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A Stationary, Pairwise Independent, Absolutely Regular Sequence for which the Central Limit Theorem Fails*

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Summary. A strictly stationary finite-state non-degenerate random sequence is constructed which satisfies pairwise independence and absolute regularity but fails to satisfy a central limit theorem. The mixing rate for absolute regularity is only slightly slower than that in a corresponding central limit theorem of Ibragimov.

1. Introduction

Suppose $(X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of pairwise independent random variables. By Etemadi [5, Theorem 1], if $E|X_0| < \infty$ then (X_k) satisfies the strong law of large numbers. However, if $EX_0^2 < \infty$ and $\operatorname{Var} X_0 > 0$, it does not follow that (X_k) satisfies the central limit theorem. Indeed, Janson [8, Example 3] has constructed counterexamples with X_0 having an arbitrary distribution with finite second moment. The purpose of this note is to construct a counterexample which (in addition to pairwise independence) has some strong mixing properties. Our main result is Theorem 1 below.

The author owes a big debt to Professor Robert Burton. Two years ago Burton showed the author a (pairwise independent, non-ergodic) apparent 2-state counterexample (for which the failure to satisfy the CLT intuitively seemed likely but was not rigorously verified). This helped to lead the author to a two-state ergodic counterexample (constructed in Sect. 2 below). (According to a trusted source, such examples were apparently already known but not well publicized.) This in turn led to the main result of this note.

First let us define the mixing condition. Suppose $X := (X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \leqq J \leqq L \leqq \infty$ let $\mathscr{F}_J^L$ denote the σ-field generated by $(X_k, J \leqq k \leqq L)$. For $n = 1, 2, 3, \ldots$ define

$$\beta(n) := \beta(X, n) := \sup \tfrac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i) P(B_j)| \qquad (1.1)$$

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where this sup is taken over all pairs of finite partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω such that $A_i \in \mathscr{F}_{-\infty}^{0}$ for all i and $B_j \in \mathscr{F}_n^{\infty}$ for all j. The sequence X is said to satisfy "absolute regularity" [10, 11] if $\beta(n) \to 0$ as $n \to \infty$.

For partial sums we use the notation $S_n := X_1 + \ldots + X_n$.

Also, the notation "$\ll$" means $O(\cdot)$.

Theorem 1. *There exists a strictly stationary sequence $X := (X_k, k \in \mathbb{Z})$ such that the following statements hold:*

$$X_0 \text{ takes only three values } (-1,\ 0, \text{ and } 1, \text{ with } P(X_0=0)=1/2 \quad \text{and} \quad P(X_0=-1)=P(X_0=1)=1/4); \tag{1.2}$$

$$\forall\, n \neq 0,\ X_0 \text{ and } X_n \text{ are independent;} \tag{1.3}$$

$$\beta(n) \ll 1/n \text{ as } n \to \infty; \tag{1.4}$$

$$[\inf_{n \geq 1} P(S_n = 0)] > 0; \text{ and} \tag{1.5}$$

$$\text{the family of r.v.'s } (S_n,\ n=1, 2, \ldots) \text{ is tight.} \tag{1.6}$$

By (1.5), S_n cannot become asymptotically normal under any kind of normalization.

Let us briefly consider the class of known stationary absolutely regular sequences (X_k) which fail to satisfy the CLT. In the examples of Herrndorf [6] and Bradley [2, Theorems 1, 2, 7] the X_k's are uncorrelated (but not pairwise independent). In the case where the X_k's are bounded (with or without being uncorrelated), the mixing rate in (1.4) is the fastest that has been obtained so far – but only slightly faster than the rates obtained earlier in Davydov [4, Example 2] and Bradley [2, Theorem 2]. As a special case of a result of Ibragimov in [7, Theorem 18.5.4], the mixing rate $\sum_{n=1}^{\infty} \beta(n) < \infty$ together with (1.2)–(1.3) would imply the CLT. There still remains a slight gap between Ibragimov's rate and (1.4). (It should perhaps be mentioned that Davydov [4] and Herrndorf [6] discussed the "strong mixing" condition instead of absolute regularity, but their arguments extend directly to absolute regularity. Also, a mistake in Bradley [2] – the attributing to M.I. Gordin of a misstatement of one of his results – is corrected in [3].)

The main idea of our construction – the use of a function of a "renewal type" Markov chain – comes from Davydov [4]. We shall adapt Davydov's [4] method of estimating dependence coefficients. We shall adapt an idea of H.C.P. Berbee (from his own construction given in [1, Theorem 11]) which converted [4, Example 2] from a bounded countable-state example to a finite-state one. Also we shall adapt an idea of Herrndorf [6] which achieved (1.5) and (1.6) in his own counterexample.

The proof of Theorem 1 will be given in Sect. 3, based on preliminary work done in Sect. 2.

2. Preliminaries

In this section some random sequences will be constructed and some lemmas will be given. This material will be used in Sect. 3 in the proof of Theorem 1.

Definition 2.1. (a) Let $\mathscr{S}$ denote the set of all ordered pairs (t, u) such that

$$t \in \{2, 4, 8, 16, 32, \dots\} \cup \{-2, -4, -8, -16, -32, \dots\}$$

and

$$u \in \{0, 1, 2, \dots, |t|-1\}.$$

(b) Let μ denote the probability measure on $\mathscr{S}$ defined by

$$\mu(\{(t, u)\}) = 1/(2t^2) \qquad \forall (t, u) \in \mathscr{S}.$$

In what follows, for $s \in \mathscr{S}$, the quantity $\mu(\{s\})$ will be written simply as $\mu(s)$.

(c) Let $(V_k, k \in \mathbb{Z})$, with $V_k := (T_k, U_k)\ \forall k \in \mathbb{Z}$, be a strictly stationary Markov chain with state space $\mathscr{S}$, with invariant marginal probability measure μ, and with one-step transition probabilities given by the following equations:

$$P(V_1 = (t, |t|-1) \mid V_0 = (T, 0)) = 3/(2t^2) \qquad \forall t, T = \pm 2, \pm 4, \pm 8, \pm 16, \dots;$$

$$P(V_1 = (t, u-1) \mid V_0 = (t, u)) = 1 \qquad \forall t = \pm 2, \pm 4, \pm 8, \dots\ \forall u = 1, 2, 3, \dots, |t|-1;$$

$$P(V_1 = s_1 \mid V_0 = s_0) = 0 \qquad \text{for all other pairs of states } s_0, s_1 \in \mathscr{S}.$$

(d) Define the function $f\colon \mathscr{S} \to \{-1, 1\}$ as follows:

$$f((t, u)) := \begin{cases} 1 & \text{if } t = 2, 4, 8, \dots \text{ and } t/2 \leqq u < t; \\ -1 & \text{if } t = 2, 4, 8, \dots \text{ and } 0 \leqq u < t/2; \\ -1 & \text{if } t = -2, -4, -8, \dots \text{ and } |t|/2 \leqq u < |t|; \\ 1 & \text{if } t = -2, -4, -8, \dots \text{ and } 0 \leqq u < |t|/2. \end{cases}$$

(e) Define the (strictly stationary) sequence $(W_k, k \in \mathbb{Z})$ as follows:

$$\forall k \in \mathbb{Z}, \qquad W_k = f(V_k).$$

Remark 2.2. Referring to Definition 2.1(c) we shall henceforth assume that for every $\omega \in \Omega$, every $k \in \mathbb{Z}$, the ordered pair $(V_k, V_{k+1})(\omega)$ is either $((T, 0), (t, |t|-1))$ for some numbers $T, t = \pm 2, \pm 4, \pm 8, \dots$ or else $((t, u), (t, u-1))$ for some $t = \pm 2, \pm 4, \pm 8, \dots$, $u = 1, 2, \dots, |t|-1$. For a given $\omega \in \Omega$, if $I < J$ are integers such that

$$U_I(\omega) = U_J(\omega) = 0 \quad \text{and} \quad U_k(\omega) \neq 0 \qquad \forall k = I+1, \dots, J-1,$$

then $J - I \in \{2, 4, 8, \dots\}$,

$$(V_{I+1}, V_{I+2}, \dots, V_J)(\omega) = ((t, t-1), (t, t-2), \dots, (t, 0))$$

or

$$((-t, t-1), (-t, t-2), \dots, (-t, 0)) \qquad \text{for } t = J - I,$$

and $(W_{I+1}, W_{I+2}, \ldots, W_J)(\omega) = (1, \ldots, 1, -1, \ldots, -1)$ or $(-1, \ldots, -1, 1, \ldots, 1)$ (with $(J-I)/2$ 1's and $(J-I)/2$ -1's in either case).

Note that by Definition 2.1(b)(c),

$$P(U_0 = 0) = \mu(\{(t, 0): t = \pm 2, \pm 4, \pm 8, \ldots\}) = 1/3. \tag{2.1}$$

In what follows, the notation $\sigma(\ldots)$ means the σ-field generated by $(\ldots)$. The following lemma will be helpful.

Lemma 2.3. *Suppose* $A \in \sigma(V_k, k \leqq 0)$, $B \in \sigma(V_k, k \geqq 1)$, *and* $P(A \cap \{U_0 = 0\}) > 0$. *Then* $P(B \mid A \cap \{U_0 = 0\}) = P(B \mid U_0 = 0)$.

The proof is elementary. One first verifies the lemma for the special case where $A = \{T_0 = t\}$ and $B = \{V_1 = s\}$ where $t = \pm 2, \pm 4, \pm 8, \ldots$ and $s \in \mathscr{S}$; and then one uses the Markov property.

For any integers $I < J$ define the event

$$D(I, J) := \{U_I = U_J = 0, U_k \neq 0 \ \forall k, I < k < J\}. \tag{2.2}$$

(By Definition 2.1 and Remark 2.2, this event is non-empty only if $J - I = 2, 4, 8, 16, \ldots$). It is easy to see that if L is a positive integer and $0 < n \leqq 2^L$, then

$$P(W_n = 1 \mid D(0, 2^L)) = P(W_n = -1 \mid D(0, 2^L)) = 1/2.$$

Hence by stationarity we have

Lemma 2.4. $P(W_0 = 1) = P(W_0 = -1) = 1/2$.

Lemma 2.5. *Suppose* $n \geqq 1$. *Then*

$$P(W_0 \neq W_n \mid U_k = 0 \textit{ for some } k = 0, 1, \ldots, n-1) = 1/2,$$

and

$$P(W_0 \neq W_n \mid U_k \neq 0 \ \forall k = 0, 1, \ldots, n-1) = 1/2.$$

Proof. The first equation is elementary. We shall just prove the second. Define the event $D := \{U_k \neq 0 \ \forall k = 0, 1, \ldots, n-1\}$. Let L be the positive integer such that $2^{L-1} \leqq n < 2^L$. The event D can be partitioned into events $D(I, I + 2^l)$ where $I < 0$ and $I + 2^l \geqq n$ (which forces $l \geqq L$). For such an I and l,

$$D(I, I + 2^l) = \{U_I = 0, |T_{I+1}| = 2^l\}$$

and hence

$$\begin{aligned} P(D(I, I + 2^l)) &= P(U_I = 0) \cdot P(|T_{I+1}| = 2^l \mid U_I = 0) \\ &= (1/3) \cdot (3/4^l) = 1/4^l \end{aligned} \tag{2.3}$$

by Definition 2.1(c) and (2.1). Hence

$$\begin{aligned} P(D) &= \sum_{l=L}^{\infty} \sum_{I=n-2^l}^{-1} P(D(I, I + 2^l)) \\ &= 2 \cdot [2^{-L} - (2n/3) \cdot 4^{-L}]. \end{aligned} \tag{2.4}$$

Next, the event $\{W_0 \neq W_n\} \cap D$ can be partitioned into events $D(I, I+2^l)$ where $l \geqq L$, $I<0$, $I+2^l \geqq n$ and $0 \leqq I+2^{l-1}<n$. (This last equation follows from Remark 2.2.) For $l=L$, the conditions on I are (equivalent to) $n-2^L \leqq I<0$. For $l \geqq L+1$, the conditions on I are (equivalent to) $-2^{l-1} \leqq I<n-2^{l-1}$. Hence

$$\begin{aligned} P(\{W_0 \neq W_n\} \cap D) = & \sum_{I=n-2^L}^{-1} P(D(I, I+2^L)) \\ & + \sum_{l=L+1}^{\infty} \sum_{I=-2^{l-1}}^{n-2^{l-1}-1} P(D(I, I+2^l)) \\ = & (1/2) \cdot P(D) \end{aligned}$$

by (2.3) and (2.4).

Lemma 2.6. *If $n \geqq 1$, then W_0 and W_n are independent* r.v.'s.

This follows from the previous two lemmas.

Lemma 2.7. *As $n \to \infty$, n even, one has that*

$$|P(U_n=0 \mid U_0=0)-2/3| \ll 1/n.$$

The proof of this lemma is an application of Rogozin [9, p. 665, Theorem 1]. The r.v.'s $\xi_1, \xi_2, \xi_3, \ldots$ in his result are to be defined by $\xi_k=(I_k-I_{k-1})/2$, where $I_0, I_1, I_2, \ldots$ are the successive non-negative random integers k such that $U_k=0$. The rest of the details are left to the reader.

The next three lemmas will require some more definitions.

Definition 2.8. (a) Let $\mathscr{E}$ (resp. $\mathscr{O}$) denote the set of all $s=(t, u) \in \mathscr{S}$ such that u is even (resp. odd).

(b) For $n=1, 2, 3, \ldots$ define the following subset $\Gamma(n) \subset \mathscr{S} \times \mathscr{S}$:

$$\Gamma(n) := \begin{cases} (\mathscr{E} \times \mathscr{E}) \cup (\mathscr{O} \times \mathscr{O}) & \text{if } n \text{ is even} \\ (\mathscr{E} \times \mathscr{O}) \cup (\mathscr{O} \times \mathscr{E}) & \text{if } n \text{ is odd} \end{cases}.$$

It is easy to see from Definition 2.1(c) (and Remark 2.2) that

$$\forall n \geqq 1, \quad \forall \omega \in \Omega, \quad (V_0, V_n)(\omega) \in \Gamma(n). \tag{2.5}$$

Next, referring to Lemma 2.7, define the constant

$$C := \sup_{\text{even } n \geqq 2} [n \cdot |P(U_n=0 \mid U_0=0)-(2/3)|]. \tag{2.6}$$

Lemma 2.9. *Suppose $n \geqq 8$ is an even integer. Suppose $s:=(t, u)$ and $s^*:=(t^*, u^*)$ are elements of $\mathscr{E}$ such that $|t| \leqq n/4$ and $|t^*| \leqq n/4$. Then*

$$\left| \frac{P(V_0=s, V_n=s^*)}{\mu(s)\, \mu(s^*)} - 2 \right| \leqq 6C/n. \tag{2.7}$$

Proof. By Remark 2.2,

$$\{V_0=s\}=\{V_u=(t,0)\}\subset\{U_u=0\}$$

and

$$\{V_n=s^*\}=\{V_{n+u^*-|t^*|+1}=(t^*,|t^*|-1)\}\subset\{U_{n+u^*-|t^*|}=0\}.$$

By Lemma 2.3 and elementary calculations,

$$\frac{P(V_0=s,\,V_n=s^*)}{P(V_0=s)\cdot P(V_n=s^*)}=\frac{P(U_u=0,\,U_{n+u^*-|t^*|}=0)}{P(U_u=0)\cdot P(U_{n+u^*-|t^*|}=0)}$$
$$=3\cdot P(U_{n+u^*-|t^*|}=0\,|\,U_u=0) \tag{2.8}$$

where the 3 comes from (2.1). Now $(n+u^*-|t^*|)-u>n/2>2$ (by the hypothesis of this lemma). Hence from (2.6) and stationarity we have

$$|P(U_{n+u^*-|t^*|}=0\,|\,U_u=0)-(2/3)|\leqq C/(n+u^*-|t^*|-u)\leqq 2C/n.$$

From this and (2.8) one has (2.7).

Lemma 2.10. *Suppose $n\geqq 8$ is an even integer. Then*

$$\sum_{s\in\mathscr{E}}\sum_{s^*\in\mathscr{E}}|P(V_0=s,\,V_n=s^*)-2\mu(s)\,\mu(s^*)|\leqq(6C+48)/n. \tag{2.9}$$

Proof. Let $\mathscr{C}$ (resp. $\mathscr{D}$) denote the set of all states $s=(t,u)\in\mathscr{E}$ such that $|t|\leqq n/4$ (resp. $|t|>n/4$). Then by Lemma 2.9,

$$\begin{aligned}
&[\text{L.H.S. of (2.9)}]\\
&\quad=\Big(\sum_{s\in\mathscr{C}}\sum_{s^*\in\mathscr{C}}+\sum_{s\in\mathscr{D}}\sum_{s^*\in\mathscr{C}}+\sum_{s\in\mathscr{E}}\sum_{s^*\in\mathscr{D}}\Big)|P(V_0=s,\,V_n=s^*)-2\mu(s)\,\mu(s^*)|\\
&\quad\leqq\sum_{s\in\mathscr{C}}\sum_{s^*\in\mathscr{C}}(6C/n)\cdot\mu(s)\,\mu(s^*)\\
&\qquad+\sum_{s\in\mathscr{D}}[P(V_0=s)+2\mu(s)]+\sum_{s^*\in\mathscr{D}}[P(V_n=s^*)+2\mu(s^*)]\\
&\quad\leqq(6C/n)+6\mu(\mathscr{D}).
\end{aligned}$$

Now $\mu(\mathscr{D})\leqq 8/n$ by an elementary calculation. The lemma follows.

For the next lemma, a corollary of Lemma 2.10, recall Definition 2.8(b).

Lemma 2.11. *Suppose $n\geqq 8$ is an integer (even or odd). Then*

$$\sum_{(s,\,s^*)\in\Gamma(n)}|P(V_0=s,\,V_n=s^*)-2\mu(s)\,\mu(s^*)|\leqq(12C+96)/(n-1).$$

Lemma 2.12. $[\inf_{\text{even } n\geqq 2} P(W_1+\ldots+W_n)=0]>0.$

This follows from three elementary facts: for all even $n \geqq 2$,

$$\{W_1 + \ldots + W_n = 0\} \supset \{U_0 = U_n = 0\};$$

for all even $n \geqq 2$, $P(U_0 = U_n = 0) > 0$; and $\operatorname*{Lim}_{n \to \infty,\, n \text{ even}} P(U_0 = U_n = 0) > 0$ (by (2.1) and Lemma 2.7).

Lemma 2.13. $\forall\, n \geqq 1$, $\forall\, c > 0$,

$$P(|W_1 + \ldots + W_n| \geqq c) \leqq 2 \cdot P(|T_0| \geqq c/2).$$

This follows from the elementary fact that for each n, $\left|\sum_{k=1}^{n} W_k\right| \leqq |T_0| + |T_n|$.

3. Proof of Theorem 1

We retain all of the definitions from Sect. 2, in particular Definition 2.1, Remark 2.2, and Definition 2.8.

Definition 3.1. (a) Let $(\varepsilon_k, k \in \mathbb{Z})$ be a sequence of i.i.d. r.v.'s, independent of $(T_k, U_k, V_k, W_k, k \in \mathbb{Z})$, such that $P(\varepsilon_0 = 0) = P(\varepsilon_0 = 1) = 1/2$.

(b) Define the random integers $\ldots, I_{-1}, I_0, I_1, \ldots$ by the conditions

$$\ldots < I_{-2} < I_{-1} < I_0 \leqq 0 < 1 \leqq I_1 < I_2 < I_3 < \ldots$$

and

$$\forall\, \omega \in \Omega, \quad \{k\colon \varepsilon_k(\omega) = 1\} = \{\ldots, I_{-1}(\omega), I_0(\omega), I_1(\omega), \ldots\}.$$

Deleting a null set from our probability space if necessary, we henceforth assume that each ε_k takes only the values 0 and 1 and that the random sequence $(I_j, j \in \mathbb{Z})$ is defined at all $\omega \in \Omega$.

(c) Define the random sequence $Y := (Y_k, k \in \mathbb{Z})$ as follows: For all $k \in \mathbb{Z}$, all $\omega \in \Omega$,

$$Y_k(\omega) = \begin{cases} V_j(\omega) & \text{if } k = I_j(\omega) \quad \text{for some } j \in \mathbb{Z} \\ 0 & \text{if } k \notin \{\ldots, I_{-1}(\omega), I_0(\omega), I_1(\omega), \ldots\} \end{cases}.$$

(d) Define the random sequence $X := (X_k, k \in \mathbb{Z})$ as follows: For all $k \in \mathbb{Z}$, all $\omega \in \Omega$,

$$X_k(\omega) = \begin{cases} W_j(\omega) & \text{if } k = I_j(\omega) \quad \text{for some } j \in \mathbb{Z} \\ 0 & \text{if } k \notin \{\ldots, I_{-1}(\omega), I_0(\omega), I_1(\omega), \ldots\} \end{cases}.$$

Note that each Y_k takes its values in $\mathscr{S} \cup \{0\}$, and each X_k takes its values in $\{-1, 0, 1\}$.

It is not hard to show that the sequence Y is strictly stationary. Perhaps the easiest way to accomplish this is to show that the sequence $((\varepsilon_k, Y_k), k \in \mathbb{Z})$ is strictly stationary.

Similarly, the sequence X is strictly stationary. By elementary arguments based on Lemmas 2.4, 2.6, 2.12, and 2.13, the sequence X satisfies (1.2), (1.3), (1.5), and (1.6). Here we shall just show that X also satisfies (1.4).

Using Lemma 2.11, let Q be a positive number such that $\forall\, n \geqq 1$,

$$\sum_{(s, s^*) \in \Gamma(n)} |P(V_0 = s, V_n = s^*) - 2\mu(s)\,\mu(s^*)| \leqq Q/n. \tag{3.1}$$

Let $N \geqq 2$ be arbitrary but fixed. To prove (1.4) it suffices to prove $\beta(X, N) \leqq 8Q/N$. Note that the sequence X is an "instantaneous" function of the sequence Y, and hence $\beta(X, N) \leqq \beta(Y, N)$. Hence, to prove (1.4) it suffices to prove

$$\beta(Y, N) \leqq 8Q/N. \tag{3.2}$$

The following fact will be useful: If $l \in \mathbb{Z}$, $s \in \mathscr{S}$, $A \in \sigma(Y_k, k \leqq l)$ and $B \in \sigma(Y_k, k \geqq l)$, then $P(A \cap B \mid Y_l = s) = P(A \mid Y_l = s) \cdot P(B \mid Y_l = s)$. This is trivial for $l = 0$, and hence also for all other l by stationarity.

(Of course Y is not a Markov chain; this fact does not hold with s replaced by 0.)

In our proof of (3.2) the following events will be useful:

For each $I \leqq 0$, each $s \in \mathscr{S}$, define the event

$$A(I, s) := \{(Y_I, Y_{I+1}, \ldots, Y_0) = (s, 0, \ldots, 0)\}.$$

(Define $A(0, s) := \{Y_0 = s\}$.)

For each $J \geqq N$, each $s \in \mathscr{S}$, define the event

$$B(J, s) := \{(Y_N, Y_{N+1}, \ldots, Y_J) = (0, \ldots, 0, s)\}.$$

(Define $B(N, s) := \{Y_N = s\}$.)

By an elementary argument,

$$2\beta(Y, N) = \sum_{I=-\infty}^{0} \sum_{s \in \mathscr{S}} \sum_{J=N}^{\infty} \sum_{s^* \in \mathscr{S}} |P(A(I, s) \cap B(J, s^*)) - P(A(I, s)) \cdot P(B(J, s^*))|. \tag{3.3}$$

For any given pair of states $s, s^* \in \mathscr{S}$, the notation $\sum_l'$ will mean the sum over all $l \in \{0, 1, \ldots, N-1\}$ such that $(s, s^*) \in \Gamma(l+1)$ (recall Definition 2.8(b)). Depending on s and s^*, $\sum_l'$ will thus mean either the sum over all *even* l, $0 \leqq l \leqq N-1$, or the sum over all *odd* l, $0 \leqq l \leqq N-1$.

For each $I \leqq 0$ define the event $A'(I) := \{\varepsilon_I = 1, \varepsilon_{I+1} = \ldots = \varepsilon_0 = 0\}$. (Define $A'(0) := \{\varepsilon_0 = 1\}$.) For each $J \geqq N$ define the event

$$B'(J) := \{\varepsilon_N = \ldots = \varepsilon_{J-1} = 0, \varepsilon_J = 1\}.$$

(Define $B'(N) := \{\varepsilon_N = 1\}$.)

Define the r.v. $Z := \varepsilon_1 + \ldots + \varepsilon_{N-1}$. Then by (3.3) and elementary calculations (and our stipulation $N \geqq 2$),

$$
\begin{aligned}
2\beta(Y, N) = & \sum_{I=-\infty}^{0} \sum_{s\in\mathscr{S}} \sum_{J=N}^{\infty} \sum_{s^*\in\mathscr{S}} \Big|\sum_l{}' P(A(I,s) \cap B(J,s^*) \cap \{Z=l\}) \\
& \qquad - P(A(I,s)) \cdot P(B(J,s^*)) \cdot 2\sum_l{}' P(Z=l)\Big| \\
= & \sum_{I=-\infty}^{0} \sum_{s\in\mathscr{S}} \sum_{J=N}^{\infty} \sum_{s^*\in\mathscr{S}} \Big|\sum_l{}' P(\{V_0=s\} \cap A'(I) \cap \{V_{l+1}=s^*\} \cap B'(J) \cap \{Z=l\} \\
& \qquad - P(A'(I)) \cdot \mu(s) \cdot P(B'(J)) \cdot \mu(s^*) \cdot 2\sum_l{}' P(Z=l)\Big| \\
= & \sum_I \sum_s \sum_J \sum_{s^*} \Big| P(A'(I)) \cdot P(B'(J)) \cdot \sum_l{}' P(Z=l) \\
& \qquad \cdot [P(V_0=s, V_{l+1}=s^*) - 2\mu(s)\,\mu(s^*)]\Big| \\
\leqq & \sum_I P(A'(I)) \cdot \sum_J P(B'(J)) \cdot \sum_s \sum_{s^*} \sum_l{}' P(Z=l) \\
& \cdot |P(V_0=s, V_{l+1}=s^*) - 2\mu(s)\,\mu(s^*)| \\
= & 1 \cdot 1 \cdot \sum_{l=0}^{N-1} \sum_{(s,s^*)\in\Gamma(l+1)} P(Z=l) \cdot |P(V_0=s, V_{l+1}=s^*) - 2\mu(s)\,\mu(s^*)| \\
\leqq & \sum_{l=0}^{N-1} P(Z=l) \cdot Q/(l+1) \\
\leqq & P(Z \leqq (N-1)/4) \cdot Q + P(Z > (N-1)/4) \cdot 4Q/(N-1) \\
\leqq & 4Q/(N-1) + 4Q/(N-1) = 8Q/(N-1) \leqq 16Q/N.
\end{aligned}
$$

Thus (3.2) holds, and this completes the proof of (1.4).

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