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Probability Theory and Related Fields

Martingale Difference Arrays and Stochastic Integrals

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Summary. Consider MDAs (X_{ni}) and (Y_{ni}), and stopping times $\tau_n(t)$, $0 \leqq t \leqq 1$. Denote

$$S_n(t) = a_0 + \sum_{i=1}^{\tau_n(t)} X_{ni}, \qquad T_n(t) = b_0 + \sum_{i=1}^{\tau_n(t)} Y_{ni},$$

and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a function. If the common distribution converges and if S_t, T_t denote the corresponding limiting processes then we give conditions such that the martingale transforms

$$\sum_{i=1}^{\tau_n(t)} \varphi(S_{n,i-1})\, Y_{ni}$$

converge weakly to the stochastic integral

$$\int_0^t \varphi(S)\, dT.$$

This result has important consequences for functional central limit theorems:

(1) If the MDAs are connected by a difference equation of the form

$$X_{ni} = \varphi(S_{n,i-1})\, Y_{ni},$$

then weak convergence of $T_n(t)$ implies that of $S_n(t)$, and the limit satisfies the stochastic differential equation

$$dS = \varphi(S)\, dT.$$

This observation leads to functional limit theorems for diffusion approximations. E.g. we obtain easily a result of Lindvall, [4], on the diffusion approximation of branching processes.

(2) If the MDA (X_{ni}) arises from a likelihood ratio martingale then the limit satisfies

$$S_t = 1 + \int_0^t S\, dT,$$

which leads to the representation of the limiting likelihood ratios as exponential martingale:

$$S_t = \exp(T_t - \tfrac{1}{2}[T,T]_t).$$

This approximation by an exponential martingale has been proved previously by Swensen, [9], using a Taylor expansion of the log-likelihood ratio.

(3) As a consequence we obtain a general functional central limit theorem: If $\left(\sum_{i=1}^{\tau_n(t)} X_{ni}^2\right)$ converges weakly to $([S,S]_t)$, then $\left(\sum_{i=1}^{\tau_n(t)} X_{ni}\right)$ converges weakly to (S_t), provided that the distribution of (S_t) is uniquely determined by that of $([S,S]_t)$. This assertion embraces previous central limit theorems, dealing with cases where the increasing process $([S,S]_t)$ is deterministic.

1. The Main Results

For every $n \in \mathbb{N}$ let $(\Omega_n, \mathscr{A}_n, P_n)$ be a probability space and $(\mathscr{A}_{nk})_{k \geqq 0}$ a filtration of $\mathscr{A}_n$. A double sequence $(X_{nk})_{k \geqq 1, n \geqq 1}$ of random variables is a martingale difference array (MDA) if each sequence $(X_{nk})_{k \geqq 1}$ is adapted, P_n-integrable and centered, $n \geqq 1$.

In this paper we are concerned with MDAs (X_{nk}), (Y_{nk}) which are defined on the same probability spaces $(\Omega_n, \mathscr{A}_n, P_n)$ and are adapted for the same filtrations $(\mathscr{A}_{nk})_{k \geqq 0}$, $n \in \mathbb{N}$. Let us consider the partial sum processes

$$S_n(t) = S_{n,\tau_n(t)}, \qquad T_n(t) = T_{n,\tau_n(t)}, \qquad t \in [0,1],$$

where

$$S_{nk} = \sum_{i=1}^{k} X_{ni}, \qquad k \geqq 1, \quad n \in \mathbb{N},$$

$$T_{nk} = \sum_{i=1}^{k} Y_{ni}, \qquad k \geqq 1, \quad n \in \mathbb{N},$$

and where $\tau_n(t)$, $0 \leqq t \leqq 1$, are stopping times for the filtration $(\mathscr{A}_{nk})_{k \geqq 0}$. For most applications the constant stopping times $\tau_n(t) = [k_n \cdot t]$, $0 \leqq t \leqq 1$, $n \in \mathbb{N}$, where $k_n \uparrow \infty$ is a given sequence in $\mathbb{N}$, would suffice, but our results are valid if the stopping times satisfy the following conditions, which are tacitely supposed to be fulfilled in all what follows.

(1.1) *Conditions*

(1) For each $n \in \mathbb{N}$, $(\tau_n(t))_{0 \leqq t \leqq 1}$ is a right-continuous, non-decreasing process taking values in $\mathbb{N} \cup \{0\}$.

(2) For every $n \in \mathbb{N}$ and every $\omega \in \Omega_n$ the function $t \mapsto \tau_n(t)(\omega)$, $0 \leqq t \leqq 1$, takes all values between zero and $\tau_n(1)(\omega)$.

The following condition on a MDA (X_{ni}) is used repeatedly. If it is fulfilled then by a standard truncation argument we can replace the MDA (X_{ni}) by an asymptotically equivalent MDA $(\tilde{X}_{ni})$, which is uniformly bounded, (confer Sect. 2).

(1.2) *Condition*

$$\sum_{i=1}^{\tau_n(1)} P_n(|X_{ni}|\cdot 1_{\{|X_{ni}|>\varepsilon\}}|\mathscr{A}_{n,i-1}) \xrightarrow{P_n} 0, \qquad \varepsilon>0.$$

For convenience let us introduce some simplifying terminology.

(1.3) A sequence of processes $(X_n(t))_{t\in[0,1]}$, $n\in\mathbb{N}$, with trajectories in $D([0,1])$ possesses asymptotically continuous trajectories if

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n\{\sup_{|s-t|<\delta} |X_n(s)-X_n(t)|>\varepsilon\}=0, \qquad \varepsilon>0.$$

(1.4) A sequence of processes $(X_n(t))_{t\in[0,1]}$, $n\in\mathbb{N}$, is stochastically uniformly bounded if

$$\lim_{a\to\infty} \overline{\lim_{n\to\infty}} P_n\{\sup_t |X_n(t)|>a\}=0.$$

Let $(V_{nk})_{k\geq 1}$ be adapted sequences and denote $V_n(t)=V_{n,\tau_n(t)}$, $0\leqq t\leqq 1$. Consider the martingale transforms

$$U_n(t)=\sum_{i=1}^{\tau_n(t)} V_{n,i-1}\cdot Y_{ni}, \qquad 0\leqq t\leqq 1.$$

The following is a tightness condition for the distributions of martingale transforms.

(1.5) **Theorem.** *Assume that* (Y_{ni}) *is a* MDA *satisfying condition* (1.2), *and* (V_{ni}) *are adapted sequences. If*

(1) *the processes* $\left(\sum_{i=1}^{\tau_n(t)} Y_{ni}^2\right)_{t\in[0,1]}$ *have asymptotically continuous trajectories, and*

(2) *the processes* $(V_n(t))_{t\in[0,1]}$ *are stochastically uniformly bounded,*
the martingale transforms $(U_n(t))_{t\in[0,1]}$ *have asymptotically continuous trajectories.*

Proof. It is obvious that we may assume that (Y_{ni}) is a truncated array in the sense of Sect. 2. Let $a>0$, $b>0$, and define

$$Y_{ni}^a = Y_{ni}\cdot 1_{\left\{\sum_{j=1}^{i-1} Y_{nj}^2\leqq a\right\}},$$

$$V_{ni}^b = V_{ni}\cdot 1_{\{|V_{ni}|\leqq b\}}.$$

Let

$$U_n^{a,b}(t)=\sum_{i=1}^{\tau_n(t)} V_{n,i-1}^b\cdot Y_{ni}^a, \qquad t\in[0,1].$$

Obviously, it is sufficient to prove the assertion for $(U_n^{a,b}(t))_{t\in[0,1]}$, $n\in\mathbb{N}$, and arbitrary $a>0$, $b>0$. Applying Burkholder's inequality we obtain

$$
\begin{aligned}
&P_n\{\sup_{|s-t|<\delta} |U_n^{a,b}(s)-U_n^{a,b}(t)|>\varepsilon\}\\
&\leqq P_n\left\{\max_{1\leqq j\leqq\left[\frac{1}{\delta}\right]} \sup_{\tau_n((j-1)\delta)<l\leqq\tau_n(j\delta)} \left|\sum_{k=\tau_n((j-1)\delta)+1}^{l} V_{n,k-1}^b\cdot Y_{nk}^a\right|>\varepsilon\right\}\\
&\leqq\frac{C}{\varepsilon^4} P_n\left(\sum_{j=1}^{\left[\frac{1}{\delta}\right]}\left(\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} (V_{n,k-1}^b\cdot Y_{nk}^a)^2\right)^2\right)\\
&\leqq\frac{C\cdot b^4}{\varepsilon^4} P_n\left(\sum_{j=1}^{\left[\frac{1}{\delta}\right]}\left(\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} (Y_{nk}^a)^2\right)^2\right)\\
&\leqq\frac{C\cdot b^4(a+1)}{\varepsilon^4} P_n\left(\max_{1\leqq j\leqq\left[\frac{1}{\delta}\right]}\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} (Y_{nk}^a)^2\right).
\end{aligned}
$$

By assumption (1), this proves the assertion. □

(1.6) *Remark.* If (X_{ni}), (Y_{ni}) are MDAs satisfying condition (1.2) and $\mu|\mathscr{B}^2(D([0,1]))$ is an accumulation point of the common distributions of $(S_n(t), T_n(t))_{0\leqq t\leqq 1}$ then there are a probability space $(\Omega,\mathscr{A},P)$, a filtration $(\mathscr{F}_t)_{0\leqq t\leqq 1}$ and adapted processes (S_t), (T_t), such that μ is the common distribution of $(S_t, T_t)_{0\leqq t\leqq 1}$. Take e.g. $\Omega=D^2([0,1])$, $\mathscr{A}=\mathscr{B}^2(D([0,1]))$, $P=\mu$, and define $\mathscr{F}_t$ to be the σ-field generated by $\{(p_1(s), p_2(s)): s<t\}$, but completed and regulated from the right. If we take $S_t=p_1(t)$, $T_t=p_2(t)$, $0\leqq t\leqq 1$, then these are adapted processes, having common distribution μ.

In the following any limiting processes of MDAs are always understood to be adapted relative to a common filtration.

(1.7) **Theorem.** *Assume that the* MDAs (X_{ni}) *and* (Y_{ni}) *satisfy condition* (1.2). *Let* φ: $\mathbb{R}\to\mathbb{R}$ *be a Lipschitz continuous function. If the common distributions of* $(S_n(t), T_n(t))_{0\leqq t\leqq 1}$ *converge weakly to the common distribution of a pair of continuous processes* $(S_t, T_t)_{0\leqq t\leqq 1}$, *where* $(T_t)_{0\leqq t\leqq 1}$ *is a square-integrable martingale, then the distributions of the martingale transforms*

$$\sum_{k=1}^{\tau_n(t)} \varphi(S_{n,k-1})\,Y_{nk},\qquad 0\leqq t\leqq 1,\ n\in\mathbb{N},$$

converge weakly to the distribution of the stochastic integral

$$\int_0^t \varphi(S)\,dT,\qquad 0\leqq t\leqq 1.$$

Proof. Uniform tightness of the distributions of the martingale transform processes follows from Theorem (1.5) together with the criterion of Lemma (2.4), (note Remark (2.3), (2)). Hence, it is sufficient to prove weak convergence of the finite-dimensional marginal distributions.

For every $\delta>0$ consider the sequence of processes

$$R_{n,t}(\delta)=\sum_{j=1}^{\left[\frac{t}{\delta}\right]} \varphi(S_n((j-1)\delta))\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} Y_{nk},\qquad n\in\mathbb{N}.$$

The finite-dimensional marginal distributions of these processes converge weakly to the corresponding marginal distributions of the process

$$R_t(\delta)=\sum_{j=1}^{[\frac{t}{\delta}]}\varphi(S_{(j-1)\delta})(T_{j\delta}-T_{(j-1)\delta}).$$

If $t_1<t_2<\ldots<t_m$ is a subdivision of $[0,1]$ then we have

$$\mathscr{L}((R_{n,t_i}(\delta))_{1\leq i\leq m}|P_n)\to\mathscr{L}((R_{t_i}(\delta))_{1\leq i\leq m}|P),\quad \text{weakly.}$$

But for $\delta\to 0$ we have

$$R_t(\delta)\xrightarrow{P}\int_0^t\varphi(S_s)\,dT_s,\qquad 0\leqq t\leqq 1.$$

Now, let $f\in\mathscr{C}_b(\mathbb{R}^m)$. Then for every $\delta>0$

$$\begin{aligned}&\left|\int f\left(\left(\sum_{k=1}^{\tau_n(t_i)}\varphi(S_{n,k-1})Y_{nk}\right)_{1\leqq i\leqq m}\right)dP_n-\int f\left(\left(\int_0^{t_i}\varphi(S_s)\,dT_s\right)_{1\leqq i\leqq m}\right)dP\right|\\ &\leqq\left|\int f\left(\left(\sum_{k=1}^{\tau_n(t_i)}\varphi(S_{n,k-1})Y_{nk}\right)_{1\leqq i\leqq m}\right)dP_n-\int f((R_{n,t_i}(\delta))_{1\leqq i\leqq m})\,dP_n\right|\\ &\quad+\left|\int f((R_{n,t_i}(\delta))_{1\leqq i\leqq m})\,dP_n-\int f\left(\left(\int_0^{t_i}\varphi(S_s)\,dT_s\right)_{1\leqq i\leqq m}\right)dP\right|.\end{aligned}$$

First, let $n\to\infty$ for fixed $\delta>0$. Then let $\delta\to 0$ on the right hand side. Applying Lemma (2.6) this proves the assertion. □

(1.8) **Corollary.** *Assume that the* MDAs (X_{ni}) *and* (Y_{ni}) *satisfy condition* (1.2). *If the common distributions of* $(S_n(t),T_n(t))_{0\leqq t\leqq 1}$ *converge weakly to the common distribution of a pair of continuous, square-integrable martingales* $(S_t,T_t)_{0\leqq t\leqq 1}$, *then the common distributions of*

$$\left(S_n(t),T_n(t),\sum_{k=1}^{\tau_n(t)}S_{n,k-1}Y_{nk},\sum_{k=1}^{\tau_n(t)}T_{n,k-1}X_{nk}\right)_{0\leqq t\leqq 1}$$

converge weakly to the common distribution of

$$\left(S_t,T_t,\int_0^t S\,dT,\int_0^t T\,dS\right)_{0\leqq t\leqq 1}.$$

Proof. This is done by an obvious extension of the proof of Theorem (1.7). □

(1.9) *Remark.* (1) in [8], Rootzén proves the following theorem: If $(S_n(t))_{0\leqq t\leqq 1}$ converges weakly on $D([0,1])$ to a Brownian motion $(W_t)_{0\leqq t\leqq 1}$, then

$$\sum_{i=1}^{\tau_n(t)}X_{ni}^2\xrightarrow{P_n}t,\qquad 0\leqq t\leqq 1.$$

Extensions of Rootzén's result are proved by Liptser and Shiryayev, [5] and [6], and by Jacod, [3]. A special case of Jacod's result is as follows: If

$(S_n(t))_{0 \leqq t \leqq 1}$ converges weakly on $D([0,1])$ to a square-integrable, continuous martingale $(S_t)_{0 \leqq t \leqq 1}$, then the common distribution of

$$\left(S_n(t), \sum_{k=1}^{\tau_n(t)} X_{nk}^2\right)_{0 \leqq t \leqq 1}$$

converges weakly to the common distribution of $(S_t, [S,S]_t)_{0 \leqq t \leqq 1}$ where $([S,S]_t)_{0 \leqq t \leqq 1}$ is the quadratic variation process of $(S_t)_{0 \leqq t \leqq 1}$. This result can be deduced from ours applying the integration by part formula

$$S_t^2 - 2\int_0^t S_s \, dS_s = [S,S]_t, \qquad 0 \leqq t \leqq 1,$$

and

$$S_{n\tau_n(t)}^2 - 2\sum_{k=1}^{\tau_n(t)} S_{n,k-1} X_{nk} = \sum_{k=1}^{\tau_n(t)} X_{nk}^2.$$

(2) Assume that the conditions of Corollary (1.8) are satisfied. Then the common distributions of the processes

$$\left(S_n(t), (T_n(t)), \sum_{k=1}^{\tau_n(t)} X_{nk} Y_{nk}\right)_{0 \leqq t \leqq 1}$$

converge weakly to the common distribution of

$$(S_t, T_t, [S,T]_t)_{0 \leqq t \leqq 1}.$$

$(([S,T]_t)_{0 \leqq t \leqq 1}$ denotes the mutual variation between S and T.) This is easily seen applying the integration by part formula

$$S_t T_t - \int_0^t S_s \, dT_s - \int_0^t T_s \, dS_s = [S,T]_t, \qquad 0 \leqq t \leqq 1,$$

and noting that

$$S_{n\tau_n(t)} T_{n\tau_n(t)} - \sum_{k=1}^{\tau_n(t)} S_{n,k-1} Y_{nk} - \sum_{k=1}^{\tau_n(t)} T_{n,k-1} X_{nk} = \sum_{k=1}^{\tau_n(t)} X_{nk} Y_{nk}, \qquad 0 \leqq t \leqq 1.$$

Now, we give some applications of our main theorems.

The following assertion embraces previous functional central limit theorems for MDAs.

(1.10) **Theorem.** *Assume that the* MDA (X_{ni}) *satisfies condition* (1.2) *and*

(1) $$\overline{\lim_{n\to\infty}} P_n\left(\sum_{i=1}^{\tau_n(1)} X_{ni}^2 1_{\{|X_{ni}| \leqq 1\}}\right) < \infty.$$

Assume further that the following conditions hold:

(2) $(S_t)_{0 \leqq t \leqq 1}$ *is a continuous, square-integrable martingale whose distribution is uniquely determined by the distribution of its increasing process* $([S,S]_t)_{0 \leqq t \leqq 1}$.

(3) *The finite-dimensional marginal distributions of*

$$\sum_{k=1}^{\tau_n(t)} X_{nk}^2, \qquad 0 \leqq t \leqq 1, \ n \in \mathbb{N},$$

converge weakly to the corresponding distributions of $[S,S]_t$, $0 \leqq t \leqq 1$.

Then

$$\mathcal{L}((S_n(t))_{0 \leqq t \leqq 1} | P_n) \to \mathcal{L}((S_t)_{0 \leqq t \leqq 1} | P)$$

weakly on $D([0,1])$.

Proof. Condition (3) implies by Lemma (2.4) and Remark (2.5) that the distributions of $(S_n(t))_{t \in [0,1]}$, $n \in \mathbb{N}$, are uniformly tight on $D([0,1])$. From (1) it follows by easy arguments that every weak accumulation point of $(S_n(t))_{t \in [0,1]}$, $n \in \mathbb{N}$, is a continuous, square-integrable martingale. Hence, by Remark (1.9), (1), the assertion is proved by an obvious compactness argument using the uniqueness condition (2). □

(1.11) *Remark.* The uniqueness condition (2) is certainly satisfied if $[S,S]_t = t$, $0 \leqq t \leqq 1$. Then the limiting process is a Brownian motion. The uniqueness condition is also satisfied if $[S,S]_t = A(t)$, $0 \leqq t \leqq 1$, where A is a nondecreasing, deterministic function. Thus, we obtain functional central limit theorems of the type given by Helland, [2], Sect. 3, Gaenssler and Haeusler, [1], Theorem 1 A.

The functional central limit Theorem (1.10) is proved by a combination of a compactness argument and a uniqueness condition. A uniqueness condition different from (1.10), (2), is obtained if the limiting process necessarily satisfies a stochastic differential equation. This is the idea of the following application.

(1.12) **Theorem.** *Assume that the* MDA (X_{ni}) *possesses stochastically uniformly bounded row sums* $(S_n(t))$ *satisfying the equation*

$$S_n(t) = a_0 + \sum_{i=1}^{\tau_n(t)} \varphi(S_{n,i-1}) Y_{ni}, \qquad t \in [0,1].$$

If the MDA (Y_{ni}) *satisfies condition* (1.2) *and if the distributions of* $(T_n(t))$ *converge weakly to those of a continuous, square-integrable martingale* $(T(t))$, *then the distributions of* $(S_n(t))$ *converge weakly to the solution* (S_t) *of the equation*

$$S_t = a_0 + \int_0^t \varphi(S)\, dT, \qquad 0 \leqq t \leqq 1.$$

Proof. It is easy to see that the MDA (X_{ni}) satisfies condition (1.2). Hence, Theorem (1.7) might be applied if we knew that the common distributions of $(S_n(t), T_n(t))_{0 \leqq t \leqq 1}$ converge weakly on $D^2([0,1])$ with asymptotically continuous trajectories. But we only know that those common distributions are uniformly tight on $D^2([0,1])$ with asymptotically continuous trajectories, by Theorem (1.5). The limit of each convergent subsequence satisfies the stochastic differential equation. By the weak uniqueness of solutions of stochastic differential equations the limit distributions of all convergent subsequences coincide, which proves the assertion. □

(1.13) *Example.* Consider a random walk situation where the distance of a jump depends on the position of the particle. To be precise, let φ: $\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function and assume that

$$P_n\{X_{nk}=\varphi(S_{n,k-1})n^{-\frac{1}{2}}\}=P_n\{X_{nk}=-\varphi(S_{n,k-1})n^{-\frac{1}{2}}\}=\tfrac{1}{2}.$$

Then the MDA $Y_{nk}=X_{nk}/\varphi(S_{n,k-1})$ satisfies condition (1.2) and converges weakly to a Brownian motion, since

$$\sum_{k=1}^{[nt]} Y_{nk}^2=\frac{1}{n}[nt]\to t, \qquad 0\leqq t\leqq 1.$$

We observe that

$$S_n(t)=\sum_{i=1}^{[nt]}\varphi(S_{n,i-1})\,Y_{ni}.$$

In order to apply Theorem (1.12) we have to show that the processes $(S_n(t))$ have stochastically uniformly bounded trajectories. This is easily seen from

$$\begin{aligned} P_n(S_{nk}^2|\mathscr{A}_{n,k-1}) &=\frac{1}{n}\varphi^2(S_{n,k-1})+S_{n,k-1}^2 \\ &\leqq\frac{1}{n}\varphi^2(0)+S_{n,k-1}^2(1+C^2/n) \end{aligned}$$

which implies by induction that

$$\begin{aligned} P_n(S_{nn}^2) &\leqq\varphi^2(0)\cdot\frac{1}{n}\left(1+\left(1+\frac{C^2}{n}\right)+\ldots+\left(1+\frac{C^2}{n}\right)^n\right) \\ &\leqq\varphi^2(0)\cdot\exp(C^2). \end{aligned}$$

Thus, we may apply Theorem (1.12) which shows that the limiting process (S_t) of $(S_n(t))$ satisfies

$$S_t=\int_0^t\varphi(S_s)\,dW_s, \qquad t\in[0,1].$$

(1.14) *Example.* Let us derive from Theorem (1.12) a result of Lindvall, [4], concerning the diffusion approximation of branching processes.

Let ξ_{ij}, $i\geqq 1$, $j\geqq 1$, be i.i.d. random variables taking values in $\mathbb{N}_0$. Let $P(\xi_{ij})=1$, $P([\xi_{ij}-1]^2)=\beta^2>0$, and define

$$\eta_{j0}=1, \qquad j\geqq 1,$$

$$\eta_{jk}=\sum_{i=1}^{\eta_{j,k-1}}\xi_{ij}, \qquad k\geqq 1,\ j\geqq 1.$$

Thus, for every $j\geqq 1$ the sequence $(\eta_{j0}, \eta_{j1}, \ldots, \eta_{jk}, \ldots)$ represents the progeny of a branching process. Define

$$Z_{nk}=\sum_{j=1}^{n}\eta_{jk}, \qquad 1\leqq k\leqq n,$$

to be the sum of n independent branching processes. Let us show that the distributions of the processes

$$S_n(t)=\frac{1}{n} Z_{n,[nt]}-1, \qquad 0\leqq t\leqq 1,$$

converge weakly to the distribution of a diffusion (S_t) defined by the equation

$$S_t=\beta\int_0^t \sqrt{S_s+1}\, dW_s, \qquad 0\leqq t\leqq 1.$$

For this, we denote

$$X_{ni}=\frac{1}{n}(Z_{ni}-Z_{n,i-1}),$$

$$Y_{ni}=(Z_{ni}-Z_{n,i-1})/(\beta\sqrt{nZ_{n,i-1}}).$$

The obvious relation

$$X_{ni}=Y_{ni}\cdot\beta\sqrt{Z_{n,i-1}/n}$$

gives us

$$S_n(t)=\sum_{i=1}^{[nt]}\beta\sqrt{S_{n,i-1}+1}\cdot Y_{ni}.$$

It is easy to check the assumptions of Theorem (1.12). In particular, the MDA (Y_{ni}) satisfies

$$\sum_{i=1}^{[nt]} P_n(Y_{ni}^2|\mathscr{A}_{n,i-1})\xrightarrow{P_n} t, \qquad t\in[0,1],$$

and the Lindeberg condition

$$\sum_{i=1}^{[nt]} P_n(Y_{ni}^2\cdot 1_{\{|Y_{ni}|\geqq\varepsilon\}}|\mathscr{A}_{n,i-1})\xrightarrow{P_n} 0, \qquad \varepsilon>0.$$

It follows that the distributions of $(T_n(t))$ converge weakly to those of a Brownian motion. This proves the assertion.

Our last application deals with possible limits of likelihood ratios. Let $(\Omega_n,\mathscr{A}_n,P_n)$ be probability spaces where filtrations $\{\emptyset,\Omega_n\}=\mathscr{A}_{n0}\subseteq\mathscr{A}_{n1}\subseteq\ldots\subseteq\mathscr{A}_{nn}$ are given. Let $Q_n|\mathscr{A}_n$ be probability measures and denote $P_{nk}=P_n|\mathscr{A}_{nk}$, $Q_{nk}=Q_n|\mathscr{A}_{nk}$, $1\leqq k\leqq n$, $n\in\mathbb{N}$. The following theorem is proved by Swensen, [9], applying a completely different method. Roughly speaking, Swensen's result is based on a Taylor expansion argument which is often used in statistics. Our argument is as follows: We show that the likelihood ratios have to fulfill in the limit a linear stochastic differential equation, whose solution is an exponential martingale.

(1.15) **Theorem.** *Assume that the* MDA

$$Y_{ni}=\left(\frac{dQ_{ni}}{dP_{ni}}\Big/\frac{dQ_{n,i-1}}{dP_{n,i-1}}-1\right), \qquad 1\leqq i\leqq n,$$

satisfies condition (1.2). *If*

$$\mathscr{L}\left(\left(\sum_{i=1}^{[nt]} Y_{ni}\right)_{0\leqq t\leqq 1}\middle| P_n\right)\to\mathscr{L}((T_t)_{0\leqq t\leqq 1}|P), \quad \textit{weakly}$$

on $D([0,1])$, *where* $(T_t)_{0\leqq t\leqq 1}$ *is a continuous, square-integrable martingale, then the distributions of the likelihood ratios*

$$\left(\frac{dQ_{n[nt]}}{dP_{n[nt]}}\right)_{0\leqq t\leqq 1}, \qquad n\in\mathbb{N},$$

converge weakly to the distribution of the exponential martingale

$$\exp(T_t-\tfrac{1}{2}[T,T]_t), \qquad 0\leqq t\leqq 1.$$

Proof. First, we note that the processes

$$S_n(t)=\frac{dQ_{n[nt]}}{dP_{n[nt]}}, \qquad 0\leqq t\leqq 1,\ n\in\mathbb{N},$$

are stochastically uniformly bounded since

$$P_n\left\{\max_{1\leqq i\leqq n}\left|\frac{dQ_{ni}}{dP_{ni}}\right|>2a\right\}\leqq\frac{1}{a}\int\frac{dQ_{nn}}{dP_{nn}}\,dP_n\leqq\frac{1}{a}, \qquad a>0.$$

Consider the identity

$$\frac{dQ_{n[nt]}}{dP_{n[nt]}}=1+\sum_{j=1}^{[nt]}\frac{dQ_{nj-1}}{dP_{nj-1}}\left(\frac{dQ_{nj}}{dP_{nj}}\middle/\frac{dQ_{n,j-1}}{dP_{n,j-1}}-1\right).$$

This means, that

$$S_n(t)=1+\sum_{j=1}^{[nt]}S_{n,j-1}\cdot Y_{nj}.$$

Now, it follows from Theorem (1.12) that $(S_n(t))_{0\leqq t\leqq 1}$ converges weakly to the solution of the stochastic differential equation

$$S_t=1+\int_0^t S\,dT, \qquad 0\leqq t\leqq 1,$$

which is the exponential martingale of $(T_t)_{0\leqq t\leqq 1}$. □

2. Auxiliary Lemmas

Let (X_{ni}) be a MDA which satisfies condition (1.2). We shall apply repeatedly the following truncation procedure. Let

$$\tilde{X}_{ni}=X_{ni}\cdot 1_{\{|X_{ni}|\leqq\frac{1}{2}\}}-P_n(X_{ni}\cdot 1_{\{|X_{ni}|\leqq\frac{1}{2}\}}|\mathscr{A}_{n,i-1}).$$

Then $(\tilde{X}_{ni})$ is a MDA, uniformly bounded by one such that

$$\max_{1 \leq i \leq \tau_n(1)} |\tilde{X}_{ni}| \xrightarrow{P_n} 0.$$

As is well-known the MDAs (X_{ni}) and $(\tilde{X}_{ni})$ are asymptotically equivalent in the sense that

$$\sum_{i=1}^{\tau_n(1)} |X_{ni} - \tilde{X}_{ni}| \xrightarrow{P_n} 0,$$

and

$$\sum_{i=1}^{\tau_n(1)} |X_{ni}^2 - \tilde{X}_{ni}^2| \xrightarrow{P_n} 0.$$

Applying the truncation argument to martingale transforms gives

(2.1) **Lemma.** *Assume that the* MDAs (X_{ni}) *and* (Y_{ni}) *satisfy condition* (1.2). *Let* φ: $\mathbb{R} \to \mathbb{R}$ *be a Lipschitz continuous function. If the processes* $(S_n(t))_{t \in [0,1]}$, $n \in \mathbb{N}$, *are stochastically uniformly bounded and if* $\left(\sum_{i=1}^{\tau_n(1)} Y_{ni}^2\right)_{n \in \mathbb{N}}$ *is stochastically bounded, then*

$$\max_{1 \leq k \leq \tau_n(1)} \left| \sum_{i=1}^{k} \varphi(S_{n,i-1}) Y_{ni} - \sum_{i=1}^{k} \varphi(\tilde{S}_{n,i-1}) \tilde{Y}_{ni} \right| \xrightarrow{P_n} 0.$$

Proof. Note that

$$\begin{aligned} &\left| \sum_{i=1}^{k} \varphi(S_{n,i-1}) Y_{ni} - \sum_{i=1}^{k} \varphi(\tilde{S}_{n,i-1}) \tilde{Y}_{ni} \right| \\ &\quad \leq \left| \sum_{i=1}^{k} (\varphi(S_{n,i-1}) - \varphi(\tilde{S}_{n,i-1})) \tilde{Y}_{ni} \right| \\ &\quad\quad + \max_{1 \leq i \leq k} |\varphi(S_{n,i-1})| \sum_{i=1}^{k} |Y_{ni} - \tilde{Y}_{ni}|. \end{aligned}$$

It is clear that the second term converges to zero.

To show the same for the first term, denote for arbitrary $\varepsilon > 0$, $b < \infty$

$$B_{ni} = \left\{ |\varphi(S_{n,i-1}) - \varphi(\tilde{S}_{n,i-1})| < \varepsilon, \sum_{j=1}^{i-1} \tilde{Y}_{ni}^2 < b \right\}.$$

Then for every $\delta > 0$ an application of Kolmogoroff's inequality yields

$$\begin{aligned} &P_n\left\{ \max_{1 \leq k \leq \tau_n(1)} \left| \sum_{i=1}^{k} (\varphi(S_{n,i-1}) - \varphi(\tilde{S}_{n,i-1})) \tilde{Y}_{ni} \right| > \delta \right\} \\ &\leq P_n\left\{ \max_{1 \leq k \leq \tau_n(1)} \left| \sum_{i=1}^{k} (\varphi(S_{n,i-1}) - \varphi(\tilde{S}_{n,i-1})) \tilde{Y}_{ni} \cdot 1_{B_{ni}} \right| > \delta \right\} + P_n\left(\bigcup_{i=1}^{\tau_n(1)} B'_{ni} \right) \\ &\leq \frac{1}{\delta^2} \cdot \varepsilon^2 \cdot (b+1) + P_n\left(\bigcup_{i=1}^{\tau_n(1)} B'_{ni} \right). \end{aligned}$$

Now, let $n \to \infty$. Since $\varepsilon > 0$ and $b < \infty$ can be chosen arbitrary, the assertion follows. □

The following assertion is a basic approximation lemma. A similar approximation has been used by Rootzén, [8].

(2.2) **Lemma.** *Assume that* (X_{ni}) *is a* MDA *satisfying condition* (1.2). *If the processes* $(S_n(t))_{t \in [0,1]}$ *have asymptotically continuous trajectories and if*

$$\lim_{a \to \infty} \sup_{\delta > 0} \overline{\lim_{n \to \infty}} P_n \left\{ \sum_{j=1}^{\left[\frac{1}{\delta}\right]} (S_n(j\delta) - S_n((j-1)\delta))^2 > a \right\} = 0, \tag{1}$$

then for every $\varepsilon > 0$

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P_n \left\{ \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\tau_n(t)} X_{ni}^2 - \sum_{j=1}^{\left[\frac{t}{\delta}\right]} (S_n(j\delta) - S_n((j-1)\delta))^2 \right| > \varepsilon \right\} = 0. \tag{2}$$

Although, the ideas of the proof are contained in Rootzén, [8], let us recapitulate the main steps for the reader's convenience.

Proof. W.l.g. we may assume that (X_{ni}) is a truncated array. Repeating a truncation, also used by Rootzén, [8], we define

$$\tau_{n\delta}(j) = \min \left\{ k = \tau_n((j-1)\delta) + 1, \ldots, \tau_n(j\delta) : \left| \sum_{i = \tau_n((j-1)\delta)+1}^{k} X_{ni} \right| > 1 \right\}.$$

The same arguments as in Rootzén, [8], show that, denoting

$$V_{nj}(\delta) = \sum_{k = \tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} X_{nk}^2 - \left(\sum_{k = \tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} X_{nk} \right)^2,$$

it is sufficient to show that

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P_n \left\{ \sup_{0 \leq t \leq 1} \left| \sum_{j=1}^{\left[\frac{t}{\delta}\right]} V_{nj}(\delta) \right| > \varepsilon \right\} = 0, \qquad \varepsilon > 0.$$

Let $\eta > 0$. Then by assumption there is $a(\eta)$ such that

$$P_n \left\{ \sum_{j=1}^{\left[\frac{1}{\delta}\right]} \left(\sum_{k = \tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} X_{nk} \right)^2 > a(\eta) \right\} < \eta, \qquad \delta > 0.$$

Defining

$$\theta_{n,\delta} = \min \left\{ k : \sum_{j=1}^{k} \left(\sum_{l = \tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} X_{nl} \right)^2 > a(\eta) \right\} \cap \left[\frac{1}{\delta} \right],$$

we arrive at stopping times $\tilde{\theta}_{n,\delta} = \tau_n(\delta \cdot \theta_{n,\delta})$.

Now, we perform a truncation

$$\tilde{X}_{ni} = X_{ni} \cdot 1_{\{\tilde{\theta}_{n\delta} \geq i\}},$$

obtaining a MDA $(\tilde{X}_{ni})$ such that

$$\overline{\lim_{n\to\infty}} P_n \bigcup_{j=1}^{[\frac{1}{\delta}]} \bigcup_{k=\tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} \{X_{nk} \neq \tilde{X}_{nk}\} < \eta, \qquad \delta > 0.$$

Since in this inequality the bound $\eta > 0$ is uniform in $\delta > 0$, it is sufficient to prove

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n \left\{ \sup_{0\leqq t\leqq 1} \left| \sum_{j=1}^{[\frac{t}{\delta}]} \tilde{V}_{nj}(\delta) \right| > \varepsilon \right\} = 0, \qquad \varepsilon > 0.$$

For this, let

$$\tilde{Y}_{n\delta}(j) = \sum_{k=\tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} \tilde{X}_{nk}, \qquad 1 \leqq j \leqq \left[\frac{1}{\delta}\right],$$

and

$$\tilde{Z}_n(\delta) = \max_{1\leqq j\leqq [\frac{1}{\delta}]} |\tilde{Y}_{n\delta}(j)|.$$

Then $(\tilde{Z}_n(\delta))$ is uniformly bounded and

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n\{|\tilde{Z}_n(\delta)| > \varepsilon\} = 0, \qquad \varepsilon > 0.$$

It follows that

$$P_n \left\{ \sup_{0\leqq t\leqq 1} \left| \sum_{j=1}^{[\frac{t}{\delta}]} \tilde{V}_{nj}(\delta) \right| > \varepsilon \right\} \leqq \frac{C}{\varepsilon^2} \cdot P_n \left(\sum_{j=1}^{[\frac{1}{\delta}]} \tilde{Y}_{n\delta}(j)^4 \right)$$

$$\leqq \frac{C}{\varepsilon^2} \cdot P_n(\tilde{Z}_n(\delta)^2 (a(\eta) + \tilde{Z}_n(\delta))^2). \quad \square$$

(2.3) *Remarks.* Assume that (X_{ni}) is a MDA satisfying condition (1.2).

(1) It is easy to see that condition (1) of Lemma (2.2) is satisfied, if $\left(\sum_{i=1}^{\tau_n(1)} X_{ni}^2\right)_{n\in\mathbb{N}}$ is stochastically bounded.

(2) If the distributions of the processes $(S_n(t))_{t\in[0,1]}$, $n\in\mathbb{N}$, converge weakly to those of a continuous, square-integrable martingale, then the conditions of Lemma (2.2) are satisfied and it follows that $\left(\sum_{i=1}^{\tau_n(1)} X_{ni}^2\right)_{n\in\mathbb{N}}$ is stochastically bounded.

We apply repeatedly the following tightness criterion for MDAs.

(2.4) **Lemma.** *Assume that* (X_{ni}) *is a* MDA *satisfying condition* (1.2). *The following assertions* (1) *and* (2) *are equivalent:*

(1) *The processes* $(S_n(t))_{t\in[0,1]}$, $n\in\mathbb{N}$, *have asymptotically continuous trajectories and* $\left(\sum_{i=1}^{\tau_n(1)} X_{ni}^2\right)_{n\in\mathbb{N}}$ *is stochastically bounded.*

(2) *The processes* $\left(\sum_{i=1}^{\tau_n(t)} X_{ni}^2\right)_{t\in[0,1]}$, $n\in\mathbb{N}$, *have asymptotically continuous trajectories.*

Proof. (1) ⇒ (2): This is almost obvious by Lemma (2.2) and Remark (2.3), (1).

(2) ⇒ (1): W.l.g. we may assume that (X_{ni}) is uniformly bounded by one. Since $\left(\sum_{i=1}^{\tau_n(1)} X_{ni}^2\right)_{n\in\mathbb{N}}$ is necessarily stochastically bounded for every $\varepsilon>0$ there is $a(\varepsilon)$ such that

$$\overline{\lim_{n\to\infty}}\, P_n\left\{\sum_{i=1}^{\tau_n(1)} X_{ni}^2 > a(\varepsilon)\right\} < \varepsilon.$$

Let

$$X_{ni}^{\varepsilon} = X_{ni}\cdot 1_{\left\{\sum_{j=1}^{i-1} X_{ni}^2 \leq a(\varepsilon)\right\}}.$$

It is sufficient to prove the assertion for the MDA (X_{ni}^{ε}). By Burkholder's inequality we have for every $\delta>0$

$$P_n\left\{\max_{1\leq j\leq\left[\frac{1}{\delta}\right]}\ \sup_{\tau_n((j-1)\delta)<l\leq\tau_n(j\delta)}\left|\sum_{k=\tau_n((j-1)\delta)+1}^{l} X_{nk}^{\varepsilon}\right|>\eta\right\}$$
$$\leq \frac{C}{\eta^3}\sum_{j=1}^{\left[\frac{1}{\delta}\right]} P_n\left[\left(\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} (X_{nk}^{\varepsilon})^2\right)^{\frac{3}{2}}\right].$$

It is not difficult to see that

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ \sum_{j=1}^{\left[\frac{1}{\delta}\right]} P_n\left[\left(\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} (X_{nk}^{\varepsilon})^2\right)^{\frac{3}{2}}\right]=0.$$

This proves the assertion. □

(2.5) *Remark.* Condition (2) of Lemma (2.4) is satisfied if

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ P_n\left\{\max_{1\leq j\leq\left[\frac{1}{\delta}\right]}\ \sum_{k=\tau_n((j-1)\delta)+1}^{\tau_n(j\delta)} X_{nk}^2>\varepsilon\right\}=0.$$

Hence, it is sufficient for the validity of condition (2) that the finite-dimensional distributions of $\left(\sum_{i=1}^{\tau_n(t)} X_{ni}^2\right)_{t\in[0,1]}$, $n\in\mathbb{N}$, converge to those of a continuous process.

Our last lemma contains an approximation of martingale transforms, which is similar to the approximation given in Lemma (2.2).

(2.6) **Lemma.** *Assume that (X_{ni}) and (Y_{ni}) are* MDAs *satisfying condition* (1.2). *Let φ: $\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function. If the processes $(S_n(t))_{t\in[0,1]}$ have asymptotically continuous trajectories and if $\left(\sum_{i=1}^{\tau_n(1)} Y_{ni}^2\right)_{n\in\mathbb{N}}$ is stochastically bounded, then the processes*

$$U_n(t)=\sum_{i=1}^{\tau_n(t)} \varphi(S_{n,i-1})\,Y_{ni},\qquad t\in[0,1],\ n\in\mathbb{N},$$

satisfy for every $\varepsilon > 0$

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n \left\{ \sup_{0\leq t\leq 1} \left| U_n(t) - \sum_{j=1}^{[\frac{t}{\delta}]} \varphi(S_n((j-1)\delta))(T_n(j\delta) - T_n((j-1)\delta)) \right| > \varepsilon \right\} = 0.$$

Proof. In view of Lemma (2.1) we may assume that (X_{ni}) and (Y_{ni}) are truncated arrays. For $\varepsilon > 0$ let $a(\varepsilon)$ be such that

$$\overline{\lim_{n\to\infty}} P_n \left\{ \sum_{i=1}^{\tau_n(1)} Y_{ni}^2 > a(\varepsilon) \right\} < \varepsilon,$$

and define

$$Y_{ni}^{\varepsilon} = Y_{ni} \cdot 1_{\left\{\sum_{j=1}^{i-1} Y_{nj}^2 \leq a(\varepsilon)\right\}}.$$

It is sufficient to prove the assertion for (Y_{ni}^{ε}) instead of (Y_{ni}). Let $\tau_{n\delta}(j)$, $1 \leq j \leq \left[\frac{1}{\delta}\right]$, $\delta > 0$, be the stopping times which have been defined in the proof of Lemma (2.2). Denoting

$$U_{nj}(\delta) = \sum_{k=\tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} (\varphi(S_{n,k-1}) - \varphi(S_n((j-1)\delta)))\, Y_{nk}^{\varepsilon},$$

we have to show that

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n \left\{ \sup_{0\leq t\leq 1} \left| \sum_{j=1}^{[\frac{t}{\delta}]} U_{nj}(\delta) \right| > \eta \right\} = 0, \qquad \eta > 0.$$

By Kolmogoroff's inequality we have

$$P_n \left\{ \sup_{0\leq t\leq 1} \left| \sum_{j=1}^{[\frac{t}{\delta}]} U_{nj}(\delta) \right| > \eta \right\} \leq \frac{1}{\eta^2} \sum_{j=1}^{[\frac{1}{\delta}]} P_n(U_{nj}^2(\delta)).$$

Since

$$P_n(U_{nj}^2(\delta)) = P_n \left(\sum_{k=\tau_n((j-1)\delta)+1}^{\tau_{n\delta}(j)} (\varphi(S_{n,k-1}) - \varphi(S_n((j-1)\delta)))^2 \cdot (Y_{nk}^{\varepsilon})^2 \right),$$

we have

$$\begin{aligned} \sum_{j=1}^{[\frac{1}{\delta}]} P_n(U_{nj}^2(\delta)) &\leq C \cdot P_n \left(Z_n(\delta)^2 \cdot \sum_{i=1}^{\tau_n(1)} (Y_{ni}^{\varepsilon})^2 \right) \\ &\leq C \cdot (a(\varepsilon) + 1) \cdot P_n(Z_n(\delta)^2), \end{aligned}$$

where

$$Z_n(\delta) = \max_{1\leq j\leq [\frac{1}{\delta}]} \sup_{\tau_n((j-1)\delta) < k \leq \tau_{n\delta}(j)} |S_{n,k-1} - S_n((j-1)\delta)|.$$

This proves the assertion. □

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