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Probability Theory and Related Fields

The Survival of the Large Dimensional Basic Contact Process

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Summary. We consider the d-dimensional basic contact process obtaining the limit value of the probability of survival when $d \to +\infty$, and showing that the finite dimensional distributions of the upper invariant measure become of the product form as $d \to +\infty$.

The basic d-dimensional contact process of Harris is a Markov (and Feller) process $(\xi^A_{d,\lambda}(t)\colon t \geqq 0)$ with states in $\{0,1\}^{\mathbb{Z}^d}$. Here $A \in \{0,1\}^{\mathbb{Z}^d}$ denotes the initial state, and when the process is at state $\xi \in \{0,1\}^{\mathbb{Z}^d}$ the "transition" or "flip" rates are given by

$$\begin{array}{cc} \text{at } x \in \mathbb{Z}^d & \text{rate} \\ \hline 1 \to 0 & 1 \\ 0 \to 1 & \lambda \sum\limits_{y \in N(x)} \xi(y) \end{array} \tag{1}$$

where $N(x) = \{y \in \mathbb{Z}^d\colon |y-x| = 1\}$. $\left(|x| = \sum\limits_{i=1}^{d} |x_i|, \text{ for } x = (x_1, \ldots, x_d) \in \mathbb{Z}^d.\right)$

As usual we identify $\{0,1\}^{\mathbb{Z}^d}$ with the set of all subsets of $\mathbb{Z}^d$, so that each $A \in \{0,1\}^{\mathbb{Z}^d}$ is identified with $\{x \in \mathbb{Z}^d\colon A(x) = 1\}$, and this set will also be denoted by A. We shall be using $\xi^A_{d,\lambda}(t,y)$ to denote $\xi^A_{d,\lambda}(t)(y)$. The sites x so that $\xi^A_{d,\lambda}(t,x) = 1$ $(=0)$ are then said to be "occupied" ("empty", resp.). For contact processes the usual expression is "infected" if $\xi^A_{d,\lambda}(t,x) = 1$ and "non-infected" otherwise, due to the origin of these models, and we use them.

It is well known that there exists a finite and positive critical value λ_d, defined by

$$\lambda_d = \sup\{\lambda > 0\colon P(\xi^{\{0\}}_{d,\lambda}(t) \neq \phi, \text{ for all } t \geqq 0) = 0\}. \tag{2}$$

Let $\nu_{d,\lambda}$ be the upper invariant measure, defined as the weak limit of $\xi^{\mathbb{Z}^d}_{d,\lambda}(t)$ as $t \to +\infty$.

We make extensive use of the selfduality of the process, given by:

$$P(\xi^B_{d,\lambda}(t) \cap A \neq \phi) = P(\xi^A_{d,\lambda}(t) \cap B \neq \phi) \tag{3}$$

if $A, B \subseteq \mathbb{Z}^d$ with at least one of them being finite; this implies that

$$\nu_{d,\lambda}\{B: B \cap A \neq \phi\} = P(\xi^A_{d,\lambda}(t) \neq \phi, \text{ for all } t) \tag{4}$$

for $A \subseteq \mathbb{Z}^d$, A finite. In particular

$$\nu_{d,\lambda}\{B: 0 \in B\} = P(\xi^{\{0\}}_{d,\lambda}(t) \neq \phi, \text{ for all } t) \stackrel{\text{def}}{=} \rho(d, \lambda). \tag{5}$$

Thus

$$\begin{aligned} \lambda_d &= \sup\{\lambda \geqq 0: \rho(d, \lambda) = 0\} \\ &= \sup\{\lambda \geqq 0: \nu_{d,\lambda} = \delta_\phi\}. \end{aligned} \tag{6}$$

In this work we are concerned with the survival of the process (propagation of "infection") when $d \to +\infty$. First let us recall that

$$d\lambda_d \to 1/2 \quad \text{as} \quad d \to +\infty. \tag{7}$$

This result has been proven (with different methods) in [4] and [7]. As pointed out in [8, p. 309] the result is quite intuitive: the reason being that as d grows to infinity the contact process starting with $\{0\}$ should behave like a continuous time branching process in which each individual after a random exponential time with mean $(1+2\lambda d)^{-1}$ either dies, which happens with probability $(1+2\lambda d)^{-1}$, or survives and gives birth to another individual with probability $2\lambda d(1+2\lambda d)^{-1}$. But, this branching process has a positive probability of survival if and only if $\lambda d > 1/2$, and this probability is $1-(2\lambda d)^{-n}$ if the initial number of individuals was n ([1], p. 108, 109).

Here we obtain limiting values for the probability of survival for this contact process, when $d \to +\infty$. The expression for the probability of survival for the "corresponding" branching process immediately suggests to take λ of the form $\lambda = \gamma d^{-1}$ with $\gamma > 0$ fixed. Recall that in this case the total rate with which an infected individual may infect its neighbors is $2\lambda d = 2\gamma$.

Our result may be stated as follows: (For $A \subseteq \mathbb{Z}^d$, $|A|$ denotes the cardinality of A).

Theorem. *If* $\gamma > 1/2$

$$\begin{aligned} &\lim_{d \to +\infty} \sup_{\substack{A \subseteq \mathbb{Z}^d \\ |A| \text{ finite}}} |\nu_{d,\gamma d^{-1}}(B: B \cap A = \phi) - (1/2\gamma)^{|A|}| \\ &= \lim_{d \to \infty} \sup_{\substack{A \subseteq \mathbb{Z}^d \\ |A| \text{ finite}}} \left| P(\xi^A_{d,\gamma d^{-1}}(t) \neq \phi, \textit{for all } t) - \left(1 - \left(\frac{1}{2\gamma}\right)^{|A|}\right)\right| = 0. \end{aligned} \tag{8}$$

As a consequence, one has the following:

Corollary. *For any* $\gamma > 0$

$$\lim_{d \to +\infty} \rho(d, \gamma d^{-1}) = (1 - 1/2\gamma) \vee 0. \tag{9}$$

In fact, if $\gamma > 1/2$ this follows from the Theorem. For $\gamma \leqq 1/2$ it suffices the direct comparison with the branching process.

Therefore, besides the limiting behavior of $\rho(d, \gamma d^{-1})$ as $d \to +\infty$, the Theorem tells us that the finite dimensional distributions of $\nu_{d, \gamma d^{-1}}$ become of product form when d tends to $+\infty$, if $\gamma > 1/2$.

For the proof of (8) we first show that for each $n \in \mathbb{N}$

$$\lim_{d \to +\infty} \sup_{\substack{A \subseteq \mathbb{Z}^d \\ |A| \leqq n}} \left| P(\xi^A_{d, \gamma d^{-1}}(t) \neq \phi, \text{ for all } t) - \left(1 - \left(\frac{1}{2\gamma}\right)^{|A|}\right) \right| = 0. \tag{10}$$

It will then be very easy to obtain (8).

In the proof we shall be using several auxiliary processes. It may be good to give the general idea before we get into the proof itself. When d is large, and we look at $\xi^A_{d,\lambda}(\cdot)$ with $|A| \leqq n$ then there is a "first period" during which the process evolves, with large probability, just as the branching process mentioned before. This is clear since, if the number of "infected" individuals is small compared to d, it is very improbable to happen any attempt to "infect" someone already "infected". Then one has two possibilities: either the "infection" has been "extinguished" before the end of this period or not. We must show that the probability of survival given that the process survives this first period goes to one. In order to get appropriate lower bounds for such probabilities we will show that when $\gamma > 1/2$ it is possible to take $l \in \mathbb{N}$ so that for d sufficiently large the process can be compared with a supercritical l-dimensional contact process. To show this, and for the comparison with the branching process we need to define some auxiliary processes.

Definitions. (a) Let $\Delta_d = \{A \in \mathbb{N}^{\mathbb{Z}^d} : |\{x \in \mathbb{Z}^d : A(x) \neq 0\}| < +\infty\}$; if $A \in \Delta_d$, $|A| = \sum_x A(x) < +\infty$.

(b) $(\bar{\xi}^A_{d,\lambda}(t) : t \geqq 0)$ is the Markovian (Feller) process with states in Δ_d such that $A \in \Delta_d$ is the initial state and the transition rates, when at state $\bar{\xi} \in \Delta_d$, are given by:

at $x \in \mathbb{Z}^d$	with rate
$a \to a-1$	a
$a \to a+1$	$\lambda \sum_{y \in N(x)} \bar{\xi}(y)$

(c) If $\bar{\xi}$ is defined by (b) we let

$$X^A_{d,\lambda}(t) = \sum_x \bar{\xi}^A_{d,\lambda}(t, x).$$

Thus $X^A_{d,\lambda}(\cdot)$ is the branching process previously defined, starting with $|A|$. (In particular, its law depends on A, d and λ only through $|A|$ and λd.

(d) Given $l \in \{1, \ldots, d\}$ we let f = integer part of dl^{-1} and define $\pi : \mathbb{Z}^d \to \mathbb{Z}^l$ as the map which takes $x = (x_1, \ldots, x_d)$ into $\pi(x) = (\pi_1(x), \ldots, \pi_l(x))$, where

$$\pi_i(x) = \sum_{j=(i-1)f+1}^{if} x_j, \quad i = 1, \ldots, l.$$

Now, we let for $A \in \{0,1\}^{\mathbb{Z}^d} \cap \Delta_d$ (i.e. A finite subset of $\mathbb{Z}^d$):

$$\eta^A_{d,\lambda}(t,x) = \sum_{y:\,\pi(y)=x} \xi^A_{d,\lambda}(t,y),$$

and

$$\zeta^A_{d,\lambda}(t,x) = \eta^A_{d,\lambda}(t,x) \wedge 1$$

for $t \geqq 0$, $x \in \mathbb{Z}^l$.

Also, for $A \in \Delta_d$:

$$\bar{\eta}^A_{d,\lambda}(t,x) = \sum_{y:\,\pi(y)=x} \bar{\xi}^A_{d,\lambda}(t,y)$$

and

$$\bar{\zeta}^A_{d,\lambda}(t,x) = \bar{\eta}^A_{d,\lambda}(t,x) \wedge 1$$

for $t \geqq 0$, $x \in \mathbb{Z}^l$.

Let us also define the extinction times:

$$\tau^A_{d,\lambda} = \inf\{t \geqq 0 : \xi^A_{d,\lambda}(t) = \phi\},$$
$$\bar{\tau}^A_{d,\lambda} = \inf\{t \geqq 0 : \bar{\xi}^A_{d,\lambda}(t) = \phi\}.$$

Remarks

R1. $(\bar{\eta}^A_{d,\lambda}(t) : t \geqq 0)$ is Markovian, with rates

at $x \in \mathbb{Z}^l$	with rate
$a \to a-1$	a
$a \to a+1$	$\lambda f \sum_{y \in N(x)} \bar{\eta}(y) + 2\lambda a(d-fl)$

when $\bar{\eta} \in \Delta_l$ is the state of the process, and here $N(x) = \{y \in \mathbb{Z}^l : |y-x| = 1\}$ i.e. we omit the dimension in our notation $N(x)$.

R2. The process $(\zeta^A_{d,\lambda}(t) : t \geqq 0)$ is not Markovian, but it dominates the l-dimensional contact process with rate $\lambda' = \lambda f$. The initial state of $\zeta^A_{d,\lambda}(\cdot)$ is $C = \{x \in \mathbb{Z}^l : A \cap \pi^{-1}(x) \neq \phi\}$ and the meaning of "domination" is that we can construct a coupling of $\zeta^A_{d,\lambda}(\cdot)$ and an l-dimensional contact process with rate λ', starting at C, call it $\xi^C_{l,\lambda'}$ so that $\xi^C_{l,\lambda'}(t) \subseteq \zeta^A_{d,\lambda}(t)$ for all t. Indeed, if $\zeta^A_{d,\lambda}(t,x) = 1$ then a transition $1 \to 0$ has rate at most one and if $\zeta^A_{d,\lambda}(t,x) = 0$ the possible rates of transition $0 \to 1$ (which do not depend only on $\zeta^A_{d,\lambda}(t)$) are always at least $\lambda' \sum_{y \in N(x)} \zeta_{d,\lambda}(t,y)$. This is the generalization of the case $l=1$ introduced in [6] (see also [2, 8]).

R3. Given $\gamma > 1/2$, let us take $l \in \mathbb{N}$ so that $l\lambda_l < \gamma$, which is possible by (7). Now $\lambda' = \lambda f \geqq \lambda\left(\frac{d}{l} - 1\right) = \frac{\gamma}{l}\frac{d-l}{d} > \lambda_l$ provided $d \geqq d_0 = d_0(l,\gamma)$. In this case $\zeta^A_{d,\gamma d^{-1}}$ dominates a supercritical l-dimensional contact process.

On the other side, from Lemma 9.14 of [5] it follows that if $\lambda' > \lambda_l$

$$\sup_{\substack{B \subseteq \mathbb{Z}^l \\ |B| = m}} P(\tau^B_{l,\lambda'} < \infty) \xrightarrow[m \to \infty]{} 0.$$

Thus if $\mathscr{C}_m = \{B \subseteq \mathbb{Z}^d: |\{x \in \mathbb{Z}^l: \pi^{-1}(x) \cap B \neq \phi\}| = m\}$, one has:

$$\sup_{d > d_0} \sup_{\substack{A \subseteq \mathbb{Z}^d \\ A \in \mathscr{C}_m}} P(\tau^A_{d, \gamma d^{-1}} < \infty) \to 0 \quad \text{as} \quad m \to \infty \tag{11}$$

for $\gamma > 1/2$, l and $d_0 = d_0(l, \gamma)$ chosen as above.

R4. It will be convenient for our proof to construct coupled versions of $\xi^A_{d, \lambda}(\cdot)$ and $\bar{\xi}^A_{d, \lambda}(\cdot)$, when $A \in \{0, 1\}^{\mathbb{Z}^d} \cap \Delta_d$. For this we construct a branching process with the same law as $X^A_{d, \lambda}(\cdot)$ but distinguishing the individuals. - Now, one constructs $\bar{\xi}^A_{d, \lambda}(\cdot)$ by locating these individuals on $\mathbb{Z}^d$, and $\bar{\xi}^A_{d, \lambda}(t, x)$ will be the number of individuals at $x \in \mathbb{Z}^d$, at time t. The allocation is done according to the rules:

(a) at $t = 0$ the individuals are located in A (one in each site of A);

(b) when an individual in the branching process creates a new one, we look at the position x of the corresponding individual in $\mathbb{Z}^d$, and the new one will be randomly allocated among the $2d$ nearest neighbors of x.

(c) when an individual dies in the branching process the corresponding one in $\mathbb{Z}^d$ also dies.

To construct $\xi_{d, \lambda}(\cdot)$ we may follow rules (a) and (c) modifying (b) so that one site cannot be ocupied by more than one individual. Thus the modification is the following: if the chosen site is already ocupied nothing happens i.e. we disregard completely this "new" individual.

Consequently, we will have:

$$\xi^A_{d, \lambda}(t) \leqq \bar{\xi}^A_{d, \lambda}(t) \qquad \text{for all} \quad t \geqq 0 \tag{12}$$

for any initial A finite subset of $\mathbb{Z}^d$.

Proof of Theorem 1. We first prove (10). For this let us consider the coupling defined in (R4) and let $l, d_0 = d_0(\gamma, l)$ be as in (R3). Let us also define:

$$\theta^A_{d, \lambda}(m) = \inf\{t \geqq 0: |\zeta^A_{d, \lambda}(t)| = m\},$$
$$\bar{\theta}^A_{d, \lambda}(m) = \inf\{t \geqq 0: |\bar{\zeta}^A_{d, \lambda}(t)| = m\}$$

for $m \in \mathbb{N}$.

Due to (12) $\tau^A_{d, \gamma d^{-1}} \leqq \bar{\tau}^A_{d, \gamma d^{-1}}$ for any A finite subset of $\mathbb{Z}^d$. Thus

$$P(\tau^A_{d, \gamma d^{-1}} = \infty) = P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty) - P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty, \tau^A_{d, \gamma d^{-1}} < \infty).$$

Since $P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty) = 1 - \left(\frac{1}{2\gamma}\right)^{|A|}$, it remains to prove that

$$\sup_{|A| \leqq n} P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty, \tau^A_{d, \gamma d^{-1}} < \infty) \to 0. \tag{13}$$

But, for any $s < +\infty$, $m \in \mathbb{N}$:

$$\begin{aligned} P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty, \tau^A_{d, \gamma d^{-1}} < \infty) \leqq\; & P(\bar{\theta}^A_{d, \gamma d^{-1}}(m) \leqq s, \theta^A_{d, \gamma d^{-1}}(m) > s) \\ & + P(\bar{\tau}^A_{d, \gamma d^{-1}} = \infty, \bar{\theta}^A_{d, \gamma d^{-1}}(m) > s) \\ & + P(\tau^A_{d, \gamma d^{-1}} < \infty, \theta^A_{d, \gamma d^{-1}}(m) \leqq s). \end{aligned} \tag{14}$$

Using the Strong Markov property of $\xi^A_{d,\gamma d^{-1}}(\cdot)$ at $\theta^A_{d,\gamma d^{-1}}(m)$:

$$\begin{aligned} &P(\tau^A_{d,\gamma d^{-1}}<\infty, \theta^A_{d,\gamma d^{-1}}(m)\leqq s)\\ &\quad=\int_{(0,s]\times\mathscr{C}_m} P(\theta^A_{d,\gamma d^{-1}}(m)\in du, \xi^A_{d,\gamma d^{-1}}(u)=B)\, P(\tau^B_{d,\gamma d^{-1}}<\infty)\\ &\quad\leqq \sup_{B\in\mathscr{C}_m} P(\tau^B_{d,\gamma d^{-1}}<\infty)\end{aligned}$$

and by (11):

$$\sup_{A\subset\mathbb{Z}^d}\sup_{s\geqq 0}\sup_{d>d_0} P(\tau^A_{d,\gamma d^{-1}}<\infty, \theta^A_{d,\gamma d^{-1}}(m)\leqq s)\to 0 \quad \text{as} \quad m\to\infty. \tag{15}$$

The second term on the r.h.s. of (14) can be treated by elementary arguments of Markov chain theory; let

$$C_i=[\bar\tau^A_{d,\gamma d^{-1}}>i, \bar\theta^A_{d,\gamma d^{-1}}(m)>i] \quad \text{for} \quad i=1,2,\ldots.$$

The process $\bar\zeta^A_{d,\gamma d^{-1}}(\cdot)$ also dominates an l-dimensional contact process with infection rate $\lambda'=\lambda[f]$. Thus for $d\geqq l$, if

$$\alpha=P(|\xi^{\{0\}}_{l,\lambda'}(1)|\leqq m)<1$$

one has $P(C_1)\leqq\alpha$ and $P(C_k\,|\,C_1\cap\ldots\cap C_{k-1})\leqq\alpha$ for any $k\geqq 2$. Thus for each m fixed

$$\sup_{A\subset\mathbb{Z}^d}\sup_{d\geqq l} P(\bar\tau^A_{d,\gamma d^{-1}}=\infty, \bar\theta^A_{d,\gamma d^{-1}}(m)>s)\leqq \sup_{d\geqq l} P(C_{[s]})\leqq\alpha^{[s]}\xrightarrow[s\to\infty]{} 0. \tag{16}$$

From (15) and (16), it remains to prove that for fixed m and s the first term on the r.h.s. of (14) tends to zero as $d\to+\infty$.

But

$$P(\bar\theta^A_{d,\gamma d^{-1}}(m)\leqq s, \theta^A_{d,\gamma d^{-1}}(m)>s)\leqq P(\bar\xi^A_{d,\gamma d^{-1}}(t)\neq\xi^A_{d,\gamma d^{-1}}(t) \text{ for some } t\leqq s),$$

and if N_s = number of births in the branching process $X^A_{d,\gamma d^{-1}}(\cdot)$ up to time s then: (recall at $A\subseteq\mathbb{Z}^d$ and $|A|\leqq n$)

$$\begin{aligned}&P(\bar\xi^A_{d,\gamma d^{-1}}(t)\neq\xi^A_{d,\gamma d^{-1}}(t) \text{ for some } t\leqq s)\leqq P(N_s+n>d^{1/3})\\ &\qquad+P(\bar\xi^A_{d,\gamma d^{-1}}(t)\neq\xi^A_{d,\gamma d^{-1}}(t) \text{ for some } t\leqq s, N_s+n\leqq d^{1/3}).\end{aligned}$$

As $d\to+\infty$ $P(N_s+n>d^{1/3})\to 0$ trivially, and on the event $[N_s+n\leqq d^{1/3}]$, and during $[0,s]$, each site has at most $d^{1/3}$ occupied neighboring sites; thus each newborn individual in $\bar\xi^A_{d,\gamma d^{-1}}(\cdot)$ has probability at most $d^{1/3}/2d$ to be put on an occupied site, and it comes:

$$P(\bar\xi^A_{d,\gamma d^{-1}}(t)\neq\xi^A_{d,\gamma d^{-1}}(t) \text{ for some } t\leqq s, N_s+n\leqq d^{1/3})\leqq d^{1/3}\cdot\frac{d^{1/3}}{2d}$$

which goes to zero as $d\to\infty$. This proves (10).

It is now easy to complete the proof; given $\varepsilon>0$ let us take n so that $(1/2\gamma)^n<\varepsilon$ and then $d_1=d_1(\varepsilon,\gamma)$ so that

$$\sup_{\substack{|A|\leqq n\\ A\subseteq\mathbb{Z}^d}}\left|P(\tau^A_{d,\gamma d^{-1}}=\infty)-\left(1-\left(\frac{1}{2\gamma}\right)^{|A|}\right)\right|\leqq\varepsilon$$

for all $d \geqq d_1$. Now, if $d \geqq d_1$ and if $B \subseteq \mathbb{Z}^d$ with $|B| = m > n$, then by the additivity of the contact process:

$$P(\tau^B_{d,\,\gamma d^{-1}} = \infty) \geqq \inf_{\substack{A \subseteq \mathbb{Z}^d \\ |A| = n}} P(\tau^A_{d,\,\gamma d^{-1}} = \infty) \geqq 1 - 2\varepsilon$$

and since $(1/2\gamma)^m \leqq \varepsilon$ we have

$$\sup_{\substack{B \subseteq \mathbb{Z}^d \\ B \text{ finite}}} |P(\tau^B_{d,\,\gamma d^{-1}} = \infty) - (1 - (1/2\gamma)^{|B|})| \leqq 2\varepsilon$$

for all $d \geqq d_1$, concluding the proof. □

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