k_n(\delta_n - \log(1+\delta_n)) > (r + \frac{1}{2} + \varepsilon) \log k_{n+1}$.

This, in turn, implies that, for some $\varepsilon > 0$,

$$\begin{aligned} &k_n^{r-1/2}(k_{n+1} - k_n) \exp(-k_n(\delta_n - \log(1+\delta_n))) \\ &\quad = O\{k_{n+1}^{-1-\varepsilon}(k_{n+1} - k_n)\} = O\{k_n^{-\varepsilon} - k_{n+1}^{-\varepsilon}\}. \end{aligned}$$

The result follows, since $\sum_n (k_n^{-\varepsilon} - k_{n+1}^{-\varepsilon}) < \infty$.

**Theorem 5.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Let $c_n = k_n(1+\delta_n) \in (0, n)$. Assume that $k_n^{-1/2}(c_n - k_n) = k_n^{1/2} \delta_n \to +\infty$, and that*

$$\sum_n \frac{1}{n} k_n^{1/2} \exp(-k_n(\delta_n - \log(1+\delta_n))) < \infty.$$

*In addition, suppose that one of the following set of conditions is satisfied:*

(i) $n^{-1} c_n = k_n n^{-1}(1+\delta_n) \downarrow$;

(ii) $n^{-1}(1+\delta_n) \downarrow$; $\liminf_{n\to\infty} \frac{k_n \delta_n^2}{\log k_n} > 1$; $\lim_{n\to\infty} \frac{\log k_{n+1}}{\log k_n} = 1$;

(iii) *For all $n \geqq 1$ such that $k_n = k_{n+1}$, $(n+1)^{-1} c_{n+1} \leqq n^{-1} c_n$;*

$$\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} < \infty; \quad \liminf_{n\to\infty} \frac{k_n \delta_n^2}{\log k_n} > 3; \quad \lim_{n\to\infty} \frac{\log k_{n+1}}{\log k_n} = 1.$$

*Then $P(n U_{k,n} > c_n \text{ i.o.}) = 0$.*

*Proof.* It follows as a direct consequence of Lemmas 1–8.

*Remark* 7. 1°) By (5), if $n^{-1} k_n \sim p_n \downarrow 0$, it can be seen that

$$n U_{k,n} - k_n = O\left(\max\left\{\log_2 n, \frac{\sqrt{\log_2 n}}{\sqrt{k_n}}\right\}\right)$$

almost surely. It follows that the condition (ii) of Theorem 5 is applicable in the case where $k_n(\log k_n)^{-1} k_n^{-1} \log_2 n = (\log_2 n)/(\log k_n) > 1$ in the upper tail. This corresponds to the situation where, for some $\varepsilon > 0$,

$$k_n = o(\log^{1-\varepsilon} n).$$

Likewise the range of application of (iii) includes $k_n = o(\log^{\frac{1}{3}-\varepsilon} n)$.

2°) By Lemma 8, one could extend the conditions of Theorem 5 to the case where $\limsup_{n\to\infty} \frac{\log k_{n+1}}{\log k_n} < \infty$. We shall not state such results which correspond to an irregular behavior of $k_n$. Clearly, if $k_n\uparrow$ and $n^{-1}k_n \sim p_n\downarrow$, we have $k_{n+1}/k_n \to 1$, and hence $(\log k_{n+1})/\log k_n \to 1$.

*Proof of Theorem 1.* The first two statements of Theorem 1 (corresponding to conditions (i) and (ii)) follow directly from Theorem 5, since

$$\liminf_{n\to\infty} k_n^{-1}(c_n - k_n) = \liminf_{n\to\infty} \delta_n > 0$$

implies, if $k_n \uparrow \infty$, that $\frac{k_n \delta_n^2}{\log k_n} \to \infty$. The proof is completed by Lemma 8.

Condition (iii) corresponds to the case where $k_n = O(\log_2 n)$, and implies that $\limsup_{n\to\infty}(c_{n+1}/c_n) < \infty$. Under this condition, there exists a $\theta > 0$ such that, for $n$ large enough,

$$k_n^{1/2} \exp(-k_n(\delta_n - \log(1+\delta_n))) \leqq (\log n)^{-\theta}.$$

By Abel's lemma, we see that the series $\sum_n (k_{n+1} - k_n)(\log n)^{-\theta}$ converges, since $\sum_n k_n n^{-1}(\log n)^{-1-\theta} = O\{\sum_n n^{-1}(\log n)^{-1-\theta/2}\} < \infty$.

Lemma 7 completes the proof of Theorem 1.

## 3. The Lower-Lower Class

In the sequel, we shall use the notations and results of Section 2 and consider the events $F_n = \{U_{k,n} \leqq n^{-1} c_n\} = E_n^c$. We remark that $P(F_n^c \cap F_{n+1}) = P(E_n \cap E_{n+1}^c)$. Our proofs will be based on Lemma 1 which implies that $P(F_n \text{ i.o.}) = 0$ whenever $\sum_n P(F_n^c \cap F_{n+1}) < \infty$ and $P(F_n) \to 0$.

**Lemma 9.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n\to\infty} n^{-1} k_n < 1$. Then $P(F_n) = P(nU_{k,n} \leqq c_n) \to 0$ for $c_n \geqq 0$ iff:*

(i) $c_n \to 0$ *when* $\lim_{n\to\infty} k_n < \infty$;

(ii) $k_n^{-1/2}(c_n - k_n) \to -\infty$ *when* $\lim_{n\to\infty} k_n = \infty$.

*Proof.* Same as for Lemma 2.

Note here that if $c_n = k_n(1+\delta_n)$, the conditions above amount to (i) $\delta_n \to -1$ if $\lim_{n\to\infty} k_n < \infty$; (ii) $k_n^{1/2}\delta_n \to -\infty$ if $\lim_{n\to\infty} k_n = \infty$.

**Lemma 10.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $1 \leqq k_n \leqq n$, $n = 1, 2, \ldots$. Let $c_n \in (0, n)$, $n = 1, 2, \ldots$. Then, if $k' = k_{n+1}$,*

$$P(F_n^c \cap F_{n+1}) \leqq P\left(\frac{c_n}{n} < U_{k'-1,n} \leqq \frac{c_{n+1}}{n+1}\right) \frac{c_{n+1}}{n+1}$$
$$+ P\left(U_{k-1,n} \leqq \frac{c_n}{n} < U_{k,n}\right) \frac{c_{n+1}}{n+1} 1_{\{k_n = k_{n+1}\}} + P\left(\frac{c_n}{n} < U_{k',n} \leqq \frac{c_{n+1}}{n+1}\right).$$

*Proof.* In general, $F_n^c \cap F_{n+1} = \left\{ U_{k_n, n} > \frac{c_n}{n},\ U_{k_{n+1}, n+1} \leqq \frac{c_{n+1}}{n+1} \right\}$.

1°) Let $k_{n+1} > k_n$. In this case $U_{k_{n+1}, n+1} \geqq U_{k_n, n}$ and

$$F_n^c \cap F_{n+1} \subset \left( \left\{ \frac{c_n}{n} < U_{k_{n+1}, n} \leqq \frac{c_{n+1}}{n+1} \right\} \cap \left\{ U_{n+1} > \frac{c_{n+1}}{n+1} \right\} \right) \cup \left( \left\{ \frac{c_n}{n} < U_{k_{n+1}-1, n} \leqq \frac{c_{n+1}}{n+1} \right\} \cap \left\{ U_{n+1} \leqq \frac{c_{n+1}}{n+1} \right\} \right).$$

2°) Let $k_{n+1} = k_n$. We get now

$$F_n^c \cap F_{n+1} \subset \left( \left\{ \frac{c_n}{n} < U_{k,n} \leqq \frac{c_{n+1}}{n+1} \right\} \cap \left\{ U_{n+1} > \frac{c_{n+1}}{n+1} \right\} \right) \cup \left( \left\{ \frac{c_n}{n} < U_{k-1,n} \leqq \frac{c_{n+1}}{n+1} \right\} \cap \left\{ U_{n+1} \leqq \frac{c_{n+1}}{n+1} \right\} \right) \cup \left( \left\{ U_{k-1,n} \leqq \frac{c_n}{n} < U_{k,n} \right\} \cap \left\{ U_{n+1} \leqq \frac{c_{n+1}}{n+1} \right\} \right).$$

The result follows.

**Lemma 11.** *Let $k = k_n \geqq 1$ be a sequence such that $\limsup_{n \to \infty} n^{-1} k_n < 1$. Let $c_n = k_n(1+\delta_n) = k(1+\delta) \in (0, n)$. Let $\Delta_n$ be a sequence such that $-1 \leqq \Delta_n \leqq 0$, $n = 1, 2, \ldots$, and that $k_n \Delta_n = O(n^{1/2})$ as $n \to \infty$. Then, as $n \to \infty$, we have uniformly in $\delta \in [\Delta_n, 0]$,*

$$P(U_{k-1,n} \leqq n^{-1} c_n < U_{k,n}) = O\left\{ \frac{k^{-1/2}}{1+\delta} \exp(-k(\delta - \log(1+\delta))) \right\}.$$

*Proof.* The proof is identical to the proof of Lemma 5, up to the point where we set $\left(1 - \frac{k\delta}{n-k}\right)^{n-k} = O\{\exp(-k\delta)\}$, which holds for $k^2 \delta^2 = O(n)$.

*Remark.* 1°) If $k_n = O(n^{1/2})$, we may choose $\Delta_n = -1$.

2°) If $k_n / \log n \to \infty$, in view of (5.1), there is no loss of generality in taking $\Delta_n = -((2+\varepsilon) k_n^{-1} \log_2 n)^{1/2}$ for some $\varepsilon > 0$. In order that $k_n \Delta_n = O(n^{1/2})$, we must restrict the range of $k_n$ by assuming that $k_n = O(n / \log_2 n)$ as $n \to \infty$.

**Lemma 12.** *Let $k = k_n \geqq 1$ be a non-decreasing integer sequence such that $\limsup_{n \to \infty} n^{-1} k_n < 1$ and $\lim_{n \to \infty} k_n = \infty$. Let $c_n = k_n(1+\delta_n) \in (0, k_n)$ be such that $k_n^{1/2} \delta_n \to -\infty$. Suppose further that one of the following sets of conditions is satisfied:*

(i) $n^{-1}(1+\delta_n) = c_n / k_n \downarrow$;

$$\sum_n k_{n+1}^{-1/2} (k_{n+1} - k_n) \exp(-k_{n+1}(\delta_{n+1} - \log(1+\delta_{n+1}))) < \infty;$$

(ii) *For any* $n \geqq 1$ *such that* $k_{n+1}=k_n$, *we have* $(n+1)\,c_{n+1} \leqq n^{-1}\,c_n$;

$$\sum_n k_{n+1}^{1/2}(k_{n+1}-k_n)\exp(-k_{n+1}(\delta_{n+1}-\log(1+\delta_{n+1})))<\infty;$$

*Then* $$\sum_n \left\{ P\left(\frac{c_n}{n}<U_{k_{n+1}-1,n} \leqq \frac{c_{n+1}}{n+1}\right)\frac{c_{n+1}}{n+1}+P\left(\frac{c_n}{n}<U_{k_{n+1},n} \leqq \frac{c_{n+1}}{n+1}\right)\right\}<\infty.$$

*Proof.* The proof is identical to the proof of Lemma 7 up to minor changes. In the first place, we have, as $n\to\infty$, $n^{-1}\,c_n<\dfrac{k_n-1}{n-1}$. Next, we use the bounds

$$(n+1)^{-1}\,c_{n+1}-n^{-1}\,c_n \leqq (k_{n+1}-k_n)\left(\frac{1+\delta_{n+1}}{n+1}\right)$$
$$\leqq k_{n+1}\left(\frac{1+\delta_{n+1}}{n+1}\right) \leqq (k_{n+1}-k_n)\,k_{n+1}\left(\frac{1+\delta_{n+1}}{n+1}\right)$$

whose validity depend on assumptions (i) or (ii). Finally, in the evaluations of $P_n$ and $Q_n$, we replace throughout $\delta_n$ by $\delta_{n+1}$, $k_n$ by $k_{n+1}$ (leaving only unchanged the terms $k_{n+1}-k_n$) and $c_n$ by $c_{n+1}$. We omit further details.

We dont need here (as was the case in Lemma 7) the assumption that $c_{n+1}/c_n$ is bounded from above.

**Theorem 6.** *Let* $k=k_n \geqq 1$ *be a non-decreasing integer sequence such that* $\limsup_{n\to\infty} n^{-1}\,k_n<1$. *Let* $c_n=k_n(1+\delta_n)\in(0,k_n)$. *Assume that* $k_n^{-1/2}(c_n-k_n)=k_n^{1/2}\,\delta_n \to -\infty$ *if* $k_n\uparrow\infty$ *and that* $c_n\to 0$ *(i.e.* $\delta_n\to -1$*) if* $k_n$ *is bounded. Assume also that* $k_n\,\delta_n=O(n^{1/2})$ *and that*

$$\sum_n \frac{1}{n}\,k_n^{1/2}\exp(-k_n(\delta_n-\log(1+\delta_n)))<\infty.$$

*In addition, suppose that one of the following sets of conditions is satisfied:*

(i) $n^{-1}\,c_n=k_n\,n^{-1}(1+\delta_n)\downarrow$;

(ii) $n^{-1}(1+\delta_n)\downarrow$; $\liminf_{n\to\infty}\dfrac{k_n\,\delta_n^2}{\log k_n}>1$;

(iii) *For any* $n\geqq 1$ *such that* $k_n=k_{n+1}$, $(n+1)^{-1}\,c_{n+1}\leqq n^{-1}\,c_n$;

$$\liminf_{n\to\infty}\frac{k_n\,\delta_n^2}{\log k_n}>3.$$

*Then* $P(U_{k,n}\leqq c_n \text{ i.o.})=0$.

*Proof.* In view of Lemmas 10–12, the proof is identical to the proof of Theorem 6, hence, details will be omitted. We remark only that we dont need the assumption that $(\log k_{n+1})/\log k_n$ is bounded from above (see Lemma 8).

*Proof of Theorem 2.* Theorem 2 is a direct corollary of Theorem 6.

### 3. The Upper-Lower Class

In this section, we shall make use of the sequence $n_j = \left[\exp\left(\frac{j}{\log j}\right)\right]$.

**Lemma 13.** *Let $k = k_n \uparrow$ be a sequence such that $\lim_{n\to\infty} k_n^{-1}(k_{n+1}-k_n)=0$. Let $c_n = k(1+\delta_n)$ be a sequence such that $\liminf_{n\to\infty} \delta_n > 0$ and that $k_n^{-1/2}(c_n - k_n) = k_n^{1/2}\delta_n \uparrow$. Then the sequence $k_n(\delta_n - \log(1+\delta_n))$ is ultimately nondecreasing in n.*

*Proof.* Let $k_{n+1}^{1/2}\delta_{n+1} \geqq k_n^{1/2}\delta_n$, or equivalently, let $\delta_{n+1} \geqq \delta_n/(1+\alpha)$, where $\alpha = (k_{n+1}/k_n)^{1/2} - 1 \geqq 0$. Noting that $\delta - \log(1+\delta)$ is an increasing function of $\delta \geqq 0$, we have:

$$\begin{aligned}
&k_{n+1}(\delta_{n+1} - \log(1+\delta_{n+1}))\\
&\quad \geqq k_{n+1}\left\{\delta_n (k_n/k_{n+1})^{1/2} - \log(1+\delta_n) - \log\left(1 - \left(\frac{\delta_n}{1+\delta_n}\right)(1-(k_n/k_{n+1})^{1/2})\right)\right\}\\
&\quad \geqq k_n(\delta_n - \log(1+\delta_n))\\
&\qquad + k_n\left\{\alpha\delta_n - (2\alpha+\alpha^2)\log(1+\delta_n) + (1+\alpha)^2\left(\frac{\delta_n}{1+\delta_n}\right)\left(1-\frac{\alpha}{1+\alpha}\right)\right\}\\
&\quad = k_n(\delta_n - \log(1+\delta_n)) + k_n\alpha\left\{\log(1+\delta_n) - \frac{\delta_n}{1+\delta_n}\right\}(R(\delta_n)-\alpha)
\end{aligned}$$

where $R(\delta) = \left(\delta - 2\log(1+\delta) + \frac{\delta}{1+\delta}\right) \Big/ \left(\log(1+\delta) - \frac{\delta}{1+\delta}\right)$. Routine computations show that $R(\delta) > 0$ for all $\delta > 0$, while $R(\delta) \sim \frac{2}{3}\delta$ as $\delta \to 0$ and $R(\delta) \sim \delta/\log(1+\delta)$ as $\delta \to \infty$. Hence, our assumptions imply that $\alpha \to 0$ while $\liminf_{n\to\infty} R(\delta_n) > 0$. This suffices for proof of Lemma 13, since, for all $\delta \geqq 0$,

$$\log(1+\delta) - \frac{\delta}{1+\delta} \geqq 0.$$

*Remark.* The assumption that $n^{-1}k_n \sim p_n \downarrow$ implies that $\lim_{n\to\infty} k_n^{-1}(k_{n+1}-k_n) = 0$, *if* $k_n \uparrow$.

**Lemma 14.** *Let $k = k_n \geqq 1$ be a nondecreasing sequence such that $k_n = O(\log_2 n)$. Let $c_n = k_n(1+\delta_n) = k(1+\delta)$ be such that $\liminf_{n\to\infty} \delta_n > 0$ and that $k_n^{-1/2}(c_n - k_n) = k_n^{1/2}\delta_n \uparrow$. Assume further that there exists a sequence $p_n \downarrow$ such that $n^{-1}k_n \sim p_n$ as $n \to \infty$. Then, if*

$$\sum_{n=1}^{\infty} \frac{1}{n} k^{1/2} \exp(-k(\delta - \log(1+\delta))) = \infty, \tag{14}$$

*we have*

$$\sum_j \frac{k_{n_j}^{1/2}}{\log_2 n_j} \exp(-k_{n_j}(\delta_{n_j} - \log(1+\delta_{n_j}))) = \infty. \tag{15}$$

*Proof.* Evidently $(14)\Leftrightarrow\sum_n p_n k^{-1/2}\exp(-k(\delta-\log(1+\delta)))=\infty$. By Lemma 13, this in turn implies that

$$\sum_j (n_{j+1}-n_j)\, p_{n_j} k_{n_j}^{-1/2}\exp(-k_{n_j}(\delta_{n_j}-\log(1+\delta_{n_j})))=\infty.$$

But this suffices for (15), since $(n_{j+1}-n_j)\, p_{n_j}\sim n_j^{-1}(n_{j+1}-n_j)\, k_{n_j}\sim k_{n_j}/\log_2 n_j$.

In the sequel, we shall assume throughout that $k=k_n\geqq 1$ is nondecreasing and such that $n^{-1}k_n\sim p_n\downarrow$ as $n\to\infty$. We shall also assume that there exist fixed constants $0<c<d<\infty$ such that, for $n=3,4,\ldots$,

$$c\log_2 n\leqq c_n-k_n=k_n\delta_n\leqq c_n=k_n(1+\delta_n)\leqq d\log_2 n. \tag{16}$$

There is no loss of generality in assuming that $c_n\leqq d\log_2 n$ for some finite $d$. This follows from the fact (see (5)) that $nU_{k,n}=O(\log_2 n)$ whenever $k=k_n=O(\log_2 n)$, and that (6) holds for $c_n=d\log_2 n$ and for a large enough $d$. Hence (14) is unaffected if we replace $c_n$ by $\min(c_n, d\log_2 n)$.

Likewise, we know from (4) that $U_{k,n}\geqq U_{k,n}\geqq\frac{1}{n}\log_2 n$ i.o.a.s. Clearly if we can prove that $P(nU_{k,n}\geqq\max(c_n,\log_2 n)\text{ i.o.})=1$, we will have also prove that $P(nU_{k,n}\geqq c_n\text{ i.o.})=1$. These arguments justify (16).

**Lemma 15.** *Let* (16) *be satisfied, and assume that, for any* $n\geqq 1$ *such that* $k_{n+1}=k_n$, $\delta_{n+1}\geqq\delta_n$. *Then* $(14)\Rightarrow(15)$.

*Proof.* First observe that there exists an $\alpha>0$ such that, for $n$ large enough, $\exp(-k_n(\delta_n-\log(1+\delta_n)))\leqq(\log n)^{-\alpha}$. It follows that for the proof of Lemma 15, it suffices to show that

$$\sum_j n_j^{-1}(n_{j+1}-n_j)\, k_{n_j}(\log n_j)^{-\alpha}\, 1_{\{k_{n_{j+1}}>k_{n_j}\}}<\infty.$$

This in turn follows from

$$\sum_j (\log n_j)^{-\alpha}\, 1_{\{k_{n_{j+1}}>k_{n_j}\}}<\infty.$$

Next, we note that, as $j\to\infty$, $(\log n_j)^{-\alpha}=o(j^{-\alpha/2})$, while

$$\sum_{i<j} 1_{\{k_{n_{i+1}}>k_{n_i}\}}\leqq k_{n_j}=O(\log j).$$

The proof of Lemma 15 is completed by Abel's lemma and the fact that $\sum_j j^{-1-\alpha}(\log j)<\infty$ for all $\alpha>0$.

*Remark.* Lemma 15 implies Lemma 14 when (16) holds. Under the assumptions of Lemma 15, we see that in (15), we may restrict the summation to those $j$'s for which $k_{n_{j+1}}=k_{n_j}$.

**Lemma 16.** *Let* $k=k_n\geqq 1$ *be such that* $n^{-1}k_n\to 0$. *Let* $\delta=\delta_n$ *be such that* $k_n\delta_n=o(n^{1/2})$ *and* $0<n^{-1}c_n=n^{-1}k_n(1+\delta_n)<1$. *Then, we have, as* $n\to\infty$,

(17) $$P(G_n) = P(U_{k-1,n} \leqq n^{-1} c_n < U_{k,n}) \sim \frac{Bk^{-1/2}}{1+\delta} \exp(-k(\delta - \log(1+\delta))),$$

*where $B$ is a constant. If $k_n \to \infty$, $B = (2\pi)^{-1/2}$, while $B = k^{k-1/2}/(k-1)!$ if $k_n \uparrow k < \infty$.*

*Proof.* We proceed as in the proof of Lemma 5.

**Lemma 17.** *Under the assumptions of Lemma* 15, *if*

(14) $$\sum_{n=1} \frac{1}{n} k^{1/2} \exp(-k(\delta - \log(1+\delta))) = \infty,$$

*then*

(18) $$\sum_{j \in J} P(A_j) = \infty,$$

*where $A_j = G_{n_j}$, $G_n = \{U_{k-1,n} \leqq n^{-1} c_n < U_{k,n}\}$ and $J = \{j \geqq 1 : k_{n_j} = k_{n_{[j-(\log j)^2]}}\}$.*

*Proof.* (14), jointly with Lemmas 15 and 16, implies that $\sum_j P(A_j) = \infty$ whenever (16) holds. Let $v_j = P(A_j)$. By the arguments in the proof of Lemma 14, we know that, as $j \to \infty$, $v_j = o(j^{-\beta})$ for some $\beta > 0$. It follows that

$$\sum_{j \notin J} v_j = O\Big(\sum_j j^{-\beta} (\log j)^3 1_{\{k_{n_{j+1}} > k_{n_j}\}}\Big) < \infty,$$

by Abel's lemma, as in the proof of Lemma 14.

*Proof of Theorem 3.* By the Hewitt-Savage zero-one law, with the notations above, we know that $P(A_j \text{ i.o.}) = 0$ or 1. Theorem 3 will be proved if we show that $P(A_j \text{ i.o.}) = 1$. For this, it suffices to show that there exists a constant $C > 0$ such that, for all $j_0$, there exists a $j_1 \geqq j_0$ with

(19) $$P\Bigg(\bigcup_{\substack{j=j_0 \\ j \in J}}^{j_1} A_j\Bigg) > C.$$

Throughout, we use the assumption that $\sum_{j \in J} P(A_j) = \infty$, which holds by Lemma 17. In the proof, we shall make use of a lower bound of the probability in (19). In order to simplify notations, let us make the convention in the sequel that the summation $\sum_j$ stands for $\sum_{j \in J}$. By an inequality of Chung and Erdős (1952),

(20) $$P\Bigg(\bigcup_{j=j_0}^{j_1} A_j\Bigg) \geqq \Bigg\{\sum_{j=j_0}^{j_1} P(A_j)\Bigg\}^2 \Bigg\{2 \sum_{j_0 \leqq j < i \leqq j_1} P(A_i \cap A_j) + \sum_{j=j_0}^{j_1} P(A_j)\Bigg\}^{-1}.$$

It follows that for (19) it suffices to show that, for $j_1 > j_0 \to \infty$, we have

(21) $$\sum_{j_0 \leqq j < i \leqq j_1} P(A_i \cap A_j) \leqq \frac{C}{2} \Bigg\{\sum_{j=j_0}^{j_1} P(A_j)\Bigg\}^2,$$

where, as usual, $i \in J$ and $j \in J$. The problem reduces therefore to the derivations of adequate upper bounds for $P(A_i \cap A_j)$.

Let us introduce for sake of convenience the following notations:

$$j<i, \quad k=k_{n_j} \leqq k'=k_{n_i}, \quad \delta=\delta_{n_j}, \quad \delta'=\delta_{n_i},$$
$$u_j=n_j^{-1} c_{n_j}=n_j^{-1} k(1+\delta), \quad u_i=n_i^{-1} c_{n_i}=n_i^{-1} k'(1+\delta').$$

In general, for $u_i \leqq u_j$, we have

$$P(A_i \mid A_j)=\sum_{l=0}^{k-1} \binom{k-1}{l} \left(\frac{u_i}{u_j}\right)^{k-l-1} \cdot \left(1-\frac{u_i}{u_j}\right)^l \binom{n_i-n_j}{k'-k+l} u_i^{k'-k+l}(1-u_i)^{n_i-n_j-k'+k-l}. \tag{22}$$

In the first place, we see that, for $j \geqq L_0$ large enough,

$$(1-u_i)^{n_i-n_j-k'+k-l} \leqq 2 \exp(-(n_i-n_j) u_i). \tag{23}$$

This follows from the fact that

$$k'-k \leqq k'-k+l \leqq k'-1=O(\log_2 n_i)=O(\log i), \quad u_i=O(n_i^{-1} \log i),$$

and

$$(k'-k+l) u_i=O(n_i^{-1} \log^2 i) \to 0 \quad \text{as} \quad i \to \infty.$$

We have used here the bound $(1-u)^r \leqq \exp(-ru)$, $r>0$, $0 \leqq u<1$.

Next, we note that $\sum_{l=0}^{k-1} \binom{k-1}{l} (u_i/u_j)^{k-l-1}(1-(u_i/u_j))^l=1$, which, by (22) and (23) implies

$$P(A_i \mid A_j) \leqq 2 \sum_{l=0}^{k'-1} \frac{1}{l!} \{(n_i-n_j) u_i\}^l \exp(-(n_i-n_j) u_i), \quad i \geqq j \geqq L_0. \tag{24}$$

Observe for further use that (24) remains valid for $u_i>u_j$.

Next, we split the range of $i>j$ into two subsets:

(I) $i \geqq \alpha(j)=\min\{l>j: l-j \geqq (\log l)(\log_2 l)\}$.

(II) $j<i<\alpha(j)$.

It may be checked (see e.g. Devroye (1982)) that the following evaluations hold, for some appropriate constants $L_1$ and $L_2$.

– For $j \geqq L_1$, $\alpha(j) \leqq j+2(\log j)(\log_2 j)$ and

$$-\frac{i-j}{\log i} \leqq \log\left(\frac{n_j}{n_i}\right) \leqq -\frac{i-j}{\log i}\left(1-\frac{2}{\log i}\right); \tag{25}$$

$$\frac{i-j}{\log i}\left(1-\frac{3(i-j)}{\log i}\right) \leqq 1-\frac{n_j}{n_i} \leqq \frac{i-j}{\log i}. \tag{26}$$

– For $j \geqq L_2$ and $i \leqq \alpha(j)$, we have

$$\log\left(\frac{n_j}{n_i}\right) \leqq -(\log_2 i)\left(1-\frac{2}{\log i}\right), \tag{27}$$

and

(28) $$1-\frac{2}{\log i} \leqq 1-\frac{n_j}{n_i} \leqq 1.$$

**Lemma 18.** *Under the assumptions above, we have, uniformly in* $i \geqq \alpha(j)$, $P(A_i \mid A_j) = O(P(A_i))$.

*Proof.* By (27), $(n_i - n_j) u_i = \left(1-\frac{n_j}{n_i}\right) k'(1+\delta') \sim k'(1+\delta')$. Also, by (16), we have $\liminf_{n\to\infty} \delta_n > 0$. By (24) and using the arguments in the proof of Lemma 5, it follows that

$$P(A_i \mid A_j) = O\left\{\frac{1}{(k'-1)!}((n_i-n_j)u_i)^{k'-1} \exp(-(n_i-n_j)u_i)\right\},$$

which gives in turn, by Stirling's formula,

$$\begin{aligned} P(A_i \mid A_j) &= O\left\{\frac{k'^{-1/2}}{1+\delta'} \exp\left(-k'\left\{\left(1-\frac{n_j}{n_i}\right)(1+\delta')-1-\log\left(\left(1-\frac{n_j}{n_i}\right)(1+\delta')\right)\right\}\right)\right\} \\ &= O\left\{\frac{k'^{-1/2}}{1+\delta'} \exp(-k'(\delta' - \log(1+\delta')))\right\} = O(P(A_i)), \end{aligned}$$

where we have used Lemma 16 and se fact (see (16) and (28)) that

$$k'(1+\delta') \frac{n_j}{n_i} \leqq \frac{2c_{n_i}}{\log i} = O\left(\frac{\log_2 n_i}{\log i}\right) = O(1).$$

This completes the proof of Lemma 18.

By Lemma 18, we have

(29) $$\sum_{j_0 \leqq j < \alpha(j) \leqq i \leqq j_1} P(A_i \cap A_j) = O\left\{\left(\sum_{j=j_0}^{j_1} P(A_j)\right)^2\right\},$$

as desired. It remains to obtain similar results for case (II) to complete the proof of Theorem 3. For this, we need only consider the case where $k=k'$ (see Lemma 17), $u_i \leqq u_j$, $c_{n_i} \geqq c_{n_j}$, $i \in J$ and $j \in J$.

By (22), we have the upper bound

(30) $$P(A_i \mid A_j) \leqq 2 \sum_{l=1}^{k-1} \binom{k-1}{l} \left(\frac{u_i}{u_j}\right)^{k-l-1} \left(1-\frac{u_i}{u_j}\right)^l \frac{1}{l!} \{(n_i-n_j)u_i\}^l \exp(-(n_i-n_j)u_i).$$

Let us use now the assumption that $c_{n_i} \geqq c_{n_j}$. We have

$$\frac{n_j}{n_i} \leqq \frac{u_i}{u_j} \leqq 1 \quad \text{and} \quad 0 \leqq 1-\frac{u_i}{u_j} \leqq 1-\frac{n_j}{n_i}.$$

This, jointly with (30), yields the bound

(31) $$P(A_i \mid A_j) \leqq 2 \sum_{l=0}^{k-1} \frac{1}{l!} \left\{k\left(1-\frac{n_j}{n_i}\right)\right\}^l \frac{1}{l!} \{(n_i-n_j)u_i\}^l \exp(-(n_i-n_j)u_i).$$

**Lemma 19.** *There exists an absolute constant $R<\infty$ such that, for all $x>0$,*

$$\sum_{l=0}^{\infty} \frac{x^{2l}}{(l!)^2} \leqq R e^{2x}. \tag{32}$$

*Proof.* We have $\sum_{l=0}^{\infty} \frac{x^{2l}}{(2l)!} \frac{(2l)!}{l!l!} = \sum_{l=0}^{\infty} \frac{x^{2l} 2^{2l}}{(2l)!} \left(\frac{1+\varepsilon(l)}{1+\sqrt{\pi l}}\right)$, where $\varepsilon(l)\to 0$ as $l\to\infty$, hence result.

By (31) and (32), we have

$$P(A_i \mid A_j) \leqq 2R \exp\left(-\left(1-\frac{n_j}{n_i}\right) n_i u_i \left(1-2\left(\frac{k}{n_i u_i}\right)^{1/2}\right)\right). \tag{33}$$

If we assume now that $k_n = o(\log_2 n)$ as $n\to\infty$, it follows from (16) and (33) that, for $i$ large enough,

$$\begin{aligned} P(A_i \mid A_j) &\leqq 2R \exp\left(-\left(1-\frac{n_j}{n_i}\right)(c\log i)(1+o(1))\right) \\ &\leqq 2R \exp\left(-\frac{c}{2}\left(1-\frac{n_j}{n_i}\right)\log i\right), \end{aligned} \tag{34}$$

where we have used the fact that $\log_2 n_i \sim \log i$.

Let us now choose $d>0$, and split $(II)=\{i\in J: j<i<\alpha(j)\}$ into two subsets:

$$(II)'_d = \{i\in J: j<i<\beta(j)\} \quad \text{and} \quad (II)''_d = \{i\in J: \beta(j)\leqq i<\alpha(j)\},$$

where $\beta(j)=\max\{l>j: l-j\leqq d\log l\}$. There exists an $L_4$ such that $j\geqq L_4$ implies $\beta(j)<\alpha(j)$, which will be assumed to hold from now on.

By (25) there exists $q=q(d)$ such that, for $j\geqq L_1$, if $j<i<\beta(j)$, we have

$$1-\frac{n_j}{n_i} \geqq q\left(\frac{i-j}{\log i}\right) > 0.$$

Likewise, there exists $r=r(d)$ such that, for $\beta(j)\leqq i<\alpha(j)$, we have

$$1-\frac{n_j}{n_i} \geqq r > 0.$$

By all this and (34), there exists $L_5$ such that $j\geqq L_5$ implies (for $i, j\in J$):

$$P(A_i \mid A_j) \leqq 2R \exp\left(-\frac{cq}{2}(i-j)\right), \quad \text{for } j<i<\beta(j),$$

and

$$P(A_i \mid A_j) \leqq 2R \exp\left(-\frac{cr}{2}\log i\right), \quad \text{for } \beta(j)\leqq i<\alpha(j).$$

It follows that, as $j\to\infty$,

$$\sum_{j<i<\alpha(j)} P(A_i \mid A_j) \leqq \frac{2\,\mathrm{Re}^{-cq/2}}{1-e^{-cq/2}} + 4R(\log j)(\log_2 j)\, j^{-cr/2} = O(1). \tag{35}$$

The proof of Theorem 3 follows from (21), (29) and (35).

*Proof of Corollary 1.* The result is known for $k=$ constant (see e.g. (4)), hence we shall limit ourselves to $k=k_n \uparrow \infty$. We shall make use of the following lemma.

**Lemma 20.** *Let $k=k_n \uparrow \infty$ and suppose that $k_n = o(\log_2 n)$ as $n \to \infty$. Then, as $n \to \infty$,*

$$(36) \qquad (k+1) \log\left(\frac{\log_2 n}{k}\right) = o(\log_2 n);$$

$$(37) \qquad \log_3 n = o\left((k+1) \log\left(\frac{\log_2 n}{k}\right)\right);$$

$$(38) \qquad \frac{k+1}{\log_2 n}(k+1) \log\left(\frac{\log_2 n}{k}\right) = o\left(k \log\left(\frac{\log_2 n}{k}\right)\right).$$

*Proof.* See Deheuvels and Devroye (1984), Lemma 1.

Let $c_k^a = k_n(1+\delta_n) = \log_2 n + k_n(1+a) \log\left(\frac{\log_2 n}{k_n}\right)$, where $a > -1$ is constant. By Lemma 20, we see that $c_k^a$ satisfies condition (iii) of Theorem 1 and conditions $(a-b)$ of Theorem 3.

Using Lemma 20 again, it is straightforward that

$$\sum_n \frac{1}{n} k_n^{1/2} \left(\frac{e}{k_n} c_n^a\right)^{k_n} \exp(-c_n^a) < \infty \text{ or } = \infty,$$

according as $a>0$ or $a<0$. The proof of Corollary 1 is completed by Theorems 1–3.

*Proof of Theorem 4.* The proof is identical to that of Theorem 3, up to the point where one has to show that

$$\sum_{i=j+1}^{\alpha(j)} P(A_i \mid A_j) < \infty.$$

For this, assuming as usual that $i, j \in J$, $k=k'=K+1$, $u_i \leqq u_j$, $c_{n_i} \geqq c_{n_j}$, we deduce from (30) that

$$\begin{aligned}(39) \qquad P(A_i \mid A_j) &\leqq 2\left\{1-\left(1-\frac{u_j}{u_i}\right)\right\}^K \sum_{l=0}^{K} \frac{1}{l!\,l!}\{K(n_i-n_j)(u_j-u_i)\}^l \exp(-(n_i-n_j)u_i) \\ &\leqq 2\exp\left(-K\left(1-\frac{u_i}{u_j}\right)-(n_i-n_j)u_i+2\{K(n_i-n_j)(u_j-u_i)\}^{1/2}\right) \\ &= 2\exp\left(-\{\sqrt{(n_i-n_j)u_i}-\sqrt{K(u_j/u_i-1)}\}^2+K\frac{(u_j-u_i)^2}{u_i u_j}\right).\end{aligned}$$

Let us assume in the first place that $i \in (II)'_d$, i.e.

$$j < i < \beta(j) = \max\{l > j : l-j \leqq d \log l\},$$

where $d>0$ will be precised later on. We have by (25), as in the proof of (30),

$$\frac{(u_j-u_i)^2}{u_i u_j} \leqq \left(\frac{u_j}{u_i}-1\right)^2 \leqq \left(\frac{n_i}{n_j}-1\right)^2 \leqq (e^d-1)\, e^d \left(\frac{n_i-n_j}{n_i}\right). \tag{40}$$

There exists an $L_6$ such that $j \geqq L_6$ implies $\delta=\delta_{n_i} \geqq \Delta>0$. It follows that

$$(n_i-n_j)\, u_i \geqq \frac{n_i-n_j}{n_i} K(1+\Delta).$$

Also, we have

$$K\left(\frac{u_j}{u_i}-1\right) \leqq K\left(\frac{n_i}{n_j}-1\right) \leqq K e^d \left(\frac{n_i-n_j}{n_i}\right).$$

We now choose $d>0$ such that

$$(\sqrt{1+\Delta}-e^{d/2})^2-e^d(e^d-1)=\frac{\Delta}{2} \quad \text{and} \quad e^d<1+\Delta.$$

We obtain that, for all $j<i<\beta(j)$,

$$P(A_i \mid A_j) \leqq \exp\left(-\frac{\Delta}{4}\left(\frac{n_i-n_j}{n_i}\right) k_{n_i}\right) \leqq \exp(-D(i-j)),$$

for some conveniently chosen $D>0$. Here, we have used the fact that $\log_2 n = O(k_n)$ and (26). By all this, we have

$$\sum_{i=j+1}^{\beta(j)} P(A_i \mid A_j) \leqq \frac{e^{-D}}{1-e^{-D}} = O(1). \tag{41}$$

Let us now use the assumption that $u_n \sim \frac{A}{n} \log_2 n$, where $A$ is a constant. We now choose $E>d$, to be precised in the sequel, and let $\theta(j)=\max\{l>j: l-j \geqq E \log l\}$. Let $p=1-\frac{u_i}{u_i}$, and consider in the first place the case where $\beta(j)<i \leqq \theta(j)$. We get easily from (25) and (30) that

$$P(A_i \mid A_j) \leqq 2 \sum_{l=0}^{k-1} \binom{k-1}{l} (1-p)^{k-l-1} \frac{p^l}{l!} \{pk(1+\lambda)\}^l \exp(-pk(1+\lambda)), \tag{42}$$

where $1+\lambda=\left(1-\frac{n_j}{n_i}\right)\left(1-\frac{u_i}{u_j}\right)^{-1}(1+\delta_{n_i}) \sim 1+\delta_{n_i}$. Furthermore, there exists $L_7$, jointly with $\Lambda>0$ and $0<p_1 \leqq p_2<\infty$ such that $i \geqq L_7$ implies that $p_1 \leqq p \leqq p_2$ and $\lambda \geqq \Lambda$.

Next, we denote by $U$ a random variable with a Binomial $B(k-1, p)$ distribution and by $V$ a random variable with a Poisson $P(kp(1+\lambda))$ distribution.

By (42), we obtain the bound

$$P(A_i \mid A_j) \leqq 2\{P(U-kp \geqq k\Lambda/2)+P(V-kp(1+\lambda) \leqq k\Lambda/2)\}. \tag{43}$$

By Chernoff's theorem or by Jensen's inequality, it can be verified that we can find $\rho_1$ depending continuously in $p$ and $\Lambda$ only and $\rho_2$ depending continuously on $p(1+\lambda)$ and $\Lambda$ only, such that, by (44),

$$P(A_i \mid A_j) \leqq 2\{\rho_1^k + \rho_2^k\}, \quad 0 < \rho_1, \ \rho_2 < 1. \tag{45}$$

By continuity, we see that the supremum $\rho$ of $\max\{\rho_1, \rho_2\}$ when $p$ and $\lambda$ vary (in a compact set) is such that $\rho = e^{-t} < 1$.

It follows that for $\beta(j) < i \leqq \theta(j)$ we can use the bound

$$P(A_i \mid A_j) \leqq 4\rho^k = 4\exp(-tk) = O(i^{-a}) \quad \text{for some } a > 0.$$

Here, we have used the fact that $k = k_{n_i}$ and that $\liminf_{n\to\infty} k_n/\log_2 n > 0$. It follows that, as $j \to \infty$,

$$\sum_{\beta(j) < i \leqq \theta(j)} P(A_i \mid A_j) = O(j^{-a} \log j) = o(1). \tag{46}$$

Up to now, the choice of $E > d$ has remained open. Let us choose $E$ such that, for any $i \geqq \theta(j)$, $(n_i - n_j)\, u_i = \left(1 - \frac{n_j}{n_i}\right) k(1+\delta') > k$. This is always possible since $\liminf_{n\to\infty} \delta_n > 0$.

By (30), we have then, for $i \geqq \theta(j)$,

$$\begin{aligned} P(A_i \mid A_j) &\leqq 2 \max_{1 \leqq l < k} \left\{ \frac{1}{l!} ((n_i - n_j)\, u_i)^l \exp(-(n_i - n_j)\, u_i) \right\} \\ &= \frac{2}{(k-1)!} \{(n_i - n_j)\, u_i\}^{k-1} \exp(-(n_i - n_j)\, u_i). \end{aligned}$$

By the same arguments as used in the proof of (29) it follows that

$$P(A_i \mid A_j) = O\left\{ \frac{k^{-1/2}}{1+\delta'} \exp\left( -k(\delta' - \log(1+\delta')) + k(1+\delta') \frac{n_j}{n_i} (1 + o(1)) \right) \right\}.$$

Choose now $D$ so large that, for $i \geqq \theta(j)$, $k(1+\delta') \geqq \frac{1}{2}(\delta' - \log(1+\delta')) \frac{n_i}{n_j}$. In this case, there exists a $b > 0$ such that $P(A_i \mid A_j) = O(i^{-b})$. Here again

$$\sum_{\theta(j) < i \leqq \alpha(j)} P(A_i \mid A_j) = O(j^{-b} (\log j)(\log_2 j)) = o(1). \tag{47}$$

(47), jointly with (29), (41) and (46) suffices for proof of Theorem 4.

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