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Probability Theory and Related Fields

Random Capacities and Their Distributions

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Summary. We formalize the notion of an increasing and outer continuous random process, indexed by a class of compact sets, that maps the empty set on zero. Existence and convergence theorems for distributions of such processes are proved. These results generalize or are similar to those known in the special cases of random measures, random (closed) sets and random (upper) semicontinuous functions. For the latter processes infinite divisibility under the maximum is introduced and characterized. Our result generalizes known characterizations of infinite divisibility for random sets and max-infinite divisibility for random vectors. Also discussed is the convergence in distribution of the row-vise maxima of a null-array of random semicontinuous functions.

1. Introduction

Our aim is mainly to formalize the notation of an increasing and outer continuous random process indexed by a class of compact sets. More specifically, we study random processes $\xi=\{\xi(K), K\in\mathscr{K}\}$, where $\mathscr{K}$ is the class of compact sets in a topological space $\mathfrak{S}$, satisfying

$$\xi(\emptyset)=0,$$

$$\xi(K)\leqq\xi(L), \qquad K\subseteq L,$$

$$\xi(K_n)\downarrow \xi(K), \qquad K_1\supseteq K_2\supseteq\ldots, K=\bigcap_n K_n$$

on a set of probability one. Below any such process will be called a *random capacity* on $\mathfrak{S}$, which will henceforth be taken to be an arbitrary locally compact second countable Hausdorff space.

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The present theory turns out to be useful in connection with some problems about extremes of random processes on multidimensional spaces. Applications to this field will be discussed in a forthcoming paper.

Clearly any random measure on $\mathfrak{S}$ (cf. e.g. [10]) may be regarded as a random capacity. Let φ be a random (closed) set in $\mathfrak{S}$ (cf. [12]), let $K\in\mathscr{K}$ and put

$$\xi(K)=\begin{cases}1, & \text{if } \varphi\cap K\neq\emptyset,\\ 0, & \text{otherwise.}\end{cases}$$

It is not hard to verify that ξ is a random capacity on $\mathfrak{S}$ satisfying

$$\xi(K\cup L)=\xi(K)\vee\xi(L), \qquad K,L\in\mathscr{K}. \tag{1.1}$$

We shall see that not only random sets but also random (upper) semicontinuous functions (cf. [14] and [16]) may be regarded as random capacities satisfying (1.1).

Another aim of this paper is to develop the special theory for distributions of random capacities satisfying (1.1). It will be seen that it closely parallels the corresponding theory for random measures.

Let η be an increasing and inner continuous random process on the class $\mathscr{G}$ of open sets in $\mathfrak{S}$ fulfilling $\eta(\emptyset)=0$. Put

$$\xi(K)=\inf_{G\supset K}\eta(G), \qquad K\in\mathscr{K}.$$

It is easy to see that ξ is a random capacity on $\mathfrak{S}$. Furthermore

$$\eta(G)=\sup_{K\subset G}\xi(K), \qquad G\in\mathscr{G}.$$

Thus our results for random capacities apply to increasing and inner continuous random processes on $\mathscr{G}$ as well.

We will see that any random capacity may be regarded as a random element in a metric space of capcities. Thus the general theory of weak convergence of probability measures on metric spaces in [2] is at our disposal. In particular it produces a short proof of a characterization of convergence in distribution for random capacities, extending a well-known fact for random measures. In addition this result generalizes a convergence theorem by Vervaat for random semicontinuous functions, see [16]. By simple manipulations we obtain from this result a characterization of convergence in distribution for random sets by the author [13].

We also prove a new characterization of convergence in distribution for random capacities satisfying (1.1), emphasizing the similarity with random measures. The corollary for random sets is new too.

Convergence in distribution for maxima of null arrays of random capacities satisfying (1.1) is characterized. The related notion of infinite divisibility is discussed. Our results on infinite divisibility generalize Balkema and Resnick's characterization of the max infinitely divisible distribution functions on $\mathbf{R}^n$, see [1]. They also generalize Matheron's results on infinite divisibility for random sets, see [12]. Finally we discuss a subclass of the infinitely divisible distributions, which we believe has important applications in the asymptotic theory of extremes of random fields.

Here are the titles of the following sections of this paper: 2. *Spaces of Capacities*, 3. *Random Capacities*, 4. *Convergence in Distribution*, 5. *Null-Arrays of Maxitive Random Capacities*.

Most of the notation is introduced when the need for it arises. Here we just note that $\mathbf{R}_+=[0,\infty)$ and $\bar{\mathbf{R}}_+=\mathbf{R}_+\cup\{\infty\}$. Also, $(0,\infty]$ is regarded as a topological space, in which the sets $(x,\infty]$, $x>0$ form a base for the neighborhoods of ∞. Note that $(0,\infty]$ is locally compact, and that each bounded (i.e. relatively compact) subset of $(0,\infty]$ is included in $[x,\infty]$ for some $x>0$.

2. Spaces of Capacities

Here we shall discuss various sets of capacities, and provide them with a topology, to be called the vague topology.

Fix an arbitrary locally compact second countable Hausdorff space $\mathfrak{S}$. Write $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{K}$ for the classes of closed, open and compact sets in $\mathfrak{S}$, respectively. Also, write $\mathscr{S}$ for the Borel sets in $\mathfrak{S}$, and $\mathscr{B}$ for the bounded (i.e. relatively compact) sets in $\mathscr{S}$. The letters G and K, with or without subscripts, are reserved for sets in $\mathscr{G}$ and $\mathscr{K}$, respectively. Unless otherwise is stated locally, the letter x, with or without subscripts, denotes a strictly positive real number. A large amount of the notation in this paper depends on the chosen $\mathfrak{S}$, and the need may occasionally arise to indicate this dependence explicitely. In such cases we just add a pair of parentheses and put the space under consideration between them. For instance, $\mathscr{F}((0,\infty])$ denotes the class of closed sets in $(0,\infty]$. Recall that a class of sets is said to be *separating* if whenever $K\subseteq G$, we have $K\subseteq A\subseteq G$ for some $A\subseteq\mathscr{A}$ (cf. [13]). We write $B_n\uparrow B$ when $B_1\subseteq B_2\subseteq\ldots\subseteq\mathfrak{S}$ and $B=\bigcup_n B_n$. If the B_n's are bounded and $B_1^-\subseteq B_2^0\subseteq B_2^-\subseteq B_3^0\subseteq\ldots$ then we write $B_n\uparrow\uparrow B$. Interprete $B_n\downarrow B$ and $B_n\downarrow\downarrow B$ analogously.

By a *capacity* on $\mathfrak{S}$ we understand an increasing and outer continuous function on $\mathscr{K}$ into $\bar{\mathbf{R}}_+$, which maps $\emptyset$ on 0. Thus, for any capacity c on $\mathfrak{S}$ we have $c(K_n)\downarrow c(K)$ as soon as $K_n\downarrow K$. The set of capacities on $\mathfrak{S}$ is denoted $\mathfrak{U}_1$. Let $c\in\mathfrak{U}_1$. By putting

$$c(G)=\sup_{K\subset G} c(K),\qquad G\in\mathscr{G}\tag{2.1}$$

the domain of c is extended to $\mathscr{K}\cup\mathscr{G}$. Note that c is increasing on $\mathscr{K}\cup\mathscr{G}$, and inner continuous on $\mathscr{G}$. Thus $c(G_n)\uparrow c(G)$ whenever $G_n\uparrow G$. Note also that

$$c(K)=\inf_{G\supset K} c(G),\qquad K\in\mathscr{K}.\tag{2.2}$$

Let $\mathscr{A}$ be a separating class, and let $c\colon\mathscr{A}\to\bar{\mathbf{R}}_+$ be increasing and such that $c(\emptyset)=0$. Define

$$\bar{c}(K)=\inf\{c(A), A\in\mathscr{A}, K\subseteq A^0\},\qquad K\in\mathscr{K}.\tag{2.3}$$

Then $\bar{c}\in\mathfrak{U}_1$. By (2.1) we get

$$\bar{c}(G)=\sup\{c(A), A\in\mathscr{A}, A^-\subseteq G\},\qquad G\in\mathscr{G}.\tag{2.4}$$

Recall that c is said to be *subadditive* if

$$c(A_1 \cup A_2) \leqq c(A_1) + c(A_2), \qquad A_1, A_2, A_1 \cup A_2 \in \mathscr{A} \tag{2.5}$$

and *strongly subadditive* if

$$c(A_1 \cup A_2) + c(A_1 \cap A_2) \leqq c(A_1) + c(A_2), \qquad A_1, A_2, A_1 \cup A_2, A_1 \cap A_2 \in \mathscr{A}. \tag{2.6}$$

Let us say that c is *additive* if we have equality in (2.6), and *maxitive* if

$$c(A_1 \cup A_2) = c(A_1) \vee c(A_2), \qquad A_1, A_2, A_1 \cup A_2 \in \mathscr{A}. \tag{2.7}$$

It is shown in [11] that, when c is a capacity, c is additive iff c is subadditive and $c(K_1 \cup K_2) = c(K_1) + c(K_2)$ for all disjoint K_1, K_2.

Now suppose that $\mathscr{A}$ is closed under finite unions. Whenever $A, A_1, \ldots, A_n \in \mathscr{A}$, we define $\triangle_{A_1} \ldots \triangle_{A_n} c(A)$ by iterating the formula $\triangle_{A_1} c(A) = c(A \cup A_1) - c(A)$ (convention: $\infty - \infty = 0$). Note that $\triangle_{A_1} c(A)$ always is non-negative. Moreover c is strongly sub-additive iff $\triangle_{A_1} \triangle_{A_2} c(A)$ never exceeds 0. Recall that c is said to be *alternating* (of infinite order) [4] if

$$(-1)^{n+1} \triangle_A \ldots \triangle_{A_n} c(A) \geqq 0, \qquad n \in N, \; A, A_1, \ldots, A_n \in \mathscr{A}. \tag{2.8}$$

We write $\mathfrak{U}_{1s}$, $\mathfrak{U}_2$, $\mathfrak{U}_\infty$, $\mathfrak{U}_a$ and $\mathfrak{U}_m$ for the classes of subadditive, strongly subadditive, alternating, additive and maxitive capacities, respectively. We further let $\mathfrak{U}_s$ be the set of $\{0,1\}$-valued maxitive capacities. Note that $\mathfrak{U}_a \cup \mathfrak{U}_m \subseteq \mathfrak{U}_\infty \subseteq \mathfrak{U}_2 \subseteq \mathfrak{U}_{1s}$ ($\subseteq \mathfrak{U}_1$). The following lemma is useful when deciding to which of these classes a given capacity belongs. The proof is simple.

Lemma 2.1. *Suppose, for* $i = 1, 2$, $K_{in} \downarrow\downarrow K_i$. *Then* $K_{1n} \cup K_{2n} \downarrow\downarrow K_1 \cup K_2$.

A capacity c on $\mathfrak{S}$ is said to be *locally finite* if $c(K) < \infty$ for all K. For $j \in \{1, 1s, 2, \infty, a, m\}$ we let $\mathfrak{U}_{jf}$ be the set of locally finite $c \in \mathfrak{U}_j$. We further put $\mathfrak{U}_{j1} = \{c \in \mathfrak{U}_j, c(S) \leqq 1\}$ and $\mathfrak{U}_{jp} = \{c \in \mathfrak{U}_j, c(S) = 1\}$. Note that the latter definition has a meaning for $j = s$ too.

Let us say that an increasing function c on $\mathfrak{S}$ is *regular* if

$$c(B) = \sup_{K \subset B} c(K) = \inf_{G \supset B} c(G), \qquad B \in \mathscr{S}. \tag{2.9}$$

The restriction to $\mathscr{K}$ of such a function is a capacity if it maps $\emptyset$ on 0. Choquet's capacitability theorem (see [5]) shows that any $c \in \mathfrak{U}_2$ may be extended to a regular function on $\mathscr{S}$.

It is easy to see that (2.9) extends the additive capacities to measures. Thus we may identify $\mathfrak{U}_{af}$ with the well-known set of locally finite measures on $\mathfrak{S}$ (cf. [4]).

Let $c \in \mathfrak{U}_m$. Straightforward compactness arguments yield

$$c(K) = \sup_{s \in K} c(s), \qquad K \in \mathscr{K}. \tag{2.10}$$

(We write $c(s)$ for $c(\{s\})$ when $c \in \mathfrak{U}_m$.) Hence the extension of $c \in \mathfrak{U}_m$ to a regular function on $\mathscr{S}$ is given by

$$c(B) = \sup_{s \in B} c(s), \qquad B \in \mathscr{S}. \tag{2.11}$$

It is easily verified that (2.10) defines a bijection between $\mathfrak{U}_m$ and the set $\mathscr{F}_+$ of upper semicontinuous functions on $\mathfrak{S}$ into $\bar{\mathbf{R}}_+$. Cf. [4, 9] and [16]. Clearly this bijection maps indicators of closed sets onto $\mathfrak{U}_s$. Thus there is a natural bijection between $\mathscr{F}$ and $\mathfrak{U}_s$. Note that the image of $F \in \mathscr{F}$ under the latter maps K on 1 iff $F \cap K \neq \emptyset$.

Let us endow $\mathfrak{U}_1$ with the topology generated by the families $\{c, c(K) < x\}$, $K \in \mathscr{K}$, $x > 0$ and $\{c, c(G) > x\}$, $G \in \mathscr{G}$, $x > 0$, to be termed the *vague* topology since it extends the usual notion of vague convergence for measures. Note that $c_n \xrightarrow{v} c$ (read c_n converges vaguely to c) iff

$$\begin{aligned} &\limsup_n c_n(K) \leqq c(K), \qquad K \in \mathscr{K}, \\ &\liminf_n c_n(G) \geqq c(G), \qquad G \in \mathscr{G}. \end{aligned} \tag{2.12}$$

We will need the following condition for vague convergence.

Lemma 2.2. *Let $c, c_1, c_2, \ldots$ be increasing functions on the separating class $\mathscr{A}$, vanishing at $\emptyset$. Suppose*

$$\bar{c}(A^0) \leqq \liminf_n c_n(A) \leqq \limsup_n c_n(A) \leqq \bar{c}(A^-), \qquad A \in \mathscr{A},\ A^- \in \mathscr{K}. \tag{2.13}$$

Then $\bar{c}_n \xrightarrow{v} \bar{c}$.

Proof. Fix K and choose bounded $A_1, A_2, \ldots \in \mathscr{A}$ with $A_m \downarrow\downarrow K$. Then

$$\limsup_n \bar{c}_n(K) \leqq \limsup_n c_n(A_m) \leqq \bar{c}(A_m^-) \to \bar{c}(K).$$

A dual argument yields the other half of (2.12). □

Put $\mathscr{B}_c = \{B \in \mathscr{B}, c(B^0) = c(B^-)\}$. It is not hard to see that $\mathscr{B}_c$ is a separating class, and that $c_n \xrightarrow{v} c$ iff $c_n(B) \to c(B)$ for all $B \in \mathscr{B}_c$ (provided the c_n's are extended to increasing functions on $\mathscr{B}$ in such a way that $c_n(B^0) \leqq c_n(B) \leqq c_n(B^-)$). Also, $\mathscr{B}_c$ is closed under finite unions if $c \in \mathfrak{U}_2$ (see [5], Lemma 9.11).

The (relative) vague topology on $\mathfrak{U}_m$ is termed the sup vague topology in [16]. Here it is also noted that the vague topology on $\mathfrak{U}_s$ is homeomorphic to Fell's topology [7] on $\mathscr{F}$. See also [6] and [12]. Convergence with respect to the (induced) vague topology on $\mathscr{F}_+$ is referred to as hypo convergence in [6]. See also [16]. The reason is that the mapping

$$f \to \operatorname{hypo} f = \{(s, x), 0 < x \leqq f(s)\} \tag{2.14}$$

maps $\mathscr{F}_+$, endowed with the vague topology, homeomorphically onto a closed subset of $\mathscr{F}(\mathfrak{S} \times (0, \infty])$ endowed with Fell's topology. The paper [6] also discusses the dual notion of epi convergence.

Our next result, parts of which can be found in the references in the above paragraph, gives the main properties of the vague topology.

Theorem 2.3. *The space $\mathfrak{U}_1$ is compact and Polish (i.e. completely and separably metrizable). So are also $\mathfrak{U}_{1s}$, $\mathfrak{U}_2$, $\mathfrak{U}_\infty$, $\mathfrak{U}_a$, $\mathfrak{U}_m$ and $\mathfrak{U}_s$.*

Proof. We first note that the vague topology is Hausdorff and second countable. The easy proofs are left to the reader. Let $\mathscr{A} \subseteq \mathscr{K}$ be a countable separating class closed under finite unions. Let $\mathfrak{V}_1$ be the set of increasing functions on $\mathscr{A}$ that map $\emptyset$ on 0, endowed with the topology of pointwise convergence. Clearly $\mathfrak{V}_1$ is closed in $\bar{\mathbf{R}}_+^{\mathscr{A}}$ and therefore compact by Tychonov's theorem. By Lemma 2.2, the mapping $c \to \bar{c}$ of $\mathfrak{V}_1$ onto $\mathfrak{U}_1$ is continuous. Hence $\mathfrak{U}_1$ is compact. The compactness of $\mathfrak{U}_{1s}$, $\mathfrak{U}_2$, $\mathfrak{U}_\infty$ and $\mathfrak{U}_m$ follow similarly. Also, $\mathfrak{U}_s$ is closed in $\mathfrak{U}_m$ and therefore compact too. Suppose $c_n \xrightarrow{v} c$, where $\{c_n\} \subseteq \mathfrak{U}_a$ and $c \in \mathfrak{U}_2$. If we can prove

$$c(K_1 \cup K_2) = c(K_1) + c(K_2), \quad K_1, K_2 \in \mathscr{K},\ K_1 \cap K_2 = \emptyset, \tag{2.15}$$

then it will follow that $c \in \mathfrak{U}_a$ and we can conclude that $\mathfrak{U}_a$ is compact. Now, if the left side of (2.15) is infinite, then, by subadditivity, so is the right. Assume $c(K_1 \cup K_2) < \infty$. If $K_1, K_2 \in \mathscr{B}_c$ then $K_1 \cup K_2 \in \mathscr{B}_c$, and (2.15) follows by convergence. The truth of (2.15) in general now follows by Lemma 2.1. Now the theorem follows from the well-known fact that compact second countable Hausdorff spaces are metrizable. □

The following two propositions extend well-known results on relative compactness for measures. The easy proofs are omitted.

Proposition 2.4. *Let $j \in \{1, 1s, 2, \infty, a, m\}$. A subset M of $\mathfrak{U}_{jf}$ is relatively compact iff*

$$\sup_{c \in M} c(K) < \infty, \quad K \in \mathscr{K}. \tag{2.16}$$

Proposition 2.5. *Let $j \subseteq \{1, 1s, 2, \infty, a, m, s\}$. A subset M of $\mathfrak{U}_{jp}$ is relatively compact iff it is tight, i.e. iff whenever $\varepsilon > 0$ we have*

$$\inf_{c \in M} c(K) > 1 - \varepsilon \tag{2.17}$$

for some $K \in \mathscr{K}$.

Note that $\mathfrak{U}_{sp}$ is homeomorphic to $\mathscr{F}' = \mathscr{F} \setminus \{\emptyset\}$. Hence a subset M of $\mathscr{F}'$ is relatively compact iff there is a K with $F \cap K \neq \emptyset$ for all $F \in M$.

As already noted, the vague topology on $\mathfrak{U}_{af}$ has its usual meaning. In particular this means that the mapping $\mathfrak{U}_{af} \ni c \to \int f dc$ is continuous whenever $f \in \mathscr{C}_0$ – the class of compactly supported continuous functions on $\mathfrak{S}$ into $\mathbf{R}_+$ – and that this class of mappings generates the vague topology on $\mathfrak{U}_{af}$. A similar statement holds for $\mathfrak{U}_{mf}$.

Theorem 2.6. *The vague topology on $\mathfrak{U}_{mf}$ is generated by the mappings*

$$c \to \sup_{x > 0} x c(f \geqq x), \quad f \in \mathscr{C}_0.$$

The proof of Theorem 2.6 uses three lemmata, the proofs of which we leave to the reader. We do not claim any originality. Write $\mathscr{S}_+$ for the Borel functions on $\mathfrak{S}$ into $\bar{\mathbf{R}}_+$. Also write $\mathscr{G}_+$ and $\mathscr{K}_+$ for the lower semicontinuous and compactly supported finite upper semicontinuous functions in $\mathscr{S}_+$. Note that $\mathscr{C}_0 = \mathscr{K}_+ \cap \mathscr{G}_+$.

Lemma 2.7. *For $f \in \mathscr{S}_+$ and $c \in \mathfrak{U}_m$, we have*

$$\sup_{x > 0} x c(f \geqq x) = \sup_{x > 0} x c(f > x) = \sup_{s \in S} f(s) c(s). \tag{2.18}$$

Thus the mapping $f \to \sup_x x c(f \geqq x)$ is continuous from below on $\mathscr{S}_+$. Continuity from above on $\mathscr{K}_+$ is now a consequence of the following two lemmata.

Lemma 2.8. *If $f_n \downarrow f$, where $f, f_1, f_2, \ldots \in \mathscr{K}_+$, then* $\sup f_n \downarrow \sup f$.

Lemma 2.9. *If $f_1, f_2 \in \mathscr{F}_+$, then $f_1 f_2 \in \mathscr{F}_+$.*

Proof of Theorem 2.6. Clearly $\{c, \sup_{y>0} y c(f \geqq y) < x\}$ is a finite intersection of open sets, and therefore open, if $f \in \mathscr{K}_+$ is simple. By approximation we see that this set is open also for arbitrary $f \in \mathscr{K}_+$. It follows similarly that $\{c, \sup_{y>0} y c(f \geqq y) > x\}$ is open whenever $f \in \mathscr{G}_+$. Thus the mappings $c \to \sup_x x c(f \geqq x)$, $f \in \mathscr{C}_0$ are continuous. Equivalently, the vague topology is stronger than the topology generated by these mappings. The converse follows from

$$\{c, c(K) < x\} = \bigcup_{f \geqq 1_K} \{c, \sup_y y c(f \geqq y) < x\},$$
$$\{c, c(G) > x\} = \bigcup_{f \leqq 1_G} \{c, \sup_y y c(f \geqq y) > x\}. \quad \square$$

Corollary 2.10. *Fell's topology on $\mathscr{F}$ is generated by the mappings*

$$F \to \sup_{s \in F} f(s), \qquad f \in \mathscr{C}_0.$$

3. Random Capacities

Our main purpose with this section is to discuss some measurability questions. We shall also give a simple, yet useful, existence theorem for random capacities.

Let $\mathfrak{U}_j$ be one of the spaces of capacities introduced in Sect. 2, and write $\mathscr{U}_j$ for its Borel sets. It is easily seen that $\mathscr{U}_j$ is generated by the mappings $c \to c(K)$, $K \in \mathscr{K}$. Also, $\mathscr{U}_j \subseteq \mathscr{U}_i$ if $\mathfrak{U}_j \subseteq \mathfrak{U}_i$. Thus we may identify random elements in $\mathfrak{U}_1$ with random capacities on $\mathfrak{S}$. Say that a random capacity is *subadditive* if it is a.s. $\mathfrak{U}_{1s}$-valued. *Strongly subadditive, alternating, additive* and *maxitive* random capacities are defined similarly. A.s. $\mathfrak{U}_{1f}$-valued random capacities are below referred to as being *locally finite.*

Note that any random capacity ξ on $\mathfrak{S}$ trivially extends to an increasing random process η on $\mathscr{B}$ with $\bar{\eta} = \xi$. Some of our results are stated in terms of such extensions. Note however that we do not distinguish typographically between ξ and the extension. We are aware of two cases where there is a regular extension to $\mathscr{S}$.

Proposition 3.1. *Any locally finite additive random capacity extends by* (2.9) *to a random measure.*

Proposition 3.2. *Let ξ be a maxitive random capacity on $\mathfrak{S}$, defined on a complete probability space. Then $\sup_x x \xi(f \geqq x)$ is a random variable whenever $f \in \mathscr{S}_+$. In particular so is also $\xi(B)$ for all $B \in \mathscr{S}$.*

Proof of Proposition 3.1. Proceed as in the proof of [10, Lemma 1.4]. □

Proof of Proposition 3.2. Use hypo (see (2.14)) to conclude that the last assertion is equivalent to a well-known fact for random sets (see [12, p. 30]). Then show the first assertion for simple $f \in \mathscr{S}_+$, and extend to arbitrary $f \in \mathscr{S}_+$ by approximation from below. □

Our existence theorem for random capacities is as follows.

Proposition 3.3. *Let $\mathscr{A} \subseteq \mathscr{B}$ be a separating class, and let η be a random process on $\mathscr{A}$. Suppose*

$$\eta(\emptyset)=0 \quad \text{a.s.,} \tag{3.1}$$

$$\mathbb{P}\{\eta(A_1) \leqq x, \eta(A_2) \leqq x\} = \mathbb{P}\{\eta(A_2) \leqq x\}, \quad x>0,\ A_1, A_2 \in \mathscr{A},\ A_1 \subseteq A_2, \tag{3.2}$$

$$\eta(A_n) \xrightarrow{d} \eta(A), \quad A, A_1, A_2, \ldots \in \mathscr{A},\ A_n \downarrow\downarrow A^-, \tag{3.3}$$

$$\eta(A_n) \xrightarrow{d} \eta(A), \quad A, A_1, A_2, \ldots \in \mathscr{A},\ A_n \uparrow\uparrow A^0. \tag{3.4}$$

Then there is a random capacity ξ on $\mathfrak{S}$ such that $\mathscr{A} \subseteq \mathscr{B}_\xi$ and

$$\xi(A)=\eta(A) \quad \text{a.s.,} \quad A \in \mathscr{A}. \tag{3.5}$$

If $\mathscr{A} \subseteq \mathscr{K}$ then (3.1)–(3.3) *is sufficient for the existence of a random capacity ξ satisfying* (3.5). *If $\mathscr{A} \subseteq \mathscr{G}$ then this result holds if* (3.3) *is replaced by* (3.4).

Proof. Let $\mathscr{A}_0$ be a countable separating subclass of $\mathscr{A}$, and write η_0 for η restricted to $\mathscr{A}_0$. By (3.1) and (3.2), $\eta_0(\emptyset)=0$ and η_0 is increasing on some set of probability one. On this set we put $\xi=\bar{\eta}_0$. Fix $A \subseteq \mathscr{A}$. By (3.3) and (3.4), we get $\eta(A)=\xi(A^-)$ and $\eta(A)=\xi(A^0)$, respectively. This proves (3.5). The rest of the proof is trivial. □

In applications this results should be combined with the following proposition.

Proposition 3.4. *Let ξ be a random capacity on $\mathfrak{S}$, and let $\mathscr{K}_0 \subseteq \mathscr{K}$ be a separating class closed under finite unions. Consider the following five conditions.*

$$\xi(K)<\infty \quad \text{a.s.,} \quad K \in \mathscr{K}_0, \tag{3.6}$$

$$\xi(K) \in \{0,1\} \quad \text{a.s.,} \quad K \in \mathscr{K}_0, \tag{3.7}$$

$$\xi(K_1 \cup K_2) \leqq \xi(K_1)+\xi(K_2) \quad \text{a.s.,} \quad K_1, K_2 \in \mathscr{K}_0, \tag{3.8}$$

$$\xi(K_1 \cup K_2) \geqq \xi(K_1)+\xi(K_2) \quad \text{a.s.,} \quad K_1, K_2 \in \mathscr{K}_0,\ K_1 \cap K_2=\emptyset, \tag{3.9}$$

$$\xi(K_1 \cup K_2) \leqq \xi(K_1) \vee \xi(K_2) \quad \text{a.s.,} \quad K_1, K_2 \in \mathscr{K}_0. \tag{3.10}$$

We have: ξ is locally finite iff (3.6) *holds; ξ is additive iff* (3.8) *and* (3.9) *hold; ξ is maxitive iff* (3.10) *holds; $\xi \in \mathfrak{U}_s$ a.s. iff* (3.7) *and* (3.8) *hold.*

Proof. The first assertion is trivial. Let $\mathscr{K}_1 \subseteq \mathscr{K}_0$ be countable, separating and closed under finite unions. Suppose (3.8). Clearly, $\xi(K_1 \cup K_2) \leqq \xi(K_1)+\xi(K_2)$,

$K_1, K_2 \in \mathscr{K}_1$ holds with probability one. By Lemma 2.1 we conclude that ξ is subadditive. It follows similarly that (3.9) implies $\mathbb{P}\{\xi(K_1 \cup K_2) \geqq \xi(K_1) + \xi(K_2),$ $K_1, K_2 \in \mathscr{K},\ K_1 \cap K_2 = \emptyset\} = 1$, and that (3.10) implies that ξ is maxitive. Note also that (3.10) follows from (3.9) if (3.7) is at hand. This proves the sufficiency. The necessity is obvious. □

Existence criteria for random measures (see e.g. [10], Th. 5.3) and random semicontinuous functions (see [16]) are easily derived from the Propositions 3.3 and 3.4.

4. Convergence in Distribution

Here we shall discuss necessary and sufficient conditions for convergence in distribution of random capacities. We begin with a general theorem, extending a well-known fact for random measures (see [10], Th. 4.2) to random capacities. This result also generalizes a characterization of convergence in distribution for random semicontinuous functions by Vervaat [16]. Then we specialize to the maxitive case.

Theorem 4.1. *Let $\xi, \xi_1, \xi_2, \dots$ be random capacities on $\mathfrak{S}$. If $\xi_n \xrightarrow{d} \xi$ and if $\mathscr{A}_0 \subseteq \mathscr{B}_\xi = \{B \in \mathscr{B},\ \xi(B^0) \stackrel{d}{=} \xi(B^-)\}$ is finite, then*

$$(\xi_n(A), A \in \mathscr{A}_0) \xrightarrow{d} (\xi(A), A \in \mathscr{A}_0). \tag{4.1}$$

Conversely, $\xi_n \xrightarrow{d} \xi$ if there is a separating $\mathscr{A} \subseteq \mathscr{B}$ (e.g. $\mathscr{B}_\xi$) such that (4.1) *holds for all finite $\mathscr{A}_0 \subseteq \mathscr{A}$.*

Proof. The necessity of (4.1) follows from [2, Th. 5.5], while the sufficiency is a consequence of the continuity of the mapping $c \to \bar{c}$, proved in Lemma 2.2 (cf. the proof of Th. 2.3). That $\mathscr{B}_\xi$ is separating may be seen by a straightforward extension of the arguments in the proof of Lemma 4.3 in [10], which treats the case where ξ is a random measure. □

A collection $\{\xi_n\}$ of locally finite random capacities on $\mathfrak{S}$ has a locally finite limit point with respect to convergence in distribution iff $\{\xi_n(K)\}$ is tight for all $K \in \mathscr{K}$. The necessity of the latter statement is obvious. To see the sufficiency, suppose that $\{\xi_n(K)\}$ is tight for all K in a countable separating class $\mathscr{K}_0 \subseteq \mathscr{K}$. Then $\{(\xi_n(K_1), \dots, \xi_n(K_m))\}$ is tight for all $m \in N$, $K_1, \dots, K_m \in \mathscr{K}_0$. A variant of the diagonal procedure now produces a subsequence $\{\xi_{n_k}\}$ and a random process $\eta = \{\eta(K), K \in \mathscr{K}_0\}$, such that

$$(\xi_{n_k}(K_1), \dots, \xi_{n_k}(K_m)) \xrightarrow{d} (\eta(K_1), \dots, \eta(K_m)), \qquad m \in N,\ K_1, \dots, K_m \in \mathscr{K}_0.$$

Clearly $\eta(\emptyset) = 0$. Furthermore it is not hard to verify that η is increasing. Now $\xi_{n_k} \xrightarrow{d} \bar{\eta}$ follows from Theorem 4.1, and we conclude that $\bar{\eta}$ is a locally finite limit point of $\{\xi_n\}$.

More can be said if additivity or maxitivity are at hand. In the former case we refer to [10]. The latter case is treated below.

It has been known since Choquet [4] (cf. [12] or [13]) that the mapping $\mathfrak{U}_{ap}(\mathscr{F})\ni\mu\to T_\mu\in\mathfrak{U}_{\infty 1}$, where $T_\mu(K)=\mu\{F, F\cap K\neq\emptyset\}$, is bijective. Straightforward calculations using Theorem 4.1 and the homeomorphism between $\mathfrak{U}_s$ and $\mathscr{F}$, now show that this mapping is a homeomorphism. This reestablishes [13, Th. 2.1]. Clearly $\mathfrak{U}_{ap}(\mathscr{F}')$ and $\mathfrak{U}_{\infty p}$ are homeomorphic too. Hence, by Proposition 2.5, a family $\{\varphi_t\}$ of a.s. non-empty random sets in $\mathfrak{S}$, has an a.s. non-empty limit point with respect to convergence in distribution iff whenever $\varepsilon>0$ there is a K such that $\inf_t \mathbb{P}\{\varphi_t\cap K\neq\emptyset\}>1-\varepsilon$. Also, a family $\{\xi_t\}$ of maxitive random capacities on $\mathfrak{S}$, satisfying $\xi_t\neq 0$ a.s. for all t, has an a.s. non-zero limit point with respect to convergence in distribution iff whenever $\varepsilon>0$ there is a pair (K,x) with $\inf_t \mathbb{P}\{\xi_t(K)\geqq x\}>1-\varepsilon$. To see the latter statement, use the former and the homeomorphism between $\mathfrak{U}_m(\mathfrak{S})$ and a closed subset of $\mathscr{F}(\mathfrak{S}\times(0,\infty])$.

Theorem 4.2. *Let $\xi,\xi_1,\xi_2,\ldots$ be maxitive random capacities on $\mathfrak{S}$. Then $\xi_n\xrightarrow{d}\xi$ iff*

$$\mathbb{P}\bigcap_{i=1}^{m}\{\xi_n(B_i)\leqq x_i\}\to\mathbb{P}\bigcap_{i=1}^{m}\{\xi(B_i)\leqq x_i\},$$

$$m\in N,\ x_i>0,\ B_i\in\mathscr{B},\qquad \mathbb{P}\{\xi(B_i^-)<x_i\}=\mathbb{P}\{\xi(B_i^0)\leqq x_i\},\quad 1\leqq i\leqq m. \tag{4.2}$$

In the locally finite case, we further have $\xi_n\xrightarrow{d}\xi$ iff

$$\sup_x x\,\xi_n(f\geqq x)\xrightarrow{d}\sup_x x\,\xi(f\geqq x),\qquad f\in\mathscr{C}_0. \tag{4.3}$$

Corollary 4.3. *Let $\varphi,\varphi_1,\varphi_2,\ldots$ be random sets in $\mathfrak{S}$. Then $\varphi_n\xrightarrow{d}\varphi$ iff*

$$\sup_{s\in\varphi_n} f(s)\xrightarrow{d}\sup_{s\in\varphi} f(s),\qquad f\in\mathscr{C}_0. \tag{4.4}$$

Proof of Theorem 4.2. Note that (4.2) is equivalent to

$$\mathbb{P}\{\text{hypo}\,\xi_n\cap\bigcup_i B_i\times(x_i,\infty]=\emptyset\}\to\mathbb{P}\{\text{hypo}\,\xi\cap\bigcup_i B_i\times(x_i,\infty]=\emptyset\}.$$

So the necessity and sufficiency of (4.2) is a consequence of the homeomorphism between $\mathfrak{U}_{ap}(\mathscr{F}(\mathfrak{S}\times(0,\infty]))$ and $\mathfrak{U}_{\infty 1}(\mathfrak{S}\times(0,\infty])$. We leave the details to the reader.

Now suppose that ξ and the ξ_n's are locally finite. By Theorem 2.6 we see that (4.3) is necessary for $\xi_n\xrightarrow{d}\xi$. Assume (4.3) is at hand. By approximation with functions in $\mathscr{C}_0$, we get

$$\liminf_n \mathbb{P}\{\sup_x x\,\xi_n(k\geqq x)<1\}\geqq\mathbb{P}\{\sup_x x\,\xi(k\geqq x)<1\},\qquad k\in\mathscr{K}_+,$$

$$\limsup_n \mathbb{P}\{\sup_x x\,\xi_n(g>x)\leqq 1\}\leqq\mathbb{P}\{\sup_x x\,\xi(g>x)\leqq 1\},\qquad g\in\mathscr{G}_+.$$

Fix $m\in N$. Choose, for $1\leqq i\leqq m$, x_i and B_i according to the requirements in (4.2). Assume that $x_1\leqq\ldots\leqq x_m$. This is no restriction, since we may always renumber the B_i's. Put $y_i = 1/x_i$. We get

$$\limsup_n \mathbb{P}\bigcap_i \{\xi_n(B_i^0) \leqq x_i\} = \limsup_n \mathbb{P}\left\{\bigvee_{i=1}^m y_i \xi_n\left(\bigcup_{j=1}^i B_j^0\right) \leqq 1\right\}$$
$$\leqq \mathbb{P}\left\{\bigvee_{i=1}^m y_i \xi\left(\bigcup_{j=1}^i B_j^0\right) \leqq 1\right\} = \mathbb{P}\bigcap_i \{\xi(B_i^0) \leqq x_i\}$$
$$= \mathbb{P}\bigcap_i \{\xi(B_i^-) < x_i\} = \mathbb{P}\left\{\bigvee_{i=1}^m y_i \xi\left(\bigcup_{j=1}^i B_j^-\right) < 1\right\}$$
$$\leqq \liminf_n \mathbb{P}\left\{\bigvee_{i=1}^m y_i \xi_n\left(\bigcup_{j=1}^i B_j^-\right) < 1\right\} = \liminf_n \mathbb{P}\bigcap_i \{\xi_n(B_i^-) < x_i\}.$$

This proves (4.2), and $\xi_n \xrightarrow{d} \xi$ follows from the first assertion of the theorem. □

5. Null-Arrays of Maxitive Random Capacities

We shall now discuss convergence in distribution of maxima $\bigvee_j \xi_{nj}$ of independent maxitive random capacities on $\mathfrak{S}$. The ξ_{nj}'s are supposed to form a *null-array* in the sense that

$$\sup_j \mathbb{P}\{\xi_{nj}(K) \geqq x\} \to 0, \qquad K \in \mathscr{K},\ x > 0. \tag{5.1}$$

In the $\{0, 1\}$-valued case this condition reduces to

$$\sup_j \mathbb{P}\{\varphi_{nj} \cap K \neq \emptyset\} \to 0, \qquad K \in \mathscr{K}, \tag{5.2}$$

where $\varphi_{nj} = \{s \in \mathfrak{S},\ \xi_{nj}(s) = 1\}$ (cf. [12] and [13]). Of course we have a particular interest in the class of possible limit laws.

Let us say that (the distribution of) a maxitive random capacity ξ is *infinitely divisible* if, whenever $n \in N$, we have $\xi \overset{d}{=} \bigvee_{i=1}^n \xi_i$ for some independent and identically distributed maxitive random capacities $\xi_1, \ldots, \xi_n$. In the $\{0, 1\}$-valued case this reduces to the notion of infinite divisibility for random sets (cf. [12] and [13]). We shall write ∞ for the capacity which is infinite at all non-empty sets.

Theorem 5.1. *The formulae*

$$h(K) = \sup\{x, \xi(K) \geqq x \text{ a.s.}\}, \qquad K \in \mathscr{K}, \tag{5.3}$$

$$\mu \bigcup_{i=1}^n \{c \geqq h, c(K_i) \geqq x_i\} = -\log \mathbb{P} \bigcap_{i=1}^n \{\xi(K_i) < x_i\},$$
$$n \in N,\ K_i \in \mathscr{K},\ x_i > h(K_i),\ 1 \leqq i \leqq n \tag{5.4}$$

define a bijection between the set of all infinitely divisible distributions $\mathbb{P}\xi^{-1}$, not carried by $\{\infty\}$, and the set of all pairs (h, μ), where $h \in \mathfrak{U}_m \setminus \{\infty\}$ and μ is a *locally finite measure on* $\mathfrak{U}_m^h=\{c\in\mathfrak{U}_m,\ c\geqq h,\ c\neq h\}$. *Also, each infinitely divisible* ξ *on S has a representation*

$$\xi \stackrel{d}{=} h \vee \bigvee_i \eta_i, \tag{5.5}$$

where $\eta_1, \eta_2, \ldots$ *are the atoms of a Poisson process on* $\mathfrak{U}_m^h$ *with intensity* μ. *Let* ξ *be a maxitive random capacity on* $\mathfrak{S}$. *Then* ξ *is infinitely divisible iff, whenever* $n\in N$ *and* $x_1, \ldots, x_n > 0$, *the capacity c on* $\{1, \ldots, n\} \times \mathfrak{S}$, *defined by*

$$c\left(\bigcup_{i=1}^{n} \{i\} \times K_i\right) = -\log \mathbb{P} \bigcap_{i=1}^{n} \{\xi(K_i) < x_i\}, \qquad K_i \in \mathscr{K},\ 1 \leqq i \leqq n, \tag{5.6}$$

is alternating.

Proof. Let ξ be a maxitive random capacity on $\mathfrak{S}$. It is easily seen that h and the c's are capacities. Suppose ξ is infinitely divisible. Then $\bigcup_{i=1}^{n} \{i\} \times \{\xi \geqq x_i\}$ is an infinitely divisible random set in $\{1, \ldots, n\} \times \mathfrak{S}$ for each $n\in N$ and $x_1, \ldots, x_n > 0$. Thus, by [13, Th. 2.4], the c's in (5.6) are alternating.

In particular so is $c_x = -\log \mathbb{P}\{\xi(\cdot) < x\}$. Note that $h(K) \geqq x$ iff $c_x(K) = \infty$. Now $h\in\mathfrak{U}_m$ follows by subadditivity. Also note that $h(K) < \infty$ if $\xi(K) < \infty$ with positive probability.

Clearly hypo ξ is an infinitely divisible random set in $\mathfrak{S} \times (0, \infty]$. Put $H = \{(s,x),\ (s,x)\in \text{hypo}\,\xi \text{ a.s.}\}$. By [12, Lemma 3-1-1], $H\in\mathscr{F}(\mathfrak{S}\times(0,\infty])$ and, for all $K\in\mathscr{K}(\mathfrak{S}\times(0,\infty])$, we have $K\cap H=\emptyset$ iff $\mathbb{P}\{\text{hypo}\,\xi \cap K = \emptyset\} > 0$. It is not hard to verify that $H = \text{hypo}\,h$. Suppose now that $h \neq \infty$, or, equivalently, $H \neq \mathfrak{S}\times(0,\infty]$. Note that $K\in\mathscr{K}(\mathfrak{S}\times(0,\infty]\setminus H)$ iff $K\in\mathscr{K}(\mathfrak{S}\times(0,\infty])$ and $K\cap H = \emptyset$. It follows that hypo $\xi\setminus H$ is an infinitely divisible random set in $\mathfrak{S}\times(0,\infty]\setminus H$ without fixed points. By [12, Prop. 3-2-1], there exists a unique locally finite measure ν on $\mathscr{F}(\mathfrak{S}\times(0,\infty]\setminus H)\setminus\{\emptyset\}$ satisfying

$$\nu\{F, F\cap K \neq \emptyset\} = -\log \mathbb{P}\{\text{hypo}\,\xi \cap K = \emptyset\}, \qquad K\in\mathscr{K}(\mathfrak{S}\times(0,\infty]),\ K\cap H=\emptyset.$$

Note that we may regard ν as a measure on $\{F\in\mathscr{F}(\mathfrak{S}\times(0,\infty]),\ F\supseteq H,\ F\neq H\}$.

Fix a set F_0 in the latter space, and suppose that $F_0 \notin \text{hypo}\,\mathscr{F}_+$. Choose points $s\in\mathfrak{S}$, $x, y > 0$ with $h(s) < x < y$ such that $(s,y)\in F_0$ while $(s,x)\notin F_0$. Then choose $K\in\mathscr{K}$, $\varepsilon > 0$ such that $h(s) < x - \varepsilon$ and $(s,x) \subseteq K^0 \times (x-\varepsilon, x+\varepsilon) \subseteq K \times [x-\varepsilon, x+\varepsilon] \subseteq F_0^c$. Clearly

$$F_0 \in \{F\supseteq H, F\neq H, F\cap K\times[x-\varepsilon, x+\varepsilon] = \emptyset, F\cap K^0\times(y-\varepsilon, y+\varepsilon) \neq 0\}.$$

The ν-measure of this open neighborhood of F_0 is easily seen to be zero. Hence ν is concentrated on hypo$\{f\in\mathscr{F}_+,\ f\geqq h,\ f\neq h\}$. Note that this space is homeomorphic to $\mathfrak{U}_m^h$. We conclude that (5.2) defines a unique locally finite measure on $\mathfrak{U}_m^h$.

Thus, the mapping defined by (5.1) and (5.2) is into. Since it is clearly one-to-one, it remains to prove that it is onto. For this, fix $h\in\mathfrak{U}_m\setminus\{\infty\}$, and let μ be a locally finite measure on $\mathfrak{U}_m^h$. Also, let $\eta_1, \eta_2, \ldots$ be the atoms of a Poisson process η on $\mathfrak{U}_m^h$ with intensity μ, and put $\xi = h \vee \bigvee_i \eta_i$. It is obvious that $\xi(\emptyset)$

$=0$, and that ξ is increasing and maxitive. Suppose that $K_n \downarrow K$ and $\xi(K) < x$. Then $\eta\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x\} = 0$. Therefore $\eta\{c \in \mathfrak{U}_m^h,\ c(K_n) \geqq x\} = 0$ for some $n \in N$. Hence $\xi(K_n) \leqq x$ for sufficiently large n. Now $\xi(K_n) \downarrow \xi(K)$ follows, and we conclude that ξ is a maxitive random capacity on S.

Clearly ξ is infinitely divisible. Suppose $x > h(K)$. Then $\xi(K) < x$ iff $\eta\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x\} = 0$. To see this, just note that $\xi(K) \geqq x$ implies that $\eta\left\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x - \frac{1}{n}\right\} \geqq 1$ for all $n \in N$ with $x - \frac{1}{n} > h(K)$. Now (5.4) is obvious.

Since $\xi \geqq h$, we must have $\mathbb{P}\{\xi(K) \geqq x\} = 1$ whenever $x \leqq h(K)$. To see the converse, suppose that $\mathbb{P}\{\xi(K) \geqq x\} = 1$. Then the event $\eta\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x\} = 0$ has probability zero. Hence $\mu\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x\} = \infty$, so the set $\{c \in \mathfrak{U}_m^h,\ c(K) \geqq x\}$ cannot be bounded, which means that we cannot have $x > h(K)$. This showes that (5.3) holds.

Suppose finally that the c's in (5.6) are alternating. Fix $n \in N$ and write $Q_n = \{k \cdot 2^{-n},\ k = 1, \ldots, 2^{2n}\}$. By [13, Th. 2.4], there exists an infinitely divisible random set φ_n in $Q_n \times \mathfrak{S}$ satisfying

$$\mathbb{P}\{\varphi_n \cap \bigcup_{q \in Q_n} \{q\} \times K_q = \emptyset\} = \mathbb{P} \bigcap_{q \in Q_n} \{\xi(K_q) < q\}, \qquad K_q \in \mathscr{K},\ q \in Q_n.$$

There is an infinitaly divisible maxitive random capacity ξ_n on $\mathfrak{S}$ with values in Q_n, satisfying $\xi_n(s) \geqq q$ iff $(q, s) \in \varphi_n$ for all $(q, s) \in Q_n \times \mathfrak{S}$. It is easily seen that $\xi_n \xrightarrow{d} \xi$. We conclude that ξ is infinitely divisible. Now all assertions of the theorem are established. □

By applying this result to the case $\mathfrak{S} = \{1, \ldots, n\}$, we obtain Balkema and Resnick's [1] (cf. [8]) characterization of the max infinitely divisible distribution functions on $\mathbf{R}^n$.

We now characterize convergence in distribution of the maxima $\bigvee_j \xi_{nj}$. The abnormal case when the distribution of these maxima is concentrated at ∞ in the limit, is easy to handle and therefore left to the reader.

Theorem 5.2. *Let ξ_{nj}, $n, j \in N$ and ξ be maxitive random capacities on $\mathfrak{S}$. Suppose that the ξ_{nj}'s form a null-array, and that $\xi \neq \infty$ with positive probability. Then $\bigvee_j \xi_{nj} \xrightarrow{d} \xi$ iff ξ in infinitely divisible and moreover*

$$\begin{aligned} &\sum_j \mathbb{P} \bigcup_{i=1}^m \{\xi_{nj}(K_i) \geqq x_i\} \to \mu \bigcup_{i=1}^m \{c \geqq h, c(K_i) \geqq x_i\}, \\ &\quad m \in N,\ K_i \in \mathscr{K},\ x_i > h(K_i),\ \mu\{c \geqq h, c(K_i) \geqq x_i, c(K_i^0) \leqq x_i\} = 0, \\ &\quad 1 \leqq i \leqq m, \end{aligned} \tag{5.7}$$

$$\sum_j \mathbb{P}\{\xi_{nj}(K) \geqq x\} \to \infty, \qquad K \in \mathscr{K},\ 0 < x \leqq h(K), \tag{5.8}$$

where (h, μ) is defined in Theorem 5.1. Let $\mathscr{A} \subseteq \mathscr{B}$ and $Q \subseteq (0, \infty)$. Suppose that the class formed by taking finite unions of sets in $\mathscr{A}$ is separating, and that Q is *dense. Also, suppose*

$$\sum_j \mathbb{P} \bigcup_{i=1}^m \{\xi_{nj}(A_i) \geqq x_i\} \to -\log \mathbb{P} \bigcap_{i=1}^m \{\xi(A_i) < x_i\}, \tag{5.9}$$

$$m \in N,\ A_i \in \mathscr{A},\ x_i \in Q,\ 1 \leqq i \leqq m.$$

Then $\bigvee_j \xi_{nj} \xrightarrow{d} \xi$.

Proof. Suppose $\xi_n = \bigvee_j \xi_{nj} \xrightarrow{d} \xi$. Fix $m \in N$, and let $\mathbb{P}\{\xi(K_i) < x_i\} = \mathbb{P}\{\xi(K_i^0) \leqq x_i\}$ for $1 \leqq i \leqq m$. By Theorem 4.2,

$$\prod_j \mathbb{P} \bigcap_{i=1}^m \{\xi_{nj}(K_i) < x_i\} \to \mathbb{P} \bigcap_{i=1}^m \{\xi(K_i) < x_i\}.$$

By well-known facts for the logarithm, this implies

$$\sum_j \mathbb{P} \bigcup_{i=1}^m \{\xi_{nj}(K_i) \geqq x_i\} \to -\log \mathbb{P} \bigcap_{i=1}^m \{\xi(K_i) < x_i\}.$$

It follows by the last assertion of Theorem 5.1 that ξ is infinitely divisible. Define h and μ by (5.5) and (5.4), respectively. It is easily seen that, whenever $x > h(K)$, we have $\mathbb{P}\{\xi(K) < x\} = \mathbb{P}\{\xi(K^0) \leqq x\}$ iff $\mu\{c \geqq h,\ c(K) \geqq x,\ c(K^0) \leqq x\} = 0$. Now the necessity and sufficiency of (5.7) and (5.8) follow easily.

To see the sufficiency of (5.9), we just note that it implies

$$\mathbb{P} \bigcap_{i=1}^m \{\xi_n(A_i) < x_i\} \to \mathbb{P} \bigcap_{i=1}^m \{\xi(A_i) < x_i\},$$

$$m \in N,\ A_i \in \mathscr{A},\ x_i \in Q,\ 1 \leqq i \leqq m.$$

Therefore

$$(\xi_n(A_1), \ldots, \xi_n(A_m)) \xrightarrow{d} (\xi(A_1), \ldots, \xi(A_m)), \qquad A_1, \ldots, \mathscr{A}_m \in \mathscr{A}.$$

This assertion obviously extends to finite unions of sets in $\mathscr{A}$, so, by Theorem 4.1, we get $\xi_n \xrightarrow{d} \xi$. □

We say that (the distribution of) a maxitive random capacity ξ on S has *independent peaks* (cf. [15]), if $\xi(K_1), \ldots, \xi(K_n)$ are independent whenever $K_1, \ldots, K_n$ are disjoint. It follows from the next result that the distributions with independent peaks are infinitely divisible.

Proposition 5.3. *Let* $h \in \mathfrak{U}_m$, $h \neq \infty$, *and let* m *be a locally finite measure on* $\mathfrak{S} \times (0.\ \infty] \setminus \operatorname{hypo} h$. *Then there exists a maxitive random capacity* ξ *on* $\mathfrak{S}$ *with independent peaks satisfying*

$$\mathbb{P} \bigcap_{i=1}^n \{\xi(K_i) < x_i\} = \exp\left(-m \bigcup_{i=1}^n K_i \times [x_i, \infty]\right), \tag{5.10}$$

$$n \in N,\ K_i \in \mathscr{K},\ x_i > h(K_i),\ 1 \leqq i \leqq n,$$

$$\mathbb{P}\{\xi(K) < x\} = 0, \qquad K \in \mathscr{K},\ 0 < x \leqq h(K). \tag{5.11}$$

Conversely, let ξ be a maxitive random capacity on $\mathfrak{S}$ with independent peaks. Then $\xi=\infty$ a.s., or (5.10) *and* (5.11) *hold for some pair* (h,m) *as above.*

Proof. Fix $h\in\mathfrak{U}_m$, $h\neq\infty$. Write $H=\operatorname{hypo} h$, and let m be a locally finite measure on $\mathfrak{S}\times(0,\infty]\setminus H$. Then $mK\times[x,\infty]<\infty$ whenever $x>h(K)$. Furthermore, let η be a Poisson process on $\mathfrak{S}\times(0,\infty)\setminus H$ with intensity m, and write $\xi(K)$ for the maximum of $h(K)$ and $\sup\{x,\eta K\times[x,\infty]\geqq 1\}$. It is not hard to verify that ξ is a maxitive random capacity on $\mathfrak{S}$, with independent peaks satisfying (5.10) and (5.11).

Conversely, let ξ be a maxitive random capacity on $\mathfrak{S}$. Suppose ξ has independent peaks and that $\mathbb{P}\{\xi=\infty\}<1$. Note that $c_x=-\log\mathbb{P}\{\xi(\cdot)<x\}\in\mathfrak{U}_a$. To see this, first check that c_x is a capacity. Next fix K_1,K_2 and choose $G\supseteq K_1$ such that

$$\mathbb{P}\{\xi(K_1)<x\}-\varepsilon\leqq\mathbb{P}\{\xi(G)<x\}\leqq\mathbb{P}\{\xi(K_1)<x\},$$

where $\varepsilon>0$ is arbitrary but fixed. Then

$$\begin{aligned}\mathbb{P}\{\xi(K_1)<x\}\mathbb{P}\{\xi(K_2)<x\}&\leqq\mathbb{P}\{\xi(G)<x\}\mathbb{P}\{\xi(K_2\setminus G)<x\}+\varepsilon\\ &\leqq\mathbb{P}\{\xi(G)<x,\xi(K_2\setminus G)<x\}+\varepsilon\leqq\mathbb{P}\{\xi(K_1\cup K_2)<x\}+\varepsilon.\end{aligned}$$

Hence $c_x\in\mathfrak{U}_{1s}$, and our claim now follows from the fact that $c_x(K_1\cup K_2)=c_x(K_1)+c_x(K_2)$ whenever $K_1\cap K_2=\emptyset$.

Define h as in Theorem 5.1. Clearly $h(K)<x$ iff $c_x(K)<\infty$. Now $h\in\mathfrak{U}_m\setminus\{\infty\}$ follows as in the proof of Theorem 5.1.

Let us define a set function $\tilde{m}$ on a semi-ring, which generates the Borel-σ-field on $\mathfrak{S}\times(0,\infty]\setminus H$, where $H=\operatorname{hypo} h$, by

$$\tilde{m}K\setminus L\times[x,y)=c_x(K)-c_y(K)+c_y(K\cap L)-c_x(K\cap L),$$
$$K,L\in\mathscr{K},\quad h(K)<x<y\leqq\infty.$$

Clearly $\tilde{m}\emptyset=0$ and $\tilde{m}$ is additive. Suppose $K\setminus L\times[x,y)\subseteq\bigcup_i K_i\setminus L_i\times[x_i,y_i)$, and fix $\varepsilon>0$. Choose $G\supseteq K\cap L$ and $u<y$ such that $0\leqq c_x(G)-c_x(K\cap L)<\varepsilon$ and $0\leqq c_u(K\setminus G)-c_y(K\setminus G)<\varepsilon$. A simple calculation yields

$$\tilde{m}K\setminus L\times[x,y)\leqq\tilde{m}K\setminus G\times[x,u)+2\varepsilon.$$

For each fixed i, we next choose $u_i>x_i$ and $M_i\in\mathscr{K}$ with $K_i\subseteq M_i^0$ such that $0\leqq c_{u_i}(K_i)-c_{x_i}(K_i)<\varepsilon 2^{-i}$ and $0\leqq c_{u_i}(M_i)-c_{u_i}(K_i)<\varepsilon 2^{-i}$. We get

$$\tilde{m}M_i\setminus L_i\times[u_i,y_i)\leqq\tilde{m}K_i\setminus L_i\times[x_i,y_i)+2\varepsilon 2^{-i}.$$

Note that $K\setminus G\times[x,u]\subseteq\bigcup_i M_i^0\setminus L_i\times(u_i,y_i)$. By compactness, $K\setminus G\times[x,y)\subseteq\bigcup_{i=1}^{n} M_i\setminus L_i\times[u_i,y_i)$. By finite subadditivity,

$$\tilde{m}K\setminus L\times[x,y)\leqq\sum_i\tilde{m}K_i\setminus L_i\times[x_i,y_i)+4\varepsilon.$$

Thus, $\tilde{m}$ is countably subadditive. By [3, Th. 11.3], $\tilde{m}$ extends to a measure on $\mathfrak{S}\times(0,\infty)\setminus H$.

Write m for the sum of this measure and the restriction to $\mathfrak{S}\times\{\infty\}\setminus H$ ("$=$"$S\setminus\{b=\infty\}$) of c_∞. Clearly

$$\mathbb{P}\{\xi(K)<x\}=\begin{cases}\exp(-mK\times[x,\infty]), & x>h(K),\\ 0, & x\leqq h(K).\end{cases}$$

In particular this showes (5.11). Let us define

$$\xi(K\setminus L)=\sup_{s\in K\setminus L}\xi(s), \qquad K,L\in\mathscr{K}.$$

Fix $K,L\in\mathscr{K}$ and suppose $G_n\downarrow L$. It is easily seen that $\xi(K\setminus G_n)\uparrow\xi(K\setminus L)$. Moreover, if $x>h(K)$ then

$$\mathbb{P}\{\xi(K\setminus L)<x\}=\exp(-mK\setminus L\times[x,\infty]).$$

Fix $n\in N$, and let $x_i>h(K_i)$ for $1\leqq i\leqq n$. Write $L_0=\emptyset$ and $L_i=\bigcup_{j=1}^{i}K_j$ for $1\leqq i\leqq n$. Suppose $x_1\leqq\ldots\leqq x_n$. This is no restriction since we may always renumber $K_1,\ldots,K_n$. Now we get

$$\begin{aligned}\mathbb{P}\bigcap_{i=1}^{n}\{\xi(K_i)<x_i\}&=\mathbb{P}\bigcap_i\{\xi(L_i)<x_i\}=\mathbb{P}\bigcap_i\{\xi(L_i\setminus L_{i-1})<x_i\}\\ &=\prod_i\mathbb{P}\{\xi(L_i\setminus L_{i-1})<x_i\}=\exp(-\sum_i mL_i\setminus L_{i-1}\times[x_i,\infty])\\ &=\exp\left(-m\bigcup_{i=1}^{n}K_i\times[x_i,\infty]\right).\end{aligned}$$

This establishes (5.10). The proposition is proved. $\square$

Let ξ be a maxitive random capacity on $(0,\infty)$ with independent peaks. Put

$$X_t=\xi((0,t]), \qquad t>0.$$

Clearly $\{X_t, t>0\}$ is increasing and right continuous. Moreover, whenever $0<t_1<\ldots<t_n$ and $0<x_1\leqq\ldots\leqq x_n$, we have

$$\mathbb{P}\bigcap_{i=1}^{n}\{X_{t_i}\leqq x_i\}=\prod_{i=1}^{n}\frac{F_{t_i}(x_i)}{F_{t_{i-1}}(x_i)},$$

where $t_0=0$ and

$$F_t(x)=\mathbb{P}\{\xi((0,t])\leqq x\}, \qquad t\geqq 0,\ x>0.$$

Hence $\{X_t\}$ is an extremal process as defined in [17]. Conversely, it can be shown that to each right continuous extremal process $\{X_t\}$ there exists a maxitive random capacity ξ on $(0,\infty)$ with independent peaks satisfying

$$(X_{t_1},\ldots,X_{t_n})\stackrel{d}{=}(\xi((0,t_1]),\ldots,\xi((0,t_n])), \qquad n\in N,\ t_1,\ldots,t_n>0.$$

The proof is left to the reader.

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