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Probability Theory and Related Fields

Transformations of Diffusion and Schrödinger Processes

To commemorate the centenary of E. Schrödinger's birth (1887–1961)

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Summary. A transformation by means of a new type of multiplicative functionals is given, which is a generalization of Doob's space-time harmonic transformation, in the case of arbitrary non-harmonic function $\phi(t, x)$ which may vanish on a subset of $[a, b] \times \mathbb{R}^d$. The transformation induces an additional (singular) drift term $\nabla \phi/\phi$, like in the case of Doob's space-time harmonic transformation. To handle the transformation, an integral equation of singular perturbations and a diffusion equation with singular potentials are discussed and the Feynman-Kac theorem is established for a class of singular potentials. The transformation is applied to Schrödinger processes which are defined following an idea of E. Schrödinger (1931).

§ 1. Introduction

In this paper we consider space-time diffusion processes in an open subset D of $[a, b] \times \mathbb{R}^d$, where $-\infty < a < b < \infty$, and their transformations by means of multiplicative functionals in connection with Schrödinger processes (1931), the definition of which will be given in Sect. 6. It is well known that the theory of transformations of Markov processes plays an important rôle in constructing new processes from known ones and analysing them. A celebrated transformation is Doob's space-time harmonic transformation (cf. e.g. Doob (1983)): Let $\{(t, X_t), W_{(s,x)}\}$ be a space-time Brownian motion, and $h(s, x)$ be a non-negative space-time harmonic function satisfying

$$h(s, x) = W_{(s,x)}[h(t, X(t)); t < T_s], \quad \text{for } a \leqq s < t \leqq b, \tag{1.1}$$

where T_s is the first hitting time to the nodal set of $h(s, x)$, $N = \{(s, x): h(s, x) = 0\}$, and $W[f; A]$ denotes the expectation of f on A with respect to a measure W. The space-time harmonic transformation is defined by

$$p(s, x; t, f) = W_{(s,x)}\left[f(t, X(t)) \frac{h(t, X(t))}{h(s, X(s))}; t < T_s\right], \quad \text{for } (s, x) \in D, \tag{1.2}$$

where $D=\{(s, x): h(s, x) \neq 0\}$ and f's are bounded continuous functions on D. The transformation induces a drift term $\nabla h/h$, that is, p satisfies

$$\frac{\partial p}{\partial s}+\frac{1}{2} \Delta p+\frac{1}{h} \nabla h \cdot \nabla p=0, \quad \text{in } D. \tag{1.3}$$

Taking account of (1.1), it follows immediately from the definition (1.2) that the transformed process does not hit the nodal set N of h.

If a drift coefficient $\mathbf{b}(s, x)$, which is not necessarily of the form $\nabla h/h$, satisfies the Novikov condition (1973)

$$W_{(s,x)}\left[\exp\left(\tfrac{1}{2} \int_s^b \|\mathbf{b}(v, X_v)\|^2 \, dv\right)\right]<\infty, \tag{1.4}$$

then we can obtain a new space-time diffusion process $\{(t, X_t), P_{(s,x)}\}$ with transformed probability measures

$$P_{(s,x)}=M_s^b \, W_{(s,x)}, \tag{1.5}$$

where M_s^t is the Maruyama density (1954),[1]

$$M_s^t=\exp\left(\int_s^t \mathbf{b}(v, X_v) \cdot dX_v-\tfrac{1}{2} \int_s^t \|\mathbf{b}(v, X_v)\|^2 \, dv\right). \tag{1.6}$$

The transition function of the transformed process defined by

$$\begin{aligned} p(s, x; t, f) &= P_{(s,x)}[f(t, X_t)] \\ &= W_{(s,x)}[f(t, X_t) M_s^t] \end{aligned} \tag{1.7}$$

satisfies in a weak sense a diffusion equation with drift $\mathbf{b}(s, x)$

$$\frac{\partial p}{\partial s}+\frac{1}{2} \Delta p+\mathbf{b} \cdot \nabla p=0. \tag{1.8}$$

In other words

$$B(t)=X(t)-X(s)-\int_s^t \mathbf{b}(v, X(v)) \, dv \tag{1.9}$$

is a Brownian motion with respect to $P_{(s,x)}$ defined by (1.5).

Therefore, if $\nabla h/h$ satisfies the Novikov condition (1.4), one can apply the drift transformation of Maruyama-Girsanov instead of Doob's harmonic transformation. However, there are difficulties in showing the Novikov condition for $\nabla h/h$ if h has zeros.

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[1] Some probabilists first called this "Girsanov formula" without refering to (or not aware of) Maruyama's paper which had been published in 1954. Girsanov's and Motoo's paper on this subject were published at the same time but six years later in 1960. For Maruyama's formula see also Ikeda-Watanabe (1981), Liptser-Shiryayev (1977)

Now, given an arbitrary non-negative function $\phi(s, x)$, which is *not* assumed to be a space-time harmonic function of a space-time Brownian motion, we consider a diffusion equation

$$\frac{\partial q}{\partial s}+\frac{1}{2} \Delta q+\frac{1}{\phi} \nabla \phi \cdot \nabla q=0, \qquad \text{in } D, \tag{1.10}$$

where $D=\{(s, x): \phi(s, x) \neq 0\}$. This equation has been investigated by many authors in connection with probabilistic treatment of Schrödinger equations (cf. e.g. Zheng-Meyer (1982, 84), Carlen (1984), Nelson (1985), Carmona (1985), Blanchard-Golin (1987)). Due to the singularity caused by the nodal set of the function $\phi(s, x)$, the drift coefficient $\nabla \phi / \phi$ does not satisfy the Novikov condition (1.4). Furthermore, Doob's space-time harmonic transformation can not be applied, since $\phi(s, x)$ is not a space-time harmonic function. Nevertheless it has been shown, under certain regularity conditions on the drift coefficient, that a diffusion process governed by (1.10) does not hit the nodal set of $\phi(s, x)$ and then the process is constructed based on this fact. However, proofs which have been given are more or less involved. It is, therefore, desirable to find out a simple transformation such as Doob's one for non-harmonic $\phi(s, x)$. In this paper it will be shown that such a transformation exists, and then it will be applied to Schrödinger processes.

In Sect. 2, a multiplicative functional N_s^t is defined and a transformation by means of N_s^t is discussed. In Sect. 3, the existence and uniqueness of solutions of an integral equation is treated and shown that the transformed process by means of N_s^t is governed by (1.10). A diffusion equation with creation and killing $c(t, x)$ is discussed in Sect. 4. Section 5 treats the inaccessibility of the transformed processes to the nodal set of $\phi(t, x)$ based on the results in Sects. 3 and 4. Sections 6 and 7 are devoted to Schrödinger processes, to which the transformation by means of N_s^t is applied.

§ 2. A Transformation of Space-Time Diffusion Processes

Let $\mathbf{a}(s, x)$ be a bounded continuous function on $[a, b] \times \mathbb{R}^d$ taking values in $\mathbb{R}^d$, which has bounded uniformly Hölder continuous derivatives in x. Then there exists a fundamental solution $g(s, x; t, y)$, $a \leqq s < t \leqq b$, $x, y \in \mathbb{R}^d$, which satisfies as a function of (s, x)

$$\frac{\partial u}{\partial s}+\frac{1}{2} \Delta u+\mathbf{a} \cdot \nabla u=0, \tag{2.1}$$

and as a function of (t, y)

$$-\frac{\partial u}{\partial t}+\frac{1}{2} \Delta u-\nabla \cdot(\mathbf{a} u)=0 \tag{2.2}$$

(cf. e.g. Friedman (1964)). Let $\{X_v, P_{(s,x)}\}$ be the diffusion process which has $g(s, x; t, y)$ as the transition probability density. In other words, $P_{(s,x)}$ can be obtained from the standard Brownian motion $\{X_v, W_{(s,x)}\}$ [2] by means of the drift transformation of Maruyama-Girsanov; or from a solution Y_t of a stochastic differential equation

$$Y_t = X_t + \int_s^t \mathbf{a}(v, Y_v)\, dv,$$

with respect to $\{X_t, W_{(s,x)}\}$. We will use these facts freely in the following without mentioning it explicitly.

From now we consider processes in space-time, which will be denoted by (t, X_t). Let N be a closed subset of $[a, b] \times \mathbb{R}^d$ and define the first hitting time to N after s by

$$T_s = \begin{cases} \inf\{v \in (s, b]: (v, X_v) \in N\}, & \text{if such } v \text{ exists,} \\ \infty, & \text{otherwise,} \end{cases} \tag{2.3}$$

where N will be specified below.

Let $\phi(s, x)$ be a non-negative continuous function on $[a, b] \times \mathbb{R}^d$, which may diverge at $s = b$, where $-\infty < a < b < \infty$. If $\phi(s, x)$ diverges at $s = b$, then taking $b' < b$, we can consider ϕ on $[a, b'] \times \mathbb{R}^d$ and hence we will assume that $\phi(s, x)$ is defined on $[a, b] \times \mathbb{R}^d$. Let us denote

$$\begin{aligned} D &= \{(s, x): \phi(s, x) \neq 0, (s, x) \in [a, b] \times \mathbb{R}^d\}, \\ N &= \{(s, x): \phi(s, x) = 0, (s, x) \in [a, b] \times \mathbb{R}^d\}, \end{aligned} \tag{2.4}$$

and assume that $\phi \in C^{1,2}(D)$. [3]

We shall discuss a transformation of space-time diffusion processes $\{(v, X_v), P_{(s,x)}\}$ by means of a multiplicative functional which will be given in the following

Definition 2.1. A multiplicative functional $\{N_s^t: a \leqq s \leqq t \leqq b\}$ is defined by

$$N_s^t = \exp\left\{ -\int_s^t \frac{L\phi}{\phi}(v, X(v))\, dv \right\} \frac{\phi(t, X(t))}{\phi(s, X(s))} 1_{\{t < T_s\}}, \text{ [4]} \tag{2.5}$$

where L is a parabolic differential operator

$$L = \frac{\partial}{\partial s} + \frac{1}{2}\Delta + \mathbf{a}(s, x) \cdot \nabla \tag{2.6}$$

which governs the diffusion process $\{X_v, P_{(s,x)}\}$.

It is clear that N_s^t is a multiplicative functional which is continuous in $t < T_s$, but may have a jump at $t = T_s$ to zero.

[2] $W_{(s,x)}$ is the Wiener measure starting at (s, x)

[3] $C^{1,2} = \{f(s, x)$: continuously differentiable once in s and twice in $x\}$. This differentiability requirement on ϕ has been loosened by R. Aebi

[4] It should be read that $N_s^t = 0$, on $\{t \geqq T_s\}$

Theorem 2.1. *Let N_s^t be defined by* (2.5). *Then*

$$P_{(s,x)}[N_s^t] \leqq 1, \quad \textit{for } (s,x)\in D \textit{ and } s\leqq t\leqq b, \tag{2.7}$$

and

$$N_s^t = \exp\left\{1_{\{t<T\}} \circ \int_s^t \frac{\nabla\phi}{\phi}(v, X_v)\cdot dB_v - \frac{1}{2}\int_s^t \left\|\frac{\nabla\phi}{\phi}(v, X_v)\right\|^2 dv\right\} 1_{\{t<T_s\}}, \ P_{(s,x)}\text{-a.s.}, \tag{2.8}$$

where B_t is a Brownian motion with respect to $P_{(s,x)}$ defined by

$$B_t = X_t - X_s - \int_s^t \mathbf{a}(v, X_v)\, dv, \tag{2.9}$$

$$1_{\{t<T\}} \circ \int_s^t \frac{\nabla\phi}{\phi}(v, X_v)\cdot dB_v = P\text{-}\lim_{n\to\infty} 1_{\{t<T\}} \int_s^t 1_{\{v<T_n\}} \frac{\nabla\phi}{\phi}(v, X_v)\cdot dB_v,^5 \tag{2.10}$$

$$T_n = \begin{cases} \inf\left\{u\colon \int_s^u \left\|\dfrac{\nabla\phi}{\phi}(v, X_v)\right\|^2 dv \geqq n\right\}, & \textit{if such } u\leqq b \textit{ exists},\\ \infty, & \textit{otherwise}, \end{cases} \tag{2.11}$$

and

$$T = \lim_{n\to\infty} T_n. \tag{2.12}$$

Proof. It is clear by the definition that

$$T_s \leqq T. \tag{2.13}$$

An application of Itô's formula to $\log\phi$, up to $T_n^s = T_n \wedge T_s$, yields

$$\frac{\phi(t\wedge T_n^s, X(t\wedge T_n^s))}{\phi(s,x)} \exp\left(-\int_s^{t\wedge T_n^s} \frac{L\phi}{\phi}(v, X_v)\, dv\right) = M_s^{t\wedge T_n^s}, \ P_{(s,x)}\text{-a.s.}, \tag{2.14}$$

where $M_s^{t\wedge T_n^s}$ is an exponential martingale with respect to $P_{(s,x)}$ defined by

$$M_s^{t\wedge T_n^s} = \exp\left(\int_s^{t\wedge T_n^s} \frac{\nabla\phi}{\phi}(v, X_v)\cdot dB_v - \tfrac{1}{2}\int_s^{t\wedge T_n^s} \left\|\frac{\nabla\phi}{\phi}(v, X_v)\right\|^2 dv\right), \tag{2.15}$$

where B_t is given by (2.9). Multiplying (2.14) by $1_{\{t<T_n<T_s\}}$, we have

$$N_s^{t\wedge T_n} = M_s^{t\wedge T_n} 1_{\{t<T_n<T_s\}} \leqq M_s^{t\wedge T_n^s}, \tag{2.16}$$

and because of (2.13)

$$N_s^t = \lim_{n\to\infty} N_s^{t\wedge T_n}. \tag{2.17}$$

[5] We are using a notation different from Liptser-Shiryayev (1977)

Therefore, by Fatou's lemma

$$P_{(s,x)}[N_s^t] \leqq \varliminf_{n\to\infty} P_{(s,x)}[N_s^{t\wedge T_n^s}] \leqq \varliminf_{n\to\infty} P_{(s,x)}[M_s^{t\wedge T_n^s}] = 1. \tag{2.18}$$

Thus (2.7) holds. It is known (cf. Liptser-Shiryayev (1977)) that $1_{\{t<T\}} M_s^{t\wedge T_n}$ converges in probability to the exponential factor of (2.8), and hence we have (2.8) because of (2.13).

Corollary 1. *$\{N_s^t, \mathscr{F}_s^t, t\in[s,b], P_{(s,x)}\}$ is a super-martingale, where $\mathscr{F}_s^t$ denotes the σ-field generated by X_u, $s\leqq u\leqq t$.*

Corollary 2. *For $(s,x)\in D$ it holds that*

$$P_{(s,x)}\left[\phi(t,X_t)\exp\left(-\int_s^t \frac{L\phi}{\phi}(v,X_v)\,dv\right);\, t<T_s\right] \leqq \phi(s,x). \tag{2.19}$$

Definition 2.2. The function defined by

$$c(s,x) = -\frac{L\phi}{\phi}(s,x), \qquad \text{for } (s,x)\in D, \tag{2.20}$$

will be called a *reference potential* of ϕ (with respect to the parabolic differential operator L in (2.6)), which may diverge on N.

Definition 2.3. The process with creation and killing $c(s,x)$ (as defined at (2.20)) killed at T_s will be called a *reference process.*

Corollary 2 means that $\phi(s,x)$ is a space-time superharmonic function of the reference process. The reference process is a process with creation and killing,[6] and probabilistic meanings of its super-harmonic transformation are not clear. Therefore, it is inadequate to decompose N_s^t into $\exp\left(\int_s^t c(v,X_v)\,dv\right) 1_{\{t<T_s\}}$ and $\phi(t,X_t)/\phi(s,X_s)$.

Because of (2.7) in Theorem 2.1, we can apply a Theorem of Dynkin (1961) and Kunita-Watanabe (1963), which implies

Theorem 2.2. *There exists a diffusion process $\{X_t, Q_{(s,x)}, \zeta\}$[7] on D such that for $(s,x)\in D$ and $t\in[s,b]$*

$$Q_{(s,x)}[f(t,X_t);\, t<\zeta] = P_{(s,x)}[f(t,X_t)N_s^t], \tag{2.21}$$

for $f\in\mathbb{B}(D)=\{f$: bounded measurable functions on $D\}$.

In general, the transformed process $\{X_t, Q_{(s,x)}, \zeta\}$ hits the nodal set N of ϕ and is killed there. For example, it is well known that a Bessel process on $[0,\infty)$ of index $1\leqq d<2$ hits the origin $\{0\}$. In this case ϕ is time-independent and given by $\phi(s,x)=x^{(d-1)/2}$, $x\geqq 0$, and $N=[a,b]\times\{0\}$. If $d\geqq 2$, the Bessel

[6] For this process see Nagasawa (1969), Mitro (1979), Getoor-Glover (1986), Kuznetzov (1973), Dynkin-Getoor (1985)

[7] ζ is the life time of the process and $Q_{(s,x)}[\zeta=T_s]=1$

process does not hit the origin $\{0\}$, because $\{0\}$ is an entrance boundary point of the process. In Sect. 5 it will be shown that the transformed process $\{(t, X_t), Q_{(s,x)}, \zeta\}$ does not hit the nodal set N of ϕ under certain conditions on ϕ.

Remark. For a Brownian motion and time-independent $\phi(x)$, a similar transformation was employed by Donsker-Varadhan (1975) and by Carmona (1979), and developed later by Fukushima-Takeda (1984) and Oshima-Takeda (1986) for time-homogeneous symmetric Markov processes in terms of Dirichlet spaces. The formula (2.14) appeared first in Itô-Watanabe (1965) for a Brownian motion and excessive functions $\phi(x)$ as a decomposition formula of a multiplicative functional $\phi(X_t)/\phi(X_0)$ into a Maruyama density and an exponential killing. Cf. also Jamison (1975).

§ 3. An Integral Equation

Let $\{(v, X_v), P_{(s,x)}\}$ be the diffusion process treated in Sect. 2, N be a closed subset of $[a, b] \times \mathbb{R}^d$, and T_s be the first hitting time to the subset N after s defined in (2.3).

Definition 3.1. For a closed subset $N \subset [a, b] \times \mathbb{R}^d$, a transition density $g^0(s, x; t, y)$ is defined by

$$g^0(s, x; t, y) = g(s, x; t, y) - P_{(s,x)}[g(T_s, X_{T_s}; t, y); t > T_s], \tag{3.1}$$

for $a \leqq s < t \leqq b$ and $x, y \in \mathbb{R}^d$, where $g(s, x; t, y)$ is the transition density of the process $\{(v, X_v), P_{(s,x)}\}$.

Proposition 3.1. *$g^0(s, x; t, y)$ satisfies* (2.2) *as a function of* (t, y), *and*

$$P_{(s,x)}[f(t, X_t); t < T_s] = \int g^0(s, x; t, y) f(t, y)\, dy, \quad \text{for } (s, x) \in N^c, \tag{3.2}$$

where $f \in \mathbb{B}([a, b] \times \mathbb{R}^d)$ with $\operatorname{supp}(f) \subset N^c$.

Proof. Let $g(t, y)$ denote $g(T_s, X_{T_s}; t, y)$. Then

$$\frac{1}{h}\{P_{(s,x)}[g(t+h, y); t+h > T_s] - P_{(s,x)}[g(t, y); t > T_s]\}$$

$$= \frac{1}{h} P_{(s,x)}[g(t+h, y); t+h > T_s \geqq t] + \frac{1}{h} P_{(s,x)}[g(t+h, y) - g(t, y); t > T_s].$$

The first term vanishes as $h \downarrow 0$, since $\frac{1}{h} P_{(T_s, X(T_s))}[X_{t+h} \in U] \to 0$, $P_{(s,x)}$-a.s. on $\{t+h > T_s \geqq t\}$ by the path-continuity, where U is a neighbourhood of y such that $U \times [t, t+h] \subset D$. The second term converges to $P_{(s,x)}\left[\frac{\partial}{\partial t} g(t, y); t > T_s\right]$.

Therefore the second term of (3.1) satisfies (2.2) and hence so does $g^0(s, x; t, y)$. The left-hand side of (3.2) is equal to

$$P_{(s,x)}[f(t, X_t)] - P_{(s,x)}[f(t, X_t); t > T_s] - P_{(s,x)}[f(t, X_t); t = T_s],$$

where the third term vanishes, since $\operatorname{supp}(f) \subset N^c$. By the Markov property the second term is equal to

$$P_{(s,x)}[P_{(T_s, X(T_s))}[f(t, X_t)]; t > T_s] = \int P_{(s,x)}[g(t, y); t > T_s] f(t, y)\, dy,$$

and hence (3.2) holds.

We consider an integral equation

$$p(s, x) = \int g^0(s, x; t, y)\, dy\, f(t, y) + \int_s^t dv \int g^0(s, x; v, y)\, dy\, c(v, y)\, p(v, y), \quad (3.3)$$

for $(s, x) \in D = N^c$ and $s < t \leqq b$, where $c(v, y)$ is a contiuous function on D which may diverge at N, and $f(t, x)$ is a continuous (or measurable) function in $x \in D_t = \{x: (t, x) \in D\}$. We impose two integrability conditions

$$P_{(s,x)}[|p(v, X_v)|; v < T_s] < \infty, \qquad \text{for } (s, x) \in D,\ s < v \leqq t \quad (3.4)$$

and

$$\int_s^t dv\, P_{(s,x)}[|c(v, X_v)\, p(v, X_v)|; v < T_s] < \infty, \qquad \text{for } (s, x) \in D. \quad (3.5)$$

For brevity $p(v, x)$ denotes a solution $p(v, x; t, f)$ of (3.3). It is easy to see that $p(v, x)$ satisfies

$$p(s, x) = \int g^0(s, x; u, y)\, dy\, p(u, y) + \int_s^u dv \int g^0(s, x; v, y)\, dy\, c(v, y)\, p(v, y), \quad (3.6)$$

for $s < u \leqq t$. Accordingly, for non-negative $p(v, x)$ it is enough to assume, instead of (3.4),

$$\int g^0(s, x; t, y)\, dy\, f(t, y) < \infty, \qquad \text{for } (s, x) \in D,\ s < t \leqq b, \quad (3.7)$$

where t is fixed.

Theorem 3.1. *Under conditions* (3.4), (3.5) *and*

$$\int_s^t du\, P_{(s,x)}\left[\exp\left(\int_s^u |c(v, X_v)|\, dv\right) |c\, p(u, X_u)|; u < T_s\right] < \infty, \quad (3.8)$$

the solutions of the integral equation (3.3) *are uniquely determined.*

Proof.[8] Let $p_1(s,x)$ and $p_2(s,x)$ be solutions of (3.3) with (3.4), (3.5) and (3.8), and denote $p(s,x)=|p_1(s,x)-p_2(s,x)|$. Let us denote $|c|$ by c for brevity in the following. Therefore $c \geqq 0$. Then $p(s,x)$ satisfies

$$\begin{aligned} p(s,x) &\leqq \int_s^t dv\, P_{(s,x)}[c(v,X_v)\, p(v,X_v);\, v<T_s] \\ &\leqq \int_s^t dv\, P_{(s,x)}\left[c(v,X_v)\int_v^t dv_1\, P_{(v,X_v)}[c\,p(v_1,X_{v_1});\, v_1<T_v];\, v<T_s\right] \\ &\leqq \int_s^t dv_1 \int_s^{v_1} dv\, P_{(s,x)}[c(v,X_v)\, c(v_1,X_{v_1})\, p(v_1,X_{v_1});\, v_1<T_s], \end{aligned}$$

where the Markov property and Fubini's theorem have been applied. Then we have, by induction,

$$\begin{aligned} p(s,x) &\leqq \int_s^t dv_n \int_s^{v_n} dv_{n-1} \ldots \int_s^{v_1} dv \\ &\quad \cdot P_{(s,x)}[c(v,X_v)\, c(v_1,X_{v_1}) \ldots c(v_n,X_{v_n})\, p(v_n,X_{v_n});\, v_n<T_s] \\ &= \int_s^t dv\, P_{(s,x)}\left[\frac{1}{(n-1)!}\left(\int_s^v c(u,X_u)\,du\right)^{n-1} c(v,X_v)\, p(v,X_v);\, v<T_s\right], \end{aligned}$$

which vanishes as $n\to\infty$, because of (3.8).

Theorem 3.2. (i) *Let $f(t,x)$ be a non-negative function satisfying* (3.7) *and assume that*

$$\bar{p}(s,x;t,f)=P_{(s,x)}\left[f(t,X_t)\exp\left(\int_s^t c(v,X_v)\,dv\right);\, t<T_s\right] \tag{3.9}$$

satisfies the condition (3.5).

Then, $\bar{p}(s,x;t,f)$ is a solution of (3.3) *with* (3.4) *and* (3.5).

(ii) *Let $f(t,x)$ satisfy*

$$\int g^0(s,x;t,y)\,|f(t,y)|\,dy<\infty, \qquad \textit{for } (s,x)\in D,\ a\leqq s<t\leqq b. \tag{3.10}$$

Assume that

$$\bar{p}(s,x;t,|f|)=P_{(s,x)}\left[|f(t,X_t)|\exp\left(\int_s^t c(v,X_v)\,dv\right);\, t<T_s\right] \tag{3.11}$$

satisfies (3.5).

Then, $\bar{p}(s,x;t,f)$ is well-defined and is a solution of (3.3).

Proof. We have remarked already that (3.7) is enough for non-negative $p(s,x)$ to satisfy (3.4). Let $\bar{p}(s,x)$ denote $\bar{p}(s,x;t,f)$ for brevity. Under (3.5) the second

[8] There was an error in my proof, which is corrected by Wakolbinger.

term of the right-hand side of (3.3) is well defined and equal to, by the Markov property and Fubini theorem,

$$\int_s^t du\, P_{(s,x)}\left[c(u, X_u)\, P_{(u,X_u)}\left[f(t, X_t) \exp\left(\int_u^t c(v, X_v)\, dv\right); t < T_u\right]; u < T_s\right]$$

$$= \int_s^t du\, P_{(s,x)}\left[c(u, X_u) \exp\left(\int_u^t c(v, X_v)\, dv\right) f(t, X_t); t < T_s\right]$$

$$= P_{(s,x)}\left[\int_s^t du\, c(u, X_u) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\int_u^t c(v, X_v)\, dv\right)^{n-1} f(t, X_t); t < T_s\right]$$

$$= P_{(s,x)}\left[\sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_s^t c(v, X_v)\, dv\right)^n f(t, X_t); t < T_s\right].$$

Therefore together with the first term, the right-hand side of (3.3) is equal to

$$P_{(s,x)}\left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_s^t c(v, X_v)\, dv\right)^n f(t, X_t); t < T_s\right],$$

which is nothing but $\bar{p}(s, x)$. Thus $\bar{p}(s, x)$ satisfies (3.3).

(ii) Under the stated conditions $\bar{p}(s, x; t, |f|)$ is a solution of (3.3) with $|f|$ in place of f by the first assertion of the theorem. Especially

$$\int g^0(s, x; u, y)\, dy\, \bar{p}(u, y; t, |f|) < \infty, \qquad \text{for } (s, x) \in D,\ s < u \leqq t. \tag{3.12}$$

Since

$$|\bar{p}(s, x; t, f)| \leqq \bar{p}(s, x; t, |f|), \tag{3.13}$$

$\bar{p}(s, x; t, f)$ is well defined and satisfies (3.3). Because of (3.12) and (3.13), it satisfies (3.4) and (3.5).

Remark. It is easy to see that

$$\int_s^t dv\, P_{(s,x)}[|c(v, X_v)|\, \bar{p}(v, X_v; t, |f|); v < T_s]$$
$$\leqq 2 P_{(s,x)}\left[|f(t, X_t)| \exp\left(\int_s^t c^+(v, X_v)\, dv\right); t < T_s\right],$$

where $c^+ = c \vee 0$. Therefore,

$$P_{(s,x)}\left[|f(t, X_t)| \exp\left(\int_s^t c^+(v, X_v)\, dv\right); t < T_s\right] < \infty \tag{3.14}$$

is a sufficient condition for $\bar{p}(s, x; t, |f|)$ to satisfy (3.5).

Lemma 3.1. *Assume that* $p(s, x)$ *satisfies* (3.5). *Then, for any* $\varepsilon>0$, *there exists* $\delta_0>0$ *and an open subset* U *of* D *with the compact closure* $\bar{U}\subset D$ *such that* $(v, x)\in U$ *for* $v\in(s-\delta_0, s]$ *and*

$$\frac{1}{\delta}\int_{s-\delta}^{s} d v \int g^0(s-\delta, x; v, y)\, d y\, |c p(v, y)|\, 1_{U^c}(v, y)<\varepsilon, \quad \text{for } 0<\delta<\delta_0. \tag{3.15}$$

Proof. Let $(s, x)\in D$ and choose $\delta_0>0$ small enough so that $(v, x)\in D$ for all $v\in[s-\delta_0, s]$. Define a measure μ_δ by

$$\mu_\delta(d v\, d y)=\frac{1}{\delta}\, g^0(s-\delta, x; v, y)\, 1_{(s-\delta, s]}(v)\, d v\, d y.$$

Since $c p\in L^1(\mu_{\delta_0})$ by (3.5), there exists an open subset U of D with the compact closure $\bar{U}\subset D$ such that $(v, x)\in U$ for $v\in(s-\delta_0, s]$ and

$$\iint \mu_{\delta_0}(d v\, d y)\, |c(v, y)\, p(v, y)|\, 1_{U^c}(v, y)<\varepsilon. \tag{3.16}$$

Then (3.15) follows from (3.16), since $\mu_\delta(U^c)\downarrow 0$ as $\delta\downarrow 0$ by the continuity of paths of the process $\{X_t, P_{(s,x)}\}$.

Theorem 3.3. *Let* $c(s, x)$ *be continuous in* D *and* $p(s, x)$ *be a solution of the integral equation* (3.3) *satisfying* (3.4) *and* (3.5). *Then,* $p(s, x)$ *is continuous in* x, *once differentiable in* s, *and satisfies*

$$\frac{\partial p}{\partial s}+\frac{1}{2}\Delta p+\mathbf{a}\cdot\nabla p+c p=0, \tag{3.17}$$

in the sense of distributions (locally in D*).*[9]

Proof. It is easy to see that the second term of the right-hand side of (3.3) is continuous in x, and hence $p(s, x)$ is continuous in x, since the continuity of the first term is clear. As we remarked in (3.6) we have

$$p(s-\delta, x)=\int g^0(s-\delta, x; s, y)\, d y\, p(s, y)+\int_{s-\delta}^{s} d v \int g^0(s-\delta, x; v, y)\, d y\, c\, p(v, y). \tag{3.18}$$

Let U be the open subset in Lemma 3.1, and take another open subset U_1 such that $\bar{U}\subset U_1\subset \bar{U}_1\subset D$. Let $k(v, y)$ be a continuous function taking values in $[0, 1]$, $k(v, y)=1$ for $(v, y)\in U$, and $k(v, y)=0$ for $(v, y)\in U_1^c$. Then, Lemma 3.1 implies that the second term of the right-hand side of (3.18) divided by $\delta>0$ is equal to

$$\frac{1}{\delta}\int_{s-\delta}^{s} d v \int g^0(s-\delta, x; v, y)\, d y\, c(v, y)\, p(v, y)\, k(v, y)+O(\delta), \tag{3.19}$$

[9] Since D is not cylindrical, the space of test functions depends on s. "Local" indicates this, but will not be repeated hereafter

where the integral converges to $c(s, x)\, p(s, x)\, k(s, x)=c(s, x)\, p(s, x)$ and $O(\delta)\downarrow 0$ as $\delta\downarrow 0$. Let $h(x)$ be a C^∞-function with a compact support K such that there exists $\delta>0$ and $\{(v, x): v\in[s-\delta, s], x\in K\}\subset D$. Then, combining (3.18) and (3.19), we have

$$\left\langle\frac{1}{\delta}\{h\, g^0(s-\delta, s)-h\}, p(s)\right\rangle+\left\langle h, \frac{1}{\delta}\{p(s)-p(s-\delta)\}\right\rangle+\langle h, c\, p(s)\rangle+O(\delta)=0, \quad (3.20)$$

where

$$h\, g^0(s-\delta, s)=\int h(x)\, d x\, g^0(s-\delta, x; s, \cdot)$$

and

$$\langle f, h\rangle=\int f(x)\, h(x)\, d x.$$

The first term of (3.20) converges to $\langle\frac{1}{2}\Delta h-\nabla\cdot(\mathbf{a}\, h), p(s)\rangle$ as $\delta\downarrow 0$. This implies that the limit of the second term of (3.20) exists and is equal to $\left\langle h, \frac{\partial p}{\partial s}(s)\right\rangle$. Consequently, we have

$$\left\langle h, \frac{\partial p}{\partial s}(s)\right\rangle+\langle\tfrac{1}{2}\Delta h-\nabla\cdot(\mathbf{a}\, h), p(s)\rangle+\langle h, c(s)\, p(s)\rangle=0, \quad (3.21)$$

that is, $p(s, x)$ satisfies (3.17) in the sense of distributions.

Theorem 3.4. *Let $\phi(s, x)$ be a non-negative function on D satisfying* (3.4), (3.5) *and*

$$P_{(s,x)}\left[\phi(t, X_t)\exp\left(\int_s^t c(v, X_v)\, d v\right); t<T_s\right]\leqq\phi(s, x) \quad (3.22)$$

for $(s, x)\in D$, $a\leqq s<t\leqq b$. Then, for bounded continuous f on D,

$$\bar{p}(s, x; t, f\phi)=P_{(s,x)}\left[f(t, X_t)\,\phi(t, X_t)\exp\left(\int_s^t c(v, X_v)\, d v\right); t<T_s\right] \quad (3.23)$$

is a solution of (3.3) *with $f\phi$ in place of f, satisfying* (3.4) *and* (3.5). *Moreover, it satisfies* (3.17) *in the sense of distributions.*

Proof. It is clear that $f\phi$ satisfies (3.10).

By (3.22) we have

$$\int_s^t d v\, P_{(s,x)}[|c(v, X_v)\, p(v, X_v; t, f\phi)|; v<T_s]$$

$$\leqq\|f\|\int_s^t d v\, P_{(s,x)}[|c(v, X_v)|\,\phi(v, X_v); v<T_s]<\infty.$$

Thus (3.5) is satisfied. Therefore, $\bar{p}(s, x; t, f\phi)$ is a solution of (3.3) by Theorem 3.2 and (3.17) holds in the sense of distributions by Theorem 3.3.

Remark. By Corollary 2 of Theorem 2.1, if ϕ satisfies the conditions in Sect. 2 and if $c(s, x)$ is the reference potential of ϕ, then the inequality (3.22) holds.

Now we apply Theorems 3.3 and 3.4 to the transformed process $\{(v, X_v), Q_{(s,x)}, \zeta\}$ (by the multiplicative functional N_s^t in (2.5) defined by a given $\phi(s, x)$ subject to the conditions in Sect. 2).

Theorem 3.5. *Assume that for* $(s, x) \in D$ *and* $s < v \leqq t \leqq b$

$$P_{(s,x)}[\phi(v, X_v); v < T_s] < \infty \tag{3.24}$$

and

$$\int_s^t dv \, P_{(s,x)}[|L\phi(v, X_v)|; v < T_s] < \infty. \tag{3.25}$$

Then, for bounded continuous functions $f(t, x)$ *on* D,

$$q(s, x; t, f) = Q_{(s,x)}[f(t, X_t); t < T_s] \tag{3.26}$$

is continuous in x, *differentiable in* s, *and satisfies*

$$\frac{\partial q}{\partial s} + \frac{1}{2} \Delta q + \mathbf{a} \cdot \nabla q + \frac{\nabla \phi}{\phi} \cdot \nabla q = 0, \tag{3.27}$$

in the sense of distributions, that is, the transformation by the multiplicative functional N_s^t *induces an additional drift* $\nabla \phi / \phi$, *and*

$$\lim_{s \uparrow t} q(s, x; t, f) = f(t, x), \qquad \text{for } (t, x) \in D. \tag{3.28}$$

Proof. We first remark that $\phi(s, x)$ trivially satisfies

$$L\phi + c(s, x)\,\phi = 0, \qquad \text{in } D, \tag{3.29}$$

where $c(s, x)$ is the reference potential of ϕ defined by (2.20). Let us define $p(s, x)$ by

$$\begin{aligned} p(s, x) &= \phi(s, x)\, Q_{(s,x)}[f(t, X_t); t < T_s] \\ &= P_{(s,x)}\left[f(t, X_t)\, \phi(t, X_t) \exp\left(-\int_s^t \frac{L\phi}{\phi}(v, X_v)\, dv\right); t < T_s\right]. \end{aligned} \tag{3.30}$$

Then it is clear by Corollary 2 of Theorem 2.1 that

$$|p(s, x)| \leqq \|f\|\, \phi(s, x). \tag{3.31}$$

Therefore, keeping Corollary 2 of Theorem 2.1, (3.24) and (3.25) in mind, we apply Theorem 3.4 and conclude that $p(s, x)$ satisfies (3.29) in the sense of distributions. Moreover, it is clear that

$$q(s, x) = \frac{p(s, x)}{\phi(s, x)}, \qquad \text{in } D, \tag{3.32}$$

and $q(s, x)$ is differentiable in s. Now we need a simple lemma.

Lemma 3.2. *Let $\phi(s, x)$ and $p(s, x)$ be in $C^{1,2}(D)$ and $\phi(s, x)>0$ in D. Define q by* (3.32). *Then*

$$Lq+\frac{1}{\phi}\nabla\phi\cdot\nabla q=\phi^{-1}(Lp+cp)-\phi^{-2}p(L\phi+c\phi), \quad \text{in } D, \tag{3.33}$$

where L is a parabolic differential operator in (2.6) *and $c(s, x)$ is arbitrary.*

Proof is routine and omitted.

Let $\gamma\in C^{1,2}(D)\cap C_K(D)$,[10] and p_n be a sequence of smooth functions converging to p on $\mathrm{supp}(\gamma)$. We apply Lemma 3.2 to ϕ and p_n, obtaining

$$Lq_n+\frac{1}{\phi}\nabla\phi\cdot\nabla q_n=\phi^{-1}(Lp_n+cp_n)-\phi^{-2}p_n(L\phi+c\phi), \tag{3.34}$$

where $q_n=p_n/\phi$ and c is the reference potential of ϕ. The second term of the right-hand side of (3.34) vanishes. Multiplying (3.34) by γ and integrating over D, and then passing to the limit $n\to\infty$, we have

$$\begin{aligned}&\left\langle\gamma, \frac{\partial q}{\partial s}\right\rangle+\left\langle\frac{1}{2}\Delta\gamma-\nabla\cdot\left\{\left(\mathbf{a}+\frac{1}{\phi}\nabla\phi\right)\gamma\right\}, q\right\rangle\\ &\quad=\left\langle\frac{\gamma}{\phi}, \frac{\partial p}{\partial s}\right\rangle+\left\langle\frac{1}{2}\Delta\left(\frac{\gamma}{\phi}\right)-\nabla\cdot\left(\mathbf{a}\frac{\gamma}{\phi}\right)+\frac{c\gamma}{\phi}, p\right\rangle\end{aligned} \tag{3.35}$$

where the right-hand side vanishes. Therefore $q(s, x)$ satisfies (3.27) in the sense of distributions. To show (3.28) it is enough to notice

$$\begin{aligned}|q(s, x)-f(s, x)|&\leqq P_{(s,x)}[|f(t, X_t)-f(s, x)|\,N_s^t]+O(t-s)\\ &\leqq\varepsilon P_{(s,x)}[N_s^t]+O(t-s).\end{aligned}$$

Theorem 3.6. *The transformed process $\{(t, X_t), Q_{(s,x)}, \zeta\}$ never hits the nodal set N of ϕ, if and only if*

$$P_{(s,x)}[N_s^t]=1, \quad \text{for } (s, x)\in D \text{ and } t\in[s, b], \tag{3.36}$$

that is,

$$\phi(s, x)=P_{(s,x)}\left[\phi(t, X_t)\exp\left(-\int_s^t\frac{L\phi}{\phi}(v, X_v)\,dv\right); t<T_s\right]. \tag{3.37}$$

Proof. Because of (2.21) and $1_D(t, X_t)N_s^t=N_s^t$, we have

$$\begin{aligned}Q_{(s,x)}[1_D(t, X_t); t<\zeta]&=P_{(s,x)}[1_D(t, X_t)N_s^t]\\ &=P_{(s,x)}[N_s^t].\end{aligned}$$

Therefore, (3.36) is equivalent to $Q_{(s,x)}[t<\zeta]=1$, for $t\leqq b$, which is equivalent to that the transformed process $\{(t, X_t), Q_{(s,x)}\}$ does not hit the nodal set N of ϕ.

---

[10] $C_K(D)=\{$continuous functions with compact supports in $D\}$

If we know in advance that (3.36) holds, then the additional drift term $\nabla\phi/\phi$ can be identified in terms of drift transformation as follows:[11] If we set

$$\tilde{B}_t = X_t - X_s - \int_s^t \mathbf{b}(v, X_v)\,dv, \tag{3.38}$$

where

$$\mathbf{b}(s,x) = \mathbf{a}(s,x) + \frac{\nabla\phi}{\phi}(s,x), \qquad \text{on } D, \tag{3.39}$$

then $\tilde{B}_t$ is a Brownian motion with respect to $\tilde{Q}_{(s,x)} = Z_s^b Q_{(s,x)}$, where Z_s^b is the Maruyama density with respect to $Q_{(s,x)}$ given by

$$Z_s^b = \exp\left(\int_s^b \mathbf{b}(v, X_v)\cdot dX_v - \tfrac{1}{2}\int_s^b \|\mathbf{b}(v, X_v)\|^2\,dv\right). \tag{3.40}$$

There is no problem in defining $\tilde{B}_t$ in (3.38) and Z_s^b in (3.40), because the process $\{(t, X_t), Q_{(s,x)}\}$ does not hit the nodal set N of $\phi(s,x)$ (cf. Liptser-Shiryayev (1977)).

§4. Diffusion Equations with Creation and Killing

We consider a diffusion equation with drift $\mathbf{a}(s,x)$ and with creation and killing $c(s,x)$ in an open subset $D \subset [a,b]\times\mathbb{R}^d$, where $-\infty < a < b < \infty$;

$$\frac{\partial p}{\partial s} + \frac{1}{2}\Delta p + \mathbf{a}\cdot\nabla p + c\,p = 0, \qquad \text{in } D. \tag{4.1}$$

We assume that $c(s,x)$ is continuous in $(s,x)\in D$ and

$$|c(s,x)| < \infty, \qquad \text{for } (s,x)\in D,$$

but it may diverge at $N = D^c$. An example of $c(s,x)$ is the reference potential of a function $\phi(s,x)$ given by (2.20), which diverges at the nodal set of ϕ. For a fixed $t\in(a,b]$, we consider solutions of (4.1) which do not vanish in

$$D[a,t] = \{(s,x) : (s,x)\in D,\ s \leqq t\},$$

and are continuous on $\bar{D}[a,t]$ under a terminal condition

$$\lim_{s\uparrow t} p(s,x) = f(t,x), \qquad \text{for } (t,x)\in D, \tag{4.2}$$

and a "boundary" condition

$$p(s,x) = 0, \qquad \text{on } N. \tag{4.3}$$

[11] Remarked by Wakolbinger-Stummer (preprint)

Let us define T_R^n by

$$T_R^n = \begin{cases} \inf\left\{u \in (s, b] \colon |p(u, X_u)| \leqq \frac{1}{n} \text{ or } \|X_u\| \geqq R\right\}, & \text{if such } u \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.4}$$

Then $T_R^n < T_s$ and T_R^n tends to T_s as $n \to \infty$ and $R \to \infty$, because of (4.3).

Theorem 4.1. *Let* $p(s, x)$ *be a solution of* (4.1) *with* (4.2) *and* (4.3) *subject to* (3.4) *and* (3.5). *Then* $p(s, x)$ *satisfies the integral equation* (3.3).

Proof. Let T_R^n be defined by (4.4). An application of Itô's formula to $p(s, x)$ yields

$$p(s, X_s) = p(t \wedge T_R^n, X_{t \wedge T_R^n}) - \int_s^{t \wedge T_R^n} \nabla p(v, X_v) \cdot dB_v + \int_s^{t \wedge T_R^n} -Lp(v, X_v)\, dv, \tag{4.5}$$

where B_t is a Brownian motion defined by (2.9). Therefore, we have

$$p(s, x) = P_{(s,x)}[p(t \wedge T_R^n, X_{t \wedge T_R^n})] + P_{(s,x)}\left[\int_s^{t \wedge T_R^n} -Lp(v, X_v)\, dv\right]. \tag{4.6}$$

Since p is bounded on $[a, b] \times \bar{S}_R$, where $S_R = \{x \colon \|x\| < R\}$, *and* $-Lp = cp$, tending $n \to \infty$ in (4.6) for a fixed R, we have

$$\begin{aligned} p(s, x) = {} & P_{(s,x)}[p(t \wedge T_R^\infty, X_{t \wedge T_R^\infty}); t < T_s] \\ & + P_{(s,x)}\left[\int_s^{t \wedge T_R^\infty} c\, p(v, X_v)\, dv\right] + P_{(s,x)}[p(T_R^\infty, X_{T_R^\infty}); t \geqq T_s] \end{aligned} \tag{4.7}$$

where the third term vanishes and the second one converges to

$$P_{(s,x)}\left[\int_s^{t \wedge T_s} c\, p(v, X_v)\, dv\right]$$

as $R \to \infty$, because of (3.5), and the first term can be decomposed into

$$P_{(s,x)}[p(t \wedge T_R^\infty, X_{t \wedge T_R^\infty}); X_{t \wedge T_R^\infty} \in S_R, t < T_s] \tag{4.8}$$

and

$$P_{(s,x)}[p(t \wedge T_R^\infty, X_{t \wedge T_R^\infty}); X_{t \wedge T_R^\infty} \in \partial S_R, t < T_s]. \tag{4.9}$$

If $T_R^\infty \leqq t < T_s$, then $X_{T_R^\infty} \in \partial S_R \cup N$, and hence (4.8) is equal to

$$P_{(s,x)}[p(t, X_t); X_{t \wedge T_R^\infty} \in S_R, t < T_s],$$

which converges to

$$P_{(s,x)}[p(t, X_t); t < T_s]$$

as $R \to \infty$. Moreover, (3.4) implies

$$\iint_D g^0(s, x; u, y)\, du\, dy\, |p(u, y)| < \infty.$$

Therefore, (4.9) vanishes as $R \to \infty$ and hence we have

$$p(s, x) = P_{(s,x)}[f(t, X_t);\, t < T_s] + \int_s^t dv\, P_{(s,x)}[c(v, X_v)\, p(v, X_v);\, v < T_s],$$

since $p(t, x) = f(t, x)$ by (4.2). Thus $p(s, x)$ satisfies the integral equation (3.3) under the conditions (3.4) and (3.5).

Remark. Setting $R = n$ in (4.4), we denote $T^n = T_n^n$ and assume in addition

$$\{p(t \wedge T^n, X_{t \wedge T^n}):\, n \geqq 1\} \text{ is uniformly integrable with respect to } P_{(s,x)}. \tag{4.11}$$

Then $p(s, x)$ satisfies (3.3). In fact, we can take limit in (4.6) with T^n in place of T_R^n under the assumption (4.11). A sufficient condition for (4.11) is

$$P_{(s,x)}[p(t \wedge T^n, X_{t \wedge T^n})^2;\, t < T_s] \leqq \text{const.} < \infty. \tag{4.12}$$

It is clear that the condition (4.12) is satisfied, if $p(s, x)$ is bounded.

Let us denote

$$k(s, x) = p(s, x) - \bar{p}(s, x),$$

where $p(s, x)$ is a non-negative solution of (4.1) and

$$\bar{p}(s, x) = P_{(s,x)}\left[p(t, X_t) \exp\left(\int_s^t c(v, X_v)\, dv\right);\, t < T_s\right]$$

is the minimal solution of (3.3) (cf. Corollary of Theorem 2.1). By the definition $c(s, x)$ is the reference potential of $p(s, x)$, and hence

$$k(s, x) \leqq \int_s^t dv\, P_{(s,x)}[c^+(v, X_v)\, k(v, X_v);\, v < T_s]. \tag{4.13}$$

Therefore, applying the same arguments of the proof of Theorem 3.1, we have

Theorem 4.2. *Let $p(s, x)$ be a non-negative solution of* (4.1) *with* (4.2) *and* (4.3) *subject to* (3.4) *and* (3.5). *Assume*

$$\int_s^t du\, P_{(s,x)}\left[\exp\left(\int_s^u c^+(v, X_v)\, dv\right) c^+(u, X_u)\, p(u, X_u);\, u < T_s\right] < \infty. \tag{4.14}$$

Then $p(s, x)$ is a unique solution of the integral equation (3.3) *and represented in the form of*

$$p(s, x) = P_{(s,x)}\left[p(t, X_t) \exp\left(\int_s^t c(v, X_v)\, dv\right);\, t < T_s\right]. \tag{4.15}$$

Remark. For a bounded $p(s, x)$ a sufficient condition to (4.14) is

$$P_{(s,x)}\left[\exp\left(\int_s^t c^+(v, X_v)\,dv\right); t<T_s\right]<\infty. \tag{4.16}$$

This condition is fullfilled, if $Lp(s, x)\geqq 0$ in a neighbourhood of N, because $c(s, x)=-\dfrac{Lp}{p}(s, x)$. The condition (3.5) follows from (4.16), if f is bounded (cf. (3.14)).

§ 5. Inaccessibility to the Nodal Set

Let N_s^t be the multiplicative functional defined by (2.5) in terms of a given function $\phi(s, x)$, and $\{X_t, Q_{(s,x)}, \zeta\}$ be the transformed process discussed in Sect. 2. Our problem is to show $P_{(s,x)}[N_s^t]=1$, from which follows that the transformed space-time process (t, X_t) does not hit the nodal set N of $\phi(s, x)$, $Q_{(s,x)}$-a.s., by Theorem 3.6. Let $\phi(s, x)$ satisfy conditions in Sect. 2, and define the reference potential $c(s, x)$ by (2.20). Then $\phi(s, x)$ satisfies trivially the diffusion equation (4.1) with the reference potential $c(s, x)$ of ϕ, and also (4.2) and (4.3).

Theorem 5.1. *Assume* (3.24), (3.25) *and* (4.14).[12] *Then the normalization condition of the multiplicative functional N_s^t holds:*

$$P_{(s,x)}[N_s^t]=1, \quad \text{for } (s, x)\in D \text{ and } t\in[s, b], \tag{5.1}$$

and hence the transformed process $\{X_v, Q_{(s,x)}, \zeta\}$ does not hit the nodal set N of ϕ.

Proof. By Corollary 2 of Theorem 2.1 $\phi(s, x)$ satisfies (3.9) and hence Theorem 4.2 (and its Remark) can be applied to ϕ to get

$$\phi(s, x)=P_{(s,x)}\left[\phi(t, X_t)\exp\left(\int_s^t c(v, X_v)\,dv\right); t<T_s\right], \tag{5.2}$$

for $(s, x)\in D$ and $a\leqq s<t\leqq b$. Dividing (5.2) by $\phi(s, x)$ we have (5.1), which implies the inaccessibility (cf. Theorem 3.6).

Theorem 5.2. *Under the assumption of Theorem* 5.1, $\{N_s^t, \mathscr{F}_s^t, t\in[s, b], P_{(s,x)}\}$ *is a martingale with* (5.1), *and hence it is continuous in t, $P_{(s,x)}$-a.s., and given by*

$$N_s^t=\exp\left\{1_{\{t<T\}}\circ\int_s^t \frac{\nabla\phi}{\phi}(v, X_v)\cdot dB_v-\frac{1}{2}\int_s^t\left\|\frac{\nabla\phi}{\phi}(v, X_v)\right\|^2 dv\right\}, \quad P_{(s,x)}\text{-a.s.} \tag{5.3}$$

where T is defined by (2.12) *and*

$$T_s=T, \quad P_{(s,x)}\text{-a.s.} \tag{5.4}$$

[12] (4.16) is sufficient for bounded ϕ

Proof. Because of (5.1) it is clear that $\{N_s^t, \mathscr{F}_s^t, t\in[s,b], P_{(s,x)}\}$ is a martingale which is right continuous, and moreover $\mathscr{F}_s^t=\sigma(B_v; v\in[s,t])$. Therefore, it is continuous by the martingale representation theorem of Clark (1970). This implies that T must coincide with T_s, $P_{(s,x)}$-a.s., and hence we can take off the factor $1_{\{t<T_s\}}$ in (2.8). Thus (5.3) holds.

Remark 1. (5.2) indicates that $\phi(s,x)$ is a space-time harmonic function of the reference process (cf. remark after Definition 2.3). But this does not mean that we can obtain the transformed process $\{X_v, Q_{(s,x)}\}$, applying the space-time harmonic transformation to the reference process. We should transform $\{X_t, P_{(s,x)}\}$ directly by the multiplicative functional N_s^t to get $\{X_t, Q_{(s,x)}\}$.

Remark 2. Let $\phi(s,x)$ be a function satisfying

$$L\phi(s,x)=0, \qquad \text{for } (s,x)\in D. \tag{5.5}$$

Then, by Theorem 5.1, it holds that

$$\phi(s,x)=P_{(s,x)}[\phi(t,X_t); t<T_s], \tag{5.6}$$

and ϕ is a space-time harmonic function of $\{(t,X_t), P_{(s,x)}\}$. In this case the reference potential $c(s,x)$ vanishes and the transformation in terms of N_s^t reduces to Doob's space-time harmonic transformation.

§ 6. Schrödinger Processes

In a paper entitled "Ueber die Umkehrung der Naturgesetze" (1931) Schrödinger considered diffusion processes conditioned by prescribed probability distributions at the initial and terminal time to establish a time reversible formulation of diffusion processes. He considered the case of a Brownian transition probability but one can formulate them in general for a given "transition density" $p(s,x;t,y)$, which is non-negative and satisfies the Chapman-Kolmogorov equation but

$$\int p(s,x;t,y)\,dy\leqq 1$$

is *not* required. A typical example is the transition density of a diffusion process with creation and killing.

Definition 6.1. A diffusion process $\{X_t, t\in[a,b], \mathbb{P}\}$ will be called a *Schrödinger process* prescribed by a pair of functions $\{\hat{\phi}(a,x), \phi(b,x)\}$ and a transition density $p(s,x;t,y)$, if the distribution of the process X_t is factorized by the pair of functions $\hat{\phi}(t,x)$ and $\phi(t,x)$: For a bounded measurable function f

$$\mathbb{P}[f(X_t)]=\int \hat{\phi}(t,x)\,\phi(t,x)\,f(x)\,dx, \tag{6.1}$$

where $\hat{\phi}(t, x)$ and $\phi(t, x)$ are space-time harmonic and co-harmonic functions of the transition density $p(s, x; t, y)$, respectively; namely

$$\begin{aligned} \phi(t, x) &= \int p(t, x; b, y)\, dy\, \phi(b, y) \\ \hat{\phi}(t, x) &= \int \hat{\phi}(a, z)\, dz\, p(a, z; t, x). \end{aligned} \tag{6.2}$$

The factorization (6.1) of Schrödinger process has a significant resemblance to Born's statistical interpretation of wave functions in quantum mechanics, in which probability distribution densities are factorized by complex-valued wave functions as $\bar{\psi}_t \psi_t$, where

$$\begin{aligned} \psi_t &= U(t, b)\psi_b \\ \bar{\psi}_t &= \bar{\psi}_a U(a, t), \end{aligned} \tag{6.3}$$

$U(s, t) = e^{-iH(t-s)}$ with a Hamiltonian H, and $\bar{\psi}_b = \psi_a U(a, b)$. In both cases the probability of present events (at t) is predicted by the data at an initial time a and a terminal time b.[13] In fact, $\hat{\phi}(t, x)$ (resp. $\bar{\psi}_t$) is the prediction from the past and $\phi(t, x)$ (resp. ψ_t) is the prediction from the future (i.e. time reversed prediction). In other words, the factorization (6.1) combined with (6.2) is a real-valued counterpart of Born's statistical interpretation of wave functions in quantum mechanics.

We will construct Schrödinger processes, assuming that[14] a transition density $p(s, x; t, y)$ and a pair of functions $\{\hat{\phi}(a, x), \phi(b, x)\}$ subject to

$$\iint dx\, \hat{\phi}(a, x)\, p(a, x; b, y)\, \phi(b, y)\, dy = 1, \tag{6.4}$$

are given in advance. We follow Kolmogoroff's way: For $a < t_1 < t_2 < \ldots < t_n < b$ and Borel subsets A_i, $i = 1, 2, \ldots, n$, define

$$\begin{aligned} P^{(t_1, \ldots, t_n)}(A_1 \times \ldots \times A_n) = \int \ldots \int dx\, \hat{\phi}(a, x)\, p(a, x; t_1, y_1)\, dy_1 \\ \cdot p(t_1, y_1; t_2, y_2)\, dy_2 \ldots p(t_n, y_n; b, y)\, dy\, \phi(b, y) \prod_{i=1}^{n} 1_{A_i}(y_i). \end{aligned} \tag{6.5}$$

Then, $P^{(t_1, \ldots, t_n)}$ is a probability measure on S^n, where $S = \mathbb{R}^d$, because

$$P^{(t_1, \ldots, t_n)}(S \times \ldots \times S) = \iint dx\, \hat{\phi}(a, x)\, p(a, x; b, y)\, \phi(b, y) = 1, \tag{6.6}$$

by the required condition (6.4), and hence there exists a unique probability measure $\mathbb{P}$ on $\Omega = S^\infty$ such that $\mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n]$ is equal to the right-hand side of (6.5), where $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in [a, b]$. Under certain regularity conditions on $p(s, x; t, y)$ we can show the continuity of paths, though we do not discuss it here.

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[13] This is a Markov-field like prediction in one dimension

[14] This assumption is not realistic in some applications as will be seen

Theorem 6.1. *A Schrödinger process $\{X_t, t \in [a, b], \mathbb{P}\}$ is a Markov process with a transition probability density*

$$q(s, x; t, y) = \begin{cases} \dfrac{1}{\phi(s, x)} p(s, x; t, y)\, \phi(t, y), & \text{for } (s, x) \in D,\ s < t \leqq b \\ 0, & \text{otherwise,} \end{cases} \tag{6.7}$$

where $D = \{(s, x): \phi(s, x) \neq 0, s \in [a, b]\}$, and with an initial (resp. final) distribution density

$$\mu_a(x) = \hat{\phi}(a, x)\, \phi(a, x), \quad (\text{resp. } \mu_b(x) = \hat{\phi}(b, x)\, \phi(b, x)), \tag{6.8}$$

where $\hat{\phi}(t, x)$ and $\phi(t, x)$ are given in (6.2).

Proof. Multiplying and dividing by $\phi(t, x)$ at $t = a$ and t_i, $i = 1, 2, \ldots, n$, on the right-hand side of (6.5), we have

$$\begin{aligned} \mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] = \int \ldots \int d x\, \hat{\phi}(a, x)\, \phi(a, x)\, q(a, x; t_1, y_1)\, d y_1 \\ \cdot q(t_1, y_1; t_2, y_2)\, d y_2 \ldots q(t_n, y_n; b, y)\, d y \prod_{i=1}^{n} 1_{A_i}(y_i), \end{aligned} \tag{6.9}$$

which proves the assertion of the theorem.

Knowing Theorem 6.1 we can release the condition (6.4). If

$$\mu_t(A) = \int d x\, \hat{\phi}(t, x)\, \phi(t, x)\, 1_A(x) \tag{6.10}$$

is σ-finite, then a σ-finite measure $\mathbb{P}$ can be constructed, since $\{\mu_t\}$ is an entrance law at a of $q(s, x; t, y)$ of (6.7) (cf. Getoor-Glover (1986), cf. also Kuznetzov (1973), Mitro (1979), Dynkin-Getoor (1985)).

Remark 1. A class of more general processes was considered by Bernstein (1932) and Jamison (1974) in connection with Schrödinger processes.

Remark 2. Suppose that probability distribution densities $\mu_a(x)$ and $\mu_b(x)$ are given instead of $\hat{\phi}(a, x)$ and $\phi(b, x)$. Then one must find out $\hat{\phi}$ and ϕ satisfying a system of equations

$$\begin{aligned} \mu_a(x) &= \hat{\phi}(a, x) \int p(a, x; b, y)\, d y\, \phi(b, y) \\ \mu_b(x) &= \int \hat{\phi}(a, z)\, d z\, p(a, z; b, x)\, \phi(b, x), \end{aligned} \tag{6.11}$$

which was called "Schrödinger's system" and treated by Fortet (1940), Beurling (1960) and Jamison (1974) showing

Theorem 6.2 (Fortet-Beurling-Jamison). *If $p(s, x; t, y)$ is positive and continuous in $(x, y) \in S \times S$, where S is a σ-compact metric space, then there exists a unique solution $\{\hat{\phi}(a, x), \phi(b, x)\}$ of the Schrödinger's system of Eqs.* (6.11) *for a given pair $\{\mu_a, \mu_b\}$.*

Let us now assume that $\phi(s, x)$ is given and that $p(s, x; t, y)$ is a fundamental solution of

$$\frac{\partial p}{\partial s}+\frac{1}{2}\Delta p+\mathbf{a}\cdot\nabla p+c\,p=0. \tag{6.12}$$

Because of the first equation of (6.2), $\phi(s, x)$ must satisfy (6.12). In other words, $c(s, x)$ in the Eq. (6.12) must be the reference potential of $\phi(s, x)$, that is,

$$c(s, x)=-\frac{L\phi}{\phi}(s, x).$$

If $c(s, x)$ is Hölder continuous, there exists a fundamental solution $p(s, x; t, y)$, assuming $\mathbf{a}(s, x)$ is nice (cf. e.g. Friedman (1964)). However, if $\phi(s, x)$ has zeros, that is, if $c(s, x)$ diverges on a subset of $[a, b]\times\mathbb{R}^d$, there is no classical existence theorem of a fundamental solution. One possibility to get it is to apply Theorem 3.4.

Theorem 6.3. *Let* $\phi(s, x)$ *be non-negative and satisfy* (6.12) *in* $D=\{(s, x): \phi(s, x) \neq 0\}$, *that is,* $c(s, x)$ *is the reference potential of* ϕ. *If* $\phi(s, x)$ *satisfies* (3.24), (3.25) *and*

$$\phi(s, x)=P_{(s,x)}\left[\phi(t, X_t)\exp\left(\int_s^t c(v, X_v)\,dv\right); t<T_s\right],^{15} \tag{6.13}$$

for $(s, x)\in D$, $a\leqq s<t\leqq b$, *where* $\{X_v, P_{(s,x)}\}$ *is a diffusion process governed by* L *(cf. Sect. 3). Then, there exists* $p(s, x; t, y)$ *such that*

$$p(s, x; t, f\phi)=\int p(s, x; t, y)\,dy\,\phi(t, y)f(t, y) \tag{6.14}$$

satisfies the integral equation (3.3) *with* ϕf *in place of* f, *where* f *is a bounded continuous function, and*

$$p(s, x; t, y)=\int_{D_u} p(s, x; u, z)\,dz\,p(u, z; t, y), \qquad \textit{for a.e. } y\in D_t, \tag{6.15}$$

for $a\leqq s<u<t\leqq b$, *where* $D_u=\{x: \phi(u, x)\neq 0\}$. *Moreover,* $p(s, x; t, \phi f)$ *satisfies* (6.12) *in the sense of distributions.*

Proof. It remains for us to prove that there exists $p(s, x; t, y)$ which satisfies (6.14), since the remaining assertions of the theorem follows from Theorem 3.4. Let $p(s, x; t, f\phi)$ be the solution of the integral equation (3.3). Then (6.14) follows from Riesz-Markov theorem and Radon-Nykodym theorem, where we take a regular version so that $p(s, x; t, y)$ is measurable in (x, y).[16] To show (6.15) notice that $\phi(s, x)^{-1}\,p(s, x; t, f\phi)$ satisfies the integral equation (3.3) with $g^{\phi}(s, x; t, y)$ defined by

$$g^{\phi}(s, x; t, y)=\phi(s, x)^{-1}\,g^{0}(s, x; t, y)\,\phi(t, y), \qquad \text{in } D\times D \tag{6.16}$$

---

[15] (3.22) is sufficient, when we apply Theorem 3.4. We assume (6.13), because it is nothing but (6.2) necessary for Schrödinger processes

[16] Cf. e.g. Dellacherie-Meyer (1978)

in place of $g^0(s, x; t, y)$. Therefore, we have

$$\phi(s, x)^{-1}\, p(s, x; t, f\phi) = \int \phi(s, x)^{-1}\, p(s, x; u, z)\, \phi(u, z)\, dz\, \phi(u, z)^{-1}\, p(u, z; t, f\phi), \tag{6.17}$$

from which (6.15) follows.

We have shown the existence of a transition density $p(s, x; t, y)$ in order to construct a Schrödinger process. For this purpose, however, it is better to employ the transformation which we have discussed in Sect. 2, since it offers information on paths of transformed processes and on their transition probabilities.

Theorem 6.4. *Let $\phi(s, x)$ be a non-negative function satisfying* (3.24), (3.25), *and* (4.14). *Let $\{(t, X_t), Q_{(s,x)}\}$ be the transformed space-time diffusion process by means of the multiplicative functional N_s^t defined by* (2.5), *and set*

$$\mathbb{P}[\Gamma] = \int \hat{\phi}(a, x)\, \phi(a, x)\, dx\, Q_{(s,x)}[\Gamma], \quad \text{for } \Gamma \in \mathscr{F}_a^b, \tag{6.18}$$

where $\hat{\phi}(a, x)$ is a non-negative function on D. Then $\{X_t, t \in [a, b], \mathbb{P}\}$ is a Schrödinger process such that its transition probability

$$\begin{aligned} q(s, x; t, f) &= Q_{(s,x)}[f(t, X_t)] \\ &= P_{(s,x)}[f(t, X_t) N_s^t], \quad f \in C_b(D), \end{aligned} \tag{6.19}$$

is represented in terms of $q(s, x; t, y)$ in (6.7) *with $p(s, x; t, y)$ given in Theorem* 6.3 *and it satisfies*

$$Lq + \frac{1}{\phi} \nabla\phi \cdot \nabla q = 0, \tag{6.20}$$

in the sense of distributions. Moreover,

$$(t, X_t) \text{ does not hit the nodal set of } \phi, \mathbb{P}\text{-a.s.} \tag{6.21}$$

Proof. For brevity let us denote $c(s, t) = \int_s^t c(v, X_v)\, dv$. Then, for $f \in C_b(D)$,

$$\begin{aligned} &\int \hat{\phi}(a, x)\, \phi(a, x)\, dx\, Q_{(a,x)}[f(t, X_t)] \\ &= \int \hat{\phi}(a, x)\, dx\, P_{(a,x)}[f(t, X_t)\, e^{c(a,t)}\, \phi(t, X_t)\, P_{(t,X_t)}[N_t^b];\, t < T_a] \\ &= \int \hat{\phi}(a, x)\, dx\, P_{(a,x)}[f(t, X_t)\, e^{c(a,t)}\, P_{(t,X_t)}[e^{c(t,b)}\, \phi(b, X_b);\, b < T_t] \cdot t < T_a] \\ &= \iint \hat{\phi}(a, x)\, dx\, p(a, x; t, y)\, dy\, f(t, y) \int p(t, y; b, z)\, dz\, \phi(b, z) \\ &= \int \hat{\phi}(t, y)\, f(t, y)\, \phi(t, y)\, dy, \end{aligned}$$

where $\hat{\phi}(t, y)$ is defined by the second equation of (6.2), and we have used the first equation of (6.2), which is nothing else but (6.13), and

$$P_{(s,x)}[e^{c(s,t)} f\phi(t, X_t);\, t < T_s] = \int p(s, x; t, y)\, dy\, f\phi(t, y),$$

with $p(s, x; t, y)$ which is given in Theorem 6.3. (6.20) has been shown in Theorem 3.5. (6.21) follows from Theorem 3.6.

§ 7. Schrödinger Processes Associated with Schrödinger Equations

We consider a Schrödinger equation

$$i\frac{\partial\psi}{\partial s}+\frac{1}{2}\Delta\psi+i\mathbf{a}\cdot\nabla\psi-V\psi=0, \tag{7.1}$$

where $V=V(s,x)$ is a $\mathbb{R}$-valued function and $\mathbf{a}=\mathbf{a}(s,x)$ is a bounded $\mathbb{R}^d$-valued function with continuous derivatives in x satisfying a gauge condition

$$\nabla\cdot\mathbf{a}(s,x)=0.^{17} \tag{7.2}$$

Lemma 7.1. *Let $\psi=e^{\alpha+i\beta}$ be a solution of a Schrödinger equation* (7.1), *then the pair of functions $\{\alpha,\beta\}$ satisfies a system of equations*

$$\begin{aligned} V&=-\frac{\partial\beta}{\partial s}+\frac{1}{2}\Delta\alpha+\frac{1}{2}(\nabla\alpha)^2-\frac{1}{2}(\nabla\beta)^2-\mathbf{a}\cdot\nabla\beta\\ 0&=\frac{\partial\alpha}{\partial s}+\frac{1}{2}\Delta\beta+(\nabla\beta+\mathbf{a})\cdot\nabla\alpha, \end{aligned} \tag{7.3}$$

in $D=\{(s,x):\psi(s,x)\neq 0\}$.

Proof. The real (resp. imaginary) part of (7.1) divided by ψ is given by (7.3).

Lemma 7.2. *Assume the gauge condition* (7.2).

(i) *$\phi=e^{\alpha+\beta}$ satisfies a diffusion equation*

$$\frac{\partial p}{\partial s}+\frac{1}{2}\Delta p+\mathbf{a}\cdot\nabla p+cp=0, \quad \text{in } D=\{(s,x):\phi(s,x)\neq 0\}, \tag{7.4}$$

if and only if the pair of functions $\{\alpha,\beta\}$ satisfies

$$\begin{aligned} c=&-\left\{\frac{\partial\alpha}{\partial s}+\frac{1}{2}\Delta\beta+(\nabla\beta+\mathbf{a})\cdot\nabla\alpha\right\}\\ &+\left\{-\frac{\partial\beta}{\partial s}+\frac{1}{2}\Delta\alpha+\frac{1}{2}(\nabla\alpha)^2-\frac{1}{2}(\nabla\beta)^2-\mathbf{a}\cdot\nabla\beta\right\}-\{\Delta\alpha+(\nabla\alpha)^2\}. \end{aligned} \tag{7.5}$$

(ii) *$\hat\phi=e^{\alpha-\beta}$ satisfies the formal adjoint equation of* (7.4)

$$-\frac{\partial\hat p}{\partial s}+\frac{1}{2}\Delta\hat p-\mathbf{a}\cdot\nabla\hat p+c\hat p=0, \quad \text{in } D, \tag{7.6}$$

if and only if the pair of functions $\{\alpha,\beta\}$ satisfies

$$\begin{aligned} c=&\left\{\frac{\partial\alpha}{\partial s}+\frac{1}{2}\Delta\beta+(\nabla\beta+\mathbf{a})\cdot\nabla\alpha\right\}\\ &+\left\{-\frac{\partial\beta}{\partial s}+\frac{1}{2}\Delta\alpha+\frac{1}{2}(\nabla\alpha)^2-\frac{1}{2}(\nabla\beta)^2-\mathbf{a}\cdot\nabla\beta\right\}-\{\Delta\alpha+(\nabla\alpha)^2\}. \end{aligned} \tag{7.7}$$

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[17] This condition is set merely technically, because Schrödinger processes are gauge invariant (cf. Wakolbinger-Stummer (preprint)

Proof is routine and omitted.

We assume that a solution of the Schrödinger equation (7.1) exists and is represented in the form of $\psi(s,x)=e^{\alpha(s,x)+i\beta(s,x)}$ by a pair of functions $\{\alpha,\beta\}$. Let us define ϕ and $\hat{\phi}$ by

$$\begin{aligned}&\phi(s,x)=e^{\alpha(s,x)+\beta(s,x)}\\&\hat{\phi}(s,x)=e^{\alpha(s,x)-\beta(s,x)}, \qquad \text{in } D.\end{aligned}\tag{7.8}$$

Then they are continuously differentiable once in s and twice in x for $(s,x)\in D$. Let $c(s,x)$ be the reference potential of $\phi(s,x)$ with $L=\frac{\partial}{\partial s}+\frac{1}{2}\Delta+\mathbf{a}\cdot\nabla$, that is,

$$c(s,x)=-\frac{1}{\phi(s,x)}L\phi(s,x).\tag{7.9}$$

Theorem 7.1. (i) *In terms of α (resp. β) of a solution $\psi=e^{\alpha+i\beta}$ of a Schrödinger equation* (7.1) *and of a potential $V(s,x)$ in* (7.1), *the reference potential $c(s,x)$ is represented in D with*

$$\begin{aligned}c(s,x)&=-\Delta\alpha-(\nabla\alpha)^2+V\\&=-2\frac{\partial\beta}{\partial s}-(\nabla\beta)^2-2\mathbf{a}\cdot\nabla\beta-V, \quad \textit{respectively.}\end{aligned}\tag{7.10}$$

(ii) *The functions ϕ and $\hat{\phi}$ defined by* (7.8) *satisfy the diffusion equations* (7.4) *and* (7.6), *respectively, where $c(s,x)$ is the reference potential given in* (7.10).

Proof. By (7.9) $\phi=e^{\alpha+\beta}$ satisfies trivially the diffusion equation (7.4), and hence (7.5) holds for the pair $\{\alpha,\beta\}$. On the other hand the pair satisfies the system of Eqs. (7.3), because $\psi=e^{\alpha+i\beta}$ is a solution of the Schrödinger equation (7.1). Therefore, on the right-hand side of (7.5) the first line vanishes and the second bracket is equal to V. Thus we have

$$c=V-\{\Delta\alpha+(\nabla\alpha)^2\},$$

which is the first formula of (7.10), the second formula of which in terms of β follows from the first equation of (7.3). Because of the same reasoning (7.7) holds, and hence $\hat{\phi}=e^{\alpha-\beta}$ satisfies the formal adjoint Eq. (7.6).

Remark. It is shown in Carmona (1985) that $\phi=e^{\alpha+\beta}$ satisfies (7.4) with $c(s,x)$ given by (7.10) in the case of $\mathbf{a}=0$. Zambrini (1986) defined $c(s,x)$ by (7.10) and called it modified potential.

Definition 7.1. The Schrödinger process $\{X_t, t\in[a,b], \mathbb{P}\}$ determined by the pair $\{\phi,\hat{\phi}\}$ of (7.8) will be called a *Schrödinger process associated with a Schrödinger equation* (7.1).

To define a Schrödinger process we must have a transition density $p(s,x;t,y)$ with which ϕ and $\hat{\phi}$ satisfy (6.2). However, the existence of $p(s,x;t,y)$ is not trivial and we refer to Theorem 6.3. Applying Theorem 6.4, we have

Theorem 7.2. *Let $\psi=e^{\alpha+i\beta}$ be a solution of a Schrödinger equation* (7.1). *Define $\{\phi,\hat{\phi}\}$ by* (7.8) *and assume that ϕ satisfies the conditions* (3.24), (3.25), *and* (4.14) (*or bounded with* (4.16)). *Then there exists a Schrödinger process* $\{X_t, t \in [a, b], \mathbb{P}\}$ *associated with a Schrödinger equation* (7.1) *such that*

$$\begin{aligned}\mathbb{P}[X_t \in A] &= \int \bar{\psi}(t, x)\, \psi(t, x)\, 1_A(x)\, dx \\ &= \int \hat{\phi}(t, x)\, \phi(t, x)\, 1_A(x)\, dx,\end{aligned} \tag{7.11}$$

where ψ *is a solution of the Schrödinger equation* (7.1) *and* $\bar{\psi}$ *its complex conjugate; and the transition probability density* $q(s, x; t, y)$ *of the Schrödinger process is given by*

$$q(s, x; t, y) = \frac{1}{\phi(s, x)}\, p(s, x; t, y)\, \phi(t, y) \tag{7.12}$$

and satisfies

$$\frac{\partial q}{\partial s} + \frac{1}{2} \Delta q + (\mathbf{a} + \nabla \alpha + \nabla \beta) \cdot \nabla q = 0, \qquad \text{in } D, \tag{7.13}$$

in the sense of distributions. Moreover,

$$(t, X_t) \text{ does not hit the nodal set of } \psi, \mathbb{P}\text{-a.s.} \tag{7.14}$$

Remark 1. Another approach is known to get the Eq. (7.13), which is a diffusion equation with singular drift $\nabla \alpha + \nabla \beta = \nabla \phi / \phi$, or Schrödinger processes: We consider a pair of space-time diffusion processes which are in duality with respect to a measure with a density $\bar{\psi}(s, x)\, \psi(s, x)$ (then, they are time reversal of each other cf. e.g. Nagasawa (1985)). The duality implies that the transition probability density of one of the pair processes satisfies the diffusion equation (7.13) (another one of the pair satisfies the dual diffusion equation with the dual additional drift $\nabla \alpha - \nabla \beta$). Cf. Nelson (1966), Albeverio-Hoegh-Krohn (1974), Zheng-Meyer (1984/85), Carlen (1984), Nelson (1985), Carmona (1985), Zheng (1985), Zambrini (1986), Blanchard-Golin (1987).

Remark 2. We naturally raise a question whether there is any physical meaning of Schrödinger processes. One possible answer is this: Since they are diffusion processes, we interprete them as random motion of representative particles in a system of large number of interacting particles. For a trial in this direction see Nagasawa (1985), Nagasawa-Tanaka (1985, 86, 87).

Remark 3. We have assumed that a solution of a Schrödinger equation (7.1) exists. Let us assume conversely that $p(s, x; t, y)$ in $D \times D$ satisfying the condition of Theorem 6.2 is given and a pair $\{\mu_a, \mu_b\}$ of distribution densities is prescribed (or a pair of functions $\{\phi(t, x), \hat{\phi}(t, x)\}$ subject to (6.2)). Let us assume that ϕ and $\hat{\phi}$ are continuously differentiable once in s and twice in x in D. Set $\alpha = \frac{1}{2} \log \phi \hat{\phi}$ and $\beta = \frac{1}{2} \log \phi \hat{\phi}^{-1}$ in D. Then $\psi = e^{\alpha + i\beta}$ is a solution of a Schrödinger equation (7.1) with a potential $V = c + \Delta \alpha + (\nabla \alpha)^2$, where c is the reference potential of ϕ.

Remark 4. For the least action principle of Schrödinger processes see Nagasawa (preprint).

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References

Aisenman, M., Simon, M.: Brownian motion and Harnack inequality for Schrödinger operators. Commun. Pure Appl. Math. **XXXV**, 209–273 (1982)

Albeverio, S., Høegh-Krohn, R.: A remark on the connection between stochastic mechanics and the heat equation. J. Math. Phys. **15**, 1745–1747 (1974)

Bernstein, S.: Sur les liaisons entre les grandeurs aléatires. Verhandlungen Math. Kongresses, Zürich, **1**, 288–309 (1932)

Beurling, A.: An automorphism of product measures. Ann. Math. **72**, 189–200 (1960)

Blanchard, P., Golin, S.: Diffusion processes with singular drift fields. Commun. Math. Phys. **109**, 421–435 (1987)

Carlen, E.A.: Conservative diffusions. Commun. Math. Phys. **94**, 293–315 (1984)

Carmona, R.: Processus de diffusion gouverné par la forme de Dirichlet de l'opérateur de Schrödinger. Sém. de Probabilités XIII. Lect. Notes Math. **721**, 557–569 (1979). Berlin Heidelberg New York: Springer 1979

Carmona, R.: Probabilistic construction of Nelson processes. Taniguchi Symp. PMMP Katata, 55–81 (1985)

Chung, K.L., Rao, K.M.: Feynman-Kac formula and Schrödinger equation. Sem. on stoch. proc. Ed. Cinlar/Chung/Getoor, 1–29. Basel: Birkhäuser 1981

Clark, I.M.C.: The representation of functionals of Brownian motion by stochastic integrals. Ann. Math. Stat. **41**, 1282–1295 (1970)

Dellacherie, C., Meyer, P.-A.: Probabilities and potential. Amsterdam: North-Holland Publ. Co. 1978

Donsker, M.D., Varadhan, S.R.S.: Asymptotic evaluation of certain Wiener integrals for large time, in Funct. Integration and its appls. (A.M. Arthurs ed.), 15–33. Oxford London: Clarendon Press 1975

Doob, J.L.: Classical potential theory and its probabilistic counterpart. Berlin Heidelberg New York: Springer 1983

Dynkin, E.B.: Transformations of Markov processes connected with additive Functionals. Proc. 4th Berkeley Symp. on Math. Stat. Prob. Univ. of Calif. Press **II**, 117–142 (1961)

Dynkin, E.B.: Markov processes. Berlin Heidelberg New York: Springer 1965

Dynkin, E.B.: An application of flows to time shift and time reversal in stochastic processes. Tran. Am. Math. Soc. **287**, 613–619 (1985)

Dynkin, E.B., Getoor, R.: Additive functional and entrance laws. J. Funct. Anal. **62**, 211–265 (1985)

Fortet, R.: Résolution d'un système d'équations de M. Schrödinger. J. Math. Pures Appl. **IX**, 83–195 (1940)

Friedman, A.: Partial differential equation of parabolic type. Englewood Cliffs: Prentice-Hall, Inc. 1964

Fukusima, M., Takeda, M.: A transformation of symmetric Markov processes and the Donsker-Varadhan theory. Osaka J. Math. **21**, 311–326 (1984)

Getoor, R., Glover, J.: Constructing Markov processes with random times of birth and death. Semin. on Stoch. Proc. 35–69. Basel: Birkhäuser 1986

Girsanov, I.V.: On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. Theory Probab. Appl. **5**, 285–301 (1960)

Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Tokyo-Amsterdam: Kodansha Ltd. & North-Holland Publ. Co. 1981

Itô, K., Watanabe, S.: Transformation of Markov processes by multiplicative functionals. Ann. Inst. Fourier, Grenoble **15.1**, 13–30 (1965)

Jamison, B.: Reciprocal processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **30**, 65–86 (1974)

Jamison, B.: The Markov processes of Schrödinger. Z. Wahrscheinlichkeitstheor. Verw. Geb. **32**, 323–331 (1975)

Kolmogoroff, A.: Zur Theorie der Markoffschen Ketten. Math. Ann. **112**, 155–160 (1936)

Kolmogoroff, A.: Zur Umkehrbarkeit der statistischen Naturgesetze. Math. Ann. **113**, 766–772 (1937)
Kunita, H., Watanabe, T.: Note on transformations of Markov processes connected with multiplicative functionals. Mem. Fac. Sci. Kyushu Univ. Ser. A, **17**, 181–191 (1963)
Kuznetzov, S.E.: Construction of Markov processes with random time of birth and death. Theory Probab. Appl. **18**, 571–575 (1973)
Liptser, R.S., Shiryayev, A.N.: Statistics of random processes I, General theory. Berlin Heidelberg New York: Springer 1977
Maruyama, G.: On the transition probability functions of the Markov processes. Nat. Sci. Rep. Ochanomizu Univ. **5**, 10–20 (1954)
Mitro, B.J.: Dual Markov processes: Construction of a useful auxiliary process. Z. Wahrscheinlichkeitstheor. Verw. Geb. **47**, 139–156 (1979)
Motoo, M.: Diffusion process corresponding to $1/2\Sigma\,\partial^2/\partial x^{i2}+b^i(x)\,\partial/\partial x^i$. Ann. Inst. Stat. Math. **12**, 37–61 (1960)
Nagasawa, M.: The adjoint process of a diffusion process with reflecting barrier. Kodai Math. Sem. Rep. **13**, 235–248 (1961)
Nagasawa, M.: Time reversions of Markov processes. Nagoya Math. J. **24**, 177–204 (1964)
Nagasawa, M.: Markov processes with creation and annihilation. Z. Wahrscheinlichkeitstheor. Verw. Geb. **14**, 49–60 (1969)
Nagasawa, M.: Segregation of population in an environment. J. Math. Biol. **9**, 213–235 (1980)
Nagasawa, M.: Macroscopic, intermediate, microscopic and mesons. Lect. Notes Phys. **262**, 427–437. Berlin Heidelberg New York: Springer 1985
Nagasawa, M., Tanaka, H.: A diffusion process in a singular mean-drift-field. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 247–269 (1985)
Nagasawa, M., Tanaka, H.: Propagation of chaos for diffusing particles of two types with singular mean field interaction. Probab. Th. Rel. Fields **71**, 69–83 (1986)
Nagasawa, M., Tanaka, H.: Diffusion with interactions and collisions between coloured particles and the propagation of chaos. Probab. Th. Rel. Fields **74**, 161–198 (1987)
Nagasawa, M., Tanaka, H.: A proof of the propagation of chaos for diffusion processes with drift coefficients not of average form Tokyo J. Math. **10**, 403–418 (1987)
Nagasawa, M.: Stochastic variational principle of Schrödinger processes (preprint)
Nelson, E.: Derivation of Schrödinger equation from Newtonian Mechanics. Phys. Rev. **150**, 1076–1085 (1966)
Nelson, E.: Critical diffusion. Sém. de Probab. XIX. Lect. Notes Math. **1123**, 1–11. Berlin Heidelberg New York: Springer 1985
Norris, J.R.: Construction of diffusions with a given density. In: Norris, J.R. (ed.) Stochastic calculus in application, pp. 69–77. London: Longman 1988
Novikov, A.A.: On moment inequalities and identities for stochastic integrals. Proc. Second Japan-USSR Symp. Prob. Th. Lect. Notes Math. **330**, 333–339. Berlin Heidelberg New York: Springer 1973
Oshima, Y., Takeda, M.: On a transformation of symmetric Markov process and recurrence property. Lect. Notes Math. **1250**, 171–183. Berlin Heidelberg New York: Springer 1986
Schrödinger, E.: Ueber die Umkehrung der Naturgesetze, Sitzungsberichte der Preussischen Akademie der Wissenschaften Physikalisch-Mathematische Klasse, 144–153 (1931)
Schrödinger, E.: Théorie relativiste de l'électron et l'interprétation de la mécanique quantique. Ann. Inst. H. Poincaré **2**, 269–310 (1932)
Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. **7**, 447–526 (1982)
Wakolbinger, A., Stummer, W.: On Schrödinger processes and stochastic Newton equations (preprint)
Yasue, K.: The least action principle in quantum theory (Japanese) (1986)
Zambrini, J.C.: Stochastic mechanics according to Schrödinger. Phys. Rev. A. **33**, 1532–1548 (1986)
Zambrini, J.C.: Variational processes and stochastic mechanics. J. Math. Phys. **27**, 2307–2330 (1986)
Zheng, W.A., Meyer, P.A.: Quelques resultats de "Mechanique stochastique". Sém. de Probab. VIII. Lect. Notes Math. **1059**, 223–244. Berlin Heidelberg New York: Springer 1982/83
Zheng, W.A., Meyer, P.A.: Sur la construction de certain diffusions. Sém. de Probab. XX. Lect. Notes Math. **1204**, 334–337. Berlin Heidelberg New York: Springer 1984/85
Zheng, W.A.: Tightness results for laws of diffusion processes. Application to stochastic mechanics. Ann. Inst. H. Poincaré B**21**, 103–124 (1985)

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