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Probability Theory and Related Fields

On the Rate of Convergence of the Sum of the Sample Extremes

Joop Mijnheer
Department of Mathematics, University of Leiden, 2300 RA Leiden, The Netherlands

Summary. Let $X_1, X_2, \ldots, X_n$ be independent random variables having a common distribution in the domain of normal attraction of a completely asymmetric stable law with characteristic exponent $\alpha \in (0, 1)$ and support bounded below. Let $X_{n:n} \geqq X_{n:n-1} \geqq \ldots \geqq X_{n:1}$ denote the ordered sample. We obtain the rate of convergence of $n^{-1/\alpha}(X_{n:n} + \ldots + X_{n:n-k_n+1})$ to the stable limit law as *both* n and $k_n \to \infty$. As a consequence we obtain a representation of the sum $X_{n:n} + \ldots + X_{n:n-k_n+1}$.

1. Main Result

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common distribution function G in the domain of *normal* attraction of the completely asymmetric stable law $F(\cdot\,; \alpha, 1)$ with $0 < \alpha < 1$.

Let $X_{n:n} \geqq X_{n:n-1} \geqq \ldots \geqq X_{n:1}$ be the ordered sample. In our theorem we need more information on the tail behaviour of the distribution function. The restriction to completely asymmetric stable limit laws is not essential. We can obtain similar results in the case of other stable limit laws. We assume that G is continuous.

Let $S_n = X_1 + \ldots + X_n$. Darling (1952) showed that $S_n/X_{n:n}$ converges to a non-constant distribution. Hence we have

$$E(S_n/X_{n:n}) \to (1-\alpha)^{-1} \quad \text{for } n \to \infty.$$

Arov and Bobrov (1960) showed (Corollary 6), for $k_n \to \infty$ and $k_n n^{-1} \log n \to 0$ as $n \to \infty$, that, for $0 < \alpha < 1$,

$$S_n = X_{n:n} + \ldots + X_{n:n-k_n+1} + [\alpha(1-\alpha)^{-1} + o(1)]\, k_n X_{n:n-k_n+1}$$

where $o(1)$ converges in probability to zero as $n \to \infty$. It follows from Theorem 3 in Csörgő a.o. (1986) that $n^{-1/\alpha}(X_{n:n} + \ldots + X_{n:n-k_n+1})$ converges in distribution to the completely asymmetric stable law $F(\cdot\,; \alpha, 1)$. They do not make the restriction to the case $\alpha \in (0, 1)$ and $\beta = 1$. The main tool in their proof is a new Brownian bridge approximation to the uniform empirical process. In Mijnheer (1986) we sketched two simple proofs for the assertion above. In this paper we derive the rate of convergence. One result about the rate of convergence is known in the literature. See Hall (1978). He considers symmetrically distributed random variables in the domain of normal attraction of $F(\cdot\,;\alpha,0)$. He orders the sample in increasing absolute values and obtains the limit distribution for the k order statistics with extremal absolute values, where k is fixed and n tends to infinity. He derives the rate of convergence to the stable limit distribution $F(\cdot\,;\alpha,0)$ as $k\to\infty$. In this paper we obtain the rate of convergence when k_n and n tend to infinity simultaneously.

In Mijnheer (1986) we derived the characteristic function φ_{n,k_n} of $n^{-1/\alpha}(X_{n:n}+\ldots+X_{n:n-k_n+1})$. We obtained

$$\begin{aligned}\varphi_{n,k_n}(t)&=E\{\exp(itn^{-1/\alpha}(X_{n:n}+\ldots+X_{n:n-k_n+1}))\}\\&=E\{E(\exp(itn^{-1/\alpha}G^{-1}(Z)\,|\,X_{n:n-k_n}))\}^{k_n},\end{aligned}$$

where, given $X_{n:n-k_n}=x$, the random variable Z has an uniform distribution $(G(x),1)$. We make the following assumptions for $x\geqq x_0$

$$1-G(x)=x^{-\alpha}+r(x), \tag{1.1}$$

where $r(x)=\mathcal{O}(x^{-\gamma})$ for $x\to\infty$ and $\alpha<\gamma$. In the case when $0<\gamma\leqq 1$ we assume that $r(x)$ is monotone for x sufficiently large. These assumptions are comparable with those given in Cramér's paper (1963). A comparison of our results with the rate of convergence obtained by Cramér – under forgoing conditions – is given in Remark 2. We write X instead of $X_{n:n-k_n}$.

Define the function h by

$$h(t,x)=E\exp(itn^{-1/\alpha}G^{-1}(Z)\,|\,X=x). \tag{1.2}$$

Using (1.1) and by partial integration we obtain for $t>0$ and $x\geqq x_0$

$$\begin{aligned}h(t,x)&=\{1-G(x)\}^{-1}\int_{G(x)}^{1}\exp\{itn^{-1/\alpha}G^{-1}(z)\}\,dz\\&=1+\{1-G(x)\}^{-1}\int_{x}^{\infty}\{\exp(itn^{-1/\alpha}y)-1\}\,dG(y)\\&=1-c_1t^{\alpha}n^{-1}\{1-G(x)\}^{-1}\\&\quad-\alpha\{1-G(x)\}^{-1}\int_{0}^{x}\{\exp(itn^{-1/\alpha}y)-1\}\,y^{-\alpha-1}\,dy\\&\quad+\{1-G(x)\}^{-1}\{\exp(itxn^{-1/\alpha})-1\}\,r(x)\\&\quad+itn^{-1/\alpha}\{1-G(x)\}^{-1}\int_{x}^{\infty}\exp(ityn^{-1/\alpha})\,r(y)\,dy,\end{aligned}\tag{1.3}$$

where c_1 is some (known complex) constant. For $t<0$ we have $h(t,x)=\overline{h(-t,x)}$.

One easily shows, without any assumption on G, that for $0 \leqq j \leqq k_n$

$$E\{1-G(X)\}^{-j} = \frac{n!}{(n-j)!} \frac{(k_n-j)!}{k_n!}. \tag{1.4}$$

Using the asymptotic behaviour of the Γ-function (see Abramowitz and Stegun (1964)) we obtain

$$EX^2 = (n/k_n)^{2/\alpha}\{1+\mathcal{O}(k_n^{-1})+\mathcal{O}((k_n/n)^{(\gamma-\alpha)/\alpha})\} \tag{1.5}$$

for $k_n, n \to \infty$ and $k_n/n \to 0$.

Lemma 1. (a) *There exists some positive constant β (which only depends on α) such that for $k_n, n \to \infty$*

$$P(\tfrac{1}{2}(n/k_n)^{1/\alpha} \leqq X \leqq 2(n/k_n)^{1/\alpha}) = 1 - \mathcal{O}(e^{-\beta k_n})$$

if $k_n = o(n^{\frac{1}{2}})$.

(b) *Take $\varepsilon > 0$ and $t > 0$.*

$$r_n = P(tXn^{-1/\alpha} > \varepsilon) = \mathcal{O}\left(\frac{1}{k_n!}(t/\varepsilon)^{\alpha(k_n+1)}) e^{-t^\alpha \varepsilon^{-\alpha}}\right)$$

for $k_n, n \to \infty$ and $k_n/n \to 0$.

Proof. We use the following well-known relation between order-statistics and the binomial distribution. For $y_n = G(\frac{1}{2}(n/k_n)^{1/\alpha})$ we have

$$P(X \leqq \tfrac{1}{2}(n/k_n)^{1/\alpha}) = \sum_{j=0}^{k_n} \binom{n}{j} (1-y_n)^j y_n^{n-j}$$

$$\leqq 2 y_n^n \sum_{j=0}^{k_n} \frac{1}{j!} (2^\alpha k_n)^j.$$

The sequence $\frac{1}{j!}(2^\alpha k_n)^j$, $j=0, 1, \ldots, k_n$, is increasing (in j). Thus

$$P(X \leqq \tfrac{1}{2}(n/k_n)^{1/\alpha}) \leqq 2 y_n^n k_n \frac{1}{k_n!}(2^\alpha k_n)^{k_n}.$$

We obtain the given upperbound by making use of Stirling's formula and the behaviour of G as given in (1.1).

Similarly we have

$$P(X \geqq 2(n/k_n)^{1/\alpha}) = \frac{n!}{k_n!(n-k_n-1)!} \int_0^{1-y_n} y^{k_n}(1-y)^{n-k_n-1}\, dy$$

where $y_n = G(2(n/k_n)^{1/\alpha})$. It follows from the monotonicity of the integrand and Stirling's formula that

$$P(X \geqq 2(n/k_n)^{1/\alpha}) = \mathcal{O}(e^{-\beta k_n}).$$

The proof of part b is analogous. □

In the following lemmas we shall make the following set of conditions

$$C_1: \ |t| \leqq k_n^p \quad \text{with} \quad \alpha p < 1$$

and

$$C_2: \begin{cases} k_n, \ n \to \infty \\ k_n/n \to 0. \end{cases}$$

Under these conditions we have $n^{-1/\alpha}|t| EX \to 0$ and it follows from Lemma 1.b that $n^{-1/\alpha}|t| X \to 0$ in probability.

Remark 1. Under the conditions given in C_1 and C_2 we have

$$P(|t| X n^{-1/\alpha} > \varepsilon) = \mathcal{O}(k_n^{-\frac{1}{2}} e^{-\Delta k_n \log k_n})$$

for some $\Delta > 0$.

We define the function h_1 by

$$h_1(t, x) = 1 - c_1 t^\alpha n^{-1} \{1 - G(x)\}^{-1} \qquad \text{for } t \geqq 0$$

and

$$h_1(t, x) = \overline{h_1(-t, x)} \qquad \text{for } t < 0.$$

The constant c_1 is the same (known complex) constant as in (1.3).

Lemma 2. *Under the conditions given in C_1 and C_2 we have for $t > 0$*

$$E\{h_1(t, X)\}^{k_n} = e^{-c_1 t^\alpha} \{1 + \mathcal{O}(t^{2\alpha} n^{-1})\} + r_n(t)$$

where $r_n(t)$ is uniformly bounded by

$$c\, k_n^{-\frac{1}{2}} e^{-\Delta k_n \log k_n}$$

for some positive constants c and Δ.

Proof.

$$\begin{aligned} E\{h_1(t, X)\}^{k_n} &= E\{1 - c_1 t^\alpha n^{-1} \{1 - G(X)\}^{-1}\}^{k_n} \\ &= \sum_{j=0}^{k_n} (-1)^j \binom{k_n}{j} \{c_1 t^\alpha n^{-1}\}^j E\{1 - G(X)\}^{-j} \\ &= \sum_{j=0}^{k_n} (-1)^j \binom{k_n}{j} \{c_1 t^\alpha n^{-1}\}^j \frac{n!}{(n-j)!} \frac{(k_n - j)!}{k_n!} \\ &\quad \text{using (1.4)} \\ &= \sum_{j=0}^{k_n} (-1)^j \binom{n}{j} \{c_1 t^\alpha n^{-1}\}^j \\ &= (1 - c_1 t^\alpha n^{-1})^n - \sum_{j=k_n+1}^{n} (-1)^j \binom{n}{j} \{c_1 t^{\alpha_n - 1}\}^j. \end{aligned}$$

The first term in the last line is equal to

$$e^{-c_1 t^\alpha}\{1+\mathcal{O}(t^{2\alpha} n^{-1})\} \qquad \text{for } n\to\infty.$$

From the expression for the error term in the expansion of $\{1+t n^{-1}\}^n$ it follows that the absolute value of the second term on the right-hand side in expression for $E\{h_1(t, X)\}^{k_n}$ is bounded by

$$\binom{n}{k_n+1}\{c t^\alpha n^{-1}\}^{k_n+1}\{1+c t^\alpha n^{-1}\}^{n-k_n-1}$$

for some positive constant c. Under the conditions in C_1 and C_2 we easily obtain the given upperbound. □

For $t n^{-1/\alpha} x$ small and $t>0$ we have

$$\begin{aligned}\int_0^x \{\exp(i t n^{-1/\alpha} y)-1\}\, y^{-\alpha-1}\, dy &= t^\alpha n^{-1} \int_0^{t n^{-1/\alpha} x} \{\exp(i z)-1\}\, z^{-\alpha-1}\, dz\\ &= i t n^{-1/\alpha} x^{1-\alpha}(1-\alpha)^{-1}-\tfrac{1}{2} t^2 n^{-2/\alpha} x^{2-\alpha}(2-\alpha)^{-1}+\mathcal{O}(t^3 n^{-3/\alpha} x^{3-\alpha})\\ &\text{for } n\to\infty.\end{aligned}$$

We define

$$h_2(t, x)=h_1(t, x)-i\alpha(1-\alpha)^{-1}\{1-G(x)\}^{-1} t n^{-1/\alpha} x^{1-\alpha}.$$

Lemma 3. *Under the conditions given in C_1 and C_2 we have*

$$|E\{h_2(t, X)\}^{k_n}-E\{h_1(t, X)\}^{k_n}| \leqq c\,|t|\, k_n^{1-1/\alpha}\, e^{-\bar{c}|t|^\alpha}+r_n$$

where c and $\bar{c}$ are positive constants and r_n is given in Lemma 1(b).

Proof. We use the following elementary equality

$$\alpha^k-\beta^k=(\alpha-\beta)\left(\sum_{j=0}^{k-1} \alpha^j \beta^{k-1-j}\right).$$

From the definitions of h_2 and G it follows that under the conditions given in C_1 and C_2 we have

$$h_2(t, x)-h_1(t, x)\sim i\alpha(1-\alpha)^{-1}\, t x n^{-1/\alpha}.$$

Using (1.5) we obtain

$$\{E(h_2(t, X)-h_1(t, X))^2\}^{\frac{1}{2}} \leqq c\,|t|\, k_n^{-1/\alpha}$$

for some positive constant c. It follows from Lemma 1(b) that $|t| X n^{-1/\alpha}$ is small with probability larger than $1-r_n$.

On the set where $|t| X n^{-1/\alpha}$ is small we have

$$|h_1(t, X)| \leqq 1 - a\,|t|^\alpha n^{-1}\{1-G(X)\}^{-1}$$

and

$$|h_2(t, X)| \leqq 1 - a\,|t|^\alpha n^{-1}\{1-G(X)\}^{-1}.$$

Then

$$\sum_{j=0}^{k_n-1} |h_1(t, X)|^j\,|h_2(t, X)|^{k_n-j-1}$$
$$\leqq k_n\{1 - a\,|t|^\alpha n^{-1}\{1-G(X)\}^{-1}\}^{k_n-1}.$$

The result follows as in Lemma 2 using Schwarz's inequality. □

Define for $t \geqq 0$

$$h_3(t, x) = h_2(t, x) + \{1-G(x)\}^{-1}\{\exp(i t n^{-1/\alpha} x) - 1\}\, r(x)$$
$$+ i t n^{-1/\alpha}\{1-G(x)\}^{-1} \int_x^\infty \exp(i t n^{-1/\alpha} y)\, r(y)\, dy$$

and for $t < 0$

$$h_3(t, x) = \overline{h_3(-t, x)}.$$

Lemma 4. *Under the conditions given in C_1 and C_2 we have*

$$|E\{h_3(t, X)\}^{k_n} - E\{h_2(t, X)\}^{k_n}|$$
$$\leqq c\,|t|\,k_n^{(\gamma-1)/\alpha}\, n^{-(\gamma-\alpha)/\alpha}\, e^{-\bar{c}|t|^\alpha} + r_n$$

for some positive constants c an $\bar{c}$. The error r_n is given in Lemma 1(b).

Proof. Take $t > 0$.

When $t x n^{-1/\alpha}$ is small we have

$$|h_2(t, x) - h_3(t, x)| \leqq c\, t\, x^{1-\gamma+\alpha} n^{-1/\alpha}.$$

The assertion follows as in Lemma 3. □

Corollary 1. *Similarly we can obtain an estimate for $E\{h(t, X)\}^{k_n} - E\{h_3(t, X)\}^{k_n}$. It follows from the Lemmas* 2, 3 *and* 4 *and this estimate that under the conditions given in C_1 and C_2 we have*

$$|\varphi_{n,k_n}(t) - e^{-c_1|t|^\alpha}| \leqq c\, e^{-\bar{c}|t|^\alpha}\{|t|\, k_n^{1-1/\alpha} + |t|^{2\alpha} n^{-1}\} + r_n.$$

Next we consider the characteristic function φ^*_{n,k_n} of

$$n^{-1/\alpha}\{X_{n:n} + \ldots + X_{n:n-k_n+1} + k_n\,\alpha(1-\alpha)^{-1} X_{n:n-k_n}\}. \tag{1.6}$$

We have

$$\varphi^*_{n,k_n}(t) = E[\{h(t, X)\}^{k_n} \exp(i t n^{-1/\alpha}\alpha(1-\alpha)^{-1} k_n X)].$$

Note that in the last term in (1.6) we have $X_{n:n-k_n}$ and in the expression given by Arov and Bobrov we have $X_{n:n-k_n+1}$.

Lemma 5. *Under the conditions given in C_1 and C_2 we have*

$$|E\{h_1(t,X)\}^{k_n}-\varphi^*_{n,k_n}(t)| \leqq c\{|t|^{1+\alpha}k_n^{-1/\alpha}+|t|\,k_n^{(\gamma-1)/\alpha}n^{-(\gamma-\alpha)/\alpha}\}\,e^{-\bar{c}|t|^\alpha}+r_n,$$

where c and $\bar{c}$ are positive real constants and r_n is given in Lemma 1(b).

Proof. We can write

$$\varphi^*_{n,k_n}(t)=E\{h(t,X)\exp(i\,t\,n^{-1/\alpha}\alpha(1-\alpha)^{-1}X)\}^{k_n}.$$

For $t>0$ and $t\,x\,n^{-1/\alpha}$ small we have

$$\begin{aligned} &h(t,x)\exp(i\,t\,n^{-1/\alpha}\alpha(1-\alpha)^{-1}x)\\ &\quad=h_1(t,x)-c_1\,i\,\alpha(1-\alpha)^{-1}t^{1+\alpha}n^{-1-1/\alpha}\{1-G(x)\}^{-1}x\\ &\qquad+\mathcal{O}(t^2x^2n^{-2/\alpha})+\mathcal{O}(t\,n^{-1/\alpha}x^{1-\gamma+\alpha}). \end{aligned}$$

The assertion follows as in Lemma 3. □

Corollary 2. *Similarly as in Corollary* 1 *we obtain that under the conditions given in C_1 and C_2 we have*

$$|\varphi^*_{n,k_n}(t)-e^{-c_1|t|^\alpha}| \leqq c\{|t|^{1+\alpha}k_n^{-1/\alpha}+|t|\,k_n^{(\gamma-1)/\alpha}n^{-(\gamma-\alpha)/\alpha}+|t|^{2\alpha}n^{-1}\}\,e^{-\bar{c}|t|^\alpha}+r_n.$$

The following lemma describes the behaviour of the characteristic functions for larger values of t.

Lemma 6. *Let M be some positive real number. There exists some positive real constant c_2 such that for n and k_n sufficiently large, each of the functions $\varphi_{n,k_n}(t)$, $\varphi^*_{n,k_n}(t)$ and $E\{h_3(t,X)\}^{k_n}$ is in absolute value bounded by*

$$\exp(-c_2|t|^\alpha)+r_n$$

for $k_n^p\leqq|t|\leqq M\,k_n^{1/\alpha}$ and r_n is given in Lemma 1(a).

Proof. By Lemma 1(a) we may assume that $x=\mathcal{O}((n/k_n)^{1/\alpha})$ for n, $k_n\to\infty$, with an error less that r_n. Then we use that

$$\int_\tau^\infty(1-\cos y)\,y^{-\alpha-1}\,dy$$

decreases as τ increases. □

Now we state our theorem.

Theorem. *Under the conditions given in C_2 we have*

(a) $\sup_x |P(n^{-1/\alpha}(X_{n:n}+\ldots+X_{n:n-k_n+1})\leqq x)-F(x;\alpha,1)|$

$=\mathcal{O}(\max(k_n^{1-1/\alpha}, n^{-1}))$.

(b) $\sup_x |P(n^{-1/\alpha}(X_{n:n}+\ldots+X_{n:n-k_n+1}+k_n\alpha(1-\alpha)^{-1}X_{n:n-k_n}\leqq x)-F(x;\alpha,1)|$

$=\mathcal{O}(\max(k_n^{-1/\alpha}, k_n^{(\gamma-1)/\alpha}n^{-(\gamma-\alpha)/\alpha}, n^{-1}))$.

Proof. We invoke the so-called smoothing lemma. See Petrov (1975), Chap. V.

$$\sup_x |P(n^{-1/\alpha}(X_{n:n}+\ldots+X_{n:n-k_n+1})\leqq x)-F(x;\alpha,1)|$$

$$\leqq b\int_{-T}^{T} t^{-1}\,|\varphi_{n,k_n}(t)-\exp(-c_1|t|^\alpha)|\,dt+c\,r(b)\,T^{-1},$$

where $r(b)$ is a positive constant depending only on b and $b>(2\pi)^{-1}$. In part a we take $T=k_n^{-1+1/\alpha}$ and $p=1/\alpha-1$. The assertion given in part a of the theorem follows easily from Corollary 1.

In order to prove part b we take $T=M k_n^{1/\alpha}$ and p such that $\alpha p<1$. We use Corollary 2 in order to obtain an estimate for the integral over $[-k_n^p, k_n^p]$.

For $k_n^p\leqq|t|\leqq M k_n^{1/\alpha}$ we make use of Lemma 6.

$$\int_{k_n^p}^{M k_n^{1/\alpha}} t^{-1}\,|\varphi^*_{n,k_n}(t)|\,dt=\mathcal{O}(\max(k_n^{1/\alpha}r_n, k_n^{-p\alpha}\exp(-\beta k_n^{p\alpha}))).$$

Similarly we estimate

$$\int_{k_n^p}^{M k_n^{1/\alpha}} t^{-1}\,|e^{-c_1 t^\alpha}|\,dt. \quad \square$$

Remark 2. Cramér (1963) showed under the same conditions

$$\sup_x |P(n^{-1/\alpha}(X_1+\ldots+X_n)\leqq x)-F(x;\alpha,1)|=\mathcal{O}(n^{-\lambda/\alpha})$$

where $\lambda=\min(1,\gamma-\alpha)$. If k_n tends slowly to infinity, for example $k_n\sim(\log n)^p$, we obtain a very poor rate of convergence for the sample extremes. We have the rate $\mathcal{O}(k_n^{1-1/\alpha})$ in part a of our theorem and $\mathcal{O}(k_n^{-1/\alpha})$ in part b.

Remark 3. We can improve the rate of convergence. Let the random variable Y have the distribution function given by

$$1-P(Y\leqq y)=y^{-\alpha-1} \qquad \text{for } y\geqq 1.$$

The characteristic function of Y has the following expansion

$$1+it(\alpha+1)\alpha^{-1}+\alpha^{-1}\Gamma(1-\alpha)\,t^{1+\alpha}e^{\pi i(\alpha+1)/2}+\mathcal{O}(t^2) \qquad \text{for } t\to 0.$$

Here we make use of some integrals occurring in the derivation of the characteristic function of a stable distribution. (See – for example – Laha and Rohatgi Theorem 5.4.4.)

Define the random variable Y^* independent of X by

$$Y^* = c\{Y - (\alpha+1)\alpha^{-1}\} + \alpha(1-\alpha)^{-1}$$

and for $t>0$

$$h_4(t, X) = h(t, X)\exp(-itn^{-1/\alpha} X Y^*)$$

and

$$h_4(t, X) = \overline{h_4(-t, X)} \qquad \text{for } t<0.$$

Now we can choose c such that the first term in the expansion of $h_4(t, x) - h_2(t, x)$ has the form $t^2 x^2 n^{-2/\alpha}$ in the case $\frac{1}{2} \leqq \alpha < 1$ and $t^{1+2\alpha} x^{1+2\alpha} n^{-2-1/\alpha}$ in the case $0<\alpha<\frac{1}{2}$. For k_n as in Remark 1 we can improve the rate of convergence to $\mathcal{O}(k_n^{1-2/\alpha})$ in the case $\frac{1}{2} \leqq \alpha < 1$ and $\mathcal{O}(k_n^{-1-1/\alpha})$ in the case $0<\alpha<\frac{1}{2}$. We need an upperbound for the characteristic function for values of t which satisfy $|t| > M k_n^{1/\alpha}$. This upperbound is given in Lemma 3 of Mijnheer (1986).

Remark 4. It follows from our theorem that

$$n^{-1/\alpha}(X_{n:n} + \ldots + X_{n:n-k_n+1})$$

has the same distribution as

$$Y_0 + k_n n^{-1/\alpha} X_{n:n-k_n+1} Y^{**}$$

where Y^{**} is a random variable $\in \mathcal{D}_{\mathcal{N}}(1+\alpha, 1)$ and Y_0 has distribution function $F(\cdot\,; \alpha, 1)$.

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