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Probability Theory and Related Fields

Super-Brownian Motion: Path Properties and Hitting Probabilities*

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Summary. Sample path properties of super-Brownian motion including a one-sided modulus of continuity and exact Hausdorff measure function of the range and closed support are obtained. Analytic estimates for the probability of hitting balls lead to upper bounds on the Hausdorff measure of the set of k-multiple points and a sufficient condition for a set to be "polar".

1. Introduction and Statement of Results

Let $M_F = M_F(\mathbb{R}^d)$ denote the space of finite measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ furnished with the weak topology and $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space. The *super-Brownian motion* (critical multiplicative measure-valued diffusion process) X_t starting at $m \in M_F(\mathbb{R}^d)$ is a continuous $M_F(\mathbb{R}^d)$-valued adapted strong Markov process defined on $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with $X_0 = m$ a.s. which is the solution of an appropriate martingale problem (cf. Perkins (1988a; Theorem 1.1)). For $m \in M_F(\mathbb{R}^d)$ this martingale problem uniquely characterizes the law, Q^m, of X on $C([0, \infty), M_F)$. The process X can also be obtained as the limit of a system of branching Brownian motions (cf. Perkins (1988a; Theorem 2.8)).

If $\nu \in M_F$ let $S(\nu)$ denote the closed support of ν. In Perkins (1988a) it is shown that in dimensions $d \geq 2$ w.p.1. for all $t > 0$, X_t is uniformly distributed (up to constants) according to a deterministic measure function over a random Borel set. This result will be extended to the canonical closed supports $S(X_t)$ in Perkins (1989). In some sense this reduces the study of X to the study of the set-valued support process $S(X_t)$. The basic goal of this work is the derivation of path properties of $S(X_t)$ analogous to well-known path properties of standard Brownian motion. These properties include a modulus of continuity, a study of "polar sets" and "multiple points", and the derivation of an exact Hausdorff measure function for the "range" of this support process.

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Two basic methodologies are used in the study of X.

The first, which is described in more detail in Sect. 2, is based on a detailed analysis of the system of branching Brownian motions. In this approach it is also helpful to use a nonstandard version of the super-Brownian motion involving a hyperfinite branching Brownian motion. Readers unfamiliar with nonstandard analysis may wish to consult Cutland (1983) where a self-contained introduction especially suited for probabilists is given. More comprehensive treatments may be found in Hurd and Loeb (1985) or Albeverio et al. (1986). However we feel that our results will still be accessible to the reader who is unfamiliar with the elementary nonstandard techniques used here.

If $0 \leqq s \leqq t < \infty$, the *weighted occupation time process* or "man-hours process" is

$$Y_{s,t}(A) := \int_s^t X_u(A)\, du, \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

We write $Y_t(A)$ for $Y_{0,t}(A)$. If m is a measure on $\mathbb{R}^d$ and ψ is a real-valued function on $\mathbb{R}^d$, we write $\langle \phi, m\rangle$ for $\int \phi\, dm$ providing the latter exists.

The second basic methodology is based on an analytic approach in which the Laplace transforms of $\langle \phi, X(t)\rangle$ and $\langle \psi, Y_{0,t}\rangle$ with $\phi, \psi \in C_0(\mathbb{R}^d)$ (or $C_l(\mathbb{R}^d)$), the space of continuous non-negative functions vanishing at ∞ (respectively with limits at ∞), are computed explicitly from

$$E_Q^m(\exp\{-[\langle \phi, X(t)\rangle + \langle \psi, Y_{0,t}\rangle]\}) = e^{-\int u(t,x)\, m(dx)} \tag{1.1 a}$$

where u is the unique solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}(\Delta u - u^2) + \psi \tag{1.1 b}$$

$$u(0,x) = \phi(x)$$

(cf. Watanabe (1968), Dawson (1978), Iscoe (1986a; Theorem 3.1)). E_Q^m or E^m (if there is no ambiguity) denotes expectation with respect to Q^m. In Sect. 3 this approach is used to obtain estimates for the probabilities $Q^m(X_t(B(x;\varepsilon)) > 0)$ for fixed $t > 0$, $Q^m(X_t(B(x;\varepsilon)) > 0$ for some $t > \delta)$, $Q^m(X_s(B(0;\varepsilon)) > 0$ for some $s \leqq t)$ and $Q^{\delta_0}(X_s(B(0;R)^c) > 0$ for some $s \leqq t)$, which give exact asymptotics for these quantities as $\varepsilon \downarrow 0$ (see Theorems 3.1–3.3). Here $B(x;\varepsilon)$ denotes the open ball in $\mathbb{R}^d$ centered at x with radius ε.

Our first path property of the support process is a one-sided modulus of continuity for $S(X_t)$. If $A \subset \mathbb{R}^d$ and $r > 0$, let $A^r := \{x : d(x, A) \leqq r\}$. Let

$$h(t) = (t(\log t^{-1}) \vee 1))^{1/2}. \tag{1.2}$$

Theorem 1.1. *If $m \in M_F(\mathbb{R}^d)$, then for Q^m-a.a. ω and each $c > 2$ there is a $\delta(\omega, c) > 0$ such that if $s, t \geqq 0$ satisfy $0 < t - s < \delta(\omega, c)$ then*

$$S(X_t) \subset S(X_s)^{ch(t-s)}. \tag{1.3}$$

The proof given in Sect. 4 uses the approximating system of branching Brownian motions.

Clearly (1.3) may fail if $t<s$, for then one could choose t, s such that $S(X_s)=\phi$ but $S(X_t)\neq\phi$. We can show that $c=2$ is critical in the Lévy modulus for the approximating system of branching Brownian motions, used in the derivation of Theorem 1.1 (cf. Theorems 4.5, 4.6). This suggests that Theorem 1.1 is a.s. false if $c<2$.

If K_1 and K_2 are non-empty compact subsets of $\mathbb{R}^d$ let

$$\rho_1(K_1, K_2):=\min[\sup_{x\in K_1} d(x, K_2), 1] \quad \text{and}$$

$$\rho(K_1, K_2):=\max(\rho_1(K_1, K_2), \rho_1(K_2, K_1))$$

$$\rho(K_1, \phi)=1.$$

ρ is the Hausdorff metric on $\mathbb{K}(\mathbb{R}^d)$, the set of compact subsets of $\mathbb{R}^d$ (cf. Dugundji (1966, p. 205) and Cutler (1984)). The following consequence of Theorem 1.1 is proved in Sect. 4.

Theorem 1.2. *$\{S(X_t)$: $t>0\}$ is a right continuous process taking values in $(\mathbb{K}(\mathbb{R}^d), \rho)$.*

The above results show that $S(X_t)$ propagates with finite speed. This will allow us to effectively control and estimate $S(X_t)$ as opposed to the non-canonical Borel supports studied in Dawson and Hochberg (1979), Zähle (1984) and Perkins (1988 a). For example it is easy to show that for $d\geqq 2$,

$$\dim S(X_t)=2 \qquad \text{for all } t\geqq 0 \quad \text{a.s.}$$

(the lower bound is immediate from Perkins (1988 a)). In Theorem 7.1, an exact Hausdorff measure function ($\phi(x)=x^2\log\log(1/x)$) is given for $S(X_t)$ for t fixed and $d\geqq 3$. More specifically $X_t(A)$ is bounded above and below by constant multiples of $\phi-m(A\cap S(X_t))$ for all A a.s. Theorem 1.1 also plays a fundamental role in the extension of the latter result to all $t>0$ a.s. in Perkins (1988 c).

Let

$$\bar{R}(s, t):=S(Y_{s,t}) \quad \text{if } 0<s\leqq t\leqq\infty,$$

$$\bar{R}_+(0, t)=\bigcup_{s>0} S(Y_{s,t}) \quad \text{if } 0<t\leqq\infty.$$

It will be convenient to write $\bar{R}(I)$ for $\bar{R}(s, t)$ when I is an interval with end points s and t. We call $\bar{R}=\bar{R}_+(0, \infty)$ the *range* of X. It is easy to use the continuity of X to show

$$\bar{R}(s, t)=cl\Big(\bigcup_{s\leqq u\leqq t} S(X_u)\Big). \tag{1.4}$$

If $A\subset\mathbb{R}^d$, X *hits* A if and only if $A\cap\bar{R}\neq\phi$. The estimates on hitting balls from Sect. 3 will help decide which sets are hit with positive probability. For example in Sect. 3 we prove the following:

Theorem 1.3. *If $d \leqq 3$, then*

$$Q^m(X \text{ hits } \{x\}) = 1 - \exp\{-2(4-d) \int |y-x|^{-2} dm(y)\}.$$

Remarks. Let $\tau_{\{x\}} := \inf\{t > 0: x \in \bar{R}_+(0, t)\}$. As a consequence of Theorem 1.3 it is easy to show that $Q^m(\tau_{\{x\}} = 0) = 0$ or 1 according as $\int |x-y|^{-2} dm(y)$ is finite or infinite. In the case $d \leqq 3$ Iscoe (1988, Theorem 4) has established the existence of a local time process. In Sugitani (1987) this has been strengthened to show that $Y(t, dx) = Y(t, x)\, dx$ and $Y(t, x)$ has a jointly continuous version. Furthermore, Dynkin (1988) has established the existence of local times for classical superdiffusions if $d \leqq 3$. The existence of this continuous density implies that $\bar{R}$ has positive Lebesgue measure for $d \leqq 3$.

If $d \geqq 4$ then Theorem 1.5 below shows that X does not hit points and $\bar{R}$ is Lebesgue null. To estimate the size of $\bar{R}$ we introduce Hausdorff measure. Let $\mathscr{H}_0 = \{\phi \in C([0, \varepsilon]): \varepsilon > 0, \phi \text{ increasing}, \phi(0) = 0\}$. If $\phi \in \mathscr{H}_0$ and $A \subset \mathbb{R}^d$, the Hausdorff ϕ-measure of A is

$$\phi - m(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i=1}^{\infty} \phi(dJ_i): J_i \text{ is a ball of diameter } dJ_i \leqq \delta, A \subset \sum_{i=1}^{\infty} J_i \right\}.$$

This definition also makes sense if ϕ belongs to

$$\mathscr{H}_f = \{\phi \in C([0, \varepsilon]): \varepsilon > 0, \phi(0) \in (0, \infty)\}$$

or

$$\mathscr{H}_\infty = \{\phi \in C([0, \varepsilon]): \varepsilon > 0, \phi(0+) = \infty\}.$$

Let $\mathscr{H} = \mathscr{H}_0 \cup \mathscr{H}_f \cup \mathscr{H}_\infty$. If $\phi \in \mathscr{H}_f \cup \mathscr{H}_\infty$, then $\phi - m(A) = \phi(0+) \operatorname{card}(A)$ where $\infty \cdot 0 = 0$. In particular note that A is empty if it has σ-finite ϕ-measure for some $\phi \in \mathscr{H}_\infty$. Recall that the Hausdorff dimension of A is given by

$$\dim A := \inf(\alpha: x^\alpha - m(A) < \infty\}.$$

Notation. $\log^+ u = \max(\log u, 0)$, $\psi_0(x) = x^4 \log^+(1/x)$, $\psi_1(x) = x^4 \log^+ \log^+(1/x)$, $\psi_4(x) = \psi_0(x) \log^+ \log^+(1/x)$. $c_{i.1}, c_{i.2}, \ldots$ will denote positive constants introduced in Sect. i which depend only on d unless otherwise indicated. $c_1, c_2, \ldots$ are used to denote positive constants used in the course of a proof whose values are unimportant.

The following exact measure function for $\bar{R}$ is obtained in Sect. 5.

Theorem 1.4. (a) *Let $d > 4$. There are positive constants $c_{1.1}(d)$ and $c_{1.2}(d)$ such that for all $m \in M_F(\mathbb{R}^d)$ and for Q^m-a.a. ω*

(1.5) $$c_{1.1}\, \psi_1 - m(\bar{R}(r, s) \cap A) \leqq Y_{r,s}(A) \leqq c_{1.2}\, \psi_1 - m(\bar{R}(r, s) \cap A)$$

for all $A \in \mathscr{B}(\mathbb{R}^d)$, $\quad 0 < r \leqq s \leqq \infty$,

(1.6) $$c_{1.1}\, \psi_1 - m(\bar{R}_+(0, s) \cap A) \leqq Y_s(A) \leqq c_{1.2}\, \psi_1 - m(\bar{R}_+(0, s) \cap A)$$

for all $A \in \mathscr{B}(\mathbb{R}^d)$, $\quad 0 < s \leqq \infty$,

(b) *Let* $d=4$. *There is a constant* $c_{1.2}(4)$ *such that for all* $m\in M_F(\mathbb{R}^4)$ *and for* Q^m-a.s. ω

$$Y_{r,s}(A)\leqq c_{1.2}\,\psi_4-m(\bar{R}(r,s)\cap A) \quad \textit{for all } A\in\mathscr{B}(\mathbb{R}^4), \quad 0<r\leqq s\leqq\infty,$$
$$\psi_0-m(\bar{R}(r,\infty))<\infty \quad \textit{for all } r>0.$$

It is clear that the lower bound on $Y_{r,s}$ in (1.5) is false in general if $r=0$. If m has strictly positive density, then the continuity of X_t shows that $\bar{R}(0,\varepsilon)=\mathbb{R}^d$. (This is the reason that we omit $t=0$ in the definition of the range of X).

The above theorem should be compared with the corresponding result for Brownian motion. Recall (Ciesielski and Taylor (1962), Ray (1963), Taylor (1964)) that for a Brownian motion B_t if $\phi_d(x)=x^2\log\log(1/x)$ for $d>2$ and $\phi_2(x)=x^2\log(1/x)\log\log\log(1/x)$, then

$$\phi_d-m(\{B_s\colon s\leqq t\})=c_d\,t \quad \text{for all } t>0 \quad \text{a.s. for some } c_d\in(0,\infty).$$

The proof in the critical case $d=2$ is more delicate. For super-Brownian motion we have been unable to find an exact measure function in the corresponding critical case $d=4$. We conjecture that $x^4\log(1/x)\log\log\log(1/x)$ is the required function. The basic approach in our proof is that used for Brownian motion. To prove the upper bound in (1.5) we show for $d>4$

$$\text{(1.7)}\quad \limsup_{\varepsilon\downarrow 0}\frac{Y_{r,s}(B(x;\varepsilon))}{\psi_1(\varepsilon)}\leqq c'_{1.2} \quad \text{for } Y_{r,s} \text{ a.a. } x \quad \text{for all } 0<r<s\leqq\infty \text{ a.s.}$$

and then use a well-known density theorem of Rogers and Taylor (1961). This part of the argument also works for super-symmetric stable processes of index α if the power 4 in the definition of ψ_1, ψ_4 is replaced by 2α (see Theorem 5.12). The lower bound on Y is obtained by a direct covering argument as in Taylor (1964). Unfortunately our proofs are much more complicated than for Brownian motion, and should probably be omitted on a first reading.

Definition. $\bar{R}_k=\bigcup\left(\bigcap_{j=1}^{k}\bar{R}(I_j)\colon I_1,\ldots,I_k \text{ disjoint compact subintervals of } (0,\infty)\right)$.

We call $\bar{R}_k$ the set of *k-multiple points of* X. Clearly $\bar{R}_1=\bar{R}$.

Definition. $A\subset\mathbb{R}^d$ is *polar* if and only if $A\cap\bar{R}=\phi$, Q^m-a.s. for any $m\in M_F$. A is *polar for* $\bar{R}_k$ if and only if $A\cap\bar{R}_k=\phi$ Q^m-a.s. for any $m\in M_F$. A is *semipolar* (respectively semipolar for $\bar{R}_k$) if and only if $A\cap\bar{R}$ (respectively $A\cap\bar{R}_k$) is countable Q^m-a.s. for any $m\in M_F$.

In Sect. 6 the estimates of Sect. 3 are used to obtain a sufficient condition for A to be polar for $\bar{R}_k$ (see Theorem 6.2).

Theorem 1.5. *Let* $A\subset\mathbb{R}^d$ *and* $k\in\mathbb{N}$.

(a) *Let* $d>4$. *If* $x^{k(d-4)}-m(A)=0$, *then* A *is polar for* $\bar{R}_k$. *If* A *has* σ-*finite* $x^{k(d-4)}-m$, *then* A *is semipolar for* $\bar{R}_k$.

(b) *Let* $d=4$. *If* $\left(\log\frac{1}{x}\right)^{-k}-m(A)=0$, *then* A *is polar for* $\bar{R}_k$. *If* A *has* σ*-finite* $\left(\log\frac{1}{x}\right)^{-k}-m$, *then* A *is semipolar for* $\bar{R}_k$.

(c) *Let* $d\geqq 4$. *Then*

$$\dim A\cap\bar{R}_k\leqq\dim A-k(d-4),\quad \text{a.s.},$$

where a negative dimension means that the set is empty.

In Perkins (1988 b) these results are shown to be close to best possible. More specifically, it is shown that zero $x^{-k(d-4)}$-capacity if $d>4$, and zero $\left(\log\frac{1}{x}\right)^k$-capacity if $d=4$, is a necessary condition for A to be polar for $\bar{R}_k$. (See Taylor (1961) for a comparison of these necessary conditions with the above sufficient conditions.)

Taking $A=\mathbb{R}^d$ we see that $\bar{R}_k=\phi$ if $k>\frac{d}{d-4}$ and

$$\dim\bar{R}_k\leqq d-k(d-4). \tag{1.8}$$

The opposite inequality is proved in Perkins (1988 b). These results leave open the question as to whether or not $\bar{R}_k=\phi$ a.s. when $k=\frac{d}{d-4}\in N$. By combining Theorem 1.4 with the techniques used in the proof of Theorem 1.5, we prove the following results in Sect. 6.

Theorem 1.6. (a) *Let* $d>4$ *and* $k\in\mathbb{N}$. *Then* $\bar{R}_k$ *has* σ*-finite* $x^{d-k(d-4)}\log\log(1/x)-m$ a.s. *In particular*, $\bar{R}_k=\phi$ a.s. *if* $k\geqq\frac{d}{d-4}$.

(*b*) *Let* $d=4$ *and* $k\in\mathbb{N}$. *Then* $\bar{R}_k$ *has* σ*-finite* $x^4\left(\log\frac{1}{x}\right)^k-m$.

Hence we see that X fails to have quintuple points if $d=5$, triple points if $d=6$ or double points if $d=8$. The second statement in (a) is immediate from the first because of the log log factor in the measure function and our definition of $\phi-m$ for $\phi\in\mathscr{H}_\infty$. The log log factor comes from the exact measure function in Theorem 1.4.

Again the sharpness of these results is proved in Perkins (1988 b) (see also Dynkin (1988)). The measure functions in Theorem 1.6 will certainly not be the best possible (except if $k=1$ and $d\geqq 5$, so that Theorem 1.4 applies). Using Le Gall's recent results for Brownian motion (Le Gall (1987)) as a guide, we conjecture that the "exact" measure functions for $\bar{R}_k$ are

$$\phi(x)=x^{d-k(d-4)}\left(\log\log\frac{1}{x}\right)^k\quad\text{for } d>4$$

$$\phi(x)=x^4\left(\log\frac{1}{x}\log\log\log\frac{1}{x}\right)^k\quad\text{if } d=4.$$

More precisely the $\phi-m$ of $\bar{R}_k$ should be positive but σ-finite. Motivated by these results, Dynkin (1988) has constructed a self-intersection local time which is supported by $\bar{R}_k$ and should provide an effective means of proving the above conjectures (among other things).

In defining $\bar{R}$ and $\bar{R}_k$, it might be more natural to work with $R(I)=\bigcup_{t\in I} S(X_t)$ in place of $\bar{R}(I)$ (cf. (1.3)) and obtain corresponding (but smaller) sets R and R_k. In fact in Perkins (1988b) it is shown that $\bar{R}-R$ is a.s. countable. Hence Theorems 1.4, 1.5(c) and 1.6 would be unaffected. Moreover Theorem 1.5(a), (b) are stronger results when stated in terms of $\bar{R}_k$. Theorem 1.3 would also remain valid (see Perkins (1988b)).

Let $M_{\exp}:=\{m\colon m\in M(\mathbb{R}^d),\ \int_{\mathbb{R}^d} e^{-\varepsilon|x|^2}dm(x)<\infty$ for all $\varepsilon>0\}$. The following result (see Perkins (1988a, d)) can be used to obtain analogous results in the case of infinite initial measures.

Theorem 1.7. *Let $m\in M_{\exp}$, $A_n=\{x\colon n-1\leqq|x|<n\}$, $m^{(n)}(\cdot)=m(\cdot\cap A_n)$ and $\{X^{(n)}\colon n\in\mathbb{N}\}$ be a sequence of independent super-Brownian motions starting at $\{m^{(n)}\colon n\in\mathbb{N}\}$, respectively, and defined on $\{\Omega,\mathscr{F},\mathscr{F}_t,P\}$. Then*

$$X_n(t)=\sum_{k=1}^{n} X^{(k)}(t)\in M_F(\mathbb{R}^d)$$

is a super-Brownian motion on $(\Omega,\mathscr{F},\mathscr{F}_t,P\}$ starting at $m_n=\sum_{k=1}^{n} m^{(k)}\in M_F(\mathbb{R}^d)$ and

$$X(t)=\lim_{n\to\infty} X_n(t)\in M_{\exp}(\mathbb{R}^d)$$

is a super-Brownian motion on $(\Omega,\mathscr{F},\mathscr{F}_t,P)$ starting at m. Moreover for any $\phi\in C_K(\mathbb{R}^d)$ or $\phi=e^{-\varepsilon|x|^2}$ for some $\varepsilon>0$, and $T>0$,

$$\limsup_{n\to\infty\, t\leqq T} |\langle\phi,X_n(t)\rangle-\langle\phi,X(t)\rangle|=0,\quad \text{a.s. and in } L^1.$$

Theorem 1.8. *Let m, X, X_n be as in Theorem* 1.7. *For P-a.a. ω, for all R, $T>0$ $\exists N(R,T,\omega)\in\mathbb{N}$ such that*

$$X(t)(A\cap B(0;R))=X_n(t)(A\cap B(0;R))\quad \textit{for all } t\leqq T,\quad A\in\mathscr{B}(\mathbb{R}^d)$$

and $n\geqq N(R,T,\omega)$.

This allows us to obtain localized versions of the above results. For example a localized (both in space and time) version of Theorem 1.1 holds for initial measures in $M_{\exp}(\mathbb{R}^d)$ (cf. Theorem 4.10).

Remark. It is possible to derive slightly weaker estimates than those obtained in Theorems 3.1–3.3, independently of the p.d.e. literature, using probabilistic arguments. More specifically a systematic use of the Feynman-Kac arguments, used for example in Lemma 3.2, leads to estimates strong enough for the proofs of Theorems 1.4 and 1.5 (at least for $d>4$) but this does not allow one to make such precise statements as Theorem 1.3. This approach works equally well for nice diffusions other than Brownian motion but one must proceed with care in general because the natural analogues of Theorems 3.1 and 3.2 for super-symmetric stable processes of index $\alpha\in(0,2)$ are completely false (see Perkins (1988 b)).

On the other hand one can also derive Theorem 1.1 using the "analytical" estimate in Theorem 3.3(b), as opposed to the probabilistic arguments involving branching Brownian motions given in Sect. 4.

We feel that it was the *combined* use of probabilistic and analytic tools that led to the resolution of the problems discussed above and have therefore made no attempt to emphasize one approach at the expense of the other.

Some of the above results were announced in Dawson, Iscoe and Perkins (1988).

To complete the introduction we provide a list of notation that has not yet been introduced and will be used below.

Notation.

$\mathbb{N}$ = natural numbers
$Z_+=\{0,1,,2,3,\dots\}$
$M(\mathbb{R}^d)$ = all measures on $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$.

If $I\subset\mathbb{R}$, $C^{1,2}(I\times\mathbb{R}^d)$ denotes the space of functions $f(t,x)$ continuously differentiable in t and twice continuously differentiable in x (a subscript b denotes that the values of the function and corresponding derivatives are bounded).
δ_x = unit point mass at x
B_t denotes a d-dimensional Brownian motion which starts at x under the measure P_0^x defined on some $(\Omega_0,\mathscr{F}_0)$. E_0^x denotes expectation with respect to P_0^x. We write P_0 for P_0^0 and $P_0^m(A)=\int P_0^x(A)\,dm(x)$ if $m\in M_F(\mathbb{R}^d)$ (we assume Borel measurability in x of $P_0^x(A)$).
$\{p(t,x): t\geqq 0, x\in\mathbb{R}^d\}$ denotes the transition probability density function of Brownian motion and $\mathscr{T}_t f(x):=\int p(t,x-y)f(y)\,dy$ if f is a bounded measurable function. $\{\mathscr{C}_t: t\geqq 0\}$ is the canonical filtration on $C([0,\infty),\mathbb{R}^d)$ i.e. $\mathscr{C}_t=\sigma(w_s: s\leqq t)$.
If M is a metric space $D([0,\infty),M)$ is the space of cadlag paths with the Skorohod J_1 topology and Θ_t denotes the canonical shift
$\Theta_t\,\omega(\cdot)=\omega(\cdot+t)$ for $t\geqq 0$ on $C([0,\infty),M)$ or $D([0,\infty),M)$.
If $\{Z_s: s\geqq 0\}$ is a process on a complete probability space let
$\mathscr{F}_t^Z=\bigcap_{r>t}\sigma(Z_s: s\leqq r)\vee\{P\text{-null sets of }\mathscr{F}\}$.

2. Branching Brownian Motions

We construct a system of branching Brownian motions and a labelling system (borrowed from Walsh (1986)) for the various branches. Let $I=\bigcup_{n=0}^{\infty} Z_+\times\{0,1\}^n$

$(Z_+ \times \{0,1\}^0 \equiv Z_+)$. If $\beta=(\beta_0, \ldots, \beta_j)$ is a multi-index in I, we call $j=|\beta|$ the length of β and for $i \leqq j$ we let $\beta|i=(\beta_0, \ldots, \beta_i)$. Write $\gamma<\beta$ and call β a descendant of γ if $\gamma=\beta|i$ for some $i \leqq |\beta|$.

Let $\{B^\beta\colon \beta\in I\}$ be a set of independent d-dimensional Brownian motions, each starting at zero, and let $\{e^\beta\colon \beta\in I\}$ be a collection of i.i.d. random variables which take on the values 0 and 2 each with probability 1/2. Assume that these collections are mutually independent and are defined on a common probability space $(\Omega^2, \mathscr{A}^2, P^2)$. If $\bar{R}^d=R^d\cup\{\Delta\}$, where Δ is added as a discrete point, let $(\Omega^1, \mathscr{A}^1)=((\bar{R}^d)^{Z_+}, \mathscr{B}((\bar{R}^d)^{Z_+}))$ and $(\Omega, \mathscr{A})=(\Omega^1\times\Omega^2, \mathscr{A}^1\times\mathscr{A}^2)$. If $\omega=(\omega^1,\omega^2)=((x_j),\omega^2)\in\Omega$, we write $B^\beta(\omega)$ and $e^\beta(\omega)$ for $B^\beta(\omega^2)$ and $e^\beta(\omega^2)$, respectively.

We fix a parameter $\mu\in\mathbb{N}$ and suppress dependency on μ if there is no ambiguity. Let $T=T^{(\mu)}=\{j/\mu\colon j\in Z_+\}$ and use $\underline{s}, \underline{t}, \underline{u}, \ldots$ to denote elements of T. Let $\{t\}=\{t\}^\mu=\max\{\underline{t}\in T\colon \underline{t}\leqq t\}$ $(t\geqq 0)$, and $\lambda=\lambda^\mu$ denote the measure on T which assigns mass μ^{-1} to each point in T.

Given $\omega=((x_j),\omega^2)$ we next construct a branching particle system as follows: a particle starts at each $x_j\neq\Delta$; subsequently particles die or split into two particles with equal probabilities at the deterministic times in $T\backslash\{0\}$; in between these times, the particles follow independent Brownian paths.

Each trajectory on $[0,(j+1)/\mu)$ is labelled by a multi-index of length j. If $\beta\in I$, $t\geqq 0$, let

$$\hat{N}_t^\beta=\hat{N}_t^{\beta,\mu}((x_j),\omega^2)\begin{cases} =x_{\beta_0}+\sum_{i=0}^{|\beta|}\int \mathbf{1}(i/\mu\leqq s<t\wedge((i+1)/\mu))\,dB_s^{\beta|i} & \text{if } x_{\beta_0}\neq\Delta,\\ =\Delta, \quad \text{if } x_{\beta_0}=\Delta. \end{cases}$$

Next use $\{e^\beta\}$ to define death times $\tau^\beta=\tau^{\beta,\mu}((x_j),\omega^2)$ by

$$\tau^\beta\begin{cases} =0 \quad \text{if } x_{\beta_0}=\Delta\\ =\min\{(i+1)/\mu\colon e^{\beta|i}=0\} \text{ if this set is nonempty and } x_{\beta_0}\neq\Delta\\ =(|\beta|+1)/\mu \text{ if the above set is empty and } x_{\beta_0}\neq\Delta. \end{cases}$$

The basic system of branching Brownian motions $\{N^\beta\colon \beta\in I\}$ is defined by

$$N_t^\beta=N_t^{\beta,\mu}=\begin{cases}\hat{N}_t^\beta & \text{if } 0\leqq t<\tau^\beta\\ \Delta & \text{if } t\geqq\tau^\beta.\end{cases}$$

We write N^β for $N^\beta_{|\beta|/\mu}$ if there is no ambiguity.

Notation. If $\beta\in I$ and $t\geqq 0$, write $\beta\overset{\mu}{\sim}t$ (or just $\beta\sim t$) if and only if $|\beta|/\mu\leqq t<(|\beta|+1)/\mu$.

$$M_F^\mu(\mathbb{R}^d)=\left\{\mu^{-1}\sum_{i=0}^{K}\delta_{x_i}\colon K=-1,0,1,2,\ldots, x_i\in\mathbb{R}^d\right\}\subset M_F(\mathbb{R}^d).$$

Definition. $N = N^{(\mu)}$: $[0, \infty) \times \Omega \to M(\mathbb{R}^d)$ is defined by

$$N_t(\omega) = \mu^{-1} \sum_{\beta \sim t} (\delta_{N_t^\beta(\omega)} \mathbf{1}(N_t^\beta(\omega) \neq \Delta).$$

If $\phi: \mathbb{R}^d \to [0, \infty)$, we define $\phi(\Delta) = 0$ so that $\langle \phi, N_t \rangle = \mu^{-1} \sum_{\beta \sim t} \phi(N_t^\beta)$.

If Π^i denotes the projection of Ω onto Ω^i, define a filtration on $(\Omega, \mathscr{A})$ by

$$\mathscr{A}_t = \mathscr{A}_t^\mu = \sigma(\pi^1, B^\beta, e^\beta\colon\ |\beta|/\mu < \{t\}) \vee (\bigcap_{u>t} \sigma(B_s^\beta\colon\ |\beta|/\mu = \{t\}, s \leqq u)),$$
$$\mathscr{A}_\infty = \mathscr{A}.$$

Finally, we introduce a family of probabilities on $(\Omega, \mathscr{A})$. If $\omega^1 \in \Omega^1$, let $P^{\omega^1} = \delta_{\omega^1} \times P^2$. If $\omega^1 = (x_j) \in \Omega^1$ and $m = \mu^{-1} \sum_{j=0}^{\infty} \delta_{x_j} \mathbf{1}(x_j \neq \Delta)$, note that (m, μ) uniquely determines P^{ω^1} on $\sigma(N_t\colon\ t \geqq 0)$ (addition is commutative) and hence we may write $P^{m,\mu}$ for the restriction of P^{ω^1} to $\sigma(N_t\colon\ t \geqq 0)$. In fact we will usually abuse notation and write P^m for P^{ω^1} itself, thus suppressing dependency on the underlying (x_j).

Let $\mathscr{E} = \sigma(e^\beta\colon\ \beta \in I)$, and if $\beta \in I$, let

$$\mathscr{F}(\beta) = \sigma(\pi^1, B^{\beta|k}, e^{\beta|k}\colon\ 0 \leqq k \leqq |\beta|)$$
$$\mathscr{F}_1(\beta) = \sigma(N^\beta_{|\beta|/\mu} \neq \Delta).$$

If $S \subset I$, let $\mathscr{G}(S) = \sigma(B^\beta, e^\beta\colon\ \beta \in S)$. If $\phi \neq S \subset I$ and $\gamma \in I$, we let $\sigma(S; \gamma)$ be the number of generations since γ first split off from the family tree generated by S. More specifically,

$$\sigma(S; \gamma) \begin{cases} = |\gamma| - \inf\{j \leqq |\gamma|\colon\ \gamma|j \notin \{\beta|j\colon\ \beta \in S, |\beta| \geqq j\}\} & \text{if this set is non-empty} \\ = -1 \quad \text{otherwise.} \end{cases}$$

Write $\sigma(\beta; \gamma)$ for $\sigma(\{\beta\}; \gamma)$.

The following easy results are proved (in a slightly modified form) in Perkins (1988a, Lemmas 2.1 and 2.5, Proposition 2.2). $\mu \in \mathbb{N}$ is fixed.

Lemma 2.1. (a) *If $\beta \in I$, $N_\cdot^\beta \in C([0, \infty), \mathbb{R}^d)$ and $N_\cdot^\beta$ is $\mathscr{A}_t$-adapted.*

(b) *If $\beta \in I$, $t \geqq 0$, $\{t\} \leqq |\beta|/\mu$, $(x_j) \in \Omega^1$ and $A \in \mathscr{C}_t$, then on $\{N_t^\beta \neq \Delta\}$,*

$$P^{(x_j)}(N_\cdot^\beta \in A \mid \mathscr{E}) = P_0^{x_{\beta_0}}(B \in A) \quad \text{a.s.}$$

(c) $\{N_t(\phi)\colon\ t \geqq 0\}$ is $\mathscr{A}_t$-adapted and, if $m \in M_F^\mu$, has sample paths in $D([0, \infty), M_F^\mu)$ P^m-a.s.

(d) If $m \in M_F^\mu$ and $\phi: \mathbb{R}^d \to [0, \infty)$ is measurable, then

$$E^m(\langle \phi, N_t \rangle) = \int E_0^x(\phi(B_t))\, dm(x) = E_0^m(\phi(B_t)).$$

(e) If $\beta, \gamma \in I$, $\sigma(\beta; \gamma) = k \geqq 0$, $\beta \sim s$ and $\gamma \sim t$, then

$$P^m((|N_t^\gamma - N_s^\beta| \leqq a \mid \mathscr{F}(\gamma) \vee \mathscr{E}) = \mathbf{1}(N_t^\gamma \neq \Delta, N_s^\beta \neq \Delta)\, P_0^{N^\gamma_{(|\gamma|-k)/\mu} - N_s^\gamma}(|B_{s-(|\gamma|-k)/\mu}| \leqq a)$$

for $a \geqq 0$.

In (e) (and throughout this paper) the fact that $N_t^\gamma \neq \Delta$ and $N_s^\gamma \neq \Delta$ is implicitly understood whenever we write an inequality such as $|N_t^\gamma - N_s^\gamma| \leqq a$.

The next result due to Watanabe (1968, Theorem 4.1) constructs the super-Brownian motion as a limit of the branching Brownian motions, $N^{(\mu)}$.

Theorem 2.2. *If $m \in M_F(\mathbb{R}^d)$ and $m_\mu \in M_F^\mu(\mathbb{R}^d)$ converge weakly to m as $\mu \to \infty$, then*

$$P^{m_\mu, \mu}(N^{(\mu)} \in \cdot) \xrightarrow{w} Q^m(\cdot) \quad \text{on } D([0, \infty), M_F(\mathbb{R}^d)) \quad \text{as } \mu \to \infty.$$

Although only the convergence of the finite dimensional distributions is proved in Watanabe (1968), tightness is easily established using martingale methods (see e.g. Roelly-Coppoletta (1968) or Ethier-Kurtz (1986, p. 406)).

We next fromulate a nonstandard version of Theorem 2.2. We work in an ω_1-saturated enlargement of a superstructure containing $\mathbb{R}$. Our underlying internal measure space is $({}^*\Omega, {}^*\mathscr{A})$, where $(\Omega, \mathscr{A})$ is as above. Fix $\eta \in {}^*N - N$ and let $\mu = 2^\eta$. Taking μ of this form will be convenient although certainly not essential. As before we have an internal collection of branching *-Brownian motions, $\{N_\cdot^\beta : \beta \in {}^*I\}$ and an internal ${}^*M(\mathbb{R}^d)$-valued process $N_t = N_t^{(\mu)}$. Consider P as a mapping from Ω^1 to the set of probabilities on $(\Omega, \mathscr{A})$ whose nonstandard extension is of course ${}^*P$. Hence if $m_\mu = \mu^{-1} \sum_{i=0}^{K} \delta_{x_i} \in {}^*M_F^\mu(\mathbb{R}^d)$, ${}^*P^{m_\mu}$ (or more precisely ${}^*P^{(x_i)}$) is the internal probability on $({}^*\Omega, {}^*\mathscr{A})$ constructed as before. With a slight abuse of notation we write

$$({}^*\Omega, \mathscr{F}, P^{m_\mu}) = ({}^*\Omega, L({}^*\mathscr{A}), L({}^*P^{m_\mu}))$$

to denote the Loeb space constructed from $({}^*\Omega, {}^*\mathscr{A}, {}^*P^{m_\mu})$. For details about this class of *standard* complete probability spaces see Cutland (1983), Loeb (1975, 1979) or Albeverio et al. (1986).

If M is a metric space and $A \subset {}^*M$, $\mathrm{ns}(A)$ denotes the nearstandard points in A and $\mathrm{st}_M : \mathrm{ns}({}^*M) \to M$ is the standard part map. If $M = \mathbb{R}^d$, we write $\mathrm{st}(x)$ or ${}^\circ x$ for $\mathrm{st}_{\mathbb{R}^d}(x)$. An internal function $F: {}^*[0, \infty) \to {}^*M$ is S-continuous if and only if $F(t_1)$ and $F(t_2)$ are infinitesimally close in $\mathrm{ns}({}^*M)$ whenever t_1 and t_2 are infinitesimally close in $\mathrm{ns}({}^*[0, \infty))$.

If M is a complete separable metric space and $\nu \in {}^*M_F(M)$ ($M_F(M)$ is the space of finite measures on the Borel sets of M), then (see e.g. Lemma 2 of Anderson-Rashid (1978))

$$\nu \in \mathrm{ns}({}^*M_F(M)) \quad \text{if and only if } L(\nu)(\mathrm{ns}({}^*M)^c) = 0 \tag{2.1}$$

and in the case

$$\mathrm{st}_{M_F(M)}(\nu)(A)=L(\nu)(\mathrm{st}_M^{-1}(A)) \quad \text{for all } A\in\mathscr{B}(M).$$

Theorem 2.3. *Let $m\in M_F(\mathbb{R}^d)$ and choose $m_\mu\in{}^*M_F^\mu(\mathbb{R}^d)$ such that $\mathrm{st}_{M_F}(m_\mu)=m$. Then $N.$ is P^{m_μ}-a.s. S-continuous from $^*[0,\infty)$ to $^*M_F$. There is a unique (up to distinguishability) continuous M_F-valued process, X_t, on $(^*\Omega, \mathscr{F}, P^{m_\mu})$ such that*

(2.2) $\quad X_{{}^\circ t}(A)=L(N_t)(\mathrm{st}^{-1}(A))$ *for all* $t\in\mathrm{ns}(^*[0,\infty))$, $\quad A\in\mathscr{B}(\mathbb{R}^d)$ a.s.

Moreover,

(2.3) $$P^{m_\mu}(X\in C)=Q^m(C) \quad \textit{for all } C\in\mathscr{B}(C([0,\infty),M_F)).$$

Proof. This would be immediate from Theorem 2.2 and (2.1) if $m_\mu={}^*\nu(\mu)$ where $\nu(n)\xrightarrow{w}m$, $\nu(n)\in M_F^n$.

In general, for $n\in\mathbb{N}$ and $m'\in M_F^n$, let

$$Q^{m',n}(A)=P^{m'}(N^{(n)}\in A), A\in(D([0,\infty),M_F)).$$

Let d_1 be a metric on M_F and d_2 be a metric on the space of probabilities on $D([0,\infty),M_F)$ (both inducing the topologies of weak convergence). Theorem 2.2 implies that

(2.4) for every $\varepsilon>0\,\exists n_0(\varepsilon)$ such that if $n\geqq n_0$ and $m'\in M_F^n$ satisfies $d_1(m',m)<n_0^{-1}$, then $d_2(Q^{m',n},Q^m)<\varepsilon$.

Let m and m_μ be as in the theorem (the existence of m_μ is trivial) and fix $\varepsilon\in(0,\infty)$. Then $\mu\geqq n_0(\varepsilon)$ and $^*d_1(m_\mu, {}^*m)<n_0(\varepsilon)^{-1}$. (2.4) and the Transfer Principle imply that $^*d_2(Q^{m_\mu,\mu}, {}^*Q^m)<\varepsilon$. Letting $\varepsilon\downarrow 0$ we see that if st_1 denotes the standard part map on the space of finite measures on $D([0,\infty),M_F)$, then

(2.5) $$\mathrm{st}_1(Q^{m_\mu,\mu})=Q^m.$$

If st_2 denotes the standard part map on $\mathrm{ns}(^*D([0,\infty),M_F))$, then (2.5) means (cf. (2.1))

(2.6) $$L(Q^{m_\mu,\mu})(\mathrm{st}_2^{-1}(A))=Q^m(A), \quad \text{for all } A\in\mathscr{B}(D([0,\infty),M_F)).$$

A simple characterization of st_2 may be found in Hoover-Perkins (1983, Theorem 2.6). This shows that the set of S-continuous paths in $^*D([0,\infty),M_F)$ is precisely $\mathrm{st}_2^{-1}(C([0,\infty),M_F))$. Take $A=C([0,\infty),M_F)$ in (2.6) to see that $N^{(\mu)}_.$ is a.s. S-continuous. (2.6) also shows that $X=\mathrm{st}_2(N)$ satisfies (2.3). The S-continuity of N and the aforementioned characterization of st_2 implies that $X({}^\circ t)=\mathrm{st}_{M_F}(N(t))$ for all $t\in\mathrm{ns}(^*[0,\infty))$ P^{m_μ}-a.s. (2.2) is now immediate from (2.1). □

We will also need a nonstandard representation for $Y_{s,t}$.

Notation. If $0 \leqq s \leqq t \leqq \infty$ let

$$M_{s,t}^{(\mu)}(A) = M_{s,t}(A) = \int_{[s,t]} N_u(A)\, d\lambda(u), \qquad A \in {}^*\mathscr{B}(\mathbb{R}^d)$$

$$M_t^{(\mu)} = M_t = M_{0,t}.$$

By using (2.1) and the fact that $L(\lambda)$ $(\mathrm{st}^{-1}(A))$ is the Lebesgue measure of A for any Borel set $A \subset \mathbb{R}^1$ (Albeverio et al. (1986, Prop. 3.2.5)) one may readily prove

(2.7) $$Y_{{}^\circ s, {}^\circ t}(A) = L(M_{s,t})\,(\mathrm{st}^{-1}(A))$$

whenever $0 \leqq s \leqq t$ are in $\mathrm{ns}({}^*[0, \infty))$ and $A \in \mathscr{B}(\mathbb{R}^d)$ a.s.

3. Estimates of the Probability of Hitting Small Balls

3.1 Introduction and Statement of Results

The purpose of this section is to obtain estimates of the probability of hitting balls, both at fixed times and over finite and infinite time intervals. The proofs are based on the representation (1.1 a, b) for the Laplace transform of $X(t)$ and $Y(t)$, certain scaling properties of (1.1 b), and the Feynman-Kac formula. This section contains the statements of the main estimates. Section 3.2 contains a number of technical lemmas concerning solutions to Eq. 1.1 b and related equations; the proofs of the main theorems are contained in Sect. 3.3. In addition, Theorems 1.3 and 1.8 are proved in Sect. 3.3.

Notation. Let $m_{x,\varepsilon} := m|_{B(x;\varepsilon)}$ and $m_{x,\varepsilon}^c := m|_{B(x;\varepsilon)^c}$.

Theorem 3.1. *Let* $d \geqq 3$, $x_0 \in \mathbb{R}^d$, *and* $m \in M_F$.

(a) *There is a constant* $c_{3.1}$ *such that for all* $t \geqq \varepsilon^2$,

$$Q^m(X_t(B(x_0;\varepsilon)) > 0) \leqq c_{3.1}\, \varepsilon^{d-2} t^{-d/2} m(\mathbb{R}^d).$$

(b) *There is a constant* $c_{3.2}$ *such that*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{d-2}} Q^m(X_t(B(x_0;\varepsilon)) > 0) = c_{3.2}\Big(\int_{\mathbb{R}^d} p(t, x - x_0)\, dm(x)\Big),$$

and for any $\delta > 0$ *and* $0 < K < \infty$ *the convergence is uniform for* $t \geqq \delta$, $x_0 \in \mathbb{R}^d$, $m(\mathbb{R}^d) \leqq K$.

Theorem 3.2. *Let* $d \geqq 4$. *There exists a constant* $c_{3.3}$, *a positive sequence* $\{\varepsilon_k\}$ *converging to zero, a function* $\varepsilon = \varepsilon_d \colon \mathbb{N} \times (0, \infty)^2 \to (0, \infty)$ (*monotone decreasing in the first and third variables*) *and a function* $\tilde{\varepsilon} = \tilde{\varepsilon}_d \colon \mathbb{N} \times (0, \infty) \to (0, \infty)$ *such that the following hold.*

(a) *If* $d>4$, $\delta>0$, $m\in M_F(\mathbb{R}^d)$, $x\in\mathbb{R}^d$, $k\in\mathbb{N}$ *and* $0<\varepsilon<\varepsilon_d(k,\delta,m(\mathbb{R}^d))$, *then*

$$Q^m(X_t(B(x;\varepsilon)>0 \quad \text{for some } t\geqq\delta) \leqq(1+\varepsilon_k)\,c_{3.3}\,\varepsilon^{d-4}(E_0^m(|B_\delta-x|^{2-d})+2^{-k}).$$

(b) *If* $d=4$, $\delta>0$, $m\in M_F(\mathbb{R}^4)$, $x\in\mathbb{R}^4$, $k\in\mathbb{N}$ *and* $0<\varepsilon<\varepsilon_4(k,\delta,m(\mathbb{R}^4))$, *then*

$$Q^m(X_t(B(x;\varepsilon))>0 \quad \text{for some } t\geqq\delta) \leqq(1+\varepsilon_k)\,2\left(\log\frac{1}{\varepsilon}\right)^{-1}(E_0^m(|B_\delta-x|^{-2})+2^{-k}).$$

(c) *If* $d>4$, $m\in M_F(\mathbb{R}^d)$, $x\in\mathbb{R}^d$, $k\in\mathbb{N}$, $\int|y-x|^{2-d}dm(y)<\infty$, *and* $0<\varepsilon<\tilde{\varepsilon}_d(k,\int|y-x|^{2-d}dm(y))$, *then*

$$Q^m(X_t(B(x;\varepsilon))>0 \quad \text{for some } t\geqq 0) \geqq(1-\varepsilon_k)\,c_{3.3}\,\varepsilon^{d-4}\int \mathbf{1}(|y-x|\geqq k\varepsilon)\,|y-x|^{2-d}dm(y).$$

(d) *If* $d=4$, $m\in M_F(\mathbb{R}^4)$, $x\in\mathbf{R}^4$, $k\in\mathbb{N}$, $\int|y-x|^{-2}dm(y)<\infty$, *and* $0<\varepsilon<\tilde{\varepsilon}_4(k,\int|y-x|^{-2}dm(y))$, *then*

$$Q^m(X_t(B(x;\varepsilon)>0 \quad \text{for some } t\geqq 0) \geqq 2(1-\varepsilon_k)\left(\log\frac{1}{\varepsilon}\right)^{-1}\int\mathbf{1}(k\varepsilon\leqq|y-x|\leqq\varepsilon^{-1/k})\,|y-x|^{-2}dm(y).$$

Remark. In using the upper bounds in (a), (b) the following easily verified facts are sometimes used:

(3.1.1.) $$E_0^m(|B_\delta-x|^{2-d})\leqq\delta^{(2-d)/2}m(\mathbb{R}^d),$$

(3.1.2) $$E_0^m(|B_\delta-x|^{2-d})\leqq\int\frac{1}{|y-x|^{d-2}}\,dm(y).$$

Theorem 3.3. *For* $d\geqq 1$, $t>0$ *there are constants* $c_{3.4}(d,t)$ *and* $c_{3.5}(d)$ *such that*

(a) *If* $m\in M_F$ *and* $\operatorname{supp}(m)\subset\overline{B(0;R+2)}^c$, *then*

$$Q^m(X_s(B(0;R))>0 \quad \text{for some } s\leqq t) \leqq c_{3.4}\int\left(\frac{|x|-(R+1)}{\sqrt{t}}\right)^{d-2}\exp\left\{-\frac{(|x|-R-1)^2}{2t}\right\}dm(x).$$

(b) $$Q^{a\delta_0}(X_s(\overline{B(0;R)}^c)>0 \quad \text{for some } s\leqq t) \leqq c_{3.5}\,aR^{-2}\left(\frac{R}{\sqrt{t}}\right)^{d+2}\exp\left\{-\frac{R^2}{2t}\right\}$$

provided that $\dfrac{R}{\sqrt{t}}>2$.

Corollary 3.4. *If $m \in M_{\exp}$, $d \geqq 1$, then for all $t>0, R>0$, there is a constant $c_{3.6}(d, R, t)$ such that*

$$Q^{m_0^{\mathscr{E}}, N}(X_s(B(0; R))>0 \quad \text{for some } s \leqq t) \leqq c_{3.6}(d, R, t) \int_{|x|>N} \exp\left(-\frac{|x|^2}{2(t+1)}\right) dm(x)$$

for $N > R+2$.

3.2. *Some Analytic Lemmas*

$\dot{u}$ will denote the partial derivative of $u(t, x)$ with respect to t, and $B = B(0; 1)$. Let $T_r := \inf\{t: |B_t| \leqq r\}$.

We begin with an elementary comparison lemma, sufficient for our needs, which is a simple consequence of the parabolic weak maximum principle. A direct proof can be given following that of Theorem 3.1.1 of Stroock and Varadhan (1979).

Lemma 3.0. *Let $\mathscr{D}$ be an open subset of $\mathbb{R}^d$ such that $\mathscr{D}^c$ is bounded and let $c(\cdot,\cdot) \in C([0, T] \times \bar{\mathscr{D}})_+$. If $u \in C_b^{1,2}((0, T] \times \mathscr{D}) \cap C_b([0, T] \times \bar{\mathscr{D}})$ satisfies*

$$\begin{aligned} &\dot{u} - \tfrac{1}{2}\Delta u + c \cdot u \geqq 0 && \text{in } (0, T] \times \mathscr{D} \\ &u(0, x) \geqq 0 && \forall x \in \bar{\mathscr{D}} \\ &u(t, x) \geqq 0 && \forall (t, x) \in [0, T] \times \partial\mathscr{D} \end{aligned}$$

then $u(t, x) \geqq 0$ for all $(t, x) \in [0, T] \times \bar{\mathscr{D}}$.

We fix a $\phi \in C(\mathbb{R}^d)_+$ such that $0 < \phi(x) \leqq 1$ for $x \in B$ and $\phi(x) = 0$ for $x \notin B$.

Lemma 3.1. *Let $u_1 \equiv u_1(t, x; \theta)$ $(\theta > 0)$ be a unique non-negative solution of the equation*

$$\text{(3.2.1)} \qquad \begin{aligned} \dot{u}_1 &= \tfrac{1}{2}(\Delta u_1 - u_1^2) \\ u_1(0) &= \theta\phi \end{aligned}$$

Then the following estimate is valid for all $t > 0$, $|x| > 1$, $\theta \geqq 0$:

$$\text{(3.2.2)} \qquad u_1(t, x; \theta) \leqq 2t^{-1} \cdot \min\left[1, (6e) \cdot \frac{\exp(-[|x|-1]\sqrt{2/t})}{[1 - \exp(-[|x|-1]\sqrt{2/t})]^2}\right].$$

Proof. The upper bound

$$\text{(3.2.3)} \qquad u_1(t, x; \theta) \leqq \frac{2\theta}{2 + \theta t} \leqq 2t^{-1}$$

follows from Lemma 3.0 with $\mathscr{D} \equiv \mathbb{R}^d$, $u \equiv u_h - u_1$ and $c \equiv \frac{1}{2}(u_h + u_1)$, where $u_h := \dfrac{2\theta}{2 + ut}$ is the solution of the initial value problem

$$(3.2.3)_h \qquad \begin{aligned} \dot{u}_h &= \tfrac{1}{2}(\Delta u_h - u_h^2) \\ u_h(0, x; \theta) &= \theta \quad \textit{for all } x. \end{aligned}$$

Let $\lambda > 0$ and denote by u_2 the soluton of:

$$\begin{aligned} \dot{u}_2 &= \tfrac{1}{2}(\Delta u_2 - \lambda u_2 - u_2^2) \\ u_2(0) &= \theta \phi \end{aligned}$$

and set $v = e^{\frac{1}{2}\lambda t} \cdot u_2$. Then

$$\begin{aligned} \dot{v} &= \tfrac{1}{2}(\Delta v - e^{-\frac{1}{2}\lambda t} \cdot v^2) \geqq \tfrac{1}{2}(\Delta v - v^2) \\ v(0) &= \theta \phi \end{aligned}$$

so that $u_1 \leqq v = e^{\frac{1}{2}\lambda t} \cdot u_2$ by Lemma 3.0 with $\mathscr{D} \equiv \mathbb{R}^d$, $u \equiv v - u_1$ and $c \equiv \frac{1}{2}(v + u_1)$. In turn, by Lemma 3.0 again, with $\mathscr{D} \equiv \overline{B(0; 1+\delta)}^c$ (δ being sufficiently small and positive but otherwise arbitrary), $u \equiv u_3 - u_2$ and $c \equiv \frac{1}{2}[\lambda + u_2 + u_3]$, we obtain that $u_2 \leqq u_3$ on $\bar{B}^c$, where u_3 is any positive solution of

$$\begin{aligned} &-\Delta u_3 + \lambda u_3 + u_3^2 \geqq 0, \qquad |x| > 1 \\ &u_3(x) \to \infty, \qquad \text{as } |x| \to 1^+ \\ &u_3(x) \to 0, \qquad \text{as } |x| \to \infty \end{aligned}$$

or in radial form:

$$\begin{aligned} &-u_3''(r) - \frac{(d-1)}{r} u_3'(r) + \lambda u_3(r) + u_3^2(r) \geqq 0, \qquad r > 1 \\ &u_3(r) \to \infty, \qquad \text{as } r \to 1^+ \\ &u_3(r) \to 0, \qquad \text{as } r \to \infty. \end{aligned}$$

For example one can choose $u_3(r) = w(r-1)$ such that w is positive and

$$\begin{aligned} &w'' = \lambda w + w^2, \qquad r > 0 \\ &w' < 0, \qquad r > 0 \\ &w(r) \to \infty, \qquad \text{as } r \to 0^+ \\ &w(r) \to 0, \qquad \text{as } r \to \infty. \end{aligned}$$

Multiplying the o.d.e. for w by w' and taking indefinite integrals (demanding that $\lim_{r \to \infty} w'(r) = 0$ as well) yields $w' = -w[\lambda + \frac{2}{3}w]^{\frac{1}{2}}$, which can be solved in closed form $w(r) = 6\lambda e^{-\sqrt{\lambda} r}/[1 - e^{-\sqrt{\lambda} r}]^2$.

In summary, we have shown that $u_1(t,x;\theta) \leqq e^{\frac{1}{2}\lambda t} w(|x|-1)$. Since this inequality is valid for all $t>0$, $|x|>1$, $\theta>0$, and $\lambda>0$, we may set $\lambda=2/t$, whereupon we arrive at the inequality in (3.2.2). □

Lemma 3.2. *Let $u_1(t,x;\theta)$ be as in Lemma 3.1. Then $u_1(t,x;\theta)$ increases to some $u_\infty(t,x)$ as $\theta \to \infty$, and there is a constant $c_{3.2.1}$ such that for all $t>1$ and all x,*

(3.2.4) $$u_\infty(t,x) \leqq c_{3.2.1} \cdot p(t+1,x).$$

Proof. Noting that $\theta \mapsto u(\cdot,\cdot;\theta)$ is increasing (an easy consequence of Lemma 3.0) define

(3.2.5) $$u_\infty(t,x) := \lim_{\theta\to\infty} u_1(t,x;\theta) \leqq \frac{2}{t} \qquad \text{(see (3.2.3))}.$$

Also, by casting (3.2.1) into its mild form [cf. Pazy (1983, p. 106)]

(3.2.6) $$u_1(t) = \mathcal{T}_{t-\delta}(u_1(\delta)) - \tfrac{1}{2}\int_\delta^t \mathcal{T}_{t-s}(u_1^2(s))\,ds, \qquad t \geqq \delta > 0,$$

it follows from the bounded and monotone convergence of $u(\cdot,\cdot;\theta)$ to u_∞, that u_∞ also satisfies (3.2.6) and hence

(3.2.7) $$\dot{u}_\infty = \tfrac{1}{2}[\Delta u_\infty - u_\infty^2] \qquad \text{for } t>0$$

by standard regularity theory (see Ladyženskaya et al. (1968)). Lemma 3.1 implies that for $t>0$, $u_\infty(t,\cdot)$ is (Lebesgue) integrable. Also (3.2.2) implies that for $r>0$

(3.2.8) $$M(r) := \sup_{|x|\geqq 1+r} \sup_{t>0} u_\infty(t,x) < \infty.$$

The Feynman-Kac formula states that if, for example, k is continuous and bounded above, and $v \in C_b^{1,2}((0,\infty)\times\mathbb{R}^d) \cap C_b([0,\infty)\times\mathbb{R}^d)$ is a solution of

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\Delta v + k(t,x)\,v$$

then

$$v(t,x) = E_0^x\left[\exp\left\{\int_0^t k(t-s,B_s)\,ds\right\} v(0,B_t)\right].$$

(cf. Stroock and Varadhan (1979, p. 114)). Therefore

(3.2.9) $$\begin{aligned} u_1(t,x;\theta) &= E_0^x\left(u_1(0,B_t;\theta)\exp\left\{-\int_0^t \tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}\right) \\ &= \theta E_0^x\left(\phi(B_t)\exp\left\{-\int_0^t \tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}\right). \end{aligned}$$

If $1<r<|x|$, then

$$u_1(t,x;\theta)=E_0^x\left(\exp\left\{-\int_0^{T_r}\tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}\mathbf{1}(T_r<t)\right.$$
$$\left.\cdot\,\theta E_0^x\left(\phi(B_{t-T_r}\circ\Theta_{T_r})\exp\left\{-\int_{T_r}^{t}\tfrac{1}{2}u_1(t-s,B_s;\theta)\right\}|\mathscr{F}_{T_r}\right)\right)$$
$$=E_0^x\left(\exp\left\{-\int_0^{T_r}\tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}\mathbf{1}(T_r<t)\right.$$
$$\left.\cdot\,\theta E_0^x\left(\phi(B_{t-T_r}\circ\Theta_{T_r})\exp\left\{-\int_0^{t-T_r}\tfrac{1}{2}u_1(t-T_r-v,B_v\circ\Theta_{T_r};\theta)dv\right\}|\mathscr{F}_{T_r}\right)\right).$$

That is, by the strong Markov property for B_t,

(3.2.10) $$u_1(t,x;\theta)=E_0^x\left(\exp\left\{-\int_0^{T_r}\tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}\mathbf{1}(T_r<t)\cdot u_1(t-T_r,B_{T_r};\theta)\right).$$

Replace T_r by $v\in[0,t)$ in the above argument to obtain

(3.2.11) $$u_1(t,x;\theta)=E_0^x\left(\exp\left\{-\int_0^{v}\tfrac{1}{2}u_1(t-s,B_s;\theta)\,ds\right\}u_1(t-v,B_v;\theta)\right)$$
for all $t>0$, $x\in\mathbb{R}^d$.

From (3.2.8) and (3.2.10) we have for $|x|>r>1$,

$$u_1(t,x;\theta)\leqq M(r-1)\,P_0^x(T_r<t)$$

and therefore

$$\begin{aligned}u_\infty(t,x)&\leqq M(r-1)\,P_0^x(T_r<t)\\&\leqq M(r-1)\,P_0^0(\sup_{s\leqq t}|B_s|>(|x|-r))\\&\leqq 2M(r-1)\,P_0^0\left(|B_1|>\frac{|x|-r}{\sqrt{t}}\right)\ \textit{(modified reflection principle)}\\&\leqq c_1(d)\,M(r-1)\left(\frac{|x|-r}{\sqrt{t}}+1\right)^{d-2}\exp\left(-\frac{(|x|-r)^2}{2t}\right).\end{aligned}$$

Therefore, (take $r=2$) there is a constant $c_2(d,t)$ such that for $|x|>2$,

(3.2.12) $$u_\infty(t,x)\leqq c_2(d,t)\exp\left\{-\frac{|x|^2}{4t}\right\}$$

by an easy computation.

Together with (3.2.5), (3.2.12) implies that there is a constant $c_{3.2.1}(d)$ such that

(3.2.13) $$u_\infty(1, x) \leqq c_{3.2.1}\, p(2, x) \quad \text{for all } x.$$

Combining (3.2.13) with (3.2.11), for $t > 1$,

$$\begin{aligned} u_1(t, x; \theta) &\leqq E_0^x(u_1(1, B_{t-1}; \theta)) \\ &\leqq E_0^x(u_\infty(1, B_{t-1})) \\ &\leqq c_{3.2.1}\, E_0^x(p(2, B_{t-1})) \\ &= c_{3.2.1} \int_{\mathbb{R}^d} p_{t-1}(y-x)\, p(2, y)\, dy \\ &= c_{3.2.1}\, p(t+1, x) \quad \text{by Chapman-Kolmogorov.} \end{aligned}$$

Finally, this yields

$$u_\infty(t, x) \leqq c_{3.2.1}\, p(t+1, x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t > 1. \quad \square$$

Lemma 3.3. *Let $d \geqq 3$. Then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-d} u_\infty(\varepsilon^{-2} t, \varepsilon^{-1} x) = c_{3.2.2} \cdot p(t, x),$$

the convergence being uniform on $[\delta, \infty) \times \mathbb{R}^d$ for any $\delta > 0$. The constant $c_{3.2.2}(d) > 0$ is given by

$$c_{3.2.2} = \lim_{t \to \infty} \int_{\mathbb{R}^d} u_\infty(t, x)\, dx.$$

Proof. By Lemma 3.1 $u_\infty(\delta, \cdot) \in L^1(\mathbb{R}^d)_+$ for fixed $\delta > 0$. Set

(3.2.14) $$w_\varepsilon(t, x) = \varepsilon^{-d} u_\infty((\varepsilon^{-2} t) + \delta, \varepsilon^{-1} x).$$

Then (by (3.2.7))

$$\begin{aligned} \dot{w}_\varepsilon &= \tfrac{1}{2}(\Delta w_\varepsilon - \varepsilon^{d-2} w_\varepsilon^2) \\ w_\varepsilon(0, x) &= \varepsilon^{-d} u_\infty(\delta, \varepsilon^{-1} x). \end{aligned}$$

Then we may apply Proposition 3.1 of Gmira and Veron (1984) to w_ε to conclude that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-d} u_\infty(\varepsilon^{-2} t + \delta, \varepsilon^{-1} x) = c_{3.2.2}\, p(t, x)$$

where

(3.2.15) $$\begin{aligned} c_{3.2.2}(d) &= \int_{\mathbb{R}^d} u_\infty(\delta, x)\, dx - \tfrac{1}{2} \int_\delta^\infty \int_{\mathbb{R}^d} u_\infty^2(t, x)\, dx\, dt \\ &= \lim_{t \to \infty} \int_{\mathbb{R}^d} u_\infty(t, x)\, dx. \end{aligned}$$

More precisely, as $\varepsilon \downarrow 0$ $w_\varepsilon(t, x)$ converges uniformly on compact subsets of $(0, \infty) \times \mathbb{R}^d$. Together with Lemma 3.2 this yields the uniform convergence on $[a, \infty) \times \mathbb{R}^d$ for any $a > 0$. Moreover, under this uniform convergence if $\lim_{\varepsilon \downarrow 0} t_\varepsilon = t_0 > 0$, then $\lim_{\varepsilon \downarrow 0} w_\varepsilon(t_\varepsilon, x) = c_{3.2.2} \cdot p(t_0, x)$. Choosing $t_\varepsilon = t_0 - \varepsilon^2 \delta$ and relabelling x as $x - x_0$ yields that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-d} u(\varepsilon^{-2} t_0, \varepsilon^{-1}[x - x_0]) = c_{3.2.2}\, p(t_0, x - x_0)$$

and convergence is uniform on $[a, \infty) \times \mathbb{R}^d$ for $a > 0$. The fact that $c_{3.2.2}(d) > 0$ for $d \geqq 3$ is established in Dawson (1977, Theorem 3.1). □

Lemma 3.4. (Iscoe (1988)). *The singular elliptic boundary value problem*

$$\Delta u(x) = u(x)^2, \qquad x \in \overline{B(0;\varepsilon)}^c$$
$$u(x) \to +\infty, \qquad \text{as } |x| \to \varepsilon^+$$
$$u(x) \to 0, \qquad \text{as } |x| \to +\infty$$

has a unique non-negative solution $u \equiv u(\cdot\,; \varepsilon)$. It is strictly positive and radial. Moreover

$$u(x; \varepsilon) = \varepsilon^{-2} u(\varepsilon^{-1} x; 1), \tag{3.2.16}$$

$$u(x; \varepsilon) \leqq 6(|x| - \varepsilon)^{-2} \quad \text{for } |x| > \varepsilon, \tag{3.2.17}$$

$$u(x; 1) \sim f_d(|x|) = f(|x|) \quad \text{as } |x| \to \infty, \tag{3.2.18}$$

where

$$f(r) = \begin{cases} 2(4-d)\, r^{-2} & 1 \leqq d \leqq 3 \\ 2 r^{-2} (\log r)^{-1} & d = 4 \\ c_{3.3} \cdot r^{2-d} & d \geqq 5. \end{cases}$$

Moreover when $d \geqq 5$, $u(x; 1) > f(|x|)$.

Lemma 3.5. *Let $R > 1$, and $\phi_R(x) = \phi(R^{-1} x)$ where ϕ is as in Lemma 3.1. Let $u^\theta(t, x)$ $(\theta > 0)$ be the unique non-negative solution of*

$$\frac{\partial u^\theta}{\partial t} = \frac{1}{2}(\Delta u^\theta - (u^\theta)^2) + \theta \phi_R, \qquad u^\theta(0, \cdot) \equiv 0.$$

Then $u^\theta(t, x)$ increases to some $u^\infty(t, x) \in [0, \infty]$ as $\theta \uparrow \infty$. If $|x| > r > R$, then

$$u^\infty(t, x) \leqq 6(r - R)^{-2} P_0^x(T_r \leqq t).$$

Proof. By Iscoe (1986a, Theorem 3.1) we have

$$E^m\left(\exp\left\{-\theta \int_0^t \langle \phi_R, X_s \rangle\, ds\right\}\right) = \exp\left\{-\int u^\theta(t, x)\, dm(x)\right\}. \tag{3.2.19}$$

Let $m=\delta_x$ in (3.2.19) and use the monotonicity of the left-hand side to see that $u^\theta(t,x)$ is non-decreasing in θ for each (t,x). Hence we may define

$$u^\infty(t,x) := \lim_{\theta\to\infty} u^\theta(t,x) \in [0,\infty]. \tag{3.2.20}$$

Fix $t>0$ and let $\bar{u}^\theta(s,x)=u^\theta(t-s,x)$, $s\leqq t$. Define

$$\bar{A}f(s,x)=\tfrac{1}{2}\Delta f(s,x)+\frac{\partial}{\partial s}f(s,x)-\frac{\bar{u}^\theta(s,x)}{2}f(s,x). \tag{3.2.21}$$

More precisely, $\bar{A}$ is the generator of

$$\bar{X}(s) := \begin{cases} (s,B_s) & \text{if } \int_0^s \bar{u}^\theta(v,B_v)\,dv<e \quad \text{and} \quad s<t \\ \Delta & \text{if } \int_0^s \bar{u}^\theta(v,B_v)\,dv \geqq e \quad \text{or} \quad s\geqq t \end{cases}$$

where e is an independent exponential random variable with mean 1. The domain of $\bar{A}$ contains $f\in C^{1,2}([0,t)\times\mathbb{R}^d)$ such that f and $\bar{A}f$ both vanish at ∞. Then

$$\bar{A}\bar{u}^\theta(s,x)=\tfrac{1}{2}\Delta u^\theta(t-s,x)-\frac{\partial}{\partial s}u^\theta(t-s,x)-\frac{u^\theta(t-s,x)^2}{2}=-\theta\phi_R(x).$$

Hence if $\{\bar{R}_\lambda\colon \lambda\geqq 0\}$ is the resolvent of $\bar{X}$,

$$u^\theta(t,x)=\bar{u}^\theta(0,x)=\theta\bar{R}_0\,\phi_R(0,x)=\theta E^{(0,x)}\int_0^t \phi_R(\bar{X}(s))\,ds;$$

$$u^\theta(t,x)=\theta E_0^x\left(\int_0^t \phi_R(B_s)\exp\left\{-\int_0^s u^\theta(t-v,B_v)\,dv\right\}ds\right). \tag{3.2.22}$$

Since $u^\theta(t,x)\uparrow u^\theta(\infty,x)$ as $t\uparrow\infty$ and $u^\theta(\infty,x)\uparrow u(x;R)$ as $\theta\uparrow\infty$, where $u(x;R)$ is as in Lemma 3.4 (see Iscoe (1988)), then (3.2.17) yields

$$\sup_{\substack{t>0\\ |x|\geqq R+\delta}} u^\infty(t,x)\leqq 6\delta^{-2}. \tag{3.2.23}$$

If $|x|>r>R$, then use (3.2.22) and the strong Markov property at T_r to obtain

$$\begin{aligned} u^\theta(t,x)=\theta E_0^x\Bigg(&\mathbf{1}(T_r<t)\exp\left\{-\int_0^{T_r} u^\theta(t-v,B_v)\,dv\right\} \\ &\cdot E_0^x\left(\int_0^{t-T_r}\phi_R(B_s)\circ\Theta_{T_r}\cdot\exp\left\{-\int_0^s u^\theta(t-T_r-v,B_v\circ\Theta_{T_r})\,dv\right\}ds\,|\,\mathscr{F}_{T_r}\right)\Bigg) \end{aligned}$$

that is,

$$(3.2.24)\quad u^\theta(t,x)=\theta E_0^x\left(\mathbf{1}(T_r<t)\exp\left\{-\int_0^{T_r} u^\theta(t-v,B_v)\,dv\right\}\right.$$
$$\left.\cdot E_0^{B(T_r(\omega))}\left(\int_0^{t-T_r(\omega)}\phi(B_s)\exp\left\{-\int_0^s u^\theta(t-T_r(\omega)-v,B_v)\,dv\right\}ds\right)\right)$$
$$\leqq E_0^x(\mathbf{1}(T_r<t)\,u^\theta(t-T_r,B(T_r)))\quad\text{by (3.2.22)}$$
$$\leqq 6(r-R)^{-2}P_0^x(T_r<t)\quad\text{by (3.2.23).}$$

Therefore

$$u^\infty(t,x)\leqq 6(r-R)^{-2}P_0^x(T_r<t).\quad\square$$

Remark. Equation (3.2.22) can also be proved by using the representation (3.2.11) of the Feynman-Kac semigroup and the usual convolution integral solution of the inhomogeneous equation (cf. Pazy (1983, 4.2)).

Lemma 3.6. *Let $u(\cdot;R)$ be the unique positive solution to the boundary value problem*

$$(3.2.25)\quad \Delta u(x)=u^2(x)\quad \textit{for}\ x\in B(0;R)$$
$$u(x)\to\infty\quad \textit{as}\ |x|\uparrow R.$$

Then

$$(3.2.26)\quad u(x;R)=R^{-2}u(x/R;1)$$

and

$$(3.2.27)\quad \lim_{|x|\uparrow 1}\frac{u(x;1)}{[6/(1-|x|)^2]}=1.$$

Proof. The first half of the lemma (as well as the existence and uniqueness of u) was proved in Proposition (3.15) of Iscoe (1988). To obtain (3.2.27) we adapt the technique used in Sawyer and Fleischman (1979) wherein the asymptotics at infinity of positive, bounded, radial solutions of $\Delta u=u^2$ was determined.

We write simply u for $u(\cdot;1)$. Rewriting the differential equation of (3.2.25) in radial form,

$$[r^{d-1}u'(r)]'=r^{d-1}u(r)^2\quad \text{(where the prime denotes } d/dr)$$

and integrating twice yields:

$$(3.2.28)\quad u(r)=\psi(r)+\int_\rho^r s^{1-d}\int_\rho^s u^2(t)\,t^{d-1}\,dt\,ds,\qquad \rho\leqq r<1$$

where ρ denotes some fixed point in (0, 1) and

$$\psi(r)=u(\rho)+\rho^{d-1}u'(\rho)\int_{\rho}^{r} s^{1-d}\,ds,$$

which is a bounded, positive function of $r\in[\rho, 1)$. Let $w(r)=6(1-r)^{-2}$ and set $v=u/w$. From (3.2.28), we see that v is a solution of the integral equation:

$$(3.2.29)\qquad v(r)=\psi(r)\cdot[w(r)]^{-1} + [w(r)]^{-1}\cdot\int_{\rho}^{r} s^{1-d}\int_{\rho}^{s} v(t)^2 w(t)^2 t^{d-1}\,dt\,ds,\qquad \rho\leqq r<1.$$

With $g(r):=[w(r)]^{-1}\cdot\int_{\rho}^{r} s^{1-d}\int_{\rho}^{s} w(t)^2 t^{d-1}\,dt\,ds$, it follows from L'Hôpital's rule that $\lim_{r\uparrow 1} g(r)=1$.

We must first establish that $\limsup_{r\uparrow 1} v(r)\neq 0$ and that $\liminf_{r\uparrow 1} v(r)\neq\infty$. Concerning the former, it can be shown, as in Iscoe (1988) (see (3.11), (3.13) therein), that since $u(r)^2=u''(r)+[(d-1)/r]\,u'(r)\geqq u''(r)$ for $0<r<1$, then

$$\int_{[u(r)/u(0)]}^{\infty} [(2/3)(v^3-1)]^{-\frac{1}{2}}\,dv\leqq\sqrt{u(0)}\cdot(1-r).$$

Changing variables in the integral by $v=(1-r)^{-2}y$ leads to

$$I(r):=\int_{l(r)}^{\infty} \{(2/3)\,[y^2-(1-r)^6]\}^{-\frac{1}{2}}\,dy\leqq\sqrt{u(0)},$$

$$l(r):=(1-r)^2 u(r)/u(0).$$

If $\lim_{r\uparrow 1} l(r)=0$, then by Fatou's lemma,

$$+\infty=\int_{0}^{\infty}\sqrt{3/2}\,y^{-3/2}\,dy\leqq\liminf_{r\to 1} I(r)\leqq\sqrt{u(0)},$$

which is absurd. Therefore $\limsup_{r\to 1} v(r)>0$. Also if $\lim_{r\to 1} v(r)=+\infty$, then with r_0 chosen such that $v(r_0)=N>2$, and $v(r)\geqq v(r_0)$, $g(r)>1/2$ for $r\geqq r_0$, it follows from (3.2.29) that $N=v(r_0)>N^2/2>N$ which is absurd. Therefore $\liminf_{r\to 1} v(r)<+\infty$.

If eventually (i.e. for all r sufficiently near 1) $v(r)\leqq c<1$, then from (3.2.29) it follows that $v(r)\leqq o(1)+c^2[1+o(1)]$; so that $\limsup_{r\uparrow 1} v(r)\leqq c^2$. Therefore if we assume that $c_0:=\limsup_{r\uparrow 1} v(r)<1$, then with $c=c_0[1+\varepsilon]$, where $0<\varepsilon<c_0^{-\frac{1}{2}}-1$,

we obtain absurdity: $c_0 \leqq c^2 < c_0$. Similarly the assumption that $1 < \liminf_{r \uparrow 1} v(r)$ leads to a contradiction. Thus we have established that

$$\limsup_{r \uparrow 1} \frac{u(r)}{w(r)} \geqq 1 \quad \text{and} \quad \liminf_{r \uparrow 1} \frac{u(r)}{w(r)} \leqq 1.$$

We now change the notation and set, for $c > 0$, $v = u - cw = w[u/w - c]$. Note that $w''(r) + \frac{d-1}{r} w'(r) = f(r)\, w(r)^2$ where $f(r) = 1 + [(d-1)(1-r)/(3r)]$; so that $\lim_{r \uparrow 1} f(r) = 1$. Therefore

$$(3.2.30) \quad v''(r) + \frac{d-1}{r} v'(r) = u(r^2) - c f(r)\, w(r)^2 = w(r)^2 \cdot \left(\left[\frac{u(r)}{w(r)}\right]^2 - c f(r) \right).$$

With $c > 1$ we have that v is negative infinitely often since $\liminf_{r \uparrow 1} \frac{u(r)}{w(r)} \leqq 1$. If $v(r)$ were also positive infinitely often as $r \uparrow 1$, then v would possess a sequence of positive local maxima, attained at $(r_n)_{n \in \mathbb{N}}$ where $\lim_{n \to +\infty} r_n = 1$. At such points, the left hand side of (3.2.30) is nonpositive while for sufficiently large n, the right hand side is positive since $\left[\frac{u(r_n)}{w(r_n)}\right]^2 - c f(r_n) > c^2 - c f(r_n) \to c^2 - c > 0$ as $n \to +\infty$. Therefore v is eventually (and permanently) nonpositive, i.e. $\limsup_{r \uparrow 1} \frac{u(r)}{w(r)} \leqq c$. Since $c > 1$ but otherwise arbitrary, $\limsup_{r \uparrow 1} \frac{u(r)}{w(r)} \leqq 1$. Similarly, arguing with $c < 1$, we deduce that $\liminf_{r \uparrow 1} \frac{u(r)}{w(r)} \geqq 1$. Therefore $\lim_{r \uparrow 1} \frac{u(r)}{w(r)} = 1$. □

3.3. *Proofs of the Theorems*

Proof of Theorem 3.1. Let $\varepsilon \in (0, 1]$, ϕ be as in Lemma 3.1, and $\phi_\varepsilon(x) := \phi([x - x_0]/\varepsilon)$. Then (see (1.1a))

$$E^m \exp(-\theta \varepsilon^{-2} \langle \phi_\varepsilon, X_t \rangle) = \exp(-\int u(t, x; \theta, \varepsilon)\, m(dx))$$

where $\dot{u} := \frac{1}{2}(\Delta u - u^2)$, $u(0) = \varepsilon^{-2} \theta \phi_\varepsilon$. If we define

$$u_1(t, x; \theta) := \varepsilon^2 u(\varepsilon^2 t, x_0 + \varepsilon x; \theta, \varepsilon)$$

then u_1 satisfies (3.2.1); and so u_1 is independent of ε. Therefore

$$Q^m(X_t(B(x_0; \varepsilon)) > 0) = 1 - \lim_{\theta \to \infty} E^m \exp(-\theta X_t(B(x_0; \varepsilon)))$$

$$= 1 - \exp\left(- \int_{\mathbb{R}^d} \varepsilon^{-2} \cdot u_\infty \left(\varepsilon^{-2} t, \frac{x - x_0}{\varepsilon} \right) m(dx) \right)$$

where $u_\infty(t,x)=\lim_{\theta\to\infty} u_1(t,x;\theta)$, as in Lemma 3.2, the passage to the limit under the integral being justified by the monotone convergence of $u_1(\cdot,\cdot;\theta)$ to u_∞ as $\theta\uparrow\infty$. By Lemma 3.2 there is a constant $c_{3.2.1}$ such that for all $t\geqq 1$,

$$u_\infty(t,x)\leqq c_{3.2.1}\, p(t+1,x).$$

From this it follows that for $t\geqq\varepsilon^2$,

$$\begin{aligned}
Q^m(X_t(B(x_0;\varepsilon))>0) &\leqq \varepsilon^{-2}\int u_\infty\left(\varepsilon^{-2}t,\frac{x-x_0}{\varepsilon}\right) m(dx)\\
&\leqq \varepsilon^{-2}c_{3.2.1}\int p\left(\varepsilon^{-2}t+1,\frac{x-x_0}{\varepsilon}\right) m(dx)\\
&\leqq \varepsilon^{-2}c_{3.1}\int \frac{1}{(\varepsilon^{-2}t+1)^{d/2}}\exp\left(\frac{-(x-x_0)^2}{2(t+\varepsilon^2)}\right) m(dx)\\
&\leqq c_{3.1}\,\varepsilon^{d-2} m(\mathbb{R}^d)\frac{1}{(t+\varepsilon^2)^{d/2}}\\
&\leqq c_{3.1}\,\varepsilon^{d-2}t^{-d/2}\, m(\mathbb{R}^d)
\end{aligned}$$

thus yielding (a).

Further, from Lemma 3.3 (relabelling $c_{3.2.2}$ as $c_{3.2}$),

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-d}u_\infty(\varepsilon^{-2}t,\varepsilon^{-1}x)=c_{3.2}\cdot p(t,x),$$

uniformly on $[\delta,\infty)\times R^d$ for any $\delta>0$, where

$$c_{3.2}=\lim_{t\to\infty}\int_{R^d} u_\infty(t,x)\,dx.$$

This yields

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-d}\int u_\infty(\varepsilon^{-2}t,\varepsilon^{-1}[x-x_0])\,m(dx)=c_{3.2}\int p(t,x-x_0)\,m(dx)$$

uniformly for $t>\delta$, and $m(\mathbb{R}^d)<K$, and consequently the uniform convergence of

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{d-2}}\left[1-\exp\left(-\int_{\mathbb{R}^d}\varepsilon^{-2}u_\infty(\varepsilon^{-2}t,\varepsilon^{-1}[x-x_0])\,m(dx)\right)\right]\\
&\quad=c_{3.2}\cdot\left[\int_{\mathbb{R}^d} p(t,x-x_0)\,m(dx)\right]
\end{aligned}$$

which completes the proof of (b). □

Proof of Theorem 3.2. Let $u(x;\varepsilon)$ be as in Lemma 3.4. It will be convenient to set $u(x;\varepsilon)=+\infty$ if $|x|\leqq\varepsilon$. Argue as in the proof of Theorem 2 in Iscoe (1988) but with $m\in M_F(\mathbb{R}^d)$ in place of δ_x to see that

$$\begin{aligned}
(3.3.1)\ Q^m(X_t(B(x;\varepsilon))&=0 \text{ for all } t\geqq 0)\\
&=Q^{m_x}(X_t(B(0;\varepsilon))=0 \text{ for all } t\geqq 0)\qquad (m_x(A)\equiv m(A+x))\\
&=\exp\{-\textstyle\int u(y;\varepsilon)\,dm_x(y)\}\\
&=\exp\{-\varepsilon^{-2}\textstyle\int u(\varepsilon^{-1}(y-x);1)\,dm(y)\}\quad \text{by (3.2.16).}
\end{aligned}$$

Let $f(\cdot)$ be as in (3.2.18), and denote

$$(1+\varepsilon_k^{(1)}):=\sup\left\{\frac{u(x;1)}{f(|x|)}:|x|\geqq k\right\}\to 1\qquad \text{as } k\to\infty$$

$$(1-\varepsilon_k^{(2)}):=\inf\left\{\frac{u(x;1)}{f(|x|)}:|x|\geqq k\right\}\to 1\qquad \text{as } k\to\infty.$$

Note that by Lemma 3.4, when $d>4$, $\varepsilon_k^{(1)}>0$ and $\varepsilon_k^{(2)}=0$. In the case $d=4$ neither $\varepsilon_k^{(1)}$ nor $\varepsilon_k^{(2)}$ are assumed to be positive.

(a) Assume $d>4$ and $k\in\mathbb{N}$. If $d(\text{supp}(m),x)>\varepsilon k$, then (3.3.1) yields

$$\begin{aligned}
(3.3.2)\qquad Q^m(X_t(B(x;\varepsilon))&>0 \text{ for some } t\geqq 0)\\
&\leqq\varepsilon^{-2}\textstyle\int u(\varepsilon^{-1}(y-x);1)\,dm(y)\\
&\leqq(1+\varepsilon_k^{(1)})\,c_{3.3}\,\varepsilon^{d-4}\textstyle\int|y-x|^{2-d}dm(y).
\end{aligned}$$

Now use the Markov property at $t=\delta$ to get that for $\varepsilon^2<\delta$,

$$\begin{aligned}
&Q^m(X_t(B(x;\varepsilon)>0 \text{ for some } t\geqq\delta)\\
&\quad\leqq Q^m(X_\delta(B(x;k\varepsilon))>0)\\
&\qquad+E^m(\mathbf{1}(X_\delta(B(x;k\varepsilon))=0)\,Q^{X_\delta}(X_t(B(x;\varepsilon))>0 \text{ for some } t\geqq 0))\\
&\quad\leqq c_{3.1}\,\delta^{-d/2}m(\mathbb{R}^d)\,(k\,\varepsilon)^{d-2}+(1+\varepsilon_k^{(1)})\,c_{3.3}\,\varepsilon^{d-4}E^m(\textstyle\int|y-x|^{2-d}dX_\delta(y))\\
&\quad\text{by Theorem 3.1.a and (3.3.2)}\\
&\quad\leqq(1+\varepsilon_k^{(1)})\,\varepsilon^{d-4}c_{3.3}(E_0^m(|B_\delta-x|^{2-d})+(c_{3.1}/c_{3.3})\,\delta^{-d/2}k^{d-2}m(\mathbb{R}^d)\,\varepsilon^2).
\end{aligned}$$

The last term within the parentheses is less than 2^{-k} if ε is taken smaller than a certain quantity depending only on $(k,\delta,m(\mathbb{R}^d))$. This completes the proof of (a).

(b) Assume $d=4$, $k\in\mathbb{N}$, $0<\varepsilon<1$ and $d(\text{supp}(m),x)>(k\varepsilon)\vee\varepsilon^{1/k}$. We will first show that

$$(3.3.3)\quad Q^m(X_t(B(x;\varepsilon))>0 \text{ for some } t\geqq 0)\leqq(1+\varepsilon_k^{(1)})\,2\left(\log\frac{1}{\varepsilon}\right)^{-1}\int|y-x|^{-2}dm(y).$$

To verify (3.3.3) note that if $y \in \mathrm{supp}(m)$, then

$$(3.3.4) \qquad (\log(|y-x|/\varepsilon))^{-1} \leqq \left(\left(1-\frac{1}{k}\right)\log\frac{1}{\varepsilon}\right)^{-1}.$$

Therefore (3.3.1) implies that

$$\begin{aligned} &Q^m(X_t(B(x;\varepsilon))>0 \text{ for some } t \geqq 0) \\ &\quad \leqq 1-\exp\left\{-2(1+\varepsilon_k^{(1)})\,\varepsilon^{-2}\varepsilon^2 \int |x-y|^{-2}\left(\log\frac{|x-y|}{\varepsilon}\right)^{-1} dm(y)\right\} \\ &\quad \leqq 2(1+\varepsilon_k^{(1)})(1-1/k)^{-1}\left(\log\frac{1}{\varepsilon}\right)^{-1}\int |x-y|^{-2}dm(y) \quad \text{by (3.3.4).} \end{aligned}$$

This proves (3.3.3). Now applying the Markov property at $t=\delta$,

$$\begin{aligned} &Q^m(X_t(B(x;\varepsilon))>0 \text{ for some } t \geqq \delta) \\ &\quad \leqq Q^m(X_\delta(B(x;k\varepsilon \vee \varepsilon^{1/k}))>0) \\ &\qquad + E^m(\mathbf{1}(X_\delta(B(x;k\varepsilon\vee\varepsilon^{1/k}))=0)\, Q^{X_\delta}(X_t(B(x;\varepsilon))>0 \text{ for some } t\geqq 0)) \\ &\quad \leqq c_{3.1}\,\delta^{-2} m(\mathbb{R}^4)(k\varepsilon\vee\varepsilon^{1/k})^2 + (1+\varepsilon_k^{(1)}) \\ &\qquad \cdot\left(1-\frac{1}{k}\right)^{-1} 2(\log 1/\varepsilon)^{-1} E^m\left(\int |y-x|^{-2} dX_\delta(y)\right) \\ &\quad \text{by Theorem 3.1.a and (3.3.3)} \\ &\quad \leqq (1+\varepsilon_k^{(1)})\left(1-\frac{1}{k}\right)^{-1} 2(\log 1/\varepsilon)^{-1}(E_0^m(|B_\delta - x|^{-2}) \\ &\qquad + (c_{3.1}/2)\,\delta^{-2} m(\mathbb{R}^4)(k\varepsilon\vee\varepsilon^{1/k})^2 \log 1/\varepsilon). \end{aligned}$$

The proof is completed as for (a).

(c) Let $d>4$ and $k \in \mathbb{N}$. Then (3.3.1) implies that

$$\begin{aligned} &Q^m(X_t(B(x;\varepsilon))>0 \text{ for some } t\geqq 0) \\ &\quad = 1-\exp\left\{-\int u(y;\varepsilon)\, dm_x(y)\right\} \\ &\quad \geqq 1-\exp\left\{-\varepsilon^{-2}\int \mathbf{1}(|y-x|\geqq k\varepsilon)\, u\left(\frac{y-x}{\varepsilon};1\right) dm(y)\right\} \\ &\quad \geqq 1-\exp\left\{-c_{3.3}\,\varepsilon^{d-4}\cdot\int \mathbf{1}(|y-x|\geqq k\varepsilon)\,|y-x|^{2-d} dm(y)\right\}. \end{aligned}$$

Therefore we can choose $\tilde{\varepsilon}_d$ depending only on k and $\int |y-x|^{2-d} dm(y)$ such that if $\varepsilon < \tilde{\varepsilon}_d$, then

$$\begin{aligned} (3.3.4) \qquad &Q^m(X_t(B(x;\varepsilon))>0 \text{ for some } t\geqq 0) \\ &\quad \geqq (1-\varepsilon_k)\, c_{3.3}\,\varepsilon^{d-4}\int \mathbf{1}(|y-x|\geqq k\varepsilon)\,|y-x|^{2-d} dm(y) \end{aligned}$$

where $\varepsilon_k \to 0$ as $k\to\infty$. This completes the proof of (c).

(d) Let $d=4$ and $k\in\mathbb{N}$. Note that if $0<\varepsilon<1$ and $|x-y|\leqq\varepsilon^{-1/k}$, then

(3.3.5) $$\left(\log\frac{|y-x|}{\varepsilon}\right)^{-1}\geqq(k/k+1)\left(\log\frac{1}{\varepsilon}\right)^{-1}.$$

Let

$$\psi_\varepsilon(y-x):=\mathbf{1}(k\varepsilon\leqq|y-x|\leqq\varepsilon^{-1/k}).$$

Then by (3.3.5)

(3.3.6) $$\int u\left(\frac{y-x}{\varepsilon};1\right)dm(y)\geqq(1-\varepsilon_k^{(2)})\int\psi_\varepsilon(y-x)\frac{2\varepsilon^2}{|y-x|^2\log\frac{|y-x|}{\varepsilon}}dm(y)$$
$$\geqq 2\varepsilon^2(1-\varepsilon_k^{(2)})\left(\log\frac{1}{\varepsilon}\right)^{-1}(k/(k+1))\int\psi_\varepsilon(y-x)\,|y-x|^{-2}dm(y).$$

Then (3.3.1) and (3.3.6) imply that

(3.3.7) $$Q^m(X_t(B(x;\varepsilon))>0\text{ for some }t\geqq0)$$
$$\geqq 1-\exp\left\{-2(k/k+1)(1-\varepsilon_k^{(2)})\left(\log\frac{1}{\varepsilon}\right)^{-1}\int\psi_\varepsilon(y-x)\,|y-x|^{-2}dm(y)\right\}.$$

By (3.3.7) we can choose $\tilde{\varepsilon}_4$ and depending only on k and $\int|y-x|^{-2}dm(y)$ such that if $0<\varepsilon<\tilde{\varepsilon}_4$, then

(3.3.8) $$Q^m(X_t(B(x;\varepsilon))>0\text{ for some }t\geqq0)$$
$$\geqq 2(1-\varepsilon_k)\left(\log\frac{1}{\varepsilon}\right)^{-1}\int\psi_\varepsilon(y-x)\,|y-x|^{-2}dm(y)$$

where $\varepsilon_k\to0$ as $k\to\infty$. This completes the proof of (d). □

Proof of Theorem 1.3. If $x\notin\operatorname{supp}(m)$, then the continuity of X_t implies that

(3.3.9) $$Q^m(X\text{ hits }\{x\})=Q^m(\text{for all }\varepsilon>0\ \exists t>0\text{ such that }X_t(B(x;\varepsilon))>0),$$

this follows from $x\notin\operatorname{supp}(m)$ and Theorem 1.1, whose proof will be independent of this result.

By (3.3.1)

(3.3.10) $$Q^m(X_t(B(x;\varepsilon)>0\text{ for some }t>0)$$
$$=1-\exp\{-\varepsilon^{-2}\int u(\varepsilon^{-1}(y-x);1)\,dm(y)\},$$

and by (3.2.18) if $y\neq x$.

(3.3.11) $$\lim_{\varepsilon\to0}\varepsilon^{-2}u(\varepsilon^{-1}(y-x);1)=2(4-d)\,|y-x|^{-2}.$$

Therefore if $d \leqq 3$ and $d(x, \operatorname{supp}(m)) > 0$, then by (3.2.17), and bounded convergence,

$$Q^m(X \text{ hits } \{x\}) = 1 - \exp\{-2(4-d) \int |y-x|^{-2} \, dm(y)\}.$$

On the other hand if $x \in \operatorname{supp}(m)$, and $m(\{x\}) = 0$ then

$$Q^m(X \text{ hits } \{x\}) \geqq \lim_{\varepsilon \downarrow 0} Q^{m^c_{x,\varepsilon}}(X \text{ hits } \{x\}) = 1 - \exp\{-2(4-d) \int |y-x|^{-2} \, dm(y)\}.$$

If $\int |y-x|^{-2} \, dm(y) = +\infty$, then $Q^m(X$ hits $\{x\}) = 1$ and we are finished. If $\int |y-x|^{-2} \, dm(y) < +\infty$ and $\delta > 0$, then for any $\varepsilon > 0$

$$\begin{aligned} Q^m(x \in \bar{R}([\delta, \infty))) \leqq 1 - \exp\{-2(4-d) &\int_{B(x;\varepsilon)^c} |y-x|^{-2} \, dm(y)\} \\ &+ Q^{m_{x,\varepsilon}}(X_\delta(\mathbb{R}^d) > 0) \end{aligned}$$

since $Q^m = Q^{m_{x,\varepsilon}} Q^{m^c_{x,\varepsilon}}$ (cf. (1.1 a)).
The result follows since by (1.1 a) and $(3.2.3)_h$ (letting $\theta \uparrow \infty$):

$$Q^{m_{x,\varepsilon}}(X_\delta(\mathbb{R}^d) > 0) = 1 - \exp\left\{\frac{-2m(B(x;\varepsilon))}{\delta}\right\} \leqq \frac{2m(B(x;\varepsilon))}{\delta} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Finally if $m(\{x\}) > 0$ note that

$$\begin{aligned} P^m(X \text{ hits } \{x\}) &\geqq E^m(P^{X_\delta}(X \text{ hits } \{x\})) \\ &= 1 - E^m(\exp\{-2(4-d) \int |y-x|^{-2} X_\delta(dy)\}) \\ &\quad \text{(since } E^m X_\delta(\{x\}) = 0 \text{ we may apply the previous} \\ &\quad \text{case with } m = X_\delta \text{ a.s.).} \\ &\to 1 \quad \text{as } \delta \downarrow 0, \end{aligned}$$

the last by an elementary argument using weak continuity at $t=0$. □

Proof of Theorem 3.3.
(a) Let ϕ_R, $u^\theta(t, \cdot)$, $u^\infty(t, \cdot)$ be as in Lemma 3.5. Then

$$E^m\left(\exp\left[-\theta \int_0^t \langle \phi_R, X_s \rangle \, ds\right]\right) = \exp(-\int u^\theta(t,x) \, dm(x)).$$

Letting $\theta \uparrow \infty$, yields

(3.3.12)
$$\begin{aligned} &Q^m(X_s(B(0;R)) > 0 \text{ for some } s \leqq t) \\ &\quad = 1 - \exp(-\int u^\infty(t,x) \, dm(x)) \leqq \int u^\infty(t,x) \, dm(x). \end{aligned}$$

But by Lemma 3.5, if $|x|>r=R+1$,

$$u^{\infty}(t,x)\leqq 6P_0^x(T_{R+1}<t)\leqq 12P_0^0(|B_t|>|x|-(R+1)) \tag{3.3.13}$$
$$=12P_0^0\left(|B_1|>\frac{|x|-(R+1)}{\sqrt{t}}\right).$$

For $\delta>0$ there exists a constant $c_{3.6}(d,\delta)$ such that for $r\geqq\delta$

$$P_0^0(|B_1|>r)\leqq c_{3.6}(d,\delta)\,r^{d-2}e^{-r^2/2}. \tag{3.3.14}$$

Therefore if $|x|>R+2$, and $t>0$, there is a constant $c_{3.4}(d,t)$ such that

$$u^{\infty}(t,x)\leqq c_{3.4}\left(\frac{|x|-(R+1)}{\sqrt{t}}\right)^{d-2}\exp\left(-\frac{(|x|-(R+1))^2}{2t}\right). \tag{3.3.15}$$

If $\operatorname{supp}(m)\subset\overline{B(0;R+2)}^c$, then (a) follows from (3.3.12) and (3.3.15).

(b) Let $\tilde{u}^\theta\equiv\tilde{u}^\theta(t,x;R)$ $(\theta>0)$ denote the solution to

$$\frac{\partial\tilde{u}^\theta}{\partial t}=\frac{1}{2}(\Delta\tilde{u}^\theta-[\tilde{u}^\theta]^2)+\theta\psi \qquad \tilde{u}(0)\equiv 0$$

where

$$\psi(x)=\begin{cases}0, & |x|\leqq R\\ |x|/R-1, & R\leqq|x|\leqq 2R.\\ 1, & 2R\leqq|x|\end{cases}$$

$\tilde{u}^\theta(t,x)$ is increasing in θ, t (cf. Iscoe (1986a, Theorem 3.3) and (1988, Proposition 3.15)) for each x; and by a maximum-principle argument similar to Lemma (3.0) (working on a domain: $[0,T]\times\overline{B(0;R-\delta)}$)

$$\tilde{u}^\theta(t,x;R)\leqq u(x;R) \tag{3.3.16}$$

where u is the function of Lemma 3.6. Similarly to the proof of Lemma 3.5 one can derive, for any $0<R_1<R$:

$$\tilde{u}^\theta(t,0)\leqq E_0^0[\mathbf{1}(T_{R_1}^c<t)\,\tilde{u}^\theta(t-T_{R_1}^c,B(T_{R_1}^c))] \tag{3.3.17}$$
$$\leqq u(R_1;R)\,P_0^0(T_{R_1}^c<t) \quad \text{(by 3.3.16)}$$

where $T_{R_1}^c:=\inf\{t\colon |B_t|>R_1\}$.

As in part (a), and in analogy with (3.3.12) and (3.3.13)

$$q\equiv Q^{a\delta_0}(X_s(\overline{B(0;R)}^c)>0 \text{ for some } s\leqq t) \tag{3.3.18}$$
$$\leqq \lim_{\theta\to\infty} a\tilde{u}^\theta(t,0;R)$$
$$\leqq a\cdot u(R_1;R)\,P_0^0(T_{R_1}^c<t) \quad \text{(by 3.3.17))}$$
$$\leqq 2a\cdot u(R_1;R)\,P_0^0(|B_1|>R_1/\sqrt{t}).$$

Assume $\frac{R}{\sqrt{t}}>2$ and $R_1/R>1/2$. Then $R_1/\sqrt{t}>1$ and (3.3.14), (3.3.18), (3.2.26) and (3.2.27) yield (for some constant $c_1(d)$),

$$q \leqq c_1\, aR^{-2}\left(1-\frac{R_1}{R}\right)^{-2}\left(\frac{R_1}{\sqrt{t}}\right)^{d-2}\exp\left\{-\frac{R_1^2}{2t}\right\}$$
$$=\left[c_1\, aR^{-2}\left(\frac{R}{\sqrt{t}}\right)^{d-2}\exp\left(-\frac{R^2}{2t}\right)\right]g(R_1/R)$$

where

$$g(\alpha)=\alpha^{d-2}(1-\alpha)^{-2}\exp\{K(1-\alpha^2)\}, \quad \tfrac{1}{2}<\alpha<1, \quad K\equiv R^2/(2t)>2$$
$$\leqq 2[\beta^{-1}e^{K\beta}]^2, \quad \beta\equiv 1-\alpha \quad (1-\alpha^2=(1+\alpha)\beta\leqq 2\beta).$$

The quantity in square brackets is minimized by $\beta^*=1/K$; so if $R_1/R=1-\beta^*>\frac{1}{2}$, then $g(R_1/R)\leqq(e^2/2)\,(R/\sqrt{t})^4$; and we can choose $c_{3.5}$ so that

$$q\leqq c_{3.5}\, aR^{-2}\left(\frac{R}{\sqrt{t}}\right)^{d+2}\exp\left\{-\frac{R^2}{2t}\right\}. \quad \square$$

Proof of Corollary 3.4. Fix $N>R+2$. Let $\{X_n\}$ be as in Theorem 1.7 but with $m^c_{0,N}$ in place of m. Then

$$Q^{m^c_{0,N}}(X_s(B(0;R))>0 \text{ for some } s\leqq t)$$
$$=\lim_{n\to\infty} Q^{m^c_{0,N}}(X_n(s)(B(0;R))>0 \text{ for some } s\leqq t) \quad \text{(Theorem 1.7)}$$
$$\leqq \lim_{n\to\infty} c_{3.4}\int_{n>|x|>N}\left(\frac{|x|-(R+1)}{\sqrt{t}}\right)^{d-2}\exp\left\{-\frac{(|x|-R-1)^2}{2t}\right\}dm(x)$$
$$\leqq c_{3.6}(d,R,t)\int_{|x|>N}\exp\left\{\frac{-|x|^2}{2(t+1)}\right\}dm(x). \quad \square$$

Proof of Theorem 1.8. By Corollary 3.4

$$P(X(t)(A\cap B(0;R))=X_n(t)(A\cap B(0;R)) \text{ for all } t\leqq T \text{ and } A\in\mathscr{B}(R^d))$$
$$=Q^{m^c_{0,n}}(X(t)(B(0;R))=0 \text{ for all } t\leqq T)\to 1 \quad \text{as } n\to\infty.$$

Since the events on the left side above are increasing in n, the result follows. $\square$

4. A Lévy Modulus for Brownian Trees

Fix $\mu\in\mathbb{N}$, $m=\mu^{-1}\sum_{i=0}^{K}\delta_{x_i}\in M_F^\mu(\mathbb{R}^d)$ and work with the system of branching Brownian motions defined on $(\Omega,\mathscr{A},P^m)$ in Sect. 2.

In order to derive a one-sided modulus of continuity for the support of a super-Brownian motion, X_t, we first derive a uniform modulus of continuity for $\{N^{\beta,\mu}\colon \beta\in I\}$ which is independent of μ. This latter result when used in conjunction with Theorem 2.3 will prove to be a powerful tool in the study of path properties of X.

Notation. If $0\leqq\varepsilon\leqq t$, let

$$I(t,\varepsilon)=\{\gamma\in I\colon \gamma\sim t-\varepsilon, \exists\beta\sim t \text{ such that } \gamma<\beta \text{ and } N_t^\beta\neq\Delta\},$$

and let $Z(t,\varepsilon)$ denote the cardinality of $I(t,\varepsilon)$.

Under $P^{\mu^{-1}\delta_0}$, $j\to\mu\langle 1, N_{j/\mu}^{(\mu)}\rangle$ is a critical Galton-Watson process starting with one individual and with each individual giving birth to 0 or 2 offspring with equal probability. Classical results on branching processes (see Harris (1963, pp. 21–22) or (5.2), (5.3) below) therefore imply that for any $t>0$,

$$p(\mu,t)\equiv P^{\mu^{-1}\delta_0}(\langle 1, N_t^{(\mu)}\rangle>0)\leqq c_{4.1}(\{t\}\,\mu\vee 1)^{-1} \tag{4.1}$$

$$\lim_{\mu\to\infty}\mu p(\mu,t)=2t^{-1} \tag{4.2}$$

$$\lim_{\mu\to\infty} P^{\mu^{-1}\delta_0}(\langle 1, N_t^{(\mu)}\rangle\geqq x\,|\,\langle 1, N_t^{(\mu)}\rangle>0)=e^{-2x/t} \quad \text{for all } x\geqq 0. \tag{4.3}$$

It is important to note that (as throughout this work) the constant $c_{4.1}$ do not depend on μ. It is for this reason that we will sometimes explicitly denote the μ-dependence of certain random variables.

Lemma 4.1. *If $0\leqq\varepsilon\leqq t$, then conditional on $\mathscr{A}_{t-\varepsilon}$, $Z(t,\varepsilon)$ has a binomial (n,p) distribution, where $n=\mu\langle 1, N_{t-\varepsilon}\rangle$ and $p=p(\mu,\{t\}-\{t-\varepsilon\})$ ($p(\mu,t)$ as in (4.1)).*

Proof.

$$Z(t,\varepsilon)=\sum_{\gamma\sim t-\varepsilon}\mathbf{1}(N_{t-\varepsilon}^\gamma\neq\Delta)f(\gamma),$$

where

$$f(\gamma)=\mathbf{1}(\exists\beta\sim t, \gamma<\beta, N_t^\beta\neq\Delta).$$

The independence of the $\{e^\beta\}$ shows that conditional on $\mathscr{A}_{t-\varepsilon}$, $\{f(\gamma)\colon \gamma\sim t-\varepsilon, N_{t-\varepsilon}^\gamma\neq\Delta\}$ are i.i.d. and are equal to one with probability $p(\mu,\{t\}-\{t-\varepsilon\})$. Note that $\mu\langle 1, N_{t-\varepsilon}\rangle=\sum_{\gamma\sim t-\varepsilon}1(N_{t-\varepsilon}^\gamma\neq\Delta)$, whence the result. □

Lemma 4.2. *There is a $c_{4.2}$ such that if $0\leqq\varepsilon_1<\varepsilon_2\leqq t$ and $c>0$, then*

$$\begin{aligned}&P^m(\sup\{|N_{t-\varepsilon_1}^\beta-N_{t-\varepsilon_2}^\beta|\colon \beta\sim t, N_t^\beta\neq\Delta\}\geqq ch(\varepsilon_2-\varepsilon_1))\\&\quad\leqq c_{4.2}\, m(\mathbb{R}^d)\, c^{d-2}((\log 1/(\varepsilon_2-\varepsilon_1))\vee 1)^{(d/2)-1}(\varepsilon_2-\varepsilon_1)^{c^2/2}(\varepsilon_1^{-1}\wedge\mu).\end{aligned}$$

Proof. The above probability equals

$$\begin{aligned}&E^m(P^m(\sup\{|N_{t-\varepsilon_1}^\gamma-N_{t-\varepsilon_2}^\gamma|\colon \gamma\in I(t,\varepsilon_1)\}\leqq ch(\varepsilon_2-\varepsilon_1)|\mathscr{E}))\\&\quad\leqq E^m(Z(t,\varepsilon_1))\, P_0(|B_1|\geqq c((\log 1/\varepsilon_2-\varepsilon_1)\vee 1)^{1/2}).\end{aligned} \tag{4.4}$$

Here we have used the $\mathscr{E}$-measurability of $Z(t,\varepsilon_1)$ and Lemma 2.1(b). Lemma 4.1 and (4.1) show that

$$E^m(Z(t,\varepsilon_1)) \leqq c_{4.1}(((\{t\}-\{t-\varepsilon_1\})\mu)\vee 1)^{-1}\mu E^m(\langle 1, N_{t-\varepsilon_1}\rangle)$$
$$\leqq c_1(\varepsilon_1^{-1}\wedge\mu)\, m(\mathbb{R}^d).$$

Plug this into (4.4) and use the standard estimate

$$P_0(|B_1|\geqq R)\leqq c_2(d)\, R^{d-2} e^{-R^2/2}$$

to complete the proof. □

Notation. If $\theta\in(0,1)$, $u\in(0,1]$, $t>0$, $c>0$ and $k\in\mathbf{Z}_+$ satisfies $\theta^{k-1}u\leqq t$, let

$$A_k=A_k(\mu,t,u,\theta,c)=\{\omega\colon \sup\{|N^{\beta,\mu}_{t-\theta^k u}-N^{\beta,\mu}_{t-\theta^{k-1}u}|\colon\ \beta\sim t,\ N^{\beta,\mu}_t\neq\Delta\}$$
$$\geqq c\,h((\theta^{k-1}-\theta^k)u)\}$$
$$B_n=B_n(\mu,t,u,\theta,c)=\bigcup_{k>n,\ \theta^{k-1}u\leqq t} A_k(\mu,t,u,\theta,c).$$

Lemma 4.3. *Let θ, u, and t be as above and let $c>\sqrt{2}$.*

(a) *There is a $c_{4.3}(u,\theta)$ such that*

$$P^m(B_n(\mu,t,u,\theta,c))\leqq c_{4.3}(u,\theta)\leqq m(\mathbb{R}^d)\, c^{d-2}\, n^{(d/2)-1}\,\theta^{n(c^2/2-1)}.$$

(b) *For each $\varepsilon>0$ there is an $n_0(\varepsilon,u,\theta)$ such that if $n\geqq n_0$, $t\geqq\theta^n u$ and $\omega\notin B_n(\mu,t,u,\theta,c)$ then for any $\beta\sim t$ such that $N^\beta_t\neq\Delta$,*

$$|N^\beta_t-N^\beta_{t-\theta^n u}|\leqq c(1+\varepsilon)(1-\theta)^{\frac{1}{2}}(1-\theta^{\frac{1}{2}})^{-1}h(u\theta^n). \tag{4.4}$$

Proof. The previous lemma shows that

$$P^m(A_k)\leqq c_{4.2}\, m(\mathbb{R}^d)\, c^{d-2}(\log(\theta^{-k}u^{-1}(\theta^{-1}-1)^{-1})\vee 1)^{d/2-1}$$
$$\cdot\,\theta^{kc^2/2}((\theta^{-1}-1)u)^{c^2/2}\,\theta^{-k}u^{-1}.$$

Sum over $k>n$ to derive (a).

Let $\omega\notin B_n(\mu,t,u,\theta,c)$, $\beta\sim t$, $N^\beta_t\neq\Delta$ and assume $t\geqq\theta^n u$. Then

$$|N^\beta_t-N^\beta_{t-\theta^n u}|\leqq\sum_{k=n+1}^{\infty}|N^\beta_{t-\theta^k u}-N^\beta_{t-\theta^{k-1}u}|$$
$$\leqq c\sum_{k=n+1}^{\infty} u^{1/2}\theta^{(k-1)/2}(1-\theta)^{1/2}(\log(\theta^{1-k}(1-\theta)^{-1}u^{-1})\vee 1)^{1/2}$$
$$\leqq c\,h(\theta^n u)(1-\theta)^{1/2}\alpha_n(u,\theta),$$

where for $n \geqq n_1(u, \theta)$,

$$\alpha_n(u,\theta)$$
$$= \sum_{k=n+1}^{\infty} \theta^{(k-1-n)/2} [((k-1)(\log 1/\theta) + \log((1-\theta)u)^{-1})/(n \log 1/\theta + \log 1/u)]^{1/2}$$
$$= \sum_{j=0}^{\infty} \theta^{j/2} [((j+n)\log 1/\theta + \log((1-\theta)u)^{-1})/(n \log 1/\theta + \log 1/u)]^{1/2}.$$

By dominated convergence we have

$$\lim_{n\to\infty} \alpha_n(u,\theta) = (1-\theta^{1/2})^{-1}.$$

For any given $\varepsilon > 0$ choose $n_0(\varepsilon, u, \theta) \geqq n_1(u, \theta)$ such that $\alpha_n(u, \theta) \leqq (1+\varepsilon)(1-\theta)^{\frac{1}{2}})^{-1}$ if $n \geqq n_0$. This gives (b). □

We can now use the above estimate and a well-known method of Lévy to obtain a modulus of continuity for the system of branching Brownian motions $\{N^\beta\}$.

Proposition 4.4. *There are constants $c_{4.4}, c_{4.5}, c_{4.6} > 0$, $n_0 \in \mathbb{N}$ and for each $M \in \mathbb{Z}_+$, a $\mathbb{Z}_+$-valued random variable $K(\omega, M, \mu)$ such that*

(4.5) $P^m(K(M,\mu) \geqq n) \leqq c_{4.4} M m(\mathbb{R}^d) 2^{-nc_{4.5}}$ *if* $n > n_0$, *and*

(4.6) *if* $0 < t-s \leqq 2^{-K(M,\mu)}$, $s, t \in [0, M]$, $\beta \sim t$ *and* $N_t^{\beta,\mu} \neq \Delta$

then

$$|N_t^{\beta,\mu} - N_s^{\beta,\mu}| \leqq c_{4.6}\, h(t-s).$$

Proof. Let

$$C_n(M,\mu) = \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{M2^k} B_k(\mu, i2^{-k}, 1, 1/2, 3)$$
$$K(M,\mu) = \min\{n : \omega \notin C_n(M,\mu)\} \vee n_0(1, 1, 1/2),$$

where $\min \phi = \infty$ and n_0 is as in Lemma 4.3. Lemma 4.3(a) implies that for $n > n_0 \equiv n_0(1, 1, 1/2)$,

$$P^m(K(M,\mu) \geqq n) \leqq P^m(C_{n-1})$$
$$\leqq \sum_{k=n-1}^{\infty} M 2^k c_{4.3}(1, 1/2)\, m(\mathbb{R}^d)\, 3^{d-2} k^{(d/2)-1} 2^{-k7/2}$$
$$\leqq c_1 M m(\mathbb{R}^d)\, n^{(d/2)-1} 2^{-n5/2}$$
$$\leqq c_{4.4} M m(\mathbb{R}^d)\, 2^{-nc_{4.5}}.$$

Turning to (4.6), let $0<t-s\leqq 2^{-K(M,\mu)}$, $s, t\in[0, M]$ and $\beta\sim t$ satisfy $N_t^\beta\neq\Delta$. Choose $n\geqq K(M, \mu)\geqq n_0$ such that

$$2^{-n-1}<t-s\leqq 2^{-n}. \tag{4.7}$$

Let $i_n, j_n\in\mathbb{Z}_+, j_n-i_n\in\{0, 1\}$ and $\{i_l, j_l: l>n\}\subset\{0, 1\}$ satisfy

$$s=i_n 2^{-n}+\sum_{l=n+1}^{\infty} i_l 2^{-l}, \quad t=j_n 2^{-n}+\sum_{l=n+1}^{\infty} j_l 2^{-l},$$

and define

$$s_n=i_n 2^{-n}, \quad t_n=j_n 2^{-n}, \quad s_k=s_n+\sum_{l=n+1}^{k} i_l 2^{-l},$$

$$t_k=t_n+\sum_{l=n+1}^{k} j_l 2^{-l} \quad (k>n).$$

If $k\geqq n\geqq K(M, \mu)$, then

$$\omega\in\bigcap_{i=1}^{M2^k} B_k(\mu, i2^{-k}, 1, 1/2, 3)^c$$

and so Lemma 4.3(b) implies

$$\begin{aligned}&\sup\{|N^\gamma_{i2^{-k}}-N^\gamma_{(i-1)2^{-k}}|: \gamma\sim i2^{-k}, 2^{-k}\leqq i2^{-k}\leqq M, N^\gamma_{i2^{-k}}\neq\Delta\}\\&\quad\leqq 6(\sqrt{2}-1)^{-1}h(2^{-k})=c_2 h(2^{-k}).\end{aligned}$$

Therefore

$$\begin{aligned}|N_t^\beta-N_s^\beta|&\leqq|N^\beta_{t_n}-N^\beta_{s_n}|+\sum_{k=n+1}^{\infty}|N^\beta_{s_k}-N^\beta_{s_{k-1}}|+|N^\beta_{t_k}-N^\beta_{t_{k-1}}|\\&\leqq(j_n-i_n)c_2 h(2^{-n})+c_2\sum_{k=n+1}^{\infty} i_k h(2^{-k})+j_k h(2^{-k})\\&\leqq c_3 h(2^{-n})\\&\leqq\sqrt{2}c_3 h(t-s)\quad(\text{by } (4.7)).\end{aligned}$$

This proves (4.6). □

The important part of the above theorem is that the upper bound in (4.5) does not depend on μ. This will allow us to let $\mu\to\infty$ and obtain a Lévy modulus for $S(X_t)$. First we refine the above result by finding the best possible value of $c_{4.6}$ in (4.6). In addition, the fact that $\langle 1, N_t\rangle$ hits zero in finite time allows us to remove the dependence on M in the above result. Here then is the main result of this section.

Theorem 4.5. *If $c>2$ there are constants $c_{4.7}(c)$, $c_{4.8}(c)$ and $c_{4.9}(c)$ and a random variable $\delta(\omega, c, \mu)$ such that if $\mu\in\mathbb{N}$ and $m\in M_F^\mu(\mathbb{R}^d)$, then*

$$(4.8) \qquad P^m(\delta(c,\mu)\leqq\rho)\leqq c_{4.7}\, m(\mathbb{R}^d)\,\rho^{c_{4.8}} \quad \text{for } 0\leqq\rho\leqq c_{4.9}.$$

$$(4.9) \qquad \text{If } 0<t-s\leqq\delta(\omega,c,\mu), \quad s,t\in[0,\infty), \quad \beta\sim t \quad \text{and} \quad N_t^{\beta,\mu}\neq\Delta,$$

then

$$|N_t^{\beta,\mu}-N_s^{\beta,\mu}|<c\,h(t-s).$$

In particular, if $m_\mu\in M_F^\mu(\mu\in\mathbb{N})$ satisfy $\sup\{m_\mu(\mathbb{R}^d)\colon \mu\in\mathbb{N}\}<\infty$, *then*

$$(4.10) \qquad \lim_{\rho\downarrow 0}\limsup_{\mu\to\infty} P^{m_\mu}(\sup\{|N_t^{\beta,\mu}-N_s^{\beta,\mu}|\,h(t-s)^{-1}\colon \\ 0<t-s\leqq\rho,\, s,t\geqq 0,\, \beta\sim t,\, N_t^\beta\neq\Delta\}>c)=0.$$

Proof. Fix $c>2$, $\mu\in\mathbb{N}$ and $m\in M_F^\mu$. Let $c_1\in(2,c)$, choose ε, $\theta\in(0,1)$ small enough so that

$$c_1(1+\varepsilon)(1-\theta)^{1/2}(1-\theta^{\frac{1}{2}})^{-1}<c,$$

and then choose $M_1\in\mathbb{N}$ large enough so that

$$(4.11) \qquad e/M_1<\theta,$$

and

$$(4.12) \qquad 2c_{4.6}(\theta M_1)^{-1/2}(\log M_1/\log\theta^{-1})^{1/2}+c_1(1+\varepsilon)(1-\theta)^{1/2}(1-\theta^{\frac{1}{2}})^{-1}<c.$$

Note that ε, θ, c_1 and M_1 depend only on c. If $M\in\mathbb{Z}_+$, let

$$D_n=D_n(M,c,\mu)=\bigcup_{k=n}^{\infty}\ \bigcup_{0\leqq i\leqq\theta^{-k}M}\ \bigcup_{p=0}^{M_1-1}\ \bigcup_{q=1}^{M_1} B_k(\mu,(i+p/M_1)\,\theta^k, q/M_1,\theta,c_1),$$

$n_0(c)=\max\{n_0(\varepsilon, q/M_1,\theta)\colon q=1,\ldots,M_1\}$ ($n_0(\varepsilon,u,\theta)$ as in Lemma 4.3),

$$J(\omega,M,c,\mu)=\min\{n\colon \omega\notin D_n(M,c,\mu)\}\vee n_0(c) \qquad (\min\phi=\infty).$$

If $n>n_0(c)$, Lemma 4.3(a) implies

$$(4.13) \qquad P^m(J(M,c,\mu)\geqq n)\leqq P^m(D_{n-1})$$

$$\leqq\sum_{k=n-1}^{\infty}(1+\theta^{-k}M)\,M_1^2\sup\{c_{4.3}(q/M_1,\theta)\colon q=1,\ldots,M_1\}\,m(\mathbb{R}^d) \\ \cdot c_1^{d-2}\,k^{(d/2)-1}\,\theta^{k(c_1^2/2-1)}\leqq c_2(c)\,M\,m(\mathbb{R}^d)\,n^{(d/2-1)}\,\theta^{n(c_1^2/2-2)}.$$

If $K(M,\mu)$ is as in Proposition 4.4, let

$$\delta_1(\omega,M,c,\mu)=\theta^{J(\omega,M,c,\mu)}\wedge 2^{-K(\omega,M,\mu)}.$$

(4.5) and (4.13) imply there are constants $c_3(c)$, $c_4(c)$ and $c_5(c)$ such that

(4.14) $$P^m(\delta_1(M, c, \mu) \leqq \rho) \leqq c_3(c) M m(\mathbb{R}^d) \rho^{c_4(c)} \quad \text{if } 0 \leqq \rho \leqq c_5(c).$$

Suppose $\omega \in \Omega$, s, $t \in [0, M]$ satisfy $0 < t - s \leqq \delta_1(\omega, M, c, \mu)$ and $\beta \sim t$ is such that $N_t^\beta \neq \Delta$. Choose $n \geqq J(M, c, \mu)$ such that

(4.15) $$\theta^{n+1} < t - s \leqq \theta^n,$$

$0 \leqq i \leqq \theta^{-n} M$ and $p \in \{0, \ldots, M_1 - 1\}$ such that

$$t_0 = (i + (p/M_1))\, \theta^n \leqq t < (i + (p+1)/M_1)\, \theta^n,$$

and finally $q \in \{1, \ldots, M_1\}$ so that

$$(i + ((p - q - 1)/M_1))\, \theta^n < s \leqq (i + ((p-q)/M_1))\, \theta^n = s_0 < t_0.$$

The existence of such a q follows from (4.11) and (4.15). Since

$$(t - t_0) \vee (s_0 - s) \leqq t - s \leqq 2^{-K(M, \mu)},$$

(4.6) implies

(4.16) $$\begin{aligned} &|N_t^\beta - N_{t_0}^\beta| + |N_{s_0}^\beta - N_s^\beta| \\ &\leqq c_{4.6}(h(t - t_0) + h(s_0 - s)) \\ &\leqq 2 c_{4.6}\, h(\theta^n / M_1) \\ &\leqq 2 c_{4.6}\, h(\theta^{n+1})(M_1 \theta)^{-\frac{1}{2}} (\log M_1 \theta^{-n} / \log \theta^{-n-1})^{1/2} \quad \text{(use (4.11))} \\ &\leqq 2 c_{4.6} (M_1 \theta)^{-1/2} (\log M_1 / \log \theta^{-1})^{1/2} h(t-s) \quad \text{(by (4.11), (4.15))}. \end{aligned}$$

Now $n \geqq J(M, c, \mu)$ implies $\omega \notin D_n$ and hence $\omega \notin B_n(\mu, t_0, q/M_1, \theta, c_1)$. Lemma 4.3 therefore implies (note that $n \geqq J(M, c, \mu) \geqq n_0(\varepsilon, q/M_1, \theta)$)

(4.17) $$|N_{t_0}^\beta - N_{s_0}^\beta| \leqq c_1 (1 + \varepsilon)(1 - \theta)^{1/2} (1 - \theta^{1/2})^{-1} h(t_0 - s_0).$$

(4.16) and (4.17) give us

$$\begin{aligned} |N_t^\beta - N_s^\beta| &\leqq |N_t^\beta - N_{t_0}^\beta| + |N_{t_0}^\beta - N_{s_0}^\beta| + |N_{s_0}^\beta - N_s^\beta| \\ &\leqq (2 c_{4.6} (M_1 \theta)^{-1/2} (\log M_1 / \log \theta^{-1})^{1/2} \\ &\quad + c_1 (1 + \varepsilon)(1 - \theta)^{1/2} (1 - \theta^{\frac{1}{2}})^{-1})\, h(t - s) \\ &\leqq c\, h(t - s) \quad \text{(by (4.12))}. \end{aligned}$$

We have proved

(4.18) If $s, t \in [0, M]$, $0 < t - s \leqq \delta_1(\omega, M, c, \mu)$, $\beta \sim t$ and $N_t^\beta \neq \Delta$,

then $|N_t^\beta - N_s^\beta| < c\, h(t - s)$.

It remains to remove the dependency of δ_1 on M. Let

$$\begin{aligned} T_0 &= \inf\{i : \langle 1, N_{i/\mu} \rangle = 0\} \\ M_0 &= [T_0/\mu] + 1 \in \mathbb{N} \quad ([x] \text{ is the integer part of } x), \end{aligned}$$

and

$$\delta(\omega, c, \mu) = \delta_1(\omega, M_0(\omega), c, \mu),$$

where $\delta_1(\omega, \infty, c, \mu) = 0$. If $s, t \geqq 0$, $0 < t - s \leqq \delta(\omega, c, \mu)$, $\beta \sim t$ and $N_t^\beta \neq \Delta$, then $t < T_0(\omega)/\mu < M_0(\omega)$, and therefore (4.18) shows that $|N_t^\beta - N_s^\beta| < c h(t-s)$. This proves (4.9).

Assume $0 < \rho \leqq c_5(c)$, and

$$\rho^{c_4(c)/2} \leqq 1/4, \tag{4.19}$$

and let $M_2 = [\rho^{-c_4(c)/2}]$. An examination of the definition of $\delta_1(\omega, M, c, \mu)$, shows it is monotone non-increasing in M. Therefore

$$\begin{aligned} P^m(\delta(c,\mu) \leqq \rho) &\leqq P^m(M_0 > M_2) + P^m(\delta_1(\omega, M_2, c, \mu) \leqq \rho) \\ &\leqq P^m(T_0 > \mu(M_2 - 1)) + c_3(c)\, M_2\, m(\mathbb{R}^d)\, \rho^{c_4(c)} \quad \text{(by (4.14))} \\ &\leqq c_{4.1}(\mu m(\mathbb{R}^d))(\mu(M_2 - 1))^{-1} + c_3(c)\, m(\mathbb{R}^d)\, \rho^{c_4(c)/2} \quad \text{(by (4.1))} \\ &\leqq (2c_{4.1} + c_3(c))\, m(\mathbb{R}^d)\, \rho^{c_4(c)/2} \quad \text{(by (4.19))}. \end{aligned}$$

This establishes (4.8) with $c_{4.8} = c_4(c)/2$ and $c_{4.7} = 2c_{4.1} + c_3(c)$. (4.10) is immediate from (4.8) and (4.9). □

The next result shows Theorem, 4.5 and in particular (4.10) is false if $c < 2$.

Theorem 4.6. *If $c < 2$, and $m_\eta \in M_F^{2^\eta}$ $(n \in \mathbb{N})$ satisfy $\inf\{m_\eta(\mathbb{R}^d): \eta \in \mathbb{N}\} = K > 0$, then*

$$\lim_{n\to\infty} \liminf_{\eta\to\infty} P^{m_\eta, 2^\eta}(\sup\{|N_{j2^{-n}}^{\beta, 2^\eta} - N_{(j-1)2^{-n}}^{\beta, 2^\eta}|\, h(2^{-n})^{-1}:$$
$$1 \leqq j \leqq 2^n, \beta \sim j2^{-n}, N_{j2^{-n}}^\beta \neq \Delta\} > c) = 1.$$

Proof. Fix $c < 2$, $\mu = 2^\eta$ and $1 < n \leqq \eta$ $(n, \eta \in \mathbb{N})$. the above probability is decreased if m_η is replaced by $[K\mu]\mu^{-1}\delta_0$. To simplify the notation we may assume $K = 1$ and hence $m_\eta = \delta_0$. Let

$$I(t) = I(t, 0) = \{\beta \sim t : N_t^\beta \neq \Delta\},\ Z(t) = Z(t, 0) = \operatorname{card} I(t).$$

Let $A_1, \ldots, A_M$ be the partition of Ω obtained by specifying the finite number of possible values of $(I(j2^{-n}): j = 1, 2, \ldots, 2^n)$. Our underlying measure is $P^{(x_i)}$ where $x_i = 0$ if $0 \leqq i < \mu$ and $x_i = \Delta$ otherwise, so outside of a null set Λ, there are only a finite number of possible values for the above vector of sets. To obtain a partition of all of Ω we may simply include Λ in A_1, say. Clearly $A_i \in \mathscr{E}$. Fix

$$A_i = \{\omega : I(j2^{-n}) = \{\beta_k^{j,n} : k \leqq N_{j,n}^0 = Z(j2^{-n})\} \text{ for } 1 \leqq j \leqq 2^n\}\ (N_{j,n}^0 \in Z_+),$$

let

$$\{\gamma \sim (j-1)2^{-n} : \gamma < \beta_k^{j,n} \text{ for some } k \leqq N_{j,n}^0\} = \{\gamma_k^{j,n} : k \leqq N_{j,n}^1\},$$

and define $S(\gamma_k^{j,n}) \subset \{1, \ldots, N_{j,n}^0\}$ by

$$\{\beta_l^{j,n} : l \in S(\gamma_k^{j,n})\} = \{\beta_l^{j,n} : \gamma_k^{j,n} < \beta_l^{j,n}\}.$$

Let

$$B=B(n,\mu)$$
$$=\{\omega:\sup\{|N^{\beta}_{j2^{-n}}-N^{\beta}_{(j-1)2^{-n}}|\,h(2^{-n})^{-1}:1\leqq j\leqq 2^n,\beta\sim j2^{-n},N^{\beta}_{j2^{-n}}\neq\Delta\}\leqq c\}.$$

On A_i, $N^{\beta^{j,n}_k}(t)=\hat{N}^{\beta^{j,n}_k}(t)$ for $t\leqq j2^{-n}$ and the latter processes are independent of $\mathscr{E}$. Therefore on A_i,

$$(4.20)\quad P^{\delta_0}(B|\mathscr{E})=P^{\delta_0}(\sup\{|\hat{N}^{\beta^{j,n}_k}(j2^{-n})-\hat{N}^{\beta^{j,n}_k}((j-1)2^{-n})|:k\leqq N^0_{j,n},1\leqq j\leqq 2^{-n}\}\leqq ch(2^{-n}))$$
$$=P^{\delta_0}(\sup\{W^{j,n}_k:1\leqq k\leqq N^1_{j,n},1\leqq j\leqq 2^n\}\leqq ch(2^{-n})),$$

where
$$W^{j,n}_k=\sup\{|\hat{N}^{\beta^{j,n}_l}(j2^{-n})-\hat{N}^{\beta^{j,n}_l}((j-1)2^{-n})|:l\in S(\gamma^{j,n}_k)\}.$$
The sets

$$I^{j,n}_k=\{\beta^{j,n}_l|i:l\in S(\gamma^{j,n}_k),(j-1)2^{-n}\leqq i/\mu<j2^{-n}\},1\leqq j\leqq 2^n,1\leqq k\leqq N^1_{2,n},$$

are mutually disjoint. Therefore the σ-fields
$$\mathscr{G}^{j,n}_k=\sigma(B^{\beta}:\beta\in I^{j,n}_k),1\leqq j\leqq 2^n,1\leqq k\leqq N^1_{j,n}$$

are mutually independent. $W^{j,n}_k$ is $\mathscr{G}^{j,n}_k$ measurable, so (4.20) implies that on A_i,

$$P^{\delta_0}(B|\mathscr{E})=\prod_{j=1}^{2^n}\prod_{k=1}^{N^1_{j,n}}P^{\delta_0}(W^{j,n}_k\leqq ch(2^{-n})).$$

Each $\{\hat{N}^{\beta}_s:s<(|\beta|+1)/\mu\}$ is a P^{δ_0}-Brownian motion. Therefore on A_i,

$$P^{\delta_0}(B|\mathscr{E})\leqq P_0(|B(1)|\leqq c(n\log 2)^{1/2})^{\sum_{j=1}^{2^n}N^1_{j,n}}$$
$$\leqq(1-c_1\,c^{d-2}(n\log 2)^{(d/2)-1}2^{-nc^2/2})^{\sum_{j=1}^{2^n}N^1_{j,n}}.$$

On A_i, $N^1_{j,n}=Z(j2^{-n},2^{-n})$, so we have proven

$$(4.21)\qquad P^{\delta_0}(B|\mathscr{E})\leqq\exp\left\{-\varepsilon_n\sum_{j=1}^{2^n}Z(j2^{-n},2^{-n})\right\},$$

$$(4.22)\qquad \varepsilon_n=c_1\,c^{d-2}(n\log 2)^{d/2-1}2^{-nc^2/2}.$$

Let

$$q(\mu,n)=\mu p(\mu,2^{-n})$$

and

$$A_n(j)=q(\mu,n)(1-e^{-2\varepsilon_n})(2\varepsilon_n)^{-1}\langle 1,N_{(j-1)2^{-n}}\rangle,j\in\mathbb{N}.$$

Take expected values in (4.21) and use Hölder's inequality to see

(4.23) $$P^{\delta_0}(B) \leqq E^{\delta_0}\left(\exp\left\{-2\varepsilon_n \sum_{j=1}^{2^n} Z(j2^{-n}, 2^{-n}) - A_n(j)\right\}\right)^{1/2} \cdot E^{\delta_0}\left(\exp\left\{-2\varepsilon_n \sum_{j=1}^{2^n} A_n(j)\right\}\right)^{1/2}.$$

For $j \in \mathbb{N}$, Lemma 4.1 shows that

$$\begin{aligned} &E^{\delta_0}(\exp\{-2\varepsilon_n Z(j2^{-n}, 2^{-n})\} | \mathscr{A}_{(j-1)2^{-n}}) \\ &= (1 - p(\mu, 2^{-n})(1 - e^{-2\varepsilon_n}))^{\mu\langle 1, N_{(j-1)2^{-n}}\rangle} \\ &\leqq \exp\{-(1 - e^{-2\varepsilon_n})\, q(\mu, n)\langle 1, N_{(j-1)2^{-n}}\rangle\} \\ &= \exp\{-2\varepsilon_n A_n(j)\}. \end{aligned}$$

The first factor in (4.23) is therefore bounded by 1. (4.2) shows that $\lim_{\mu \to \infty} q(\mu, n) = 2^{1+n}$. Therefore we may let $\eta \to \infty$ in (4.23) and use Theorem 2.2 to get

(4.24) $$\limsup_{\eta \to 0} P^{\delta_0, 2^\eta}(B(n, 2^\eta)) \leqq E_Q^{\delta_0}\left(\exp\left\{-2^{n+1}(1 - e^{-2\varepsilon_n}) \sum_{j=1}^{2^n} \langle 1, X_{(j-1)2^{-n}}\rangle\right\}\right)^{1/2}.$$

The continuity of X_t shows

$$\lim_{n \to \infty} 2^{-n} \sum_{j=1}^{2^n} \langle 1, X_{(j-1)2^{-n}}\rangle = \int_0^1 \langle 1, X_s\rangle\, ds > 0 \qquad Q^{\delta_0}\text{-a.s.}$$

By (4.22) and the choice of c we have

$$\lim_{n \to \infty} 2^{2n+1}(1 - e^{-2\varepsilon_n}) = +\infty.$$

The right side of (4.24) therefore approaches zero as $n \to \infty$ and the result follows. □

In order to use Theorem 4.5 to obtain a Lévy modulus for $S(X_t)$ we first interpret it in terms of the nonstandard model used to construct X in Theorem 2.3. For the rest of this section we work in the setting of that result and assume the hypotheses of Theorem 2.3. Therefore $\mu = 2^\eta$ where $\eta \in {}^*\mathbb{N} - \mathbb{N}$, $m \in M_F(\mathbb{R}^d)$, and $m_\mu \in {}^*M_F^\mu(\mathbb{R}^d)$ satisfies $st_M(m_\mu) = m$. To simplify notation we write P and ${}^*P$ for P^{m_μ}, and ${}^*P^{m_\mu}$, respectively.

Theorem 4.7. (a) *For* P-a.a. ω *and each* $c > 2$,

(4.25) *There is a* $\delta(\omega, c) \in (0, \infty)$ *such that*
if $0 < t - s \leqq \delta$, $s, t \in {}^*[0, \infty])$, $\beta \sim t$ *and* $N_t^\beta \neq \Delta$, *then* $|N_t^\beta - N_s^\beta| \leqq c\,h(t-s)$.

(b) *For P-*a.a. ω *and each* $c<2$, (4.25) *is false.*

Proof. (a) is immediate from Theorem 4.5 and the Transfer Principle.

(b) The probability in question is unaltered if m_μ is replaced with $m_\mu({}^*\mathbb{R}^d)\,\delta_0$. The result now follows easily from Theorem 4.6 by taking $m_\eta = K\delta_0$ in that theorem where $0<K<m_\mu({}^*\mathbb{R}^d)$, K a dyadic rational. □

This is an exact modulus of continuity on the internal nonstandard process $N^{(\mu)}$ whose standard part is a super-Brownian motion, X. The result uses the notion of ancestry which exists in the richer setting of $N^{(\mu)}$ but disappears when the pass to X. As a result it is not clear how to transfer the full power of Theorem 4.7 into a theorem concerning only X although Theorem 1.1 comes close. Nonetheless we shall see that Theorem 4.7 used in conjunction with Theorem 2.3, will help in the derivation of several path properties of X.

We now will use Theorem 4.7(a) to prove Theorem 1.1. To do this we must relate $S(X_{{}^\circ t})$ to $S(N_t)$.

Lemma 4.8. *For each nearstandard* $t\in{}^*[0,\infty)$ *such that* ${}^\circ t>0$,

$$S(X_{{}^\circ t}) = st(S(N_t))\,P\text{-a.s.}$$

Proof. $st(S(N_t))$ is closed by Albeverio et al. (1986, Prop. 2.1.8) and since it clearly supports $X({}^\circ t) = L(N(t))(st^{-1}(\cdot))$, we see that

$$S(X_{{}^\circ t}) \subset st(S(N_t))\ \forall t\in ns({}^*[0,\infty)) \quad \text{a.s.} \tag{4.26}$$

Conversely fix $t\in ns({}^*[0,\infty))$ and for $\gamma\sim s<t$ let

$$N_t(\gamma) = \mu^{-1} \sum_{\beta\sim t,\,\beta>\gamma} \delta_{N_t^\beta}.$$

Then for each $\gamma\sim t-2^{-n}$ and $2^{-n}<t$,

$$\begin{aligned}
&{}^{\circ *}P(N_t(\gamma)({}^*\mathbb{R}^d)\leqq 8^{-n}\mid \gamma\in I(t,2^{-n}))\\
&= {}^{\circ *}P^{\delta_0/\mu}(\langle 1, N_{2^{-n}}\rangle \leqq 8^{-n} \mid \langle 1, N_{2^{-n}}\rangle > 0)\\
&= 1-e^{-2(4^{-n})} \quad \text{(by (4.3))}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&P(N_t(\gamma)({}^*\mathbb{R}^d)\leqq 8^{-n}\ \exists \gamma\in I(t,2^{-n}))\\
&\leqq {}^\circ\sum_{\gamma\sim t-2^{-n}} {}^*P(N_t(\gamma)({}^*\mathbb{R}^d)\leqq 8^{-n}\mid\gamma\in I(t,2^{-n}))\,{}^*P(\gamma\in I(t,2^{-n}))\\
&\leqq 2(4^{-n})\,{}^{\circ *}E(Z(t,2^{-n})) \quad \text{(by the above)}\\
&= 2^{2-n} m(\mathbb{R}^d),
\end{aligned}$$

where we have used Lemma 4.1 and (4.2) in the last. The Borel-Cantelli lemma implies

(4.27) For a.a. ω, for large enough $n\in\mathbb{N}$, $N_t(\gamma)({}^*\mathbb{R}^d) > 8^{-n}$ for all $\gamma\in I(t,2^{-n})$.

Fix ω outside a null set such that the conclusions of (4.27) and Theorem 4.7(a) hold. Choose $N(\omega)\in\mathbb{N}$ such that (4.27) holds for $n\geqq N(\omega)$ and $\delta(\omega,3)\leqq 2^{-N}$ ($\delta(\omega,c)$ as in Theorem 4.7(a)). Let $\beta\sim t$ satisfy $N_t^\beta\neq\Delta$ and let $\gamma\in I(t,2^{-n})$ satisfy $\gamma<\beta$. If $n\geqq N(\omega)$,

$$\begin{aligned}N_t(*B(N_t^\beta;6h(2^{-n})))&\geqq N_t(\gamma)(*B(N^\gamma_{t-2^{-n}};3h(2^{-n}))) \quad (\text{because } \delta(\omega,3)\leqq 2^{-n})\\ &=N_t(\gamma)(*\mathbb{R}^d) \quad (\text{same reason})\\ &>8^{-n} \quad (\text{by (4.27)}).\end{aligned}$$

Therefore $X_{\circ t}(B(^\circ N_t^\beta;7h(2^{-n})))\geqq 8^{-n}$ for $n\geqq N(\omega)$ and this implies $^\circ N_t^\beta\in S(X_{\circ t})$ and hence $st(S(N_t))\subset S(X_{\circ t})$. □

Notation. If $r\geqq 0$ *and* $B\in\mathscr{B}(\mathbb{R}^d)$ let

$$T_r(B)=\inf\{t\geqq r: X_t(B)=0\}.$$

Lemma 4.9. *For P-a.a. ω, if $s,t\in ns(*[0,\infty))$ and $\gamma\sim t$ satisfy*

$$0<{}^\circ s<{}^\circ t \quad \text{and} \quad N_t^\gamma\neq\Delta, \quad \text{then } {}^\circ N_s^\gamma\in S(X_{\circ s}).$$

Proof. Fix $r\geqq 0$ and $B=B(y;\varepsilon)$. We claim that

(4.28) w.p. $1\ \exists\delta(\omega)>0$ such that $X_t(B(y;\varepsilon/2))=0 \quad \forall t\in[T_r(B),T_r(B)+\delta]$.

By conditioning with respect to $\mathscr{F}^X_{T_r(B)}$ and using the strong Markov property it suffices to show that if

$$U=\inf\{t\geqq 0: X_t(B(y;\varepsilon/2))>0\}$$

and $m(B(y;\varepsilon))=0$ then $P^m(U>0)=1$. Take $m^\mu\in{}^*M_F(\mathbb{R}^d)$ such that $m^\mu(*B(y;\varepsilon))=0$ and $m=L(m^\mu)(st^{-1}(\,))$. Then $P^m(U>0)=1$ follows immediately from Theorem 4.7(a) and the claim is proved.

Fix ω outside a P-null set such that

(i) (4.28) holds for all $r\in\mathbb{Q}^{\geqq 0}$, $\varepsilon\in\mathbb{Q}^{>0}$ and $y\in\mathbb{Q}^d$.
(ii) $S(X_t)=st(S(N_t))$ for all $t\in\mathbb{Q}^{>0}$ (Lemma 4.8).
(iii) The conclusion of Theorem 4.7(a) holds.

Assume s, t and γ satisfy the hypotheses of the lemma but ${}^\circ N_s^\gamma\notin S(X_{\circ s})$. Choose $y\in\mathbb{Q}^d$ and $\varepsilon\in\mathbb{Q}^{>0}$ such that $|{}^\circ N_s^\gamma-y|<\varepsilon/2$ and if $B=B(y;\varepsilon)$ then $X_s(B)=0$. By (iii) $\exists\delta(\omega)\in(0,s\wedge(t-s))$ such that

$${}^\circ N_u^\gamma\in B(y;\varepsilon/2) \quad \forall u\in{}^*[s-\delta,s+\delta],$$

and therefore (ii) implies

(4.29) $$X_u(B(y;\varepsilon/2))>0 \quad \forall u\in(s-\delta,s+\delta)\cap\mathbb{Q}.$$

Let $r\in(s-\delta,s)\cap\mathbb{Q}$. Then $T_r(B)\in[r,s]$ because $X_s(B)=0$. (i) implies

(4.30) $$\exists\delta'(\omega)>0 \quad \text{such that } X_u(B(y;\varepsilon/2))=0 \quad \forall u\in[T_r(B),T_r(B)+\delta']\subset(s-\delta,s+\delta).$$

(4.29) and (4.30) lead to a contradiction. □

Proof of Theorem 1.1. Fix ω outside a P-null set such that (4.26) holds, and the conclusions of Lemma 4.9 and Theorem 4.7(a) hold. Let $c>2$, and assume $s, t>0$ satisfy $0<t-s\leqq\delta(\omega, c)$ where δ is as in (4.25). If $x\in S(X_t)$ then by (4.26) $\exists\gamma\sim t$ such that $x={}^\circ N_t^\gamma$. The choice of ω implies ${}^\circ N_s^\gamma\in S(X_s)$ and $|{}^\circ N_t^\gamma-{}^\circ N_s^\gamma|\leqq ch(t-s)$. Hence $x={}^\circ N_t^\gamma\in S(X_s)^{ch(t-s)}$. To handle $s=0$ we may, and shall, assume m^μ is chosen so that $S(X_0)=st(S(N_0))$. The argument now proceeds as above except instead of using Lemma 4.9, note that ${}^\circ N_0^\gamma\in S(X_0)$ by the choice of m^μ.

We have proven the required result on $({}^*\Omega, \mathscr{F}, P)$. In order to transfer the result onto path space we must show that

$$(4.31)\qquad \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\left\{\omega\in C([0,\infty), M_F(\mathbb{R}^d)): \forall s, t\geqq 0, 0<t-s\leqq\frac{1}{n}\Rightarrow S(\omega_t)\subset S(\omega_s)^{(2+m^{-1})h(t-s)}\right\}$$

is Borel measurable. Tedious routine arguments in fact show the set in parenthesis in (4.31) is a G_δ. We leave the details to the interested reader. □

Proof of Theorem 1.2. Let m_n and X_n $(n\in\mathbb{N})$ be as in Theorem 1.8 but with $m\in M_F$. Let $\tilde{X}_n=X-X_n$ and $\tilde{m}_n=m-m_n$, so that $\tilde{X}_n$ has law $Q^{\tilde{m}_n}$. By (3.2.3) and $(3.2.3)_h$ (letting $\theta\uparrow\infty$),

$$P(\tilde{X}_n(t)(\mathbb{R}^d)>0)\leqq 2m(B(0,n)^c)\,t^{-1}\to 0 \quad \text{as } n\to\infty.$$

It follows that for a.a. ω and any $t>0\,\exists N(\omega, t)\in\mathbb{N}$ such that $\tilde{X}_n(s)=0$ for all $s\geqq t$ and $n\geqq N$. Therefore $S(X(s))=S(X_N(s))$ for all $s\geqq t$ and the latter is a compact set for all s by Theorem 1.1 and the compactness of $S(X_N(0))$. It follows that for a.a. ω, $S(X_t)$ is compact for all $t>0$.

If ω is fixed so that $X_t(\omega)$ is weakly continuous and $x\in S(X_t)$, then $\lim_{s\to t} d(x, S(X_s))=0$ by the weak continuity. A compactness argument proves that

$$(4.32)\qquad \lim_{s\to t}\rho_1(S(X_t), S(X_s))=0.$$

Theorem 1.1 implies that if $c>2$ and $0<s-t<\delta(\omega, c)$, then $\rho_1(S(X_s), S(X_t))\leqq ch(t-s)$. Therefore $\lim_{s\downarrow t}\rho(S(X_s), S(X_t))=0$ for all $t>0$ a.s. □

By using Theorems 1.7 and 1.8 it is easy to prove a localized version of Theorem 1.1 for infinite initial measures in $M_{\exp}(\mathbb{R}^d)$.

Theorem 4.10. *Let $m\in M_{\exp}(\mathbb{R}^d)$. For Q^m-a.a. ω and all $R, T>0$, $c>2$ there is a $\delta(\omega, c, R, T)>0$ such that if $s, t\in[0, T]$ and $0<t-s\leqq\delta$, then*

$$S(X_t)\cap B(0;R)\subset S(X_s)^{ch(t-s)}.$$

Proof. Let $(\Omega, \mathscr{F}, P)$, X and X_n be as in Theorem 1.7. By Theorems 1.1, 1.7 and 1.8 we may, and shall, fix ω outside a P-null set such that

(4.33) $$\text{for any } n \in \mathbb{N} \text{ and } c > 2 \text{ there is a } \delta_n(\omega, c) > 0 \text{ such that} \\ \text{if } 0 < t - s \leqq \delta_n(\omega, c) \text{ then } S(X_n(t)) \subset S(X_n(s))^{ch(t-s)},$$

and

(4.34) $$\text{for any } R, T > 0 \text{ there is an } N(R, T, \omega) \in \mathbb{N} \text{ such that} \\ X(t)(B(0; R)) = X_n(t)(B(0; R)) \text{ for } t \leqq T \text{ and } n \geqq N(R, T, \omega).$$

Let $\delta(\omega, c, R, T) = \delta_{N(R,T,\omega)}(\omega, c)$, and suppose $s, t \in [0, T]$ satisfy $0 < t - s \leqq \delta(\omega, c, R, T)$ and $x \in S(X_t) \cap B(0; R)$. (4.34) implies $x \in S(X_{N(R,T)}(t))$ and hence (4.33) implies

$$x \in S(X_{N(R,T)}(s))^{ch(t-s)} \subset S(X(s))^{ch(t-s)}.$$

Finally the measurability result needed to transfer the result to path space is again left for the reader. □

Remark. With a bit more work one can easily extend the nonstandard construction of X described in Sect. 1 to the case where $X_0 = m \in M_{\exp}(\mathbb{R}^d)$. It is then not hard to prove a localized version of Theorem 4.7(a), analogous to the previous result. We leave the details to the interested reader because in practice sample path properties for X with an infinite initial measure in $M_{\exp}(\mathbb{R}^d)$ may be readily derived from the corresponding property for a finite initial measure by using Theorems 1.7 and 1.8 as in the above argument.

5. The Exact Hausdorff Measure of the Range of X

Consider first the derivation of the upper bound of the Hausdorff measure of $\bar{R}(r, s)$ in Theorem 1.4. Our basic approach is reminiscent of that used by Taylor (1964) to establish the upper bound of the Hausdorff measure of planar Brownian motion. The infinite-dimensional setting of our problem, however, leads to considerable complications.

$d \geqq 4$ is fixed throughout this section and $\psi_1(x) = x^4 \log^+ \log^+(1/x)$, as in Sect. 1.

Notation. $a_n = 2^{-n/2}$, $j_n = 2^{n+1}$, $b_n = 2^{-j_n}$.

Λ_n is the set of closed d-dimensional cubes of side length $2^{-n/2}$, centred at a point in $\{x_1, \ldots, x_d) \in \mathbb{R}^d$: $x_i = (n_i + e_i)\, 2^{-n/2}$, $n_i \in \mathbf{Z}$, $e_i = 0$ or $1/2\}$.

Let $c_{5.1} = P_0^0(|B_2| \leqq 1/2)/24$.

Definition. $C \in \Lambda_{j_n}$ is μ-bad (or bad, if there is no ambiguity) at $t \in [5a_{j_n}^2, \infty)$ iff $N_t(C) > 0$ and

$$\int_{[t - 5a_{2^n}^2, t]} N_s^{(\mu)}(C^{7a_k})\, d\lambda^{(\mu)}(s) \leqq c_{5.1}\, \psi_1(a_k) \quad \text{for } 2^n \leqq k \leqq 2^{n+1} - n.$$

$C \in \Lambda_{j_n}$ is μ-good at t iff it is not μ-bad at t and $N_t(C) > 0$.

Dependence on $\mu\in\mathbb{N}$ and the initial measure $m\in M_F^\mu$ will be suppressed in this section wherever possible. In particular we write N_t for $N_t^{(\mu)}$ and P and E for P^m and E^m.

Notation. If $\gamma\in I$, $\gamma\sim t$, $a>0$, and $k\in\mathbb{N}$ satisfy $k/\mu<4a^2<|\gamma|/\mu$, let

$$Y_t^{\gamma,k}(a)=\mu^{-2}\sum_{a^2\leq l/\mu\leq 2a^2}\ \sum_{\substack{\beta\sim\{t\}-k/\mu+l/\mu\equiv s\\ \sigma(\beta;\gamma)=k}} \mathbf{1}(|N_t^\gamma-N_s^\beta|\leq a),$$

$$Y_t^\gamma(a)=\sum_{2a^2\leq k/\mu<4a^2} Y_t^{\gamma,k}(a),$$

$$Z_t^{\gamma,k}(a)=\mu^{-2}\sum_{a^2\leq l/\mu\leq 2a^2}\ \sum_{\substack{\beta\sim t-k/\mu+l/\mu\equiv s\\ \sigma(\beta;\gamma)=k}} \mathbf{1}(N_t^\gamma\neq\Delta, N_s^\beta\neq\Delta).$$

$$\tilde{Z}_t^{\gamma,k}(a)=\mu^{-1}\sum_{\substack{\beta\sim t-k/\mu+[\mu a^2]/\mu\equiv s\\ \sigma(\beta;\gamma)=k}} \mathbf{1}(N_t^\gamma\neq\Delta, N_s^\beta\neq\Delta).$$

In the following result recall the notation $\delta(\omega, c, \mu)$ from Theorem 4.5, and that $h(t)=(t((\log 1/t)\vee 1))^{\frac{1}{2}}$.

Lemma 5.1. *Assume $C\in A_{j_n}$ is bad at $t\in[5a_{j_n}^2, \infty)$, $2b_n<\delta(\omega, 3, \mu)$ and $j\in\mathbb{N}$ satisfies $t\in[jb_n, (j+1)b_n)$. Then there is a $\gamma\in I(jb_n, b_n)$ such that $N_{(j-1)b_n}^\gamma\in C^{3h(2b_n)}$ and*

$$Y_{(j-1)b_n}^\gamma(a_k)\leq c_{5.1}\,\psi_1(a_k)\quad \textit{for } 2^n\leq k\leq 2^{n+1}-n.$$

Proof. Assume the above hypotheses. Choose $\beta\sim t$ such that $N_t^\beta\in C$ and let $\gamma=\beta|[\mu(j-1)b_n]$. Then $\gamma\in I(jb_n, b_n)$ and the condition on $\delta(\omega, 3, \mu)$ implies $|t-(j-1)b_n|<\delta(\omega, 3, \mu)$ and therefore

$$|N_{(j-1)b_n}^\gamma-N_t^\beta|\leq 3h(2b_n)\leq 6a_k\quad\text{whenever } k\leq 2^{n+1}-n.$$

Therefore

$$B(N_{(j-1)b_n}^\gamma; a_k)\subset B(N_t^\beta; 7a_k)\subset C^{7a_k}\qquad\text{for } k\leq 2^{n+1}-n.$$

The fact that C is bad at t implies

$$\int_{[t-5a_{2^n}^2, t]} N_s(B(N_{(j-1)b_n}^\gamma; a_k))\,d\lambda(s)\leq c_{5.1}\,\psi_1(a_k)\qquad\text{for } 2^n\leq k\leq 2^{n+1}-n$$

The result follows on noting that the left side of the above exceeds $Y_{(j-1)b_n}^\gamma(a_k)$ for $k\geq 2^n$. □

The next lemma is a slight modification of Lemma 5.1 of Perkins (1988a).

Lemma 5.2. *Assume $\gamma\sim t$, $a>0$, $k\in\mathbb{N}$, $k/\mu<4a^2<|\gamma|/\mu$. Then on $\{|N_t^\gamma-N_{\{t\}-k/\mu}^\gamma|\leq a/2\}$ $(\in\mathscr{F}(\gamma))$,*

$$P(Y_t^{\gamma,k}(a)\geq 12c_{5.1}\,Z_t^{\gamma,k}(a)\,|\,\mathscr{F}(\gamma)\vee\mathscr{E})\geq 12c_{5.1}.$$

Proof. Choose γ, t, a, k as above,

$$\{\beta_i^l: i\leq M_l\}\subset\{\beta\sim t-k/\mu+l/\mu: \sigma(\beta;\gamma)=k\},\qquad\text{for } a^2\leq l/\mu\leq 2a^2,$$

and let

$$A=\Big\{\omega: N_t^\gamma \neq \Delta, |N_t^\gamma - N^\gamma_{\{t\}-k/\mu}| \leqq a/2, \{\beta_i^l : i \leqq M_l\} \\ = \{\beta \sim t-k/\mu + l/\mu : \sigma(\beta;\gamma)=k, N^\beta_{t-k/\mu+l/\mu} \neq \Delta\} \text{ for } a^2 \leqq l/\mu \leqq 2a^2\Big\} \in \mathscr{F}(\gamma) \vee \mathscr{E}.$$

On A we have

$$\begin{aligned} &E(Y_t^{\gamma,k}(a) \mid \mathscr{F}(\gamma) \vee \mathscr{E}) \\ &= \mu^{-2} \sum_{a^2 \leqq l/\mu \leqq 2a^2} \sum_{i=1}^{M_l} P(|N_t^\gamma - N^{\beta_i^l}_{\{t\}-k/\mu+l/\mu}| \leqq a \mid \mathscr{F}(\gamma) \vee \mathscr{E}) \\ &= \mu^{-2} \sum_{a^2 \leqq l/\mu \leqq 2a^2} M_l P_0^{N^\gamma_{\{t\}-k/\mu} - N_t^\gamma}(|B_{l/\mu}| \leqq a) \quad \text{(Lemma 2.1(e))} \\ &\geqq Z_t^{\gamma,k}(a) P_0^0(|B_2| \leqq 1/2). \end{aligned}$$

In the last we have used the fact that $|N^\gamma_{\{t\}-k/\mu} - N_t^\gamma| \leqq a/2$ on A. Let $p_0 = P_0^0(|B_2| \leqq 1/2)$. Therefore on

$$B = \{|N_t^\gamma - N^\gamma_{\{t\}-k/\mu}| \leqq a/2\} \in \mathscr{F}(\gamma),$$

we have

$$E(Y_t^{\gamma,k}(a) \mid \mathscr{F}(\gamma) \vee \mathscr{E}) \geqq p_0 Z_t^{\gamma,k}(a).$$

If

$$q = P(Y_t^{\gamma,k}(a) \geqq (p_0/2) Z_t^{\gamma,k}(a) \mid \mathscr{F}(\gamma) \vee \mathscr{E}),$$

this implies that on B,

$$(1-q)(p_0/2) Z_t^{\gamma,k}(a) + q Z_t^{\gamma,k}(a) \geqq p_0 Z_t^{\gamma,k}(a) \Rightarrow q \geqq p_0/2.$$

(If $Z_t^{\gamma,k}(a)=0$ the left side is one.) Recalling that $12c_{5.1} = p_0/2$, we get the result. □

Notation. If $\gamma \in I$ and $i \in \{0, \ldots, |\gamma|\}$, let

$$S^{\gamma,i} = \{\beta \in I : \sigma(\beta;\gamma) = i \geqq |\gamma| - |\beta|\}.$$

The fact that $\sigma(\beta;\gamma) \geqq |\gamma| - |\beta|$ implies that

$$S^{\gamma,i} \cap \{\gamma | k : 0 \leqq k \leqq |\gamma|\} = \emptyset \quad \text{for each } 0 \leqq i \leqq |\gamma|. \tag{5.1}$$

We recall some well-known limit theorems for branching processes from Harris (1963, p. 21–22). Let $\{Z_n : n = 0, 1, 2, \ldots\}$ be a Galton-Watson branching process such that $Z_0 = 1$ and $P(Z_1 = 0) = P(Z_1 = 2) = 1/2$. Then

$$\lim_{n\to\infty} nP(Z_n > 0) = 2 \tag{5.2}$$

$$\lim_{n\to\infty} P(Z_n/n > z \mid Z^n > 0) = e^{-2z}, z \geqq 0. \tag{5.3}$$

Lemma 5.3. *There are $c_{5.2}$, $c_{5.3}$ and, if $0<a<c_{5.2}$, there is a $\mu_{5.1}(a)\in\mathbb{N}$ such that if $\mu\geqq\mu_{5.1}(a)$, then on $\{N_t^\gamma\neq\Delta\}$,*

$$P(Y_t^\gamma(a)>c_{5.1}\,\psi_1(a)\,|\,\mathscr{F}(\gamma))$$
$$\geqq c_{5.3}\,a^{-2}(\log 1/a)^{-\frac{1}{2}}\int_{[2a^2,4a^2)}\mathbf{1}(|N_t^\gamma-N_{\{t\}-s}^\gamma|\leqq a/2)\,d\lambda(s),$$

whenever $\gamma\sim t$ and $4a^2<|\gamma|/\mu$.

Proof. Let $\mu\in\mathbb{N}$, $\gamma\sim t$ and assume $0<a<e^{-e}$, $4a^2<|\gamma|/\mu$. Let

$$k_0(\omega)=\min\{k\geqq 2\mu a^2\colon \tilde{Z}_t^{\gamma,k}(a)>0\}.$$

On $\{N_t^\gamma\neq\Delta\}$, one has

$$\begin{aligned}(5.3)\qquad & P(Y_t^\gamma(a)>c_{5.1}\,\psi_1(a)\,|\,\mathscr{F}(\gamma))\\
&\geqq\sum_{2\mu a^2\leqq k<4\mu a^2}P(k_0=k,\,Y_t^{\gamma,k}(a)>c_{5.1}\,\psi_1(a)\,|\,\mathscr{F}(\gamma))\\
&\geqq\sum_{2\mu a^2\leqq k<4\mu a^2}\mathbf{1}(|N_t^\gamma-N_{\{t\}-k/\mu}^\gamma|\leqq a/2)\\
&\quad\cdot E(\mathbf{1}(k_0=k,\,Z_t^{\gamma,k}(a)\geqq\psi_1(a)/12)\\
&\quad\cdot P(Y_t^{\gamma,k}(a)\geqq 12\,c_{5.1}\,Z_t^{\gamma,k}(a)\,|\,\mathscr{F}(\gamma)\vee\mathscr{E}))\,|\,\mathscr{F}(\gamma))\\
&\geqq 12\,c_{5.1}\sum_{2\mu a^2\leqq k<4\mu a^2}\mathbf{1}(|N_t^\gamma-N_{\{t\}-k/\mu}^\gamma|\leqq a/2)\\
&\quad\cdot P(\tilde{Z}_t^{\gamma,i}(a)=0\text{ for all }2\mu a^2\leqq i<k,\,Z_t^{\gamma,k}(a)>\psi_1(a)/12\,|\,\mathscr{F}(\gamma)),\end{aligned}$$

where we have used Lemma 5.2 in the last. $\tilde{Z}_t^{\gamma,i}(a)$ and $Z_t^{\gamma,i}(a)$ are both $\mathscr{F}_1(\gamma)\vee\mathscr{G}(S^{\gamma,i})$-measurable for $0\leqq i<4a^2\mu$. The sets $\{S^{\gamma,i}\colon 0\leqq i<|\gamma|\}$ are mutually disjoint and are disjoint from $\{\gamma|k\colon 0\leqq k\leqq|\gamma|\}$ (by (5.1)). Therefore the σ-fields

$$\mathscr{G}(S^{\gamma,0}),\,\ldots,\,\mathscr{G}(S^{\gamma,|\gamma|}),\,\mathscr{F}(\gamma)$$

are mutually independent. It follows that $\{\tilde{Z}_t^{\gamma,i}(a)\colon 0\leqq i<k\}\cup\{Z_t^{\gamma,k}(a)\}$ are mutually conditionally independent given $\mathscr{F}(\gamma)$ for $k<4\mu a^2$ and that the conditional distribution of each of these random variables given $\mathscr{F}(\gamma)$, equals their conditional distribution given the smaller σ-field $\mathscr{F}_1(\gamma)$. Therefore on $\{N_t^\gamma\neq\Delta\}$, (5.3) is bounded below by

$$\begin{aligned}(5.4)\qquad 12\,c_{5.1}\sum_{2\mu a^2\leqq k<4\mu a^2}\mathbf{1}(|N_t^\gamma-N_{\{t\}-k/\mu}^\gamma|\leqq a/2)\prod_{i=[2\mu a^2]}^{k-1}P(\tilde{Z}_t^{\gamma,i}(a)=0\,|\,N_t^\gamma\neq\Delta)\\
\cdot P(Z_t^{\gamma,k}(a)>\psi_1(a)/12\,|\,N_t^\gamma\neq\Delta).\end{aligned}$$

If $k\in[2\mu a^2,4\mu a^2)$ is fixed then under $P(\cdot\,|\,N_t^\gamma\neq\Delta)$

$$Z_l=\sum_{\substack{\beta\sim\{t\}+(l-k+1)/\mu\equiv s\\ \sigma(\beta;\gamma)=k}}\mathbf{1}(N_s^\beta\neq\Delta,\,N_t^\gamma\neq\Delta)\qquad l=0,1,\ldots$$

is equal in law to the Galton-Watson process which satisfies (5.2) and (5.3). Therefore

$$(5.5)\quad \prod_{i=[P\mu a^2]}^{k-1} P(\tilde{Z}_t^{\gamma,i}(a)=0\,|\,N_t^\gamma \neq \Delta)\, P(Z_t^{\gamma,k}(a)>\psi_1(a)/12\,|\,N_t^\gamma \neq \Delta)$$
$$\geqq P(Z_{[\mu a^2]-1}=0)^{k-[2\mu a^2]} P(\mu^{-2} \sum_{\mu a^2 \leqq l+1 \leqq 2\mu a^2} Z_l > \psi_1(a)/12)$$
$$\geqq P(Z_{[\mu a^2]-1}=0)^{k-[2\mu a^2]} P(\min_{1-(\mu a^2)^{-1} \leqq l(\mu a^2)^{-1} \leqq 2-(\mu a^2)^{-1}} (Z_l/\mu a^2)$$
$$\geqq (\log\log 1/a)/10\,|\,Z_{[\mu a^2]-1}/\mu a^2 \geqq (\log\log 1/a)/5)$$
$$\cdot P(Z_{[\mu a^2]-1}/\mu a^2 \geqq (\log\log 1/a)/5).$$

In the last we require μa^2 sufficiently large so that $[\mu a^2]/\mu a^2 \geqq 5/6$. By (5.2) and (5.3) there is a $\mu_1 \in \mathbb{N}$ such that if $\mu a^2 \geqq \mu_1$, then (5.5) is greater than

$$(5.6)\quad (1-3/\mu a^2)^{k-[2\mu a^2]} \exp\{-(\log\log 1/a)/2\} (\mu a^2)^{-1}$$
$$\cdot P(\min_{1-(\mu a^2)^{-1} \leqq l(\mu a^2)^{-1} \leqq 2-(\mu a^2)^{-1}} Z_l/\mu a^2 \geqq (\log\log 1/a)/10\,|$$
$$\cdot Z_{[\mu a^2]-1} = [\mu a^2 (\log\log 1/a)/5]).$$

Now Feller showed that $Z_{[tN]}/N$ converges weakly as $N\to\infty$ (in $D([0,\infty),\mathbb{R})$) to the diffusion with generator $Af(x)=xf''(x)/2$ (see Ethier and Kurtz (1986, p. 388)). This, together with well-known estimates for this diffusion, allow one to easily estimate the conditional probability in (5.6). The result we need is Lemma 4.1(a) of Perkins (1988a) and shows that there are $c_{5.2}$ and $\mu_{5.1}(a)\in\mathbb{N}$ such that if $\mu \geqq \mu_{5.1}(a)$, $0<a<c_{5.2}$, and $k\in[2\mu a^2, 4\mu a^2)$, (5.6) is greater than

$$e^{-8} (\log 1/a)^{-\frac{1}{2}} (\mu a^2)^{-1} (1/2).$$

Combine this estimate with (5.3)≧(5.4) and (5.5)≧(5.6) and take $c_{5.2}<e^{-e}$ and $\mu_{5.1}(a) \geqq \mu_1$ to conclude that if $0<a<c_{5.2}$ and $\mu \geqq \mu_{5.1}(a)$, then on $\{N_t^\gamma \neq \Delta\}$,

$$P(Y_t^\gamma(a) > c_{5.1}\,\psi_1(a)\,|\,\mathscr{F}(\gamma))$$
$$\geqq 12 c_{5.1} (e^{-8}/2) (\log 1/a)^{-\frac{1}{2}} a^{-2} \int_{[2a^2, 4a^2)} 1(|N_t^\gamma - N_{(t)-s}^\gamma| \leqq a/2)\, d\lambda(s). \quad \square$$

Notation. $\tilde{B}(u) = B(e^u)\, e^{-u/2}$

$\tilde{B}$ is a stationary Ornstein-Uhlenbeck process under P_0^0.

Remark. To avoid measurability problems we will use P^m to also denote the associated Carathéodory outer measure in Lemma 5.4 and Proposition 5.6 below.

Lemma 5.4. *There are* $\eta_{5.1}: (1,\infty)\times\mathbb{N}^2 \to [0,\infty)$, $c_{5.4}$, *and for each* $L>1$ *an* $n_{5.1}(L)\in\mathbb{N}$ *and a* $c_{5.5}(L)$ *such that*

(i) $\lim_{\mu\to\infty} \eta_{5.1}(L,n,\mu)=0$ *for all* $L>1$, $n\in\mathbb{N}$,

(ii) $P^m(C \text{ bad at some } t\in[L^{-1}, L], 2b_n<\delta(\omega, 3, \mu))$
$\leqq m(\mathbb{R}^d)\, c_{5.5}(L)\, 2^{-2n(d-4)} 2^{nd/2} E_0^0(\exp\{-c_{5.4}\, 2^{-n/2}$
$\cdot \int_0^{2^n \log 2/2} \mathbf{1}(|\tilde{B}_u|\leqq 1/4)\, du\}) + (m(\mathbb{R}^d)+1)\, \eta_{5.1}(L, n, \mu)$

whenever $C\in\Lambda_{j_n}$, $n\geqq n_{5.1}(L)$, $m\in M_F^\mu$.

Proof. Fix $L>1$. Choose n large enough ($n\geqq n_{5.1}(L)$) so that $a_{2^n}\leqq c_{5.2}$ and $12a_{2^n}^2<(2L)^{-1}$, let $\mu\in\mathbb{N}$ satisfy $\mu\geqq\max\{\mu_{5.1}(a_k)\colon 2^n\leqq k\leqq 2^{n+1}-n\}\vee(2L)$. Let $m=\mu^{-1}\sum_{i=0}^{K}\delta_{x_i}\in M_F^\mu$. Lemma 5.1 implies that if $C\in\Lambda_{j_n}$ and $C'=C^{3h(2b_n)}$, then

$$\begin{aligned}(5.7)\qquad & P^m(C \text{ is bad at some } t\in[L^{-1}, L], 2b_n<\delta(\omega, 3, \mu))\\ & \leqq \sum_{L^{-1}<(j+1)b_n\leqq L+b_n}\ \sum_{\gamma\sim(j-1)b_n} E^m(\mathbf{1}(N^\gamma_{(j-1)b_n}\in C', Y^\gamma_{(j-1)b_n}(a_k)\\ & \qquad \leqq c_{5.1}\,\psi_1(a_k) \text{ for } 2^n\leqq k\leqq 2^{n+1}-n)\\ & \qquad \cdot P^m(N^\beta_{jb_n}\neq\Delta \text{ for some } \beta\sim jb_n, \beta>\gamma\,|\,\mathcal{A}_{(j-1)b_n})).\end{aligned}$$

We have used the $\mathcal{A}_t$-measurability of $Y_t^\gamma(a)$. By making μ larger, depending only on n (say $\mu\geqq\mu_1(n)$) we may use (5.2) and assume the last conditional probability on the right side of (5.7) is less than $(3/b_n\mu)$. If

$$S_k=\bigcup_{2a_k^2\leqq i/\mu<4a_k^2} S^{\gamma,i},\qquad \mathcal{G}_k=\mathcal{G}(S_k)$$

then $Y^\gamma_{(j-1)b_n}(a_k)$ is $\mathcal{F}(\gamma)\vee\mathcal{G}_k$-measurable. Note that $4a_k^2=2a_{k-1}^2$. Therefore $S_{2^n},\ldots,S_{2^{n+1}-n}$, $\{\gamma|j\colon 0\leqq j\leqq|\gamma|\}$ are disjoint (see (5.1)) and hence

$$\mathcal{G}_{2^n},\ldots,\mathcal{G}_{2^{n+1}-n},\mathcal{F}(\gamma)$$

are mutually independent. It follows that $\{Y^\gamma_{(j-1)b_n}(a_k)\colon 2^n\leqq k\leqq 2^{n+1}-n\}$ are conditionally independent given $\mathcal{F}(\gamma)$. (5.7) is therefore bounded by

$$\begin{aligned}(5.8)\qquad & \sum_{L^{-1}<(j+1)b_n\leqq L+b_n}(3/b_n\mu)\sum_{\gamma\sim(j-1)b_n}E^m(\mathbf{1}(N^\gamma_{(j-1)b_n}\in C')\\ & \cdot\prod_{2^n\leqq k\leqq 2^{n+1}-n}(1-P(Y^\gamma_{(j-1)b_n}(a_k)>c_{5.1}\,\psi_1(a_k)\,|\,\mathcal{F}(\gamma)))).\end{aligned}$$

The initial assumptions on n and μ allow us to apply Lemma 5.3 and conclude that the product in (5.8) is less than or equal to

$$\begin{aligned}&\exp\left\{-\sum_{k=2^n}^{2^{n+1}-n} c_{5.3}\, a_k^{-2}(\log 1/a_k)^{-\frac{1}{2}}\int_{[2a_k^2,4a_k^2)}\mathbf{1}(|N^\gamma_{(j-1)b_n}-N^\gamma_{\{(j-1)b_n\}-s}|\leqq a_k/2)\,d\lambda(s)\right\}\\ &\leqq\exp\{-c_{5.4}\,2^{-n/2}\int_{[2a^2_{2^{n+1}-n},4a^2_{2^n})}s^{-1}\mathbf{1}(|N^\gamma_{(j-1)b_n}-N^\gamma_{\{(j-1)b_n\}-s}|\leqq s^{\frac{1}{2}}/4)\,d\lambda(s)\}.\end{aligned}$$

Therefore (5.8) is bounded by

$$\sum_{L^{-1}<(j+1)b_n\leqq L+b_n} (3/b_n\mu) \sum_{i=0}^{K} \sum_{\substack{\gamma\sim(j-1)b_n\\ \gamma_0=x_i}} E^{\delta_{x_i}/\mu}(\mathbf{1}(N^{\gamma}_{(j-1)b_n}\in C')$$
$$\cdot\exp\{-c_{5.4}\,2^{-n/2}\int_{[2^{1+n-2^{n+1}},2^{2-2^n})} s^{-1}\mathbf{1}(|N^{\gamma}_{(j-1)b_n}-N^{\gamma}_{\{(j-1)b_n\}-s}|\leqq s^{\frac{1}{2}}/4)\,d\lambda(s)\})$$
$$\leqq(3/b_n)\sum_{L^{-1}<(j+1)b_n\leqq L+b_n} \mu^{-1}\sum_{i=0}^{K} E^0_0(\mathbf{1}(B_{(j-1)b_n}\in C'-x_i)$$
$$\cdot\exp\{-c_{5.4}\,2^{-n/2}\int_{[2^{1+n-2^{n+1}},2^{2-2^n})} s^{-1}\mathbf{1}(|B_{s+1/\mu}|\leqq s^{\frac{1}{2}}/4)\,d\lambda(s)\})$$

(Lemma 2.1 (b) and time reversal).

As $\mu\to\infty$, the i^{th} summand converges to

$$E^0_0\left\{\mathbf{1}(B_{j-1)b_n}\in C'-x_i)\exp\{-c_{5.4}\,2^{-n/2}\int_{2^{1+n-2^{n+1}}}^{2^{2-2^n}} s^{-1}\mathbf{1}(|B_s|\leqq s^{\frac{1}{2}}/4)\,ds\}\right\}.$$

For each n, it is easy to see the convergence is uniform in $x_i\in\mathbb{R}$, $C\in\Lambda_{j_n}$ and $j+1\in[L^{-1}b_n^{-1},\ Lb_n^{-1}+1]$. Hence there is an $\eta_1(L, n, \mu)$ such that $\lim_{\mu\to\infty}\eta_1(L, n, \mu)=0$ and (5.7) is bounded by

$$(5.9)\ 3/b_n\sum_{L^{-1}<(j+1)b_n\leqq L+b_n}\int\Bigg(E^0_0\Bigg(\mathbf{1}(B_{(j-1)b_n}\in C'-x)$$
$$\cdot\exp\left\{-c_{5.4}\,2^{-n/2}\int_{2^{1+n-2^{n+1}}}^{2^{2-2^n}} s^{-1}\mathbf{1}(|B_s|\leqq s^{\frac{1}{2}}/4)\,ds\right\}\Bigg)+\eta_1(L,n,\mu)\Bigg)\,dm(x).$$

If $(j-1)b_n=r$ is fixed the expected value in (5.9) equals

$$E^0_0\left(\mathbf{1}(\tilde{B}(\log r)\in(C'-x)\,r^{-\frac{1}{2}})\exp\left\{-c_{5.4}\,2^{-n/2}\int_{(-2^{n+1}+n+1)\log 2}^{(-2^n+2)\log 2}\mathbf{1}(|\tilde{B}_u|\leqq 1/4)\,du\right\}\right)$$
$$=E^0_0\Bigg(\exp\left\{-c_{5.4}\,2^{-n/2}\int_{(-2^{n+1}+n+1)\log 2}^{(-2^n+2)\log 2}\mathbf{1}(|\tilde{B}_u|\leqq 1/4)\,du\right\}$$
$$\cdot P^0_0(\tilde{B}(\log r)\in(C'-x)\,r^{-\frac{1}{2}}\,|\,\tilde{B}((-2^n+2)\log 2))\Bigg)$$
$$\leqq c_2(L)(2^{-2^n}+6h(2b_n))^d E^0_0\left(\exp\left\{-c_{5.4}\,2^{-n/2}\int_0^{(2^n-n)\log 2}\mathbf{1}(|\tilde{B}_u|\leqq 1/4)\,du\right\}\right).$$

Here we have used the facts that the transition density, $p_t(x, y)$, of B is uniformly bounded on $t\geqq 1$, and that

$$\log r+(2^n-2)\log 2\geqq-\log(2L)+(2^n-2)\log 2\geqq 1\qquad\text{for } n\geqq n_{5.1}(L).$$

Use the above in (5.9) to bound (5.7) by

$$3b_n^{-2}\, Lm(\mathbb{R}^d)\Bigg(c_3(L)\, 2^{-2^n d}\, 2^{nd/2} \cdot E_0^0\bigg(\exp\Big\{-c_{5.4}\, 2^{-n/2} \int_0^{2^n \log 2/2} \mathbf{1}(|\tilde{B}_u| \leqq 1/4)\, \mathrm{du}\Big\}\bigg) + \eta_1(\mathrm{L}, \mathrm{n}, \mu)\Bigg)$$

$$\leqq m(\mathbb{R}^d)\Bigg(c_{5.5}(L)\, 2^{-2^n(d-4)}\, 2^{nd/2} \cdot E_0^0\bigg(\exp\Big\{-c_{5.4}\, 2^{-n/2} \int_0^{2^n \log 2/2} \mathbf{1}(|\tilde{B}_u| \leqq 1/4)\, du\Big\}\bigg) + \eta_2(L, n, \mu)\Bigg)$$

where $\lim_{\mu \to \infty} \eta_2(L, n, \mu) = 0$. Finally the original restriction on μ may be incorporated into the definition of $\eta_{5.1}(L, n, \mu)$. □

To bound the expectation appearing on the right side of Lemma 5.4(ii) one may proceed in one of two ways. The unsophisticated route is to introduce the successive times the Ornstein-Uhlenbeck process exits from strip $[-1/4, 1/4]$, hits 0 and then re-exits from the strip and use Cramér's classical large deviation estimates for the times between exits. Alternatively one can annihilate the problem with the Donsker-Varadhan theory of large deviations. Exercise 8.28 of Stroock (1984, p. 178) contains more than enough for our humble needs. In any case we have

Lemma 5.5. $E_0^0\left(\exp\left\{-\theta \int_0^T \mathbf{1}(|\tilde{B}_u| \leqq 1/4)\, du\right\}\right) \leqq c_{5.6}\, e^{-c_{5.7}\theta T}$ *for all* $\theta \in [0, 1]$ *and* $T > 0$. □

Combining Lemmas 5.4 and 5.5, we obtain

Proposition 5.6. *There are constants* $c_{5.8}$ *and* $\{c_{5.9}(L)\colon L > 1\}$ *such that if* $L > 1$,

$$P^m(C \text{ bad at some } t \in [L^{-1}, L],\, 2b_n < \delta(\omega, 3, \mu)) \leqq m(\mathbb{R}^d)\, c_{5.9}(L)\, 2^{-2^n(d-4)}\, 2^{nd/2} \exp\{-c_{5.8}\, 2^{n/2}\} + (m(\mathbb{R}^d) + 1)\, \eta_{5.1}(L, n, \mu)$$

whenever $C \in A_{j_n}$, $n \geqq n_{5.1}(L)$, $m \in M_F^\mu$.

Notation. If $C \subset \mathbb{R}^d$, let dC denote the diameter of C.

Theorem 5.7. *If* $d \geqq 4$ *there is a constant* $c_{1.1}$ *such that* $\forall m \in M_F(\mathbb{R}^d)$ *and for* Q^m-a.a. ω

$$c_{1.1}\, \psi_1 - m(\bar{R}(r, s) \cap A) \leqq Y_{r,s}(A)\ \forall A \in \mathscr{B}(\mathbb{R}^d), \quad 0 < r \leqq s \leqq \infty \tag{5.10}$$

$$c_{1.1}\, \psi_1 - m(\bar{R}_+(0, s) \cap A) \leqq Y_s(A)\ \forall A \in \mathscr{B}(\mathbb{R}^d), \quad 0 < s \leqq \infty. \tag{5.11}$$

Proof. Fix $m\in M_F(\mathbb{R}^d)$. We work in the ω_1-saturated enlargement described in Sect. 2. Fix $\mu\in{}^*\mathbb{N}-\mathbb{N}$ and $m_\mu\in{}^*M_F^\mu(\mathbb{R}^d)$ such that $st_{M_F}(m_\mu)=m$. We will work on the Loeb space $({}^*\Omega, \mathscr{F}, P^{m_\mu})$ introduced in Theorem 2.3. Let $L\in\mathbb{N}^{>1}$, and

$$\Lambda_{n,L}=\{C\in\Lambda_n: C\subset[-L,L]^d\}.$$

Proposition 5.6 and the Transfer Principle of nonstandard analysis imply that if $n\geqq n_{5.1}(L)$, then

$$\begin{aligned}(5.12)\qquad & E^{m_\mu}(1(2b_n<\delta(\omega,3,\mu))\sum_{C\in\Lambda_{j_n,L}}\psi_1(dC)\\ & 1({}^*C \text{ is } \mu\text{-bad at some } \underline{t}\in{}^*[L^{-1},L]\cap T))\\ & \leqq(4L)^d 2^{d2^n} d^2 2^{-2^{n+2}} n\log 2m(\mathbb{R}^d)\,c_{5.9}(L)\,2^{-2^n(d-4)}\,2^{nd/2}\\ & \cdot e^{-c_{5.8}2^{n/2}}\qquad ({}^0\eta_{5.1}(L,n,\mu)=0)\\ & \leqq c_1(L)\,m(\mathbb{R}^d)\,2^{nd/2}ne^{-c_{5.8}2^{n/2}}\\ & \equiv\varepsilon_n(L)^2.\end{aligned}$$

$\{\varepsilon_n(L): n\in\mathbb{N}\}$ is summable. Use the Borel-Cantelli lemma, (5.12) and (4.8) to see there is an $N_L(\omega)<\infty$ a.s. for all $L\in\mathbb{N}^{>1}$ such that

$$\begin{aligned}(5.13)\qquad & \sum_{C\in\Lambda_{j_n,L}}\psi_1(dC)\,1({}^*C \text{ is } \mu\text{-bad at some } \underline{t}\in{}^*[L^{-1},L]\cap T)\\ & \leqq\varepsilon_n(L)\quad \text{for } n\geqq N_L,\quad \sum_{n=1}^{\infty}\varepsilon_n(L)<\infty.\end{aligned}$$

Fix ω outside a null set such that $N_L(\omega)<\infty$ for all $L\in\mathbb{N}^{>1}$ and (2.7) holds. Let $L^{-1}<r<s<L$ $(r,s\in\mathbb{R})$. Let

$$\begin{aligned}\Lambda^{g,1}_{j_n,L}=\{C\in\Lambda_{j_n,L}: & N_{\underline{t}}({}^*C)>0 \text{ for some } \underline{t}\in T\cap{}^*[r,s] \text{ and } {}^*C \text{ is } \mu\text{-good at}\\ & \text{each } \underline{t}\in T\cap{}^*[L^{-1},L] \text{ for which } N_{\underline{t}}({}^*C)>0\}\\ \Lambda^{b}_{j_n,L}=\{C\in\Lambda_{j_n,L}: & {}^*C \text{ is } \mu\text{-bad at some } \underline{t}\in T\cap{}^*[L^{-1},L]\}.\end{aligned}$$

Let $C\in\Lambda^{g,1}_{j_n,L}$ and choose $k\in\{2^n, 2^n+1, \ldots, 2^{n+1}-n\}$ such that

$$\int_{[r-5a^2_{2^n},s]} N_u({}^*C^{7a_k})\,d\lambda(u)>c_{5.1}\,\psi_1(a_k).$$

C^{7a_k} has side length

$$2^{-2^n}+14a_k\leqq 15a_k\leqq a_{k'}/2,$$

where $k'=k-10$ or $k-11$ (whichever is even) and so $k'\in[2^n-11, 2^{n+1}-n-10]$. Therefore $C^{7a_k}\subset C'$ for some $C'\in\Lambda_{k'}$ (by the definition of $\Lambda_{k'}$). Let $\Lambda^{g,2}_{j_n,L}$ be the set of cubes obtained by choosing one such C' for each $C\in\Lambda^{g,1}_{j_n,L}$. If C' is as above, then

$$\int_{[r-5a_{2n}^2,s]} N_u(^*C')\,d\lambda(u)>c_{5.1}\,\psi_1(a_k)\geqq c_{5.1}\,2^{-20}\psi_1(a_{k'})$$
$$\geqq c_2^{-1}\,\psi_1(dC')\,\forall C'\in\Lambda^{g,2}_{j_n,L}. \tag{5.14}$$

By the proof of Lemma 1 of Taylor (1964) there is a subclass $\Lambda^g_{j_n,L}\subset\Lambda^{g,2}_{j_n,L}$ such that

$$\bigcup_{C\in\Lambda^g_{j_n,L}} C=\bigcup_{C\in\Lambda^{g,2}_{j_n,L}} C\supset\bigcup_{C\in\Lambda^{g,1}_{j_n,L}} C, \tag{5.15}$$

and

(5.16) No point in $\mathbb{R}^d$ is covered by more than 2^d cubes in $\Lambda^g_{j_n,L}$.

Note that k' was chosen to be even so that all cubes in $\Lambda^{g,2}_{j_n,L}$ have sides of length 2^{-M} for some $M\in\mathbb{N}$ and the proof of Taylor's Lemma 1 goes through in $\mathbb{R}^d$.

Let

$$S_{r,s}=st\{N^\beta_{\underline{t}}:\beta\sim\underline{t},\,N^\beta_{\underline{t}}\neq\Delta,\,\underline{t}\in{}^*[r,s]\cap T\}.$$

$S_{r,s}$ is closed by Proposition 2.1.8 of Albeverio et al. (1986), and (2.7), and the definition of $M_{r,s}$ show that $S_{r,s}$ is a support of $Y_{r,s}$. Therefore

$$S_{r,s}\supset S(Y_{r,s})=\bar{R}(r,s). \tag{5.17}$$

Let A be a compact subset of $[-M,M]^d$. (5.15) implies that

$$A\cap S_{r,s}\subset\Bigg(\bigcup_{C\in\Lambda^b_{j_n,L}} C\Bigg)\cup\Bigg(\bigcup_{C\in\Lambda^g_{j_n,L},\,C\cap A\neq\phi} C\Bigg). \tag{5.18}$$

If $n\geqq N_L$, then

$$\begin{aligned}
&\sum_{C\in\Lambda^b_{j_n,L}}\psi_1(dC)+\sum_{C\in\Lambda^g_{j_n,L},\,C\cap A\neq\phi}\psi_1(dC)\\
&\leqq\varepsilon_n(L)+c_2\sum_{C\in\Lambda^g_{j_n,L},\,C\cap A\neq\phi}{}^\circ\!\int_{[r-5a_{2n}^2,s]} N_u(C)\,d\lambda(u)\quad\text{(by (5.13) and (5.14))}\\
&\leqq\varepsilon_n(L)+2^d c_2\,{}^\circ\!\int_{[r-5a_{2n}^2,s]} N_u(A^{\sqrt{d}a_{2n-11}})\,d\lambda(u)\quad\text{(by (5.16))}\\
&\leqq 2\varepsilon_n(L)+2^d c_2\,Y_{r-5a_{2n}^2,s}(A^{\sqrt{d}a_{2n-11}})\quad\text{(by (2.7))}.
\end{aligned}$$

As $n\to\infty$ this last expression converges to $2^d c_2\,Y_{r,s}(A)$ because A is closed.
From (5.18) we may conclude

$$\psi_1-m(A\cap S_{r,s})\leqq 2^d c_2\,Y_{r,s}(A)$$
$$\text{for all } 0<r<s<\infty,\,A\text{ compact},\,A\subset[-L,L]^d. \tag{5.19}$$

c_2 does not depend on L. Therefore we may fix ω outside a single null set so that (5.19) holds for all $L \in \mathbb{N}$. The inner regularity of $\psi_1 - m$ (Rogers (1970, Thm. 47, 48 and the ensuing Corollaries)) and (5.17) imply that if $c_{1.1} = (2^d c_2)^{-1}$, then

$$c_{1.1}\, \psi_1 - m(A \cap \bar{R}(r, s)) \leqq Y_{r,s}(A) \quad \forall A \in \mathscr{B}(\mathbb{R}^d) \quad \forall 0 < r \leqq s < \infty.$$

We may let $r \downarrow 0$ to get (5.11) and $s \uparrow \infty$ to get (5.10) with $s = \infty$ (recall the process dies out at a finite time a.s.). Finally a routine measurability argument is needed to transfer the result over to the space of continuous $M_F(\mathbb{R}^d)$-valued paths. □

Theorem 5.7 may be improved if $d = 4$ by just using the estimates given in Theorem 3.2(b). Recall $\psi_0(x) = x^4 \log^+(1/x)$.

Theorem 5.8. *Assume* $d = 4$.

(a) $\psi_0 - m(\bar{R}(r, \infty)) < \infty$ *for all* $r > 0$ Q^m-a.s. *for each* $m \in M_F(\mathbb{R}^4)$.

(b) *If* $m \in M_F(\mathbb{R}^4)$ *has compact support and satisfies*

$$\sup_x \int |y - x|^{-2}\, dm(y) < \infty,$$

then $\psi_0 - m(\bar{R}_+(0, \infty)) < \infty$ Q^m-a.s.

Proof. Let $\Gamma_{n,L}$ denote the partition of $(-L, L]^4$ into "right semi-closed" cubes of side length 2^{-n} $(n, L \in \mathbb{N})$. Fix $r > 0$ and $m \in M_F(\mathbb{R}^4)$. Then for large enough n (depending on $(r, m(\mathbb{R}^4))$), we have

$$\begin{aligned} &E^m\Big(\sum_{C \in \Gamma_{n,L}} \psi_0(dC)\, \mathbf{1}(X_t(C) > 0 \text{ for some } t \geqq r)\Big) \\ &\leqq \sum_{C \in \Gamma_{n,L}} c_1\, 2^{-4n} (\log 2^n)(\log 2^n)^{-1} \big[\sup_{x \in \mathbb{R}^4} E_0^m(|B_r - x|^{-2}) + 1\big] \quad \text{(Theorem 3.2(b))} \\ &\leqq c_1\, m(\mathbb{R}^4)(1 + r^{-1})(2L)^4 \quad \text{by (3.1.1)} \\ &< \infty. \end{aligned}$$

Let $n \to \infty$ and use Fatou's Lemma to see that $\psi_0 - m(\bar{R}(r, \infty) \cap (-L, L]^4) < \infty$ for all $L \in \mathbb{N}$ Q^m-a.s. The compactness of $S(X_r)$ (Theorem 1.2), the continuity of $S(X_r)$ (i.e. Theorem 1.1) and the fact that X_t dies out in finite time together imply $\bar{R}(r, \infty)$ is compact. This gives (a).

For (b), argue as above using Theorem 3.2(b) and (3.1.2), to get

$$E^m(\psi_0 - m(\bar{R}(r, \infty) \cap (-L, L]^4)) \leqq c_1 \sup_x \int |y - x|^{-2}\, dm(y) (2L)^4$$

for all $r > 0$. Let $r \downarrow 0$ to see that

$$E^m(\psi_0 - m(\bar{R}_+(0, \infty) \cap (-L, L]^4)) \leqq c_1 \sup_x \int |y - x|^{-2}\, dm(y) (2L)^4 < \infty.$$

Since m has compact support, Theorem 1.1 implies $\bar{R}_+(0, \infty)$ is a.s. compact and the proof is complete. □

We now turn to the lower bound on the Hausdorff measure of $\bar{R}(r, s)$. Our approach is that used in Perkins (1988a). We will see that the denstiy theorem of Rogers and Taylor (1961, Lemma 3) reduces the problem to showing

$$\limsup_{a \downarrow 0} Y_{r,s}(B(x, a))/\psi(a) \leqq c(d) \quad \text{for } Y_{r,s}\text{-a.a. } x \text{ a.s.}$$

To estimate $Y_{r,s}(B(x, a))$ "for $x \in \bar{R}(r, s)$" we will estimate $\int M_{r,s}^{(\mu)}(B(x, a))^p \, dM_{r,s}^{(\mu)}(x)$. These estimates are more delicate than those in Perkins (1988a), but there is some simplification here because we are essentially concerned with one random measure Y_∞ and not an infinite family $\{X_t : t \geqq 0\}$.

We introduce some notations which will help us prune some Brownian trees.

Notation. $I_j = \{\beta \in I : \beta_0 = j\}, j \in \mathbb{Z}_+$; $\bar{I} = I_0 \cup I_1$.

If $L, p \in \mathbb{N}$, let

$$\begin{aligned} \Lambda_1(p, L) = \{(\beta(1), \ldots, \beta(p)) \in \bar{I}^p : |\beta(i)| \leqq L\mu \text{ and } \beta(j) \\ \text{is not a descendant of } \beta(i) \text{ for all } 1 \leqq i \neq j \leqq p\}, \end{aligned}$$

$$\begin{aligned} \Lambda_2(p, L) = \{(\beta(1), \ldots, \beta(p)) \in \bar{I}^p : |\beta(i)| \leqq L\mu \text{ for all } i \leqq p, \\ \text{and } \beta(i) < \beta(j), \text{ for some } 1 \leqq i \neq j \leqq p\}, \end{aligned}$$

$$\Lambda(p, L) = \Lambda_1(p, L) \cup \Lambda_2(p, L).$$

If $\beta = (\beta(1), \ldots, \beta(p)) \in I^p$ and $j \in \mathbb{Z}_+$ let

$$A(\beta, a) = \{\omega : |N^{\beta(i)} - N^{\beta(j)}| \leqq a, \text{ for all } i, j \leqq p\} \qquad (a > 0),$$

$$I_j(\beta) = I_j \cap \{\beta(i) : i \leqq p\}, \; k_j(\beta) = \text{card}(I_j(\beta)),$$

$$l_j(\beta) = \min\{l : \beta(m)|l \neq \beta(n)|l \text{ or } l > |\beta(m)| \text{ for some } \beta(m), \beta(n) \in I_j(\beta)\},$$

$$l(\beta) = \min\{l : \beta(m)|l \neq \beta(n)|l \text{ or } l > |\beta(m)| \text{ for some } m, n \leqq p\},$$

$$\Gamma(p) = \Gamma(p, L) = \{\beta \in \Lambda_1(p, L) : l(\beta) = 0\} \qquad (L, p \in \mathbb{N}).$$

If $a > 0$, $L, \mu \in \mathbb{N}$ let

$$H(a, L) = H(a, L, \mu) = \iint_{[0,L]^2} P_0^0(|B_{s+t}| \leqq a) \, d\lambda^\mu(s) \, d\lambda^\mu(t),$$

and if $p \in \mathbb{N}$, define

$$V_p(a, L) = V_p(a, L, \mu) = \sum_{\beta \in \Lambda(p, L)} \mu^{-2p+1} \mathbf{1}(A(\beta, a))$$

$$W_p(a, L) = W_p(a, L, \mu) = \sum_{\beta \in \Gamma(p, L)} \mu^{-2(p-1)} \mathbf{1}(A(\beta, a)) \qquad \text{for } p \geqq 2,$$

and set $W_1(a, L, \mu) = 1$. We stress that whenever we write an inequality like $|N^{\beta(i)} - N^{\beta(j)}| \leqq a$, as in the definition of $A(\beta, a)$, we are implicitly assuming that $N^{\beta(i)} \neq \Delta$ and $N^{\beta(j)} \neq \Delta$.

If $x_i=0$ for $i=0$, 1 and $x_i=\Delta$ for $i>1$, denote $P^{(x_i)}$ and $E^{(x_i)}$ by P_2 and E_2, respectively. If $y_0=0$ and $y_i=\Delta$ for $i>0$ denote $P^{(y_i)}$ and $E^{(y_i)}$ by P_1 and E_1, respectively.

Lemma 5.9. *If* $a>0$, $p\in\mathbb{N}^{\geqq 2}$, L, $\mu\in\mathbb{N}$, *then*

$$E_2(W_p(a,L,\mu))\leqq(p-2)!(12H(a,L,\mu))^{p-1}.$$

Proof. Fix $a>0$, L, $\mu\in\mathbb{N}$ and suppress dependence on these parameters if there is no ambiguity. Let $p\in\mathbb{N}^{\geqq 2}$. If $\phi\neq S\subsetneqq\{1,\ldots,p\}$ and $l_j\in\{1,\ldots,L\mu+1\}$, $\gamma^j\in I_j$, $|\gamma^i|=l_j-1$ for $j=0$, 1, let

$$\begin{aligned}\Lambda(S,.\,l_0,l_1,\gamma^0,\gamma^1)=\{&\beta\in\Gamma(p,L): I_0(\beta)=\{\beta(i): i\in S\}, l_j(\beta)=l_j,\\ &\beta(i)|l_j-1=\gamma^j, \text{for all } \beta(i)\in I_j(\beta), \text{for } j=0,1\}.\end{aligned}$$

These sets form a partition of $\Gamma(p,L)$ as S, l_j and γ^j $(j=0,1)$ vary. If $\beta\in\Lambda(S,l_0,l_1,\gamma^0,\gamma^1)$ where $\operatorname{card}(S)=k$ then the $\beta(i)$'s may be permuted to give a

$$\tilde\beta\in\Lambda(k,l_0,l_1,\gamma^0,\gamma^1)=\Lambda(\{1,\ldots,k\},l_0,l_1,\gamma^0,\gamma^1).$$

This is a $\binom{p}{k}$ to 1 correspondence as there are $\binom{p}{k}$ ways to select the k components in $I_0(\beta)$. It follows that

$$(5.20)\qquad \begin{aligned}E_2(W_p)=E_2\Bigg(\mu^{-2(p-1)}\sum_{k=1}^{p-1}\binom{p}{k}\\ \cdot\sum_{l_0=1}^{L\mu+1}\sum_{l_1=1}^{L\mu+1}\sum_{\gamma^0\in I_0}\sum_{\gamma^1\in I_1}\mathbf{1}(|\gamma^j|=l_j-1)\sum_{\beta\in\Lambda(k,l_0,l_1,\gamma^0,\gamma^1)}\cdot\mathbf{1}(\Lambda(\beta,a))\Bigg).\end{aligned}$$

Fix $l_j\in\{1,2,\ldots,L\mu+1\}$ and $\gamma^j\in I_j$ such that $|\gamma^j|=l_j-1$, $j=0$, 1. We consider four cases to bound

$$\sum=\sum_{\beta\in\Lambda(k,l_0,l_1,\gamma^0,\gamma^1)}P_2(\Lambda(\beta,a)).$$

In each case β will denote a multi-index in $\Lambda(k,l_0,l_1,\gamma^0,\gamma^1)$ and

$$\begin{aligned}\mathscr{H}(\beta)=\sigma(&\hat N^{\beta(i)}_{s+l_0/\mu}-\hat N^{\beta(i)}_{l_0/\mu}: s\geqq 0, i\leqq k)\\ \vee\,\sigma(&\hat N^{\beta(i)}_{s+l_1/\mu}-\hat N^{\beta(i)}_{l_1/\mu}: s\geqq 0, k<i\leqq p)\vee\mathscr{E}.\end{aligned}$$

Case 1. $2\leqq k\leqq p-2$.

In this case both $I_0(\beta)$ and $I_1(\beta)$ contain at least two elements and so

$$(5.21)\qquad \beta=(\gamma^0\vee\tilde\beta^0,\gamma^1\vee\tilde\beta^1)\quad\text{for some } \tilde\beta^0\in\Gamma(k)\quad\text{and}\quad\tilde\beta^1\in\Gamma(p-k).$$

Here $\vee$ is interpreted componentwise. Therefore

$$(5.22)\quad P_2(A(\beta,a))=E_2(\mathbf{1}(|(N^{\beta(i)}-N^{\beta(i)}_{l_0/\mu})-(N^{\beta(j)}-N^{\beta(j)}_{l_0/\mu})|\leqq a \text{ for all } i,j\leqq k)$$
$$\cdot\mathbf{1}(|(N^{\beta(i)}-N^{\beta(i)}_{l_1/\mu})-(N^{\beta(j)}-N^{\beta(j)}_{l_1/\mu})|\leqq a \text{ for all } k<i,j\leqq p)$$
$$\cdot P_2(\hat{N}^{\gamma^0}_{l_0/\mu}-\hat{N}^{\gamma^1}_{l_1/\mu}\in\bar{B}(\hat{N}^{\beta(p)}-\hat{N}^{\beta(p)}_{l_1/\mu}-(\hat{N}^{\beta(1)}-\hat{N}^{\beta(1)}_{l_0/\mu});a)\,|\,\mathscr{H}(\beta)))$$

The centre of the ball in this last conditional probability is $\mathscr{H}(\beta)$-measurable, and $\hat{N}^{\gamma^0}_{\cdot}$ and $\hat{N}^{\gamma^1}_{\cdot}$ are independent Brownian motions, stopped at l_0/μ and l_1/μ, respectively, and are jointly independent of $\mathscr{H}(\beta)$. The conditional probability in (5.22) evaluated at ω is therefore a.s. equal to

$$P_0^0(B((l_0+l_1)/\mu)\in\bar{B}((\hat{N}^{\beta(p)}-\hat{N}^{\beta(p)}_{l_1/\mu}-(\hat{N}^{\beta(1)}-\hat{N}^{\beta(1)}_{l_0/\mu}))(\omega);a))$$
$$\leqq P_0^0(|B((l_0+l_1)/\mu)|\leqq a).$$

Note also that $\{N^{\beta(i)}_{\cdot}: i\leqq k\}$ and $\{N^{\beta(i)}_{\cdot}: k<i\leqq p\}$ are independent since $\beta(i)_0 \neq\beta(j)_0$ whenever $i\leqq k<j$. Use the above estimate, (5.22) and (5.21) to conclude

$$(5.23)\quad \sum\leqq P_0^0(|B((l_0+l_1)/\mu)\leqq a)\,2^{-l_0-l_1}$$
$$\cdot E_2\Big(\sum_{\tilde{\beta}^0\in\Gamma(k)}\mathbf{1}(|N^{\gamma^0\vee\tilde{\beta}^0(i)}-N^{\gamma^0\vee\tilde{\beta}^0(i)}_{l_0/\mu}-(N^{\gamma^0\vee\tilde{\beta}^0(j)}-N^{\gamma^0\vee\tilde{\beta}^0(j)}_{l_0/\mu})|\leqq a,$$
$$\forall i,j\leqq k\,|\prod_{0\leqq i<l_0} e^{\gamma^0|i}=1\Big)$$
$$\cdot E_2\Big(\sum_{\beta^1\in\Gamma(p-k)}\mathbf{1}(|N^{\gamma^1\vee\tilde{\beta}^1(i)}-N^{\gamma^1\vee\tilde{\beta}^2(i)}_{l_1/\mu}-(N^{\gamma^1\vee\tilde{\beta}^1(j)}-N^{\gamma^1\vee\tilde{\beta}^1(j)}_{l_1/\mu})|\leqq a$$
$$\forall i,j\leqq p-k)\,|\prod_{0\leqq i<l_1} e^{\gamma^1|i}=1\Big).$$

Note that the $P_{2,j}=P_2(\cdot\,|\prod_{0\leqq i<l_j} e^{\gamma^j|i}=1)$-distribution of $\{B^{\gamma^j\vee\beta}_{l_j/\mu+\cdot}-B^{\gamma^j\vee\beta}_{l_j/\mu},\ e^{\gamma^j\vee\beta}:$ $\beta\in\bar{I}\}$ is equal to the P_2-distribution of $\{B^\beta,\ e^\beta:\ \beta\in\bar{I}\}$ $(j=0,1)$. Therefore the $P_{2,j}$-distribution of $N^{\gamma^j\vee\beta}_{l_j/\mu+\cdot}-N^{\gamma^j\vee\beta}_{l_j/\mu}:\ \beta\in\bar{I}\}$ is equal to the P_2-distribution of $\{N^\beta_{\cdot}:$ $\beta\in\bar{I}\}$ $(j=0,1)$, and (5.23) implies that

$$(5.24)\qquad \sum\leqq P_0^0(|B((l_0+l_1)/\mu)\leqq a)\,2^{-l_0-l_1}E_2(W_k)\,E_2(W_{p-k})\,\mu^{2(p-1)-2}.$$

Case 2. $k=1$, $p\geqq 3$.

In this case $I_0(\beta)=\{1\}$ and $I_1(\beta)=\{2,\ldots,p\}$ contains at least two elements, and so

$$(5.25)\qquad \beta=(\gamma^0,\gamma^1\vee\tilde{\beta})\quad \text{for some } \tilde{\beta}\in\Gamma(p-1).$$

Therefore

$$P_2(A(\beta,a))=E_2(\mathbf{1}(|(N^{\beta(i)}-N^{\beta(i)}_{l_1/\mu})-(N^{\beta(j)}-N^{\beta(j)}_{l_1/\mu})|\leqq a \text{ for all } 1<i,j\leqq p)\,\mathbf{1}(N^{\gamma^0}\neq\Delta)$$
$$\cdot\ P_2(\hat{N}^{\gamma^0}-\hat{N}^{\gamma^0}_{l_1/\mu}\in\bar{B}(\hat{N}^{\beta(2)}-\hat{N}^{\beta(2)}_{l_1/\mu};a)\,|\,\mathscr{H}(\beta)))$$
$$\leqq P_0^0(|B((l_0-1+l_1)/\mu)|\leqq a)\,2^{-l_0+1}P_2(|N^{\gamma^1\vee\tilde{\beta}(i)}-N^{\gamma^1\vee\tilde{\beta}(i)}_{l_1/\mu}$$
$$-(N^{\gamma^1\vee\tilde{\beta}(j)}-N^{\gamma^1\vee\tilde{\beta}(j)}_{l_1/\mu})|\leqq a \text{ for all } i,j\leqq p-1\,|\prod_{0\leqq i<l_1} e^{\gamma^1|i}=1)\,2^{-l_1}$$

where we have argued as in the previous case. Sum over $\beta \in \Lambda(1, l_0, l_1, \gamma^0, \gamma^1)$, use (5.25) and the equivalence in law outlined in the previous case to see that

$$(5.26) \qquad \sum \leqq P_0^0(|B((l_0+l_1-1)/\mu)| \leqq a)\, 2^{-(l_0+l_1-1)} E_2(W_{p-1})\, \mu^{2(p-1)-2}.$$

Case 3. $k=p-1$, $p \geqq 3$.

This is the mirror image of case 2 and (5.26) holds.

Case 4. $p=2$, $k=1$.

In this case $\Lambda(1, l_0, l_1, \gamma^0, \gamma^1)=\{(\gamma^0, \gamma^1)\}$ and

$$(5.27) \qquad \begin{aligned} \sum &= P_2(|N^{\gamma^0}-N^{\gamma^1}| \leqq a \,|\, N^{\gamma^0} \neq \Delta, N^{\gamma^1} \neq \Delta)\, 2^{-(l_0-1)-(l_1-1)} \\ &= P_0^0(|B((l_0+l_1-2)/\mu) \leqq a)\, 2^{-(l_0+l_1-2)}. \end{aligned}$$

(5.24), (5.26) and (5.27) can be combined to conclude that

$$(5.28) \quad \sum \leqq P_0^0(|B((l_0+l_1-2)/\mu)| \leqq a)\, 2^{-(l_0+l_1-2)} E_2(W_k)\, E_2(W_{p-k})\, \mu^{2(p-1)-2} \quad 1 \leqq k \leqq p-1, p \geqq 2.$$

Substitute the above into (5.20) to obtain

$$\begin{aligned} E_2(W_p) \leqq \sum_{k=1}^{p-1} \binom{p}{k} \sum_{l_0=1}^{L\mu+1} \sum_{l_1=1}^{L\mu+1} & 2^{l_0-1} 2^{l_1-1} P_0^0(|B((l_0+l_1-2)/\mu)| \leqq a) \\ & \cdot 2^{-(l_0+l_1-2)} E_2(W_k)\, E_2(W_{p-k})\, \mu^{-2} \end{aligned}$$

and hence,

$$(5.29) \qquad E_2(W_p) \leqq \sum_{k=1}^{p-1} \binom{p}{k} E_2(W_k)\, E_2(W_{p-k}) H,$$

where $H=H(a, L, \mu)$ is defined above. We now proceed by induction. (5.29) implies the result if $p=2$. Let $p \in \mathbb{N}^{>2}$. Assume the required result holds for $E_2(W_k)$ where $k<p$. Substitute these bounds into (5.29) to see that, if $(-1)! \equiv 1$, then

$$\begin{aligned} E_2(W_p) &\leqq \sum_{k=2}^{p-1} \binom{p}{k} (k-2)!\,(p-k-2)!\,(12H)^{p-2} H \\ &= (p-2)!\,(12H)^{p-1}\, 12^{-1} \Big[p(p-1) \sum_{k=2}^{p-2} (k(k-1)(p-k)(p-k-1))^{-1} \\ &\qquad + 2p(p-2)^{-1} \Big] \\ &= (p-2)!\,(12H)^{p-1}\, 12^{-1} \Big[p(p-1) \sum_{k=2}^{p-2} (p(p-2))^{-1} ((k-1)^{-1} - k^{-1} \\ &\qquad + (p-k-1)^{-1} - (p-k)^{-1} + (p-1)^{-1} (k^{-1} + (p-1-k)^{-1} \\ &\qquad + (k-1)^{-1} + (p-k)^{-1})) + 2p(p-2)^{-1} \Big] \\ &\leqq (p-2)!\,(12H)^{p-1}\, 12^{-1} \Big[(p-1)(p-2)^{-1} \Big(2(1-(p-2)^{-1}) \\ &\qquad + 4(p-1)^{-1} \sum_{k=1}^{p-2} k^{-1} \Big) + 6 \Big] \leqq (p-2)!\,(12H)^{p-1}. \quad \square \end{aligned}$$

Lemma 5.10. *If* $a>0$, $p\in\mathbb{N}^{\geqq 2}$, L, $\mu\in\mathbb{N}$, *then*

$$E_1(V_p(a, L, \mu)) \leqq L(p-2)!\,(12H(a, L, \mu))^{p-1} + \mu^{-1}(L+1)^p(p!)^2 e^{12(L+1)^2}.$$

Proof. Fix $a>0$ and L, $\mu\in\mathbb{N}$ and suppress dependence on these parameters. Let $p\in\mathbb{N}^{\geqq 2}$. *Then*

$$V_p = V_{p,1} + V_{p,2},$$

where

$$V_{p,i} = \sum_{\beta\in\Lambda_i(p,L)} \mu^{-2p+1}\mathbf{1}(A(\beta, a)) \qquad i=1,2.$$

Let $\beta\in\Lambda_1(p,L)\cap I_0^p$. Then $l(\beta)=l\in\{1,2,\dots,L\mu\}$ and there is an $\alpha\in\{0\}\times\{0,1\}^{l-1}$ and a $\tilde{\beta}\in\Gamma(p,L)$ such that $\beta=\alpha\vee\tilde{\beta}$ (as before $\vee$ is interpreted componentwise). Under P_1, $N^\beta=\Delta$ unless $\beta\in I_0$, so the summation in the definition of $V_{p,1}$ may be taken over $\beta\in\Lambda_1(p,L)\cap I_0^p$. The above shows that

$$\begin{aligned}(5.30)\quad E_1(V_{p,1}) &\leqq E_1\left(\mu^{-2p+1}\sum_{l=1}^{L\mu}\ \sum_{\alpha\in\{0\}\times\{0,1\}^{l-1}}\ \sum_{\tilde{\beta}\in\Gamma(p,L)} \mathbf{1}(A(\alpha\vee\tilde{\beta}, a))\right)\\ &= \mu^{-1}\sum_{l=1}^{L\mu}\ \sum_{\alpha\in\{0\}\times\{0,1\}^{l-1}} \mu^{-2(p-1)} P_1\left(\prod_{i=0}^{l-1} e^{\alpha|i}=1\right)\\ &\quad\cdot\sum_{\tilde{\beta}\in\Gamma(p,L)} P_1\Big(|N^{\alpha\vee\tilde{\beta}(i)} - N_{l/\mu}^{\alpha\vee\tilde{\beta}(i)} - (N^{\alpha\vee\tilde{\beta}(j)} - N_{l/\mu}^{\alpha\vee\tilde{\beta}(j)})| \leqq a\\ &\quad\text{for all } i,j\leqq p \,\Big|\, \prod_{i=0}^{l-1} e^{\alpha|i}=1\Big).\end{aligned}$$

As in the proof of Lemma 5.9, the $P_1\left(\cdot\,\middle|\,\prod_{i=0}^{l-1} e^{\alpha|i}=1\right)$-distribution of $\{N_{l/\mu+\cdot}^{\alpha\vee\beta} - N_{l/\mu}^{\alpha\vee\beta}\colon \beta\in\bar{I}^p\}$ is equal to the P_2-distribution of $\{N_\cdot^\beta\colon \beta\in\bar{I}^p\}$. Therefore from (5.30) we may obtain

$$\begin{aligned}(5.31)\quad E_1(V_{p,1}) &\leqq \mu^{-1}\sum_{l=1}^{L\mu}\ \sum_{\alpha\in\{0\}\times\{0,1\}^{l-1}} 2^{-l}\mu^{-2(p-1)} \sum_{\beta\in\Gamma(p,L)} P_2(A(\beta,a))\\ &\leqq (L/2)\,E_2(W_p) \leqq (L/2)\,(p-2)!\,(12H)^{p-1} \qquad (p\geqq 2)\quad \text{(Lemma 5.9)}.\end{aligned}$$

Next we show

$$(5.32)\qquad E_1(V_{p,2}) \leqq \sum_{j=1}^{p-1} (p!)^2((p-j)!)^2(L+1)^j\mu^{-j}E_1(V_{p-j,1}) \qquad \text{for } p\in\mathbb{N},$$

by induction on p. If $p=1$, $V_{1,2}=0$ because $\Lambda_2(1,L)=\phi$ and (5.32) is trivial. If $p\geqq 2$, we may divide $\Lambda_2(p,L)$ into the $p(p-1)$ (non-disjoint) sets

$$\begin{aligned}B(p,i,j) = \{\beta\in\Lambda_2(p,L)\colon \beta(i)<\beta(j) &\text{ and there is no } k\in\{1,\dots,p\}-\{i,j\}\\ &\text{such that } \beta(i)<\beta(k)<\beta(j)\}, \qquad 1\leqq i\neq j<p.\end{aligned}$$

By symmetry we have

$$E_1(V_{p,2}) \leqq p(p-1)\, E_1\Big(\mu^{-2p+1} \sum_{\beta\in B(p,p-1,p)} \mathbf{1}(A(\beta,a))\Big)$$
$$\leqq p(p-1)\, E_1\Big(\mu^{-2(p-1)} \sum_{\beta\in \Lambda(p-1,L)} \mu^{-1} \sum_{l=0}^{L\mu} \sum_{\alpha\in\{0,1\}^l}$$
$$\cdot \mathbf{1}(A(\beta,a))\, P_1(N^{\beta(p-1)\vee\alpha} \neq \Delta \,|\, \mathscr{G}(\{\beta(i): i\leqq p-1\})\Big)$$
$$\leqq p(p-1)(L+\mu^{-1})\,\mu^{-1} E_1(V_{p-1})$$
$$\leqq p(p-1)(L+1)\mu^{-1}(E_1(V_{p-1,1})+E_1(V_{p-1,2})).$$

The obvious induction argument now gives (5.32).

To complete the proof, substitute (5.31) into (5.32) (use $E_1(V_{1,1})\leqq L+1$) and add the resulting inequality to (5.31). A bit of algebra then gives

$$E_1(V_p) \leqq (L/2)(p-2)!\,(12H)^{p-1} + \sum_{j=1}^{p-1} (p!)^2 (p-j)!^{-1} (L+1)^{j+1} \mu^{-j} (12H)^{p-1-j}$$
$$\leqq L(p-2)!\,(12H)^{p-1} + \mu^{-1}(L+1)^p (p!)^2 e^{12H}$$
$$\leqq L(p-2)!\,(12H)^{p-1} + \mu^{-1}(L+1)^p (p!)^2 e^{12(L+1)^2}. \quad \square$$

Recall $\psi_4(x)=x^4 \log^+ 1/x \log^+ \log^+ 1/x$.

Theorem 5.11. *If $d\geqq 4$ there is a constant $c_{1.2}$, such that $\forall m\in M_F(\mathbb{R}^d)$ and for Q^m-a.a. ω*

(5.33a) $\qquad Y_\infty(A) \leqq c_{1.2}\, \psi_1 - m(A) \quad \forall A\in\mathscr{B}(\mathbb{R}^d) \quad (d>4)$

(5.33b) $\qquad Y_\infty(A) \leqq c_{1.2}\, \psi_4 - m(A) \quad \forall A\in\mathscr{B}(\mathbb{R}^d) \quad (d=4).$

Proof. Let $d\geqq 4$ and set $\psi_d=\psi_1$ if $d>4$. Fix $m\in M_F(\mathbb{R}^d)$. We work in the ω_1-saturated enlargement described in Sect. 2. More specifically, fix $\mu\in{}^*\mathbb{N}-\mathbb{N}$, $m_\mu\in{}^*M_F(\mathbb{R}^d)$ such that $st_{M_F}(m_\mu)=m$ and work on the Loeb space $({}^*\Omega, \mathscr{F}, P^{m_\mu})\equiv({}^*\Omega, \mathscr{F}, P)$ of Theorem 2.3. Dependence on μ will usually be suppressed. c will denote a positive constant to be chosen later. Let $L\in\mathbb{N}^{>1}$ and $a>0$ satisfy

(5.34) $\qquad a<(2L)^{-1}.$

If $\beta\in I$ define

$$M^{\beta,1}(a,L)=\mu^{-2} \sum_{\mu L^{-1}\leqq|\gamma|\leqq\mu L,\ \gamma_0=\beta_0} \mathbf{1}(|N^\gamma-N^\beta|\leqq a)\,\mathbf{1}(N^\gamma, N^\beta\neq\Delta)$$
$$M^{\beta,2}(a,L)=\mu^{-2} \sum_{\mu L^{-1}\leqq|\gamma|\leqq\mu L,\ \gamma_0\neq\beta_0} \mathbf{1}(|N^\gamma-N^\beta|\leqq a)\,\mathbf{1}(N^\gamma, N^\beta\neq\Delta).$$

Then (2.7) implies that

$$(5.35)\quad \begin{aligned} &E(\int \mathbf{1}(Y_{L^{-1},L}(B(x;a))>c\psi_d(a))\,dY_{L^{-1},L}(x))\\ &\leqq {}^{\circ}{}^{*}E^{m_\mu}(\int \mathbf{1}(M_{L^{-1},L}({}^{*}B(x;a))>c\psi_d(a))\,dM_{L^{-1},L}(x))\\ &\quad\text{(recall } B(x;a) \text{ is a } \textit{open} \text{ ball)}\\ &\leqq {}^{\circ}{}^{*}E^{m_\mu}(\mu^{-2}\sum_{\mu L^{-1}\leqq|\beta|\leqq\mu L}\mathbf{1}(M^{\beta,1}(a,L)+M^{\beta,2}(a,L)>c\psi_d(a)))\\ &\leqq {}^{\circ}{}^{*}E^{m_\mu}(\mu^{-2}\sum_{\mu L^{-1}\leqq|\beta|\leqq\mu L}\mathbf{1}(M^{\beta,1}(a,L)>(c/2)\,\psi_d(a)))\\ &\quad+{}^{\circ}{}^{*}E^{m_\mu}(\mu^{-2}\sum_{\mu L^{-1}\leqq|\beta|\leqq\mu L}\mathbf{1}(M^{\beta,2}(a,L)>(c/2)\,\psi_d(a)))\\ &\equiv E_1+E_2. \end{aligned}$$

Let η denote an infinite natural number to be specified later and define

$$\mathscr{E}(x)=\sum_{p=1}^{\eta} x^p/p!\approx e^x-1 \text{ if } {}^{\circ}x \text{ is finite } (x\in{}^{*}\mathbb{R}).$$

If $\theta>0$ and $m_\mu=\mu^{-1}\sum_{j=0}^{K}\delta_{x_j}$, then

$$(5.36)\quad \begin{aligned} E_1&\leqq {}^{\circ}\Big(\mathscr{E}((\theta c/2)\,\psi_d(a))^{-1}\,{}^{*}E^{m_\mu}\Big(\mu^{-2}\sum_{|\beta|\leqq\mu L}\sum_{p=1}^{\eta}(\theta M^{\beta,1}(a,L))^p/p!\Big)\Big)\\ &\leqq(\exp\{c\theta\psi_d(a)/2\}-1)^{-1}\,{}^{\circ}\Big(\sum_{p=1}^{\eta}(\theta^p/p!)\,\mu^{-2(p+1)}\\ &\quad\cdot{}^{*}E^{m_\mu}\Big(\sum_{|\beta(1)|\leqq L\mu}\cdots\sum_{|\beta(p+1)|\leqq L\mu}\mathbf{1}(\beta(i)_0=\beta(1)_0\\ &\qquad\qquad\text{and } |N^{\beta(i)}-N^{\beta(1)}|\leqq a \text{ for } 1\leqq i\leqq p+1)\Big)\Big)\\ &=(\exp\{c\theta\psi_d(a)/2\}-1)^{-1}\,{}^{\circ}\Big(\sum_{p=1}^{\eta}(\theta^p/p!)\,\mu^{-1}\\ &\quad\cdot\sum_{j=0}^{K}{}^{*}E^{m_\mu}\Big(\mu^{-2p-1}\sum_{\beta\in I_j^{p+1}}\mathbf{1}(\max|\beta(i)|\leqq L\mu,\,A(\beta,2a))\Big)\Big)\\ &=(\exp\{c\theta\psi_d(a)/2\}-1)^{-1}\,{}^{\circ}\Big(\sum_{p=1}^{\eta}(\theta^p/p!)\,m_\mu({}^{*}\mathbb{R}^d)\,E_1(V_{p+1}(2a,L,\mu))\Big)\\ &\leqq(\exp\{c\theta\psi_d(a)/2\}-1)^{-1}\,m(\mathbb{R}^d)\Big({}^{\circ}\Big(\sum_{p=1}^{\eta}(\theta\,12H(2a,L,\mu))^p\,p^{-1}L\Big)\\ &\quad+{}^{\circ}\varepsilon(\eta,\mu,L,\theta)\Big), \end{aligned}$$

where

$$\varepsilon(\eta, \mu, L, \theta) = \mu^{-1} \sum_{p=1}^{\eta} ((L+1)\,\theta)^p (L+1)\,(p+1)!\,(p+1)\,e^{12(L+1)^2},$$

and we have used Lemma 5.10 in the last. Let $\theta = {}^\circ(24H(2a, L, \mu))^{-1}$ and then choose $\eta \in {}^*\mathbb{N} - \mathbb{N}$ such that for this choice of θ, ${}^\circ\varepsilon(\eta, \mu, L, \theta) = 0$. (5.36) therefore implies

$$(5.37) \quad E_1 \leqq (\exp\{c\psi_d(a)/(48\,{}^\circ H(2a, L, \mu))\} - 1)^{-1}\, m(\mathbb{R}^d)\, L.$$

Since B_1 has a bounded density we see that

$$(5.38) \qquad {}^\circ H(2a, L, \mu) = \iint_{[0,L]^2} P_0^0(|B_{s+t}| \leqq 2a)\, ds\, dt$$
$$\leqq c_1(d) \int_0^{2L} \int_0^u (a^d (u)^{-d/2}) \wedge 1\, dv\, du \leqq c_2(d)\, \psi_d^0(a),$$

where

$$\psi_d^0(a) = \begin{cases} a^4 & \text{if } d > 4, \\ a^4 \log 1/a & \text{if } d = 4 \end{cases}$$

$$c_2(d) = \begin{cases} c_1(d)\,(1/2 + (2 - d/2)^{-1}) & \text{if } d > 4, \\ c_1(d)\, 7/2 & \text{if } d = 4, \end{cases}$$

and we have used (5.34) if $d=4$. Let $c = c(d) = 96 c_2(d)$ and use (5.37) to see that

$$(5.39) \qquad E_1 \leqq m(\mathbb{R}^d)\, L(\exp\{2 \log\log 1/a\} - 1)^{-1} \qquad (a < (2L)^{-1}).$$

A simple first moment argument gives a sufficient bound on E_2 if $d > 4$ but to handle the 4-dimensional case a second moment calculation is required.

$$E_2 \leqq 4c^{-2}\psi_d(a)^{-2}\,{}^\circ{}^*E^{m_\mu}\Big(\mu^{-2} \sum_{\mu L^{-1} \leqq |\beta| \leqq \mu L} M^{\beta,2}(a, L)^2\Big)$$
$$= 4c^{-2}\psi_d(a)^{-2}\,{}^\circ{}^*E^{m_\mu}\Big(\mu^{-6} \sum_{\beta \in I^3} 1(\beta(i)_0 \neq \beta(1)_0 \text{ and } |N^{\beta(i)} - N^{\beta(1)}|$$
$$\leqq a \text{ for } i = 2, 3,\ \mu L^{-1} \leqq |\beta(i)| \leqq \mu L \text{ for } i = 1, 2, 3)\Big)$$
$$\leqq 4c^{-2}\psi_d(a)^{-2}\,{}^\circ{}^*E^{m_\mu}\Big(\mu^{-6} \sum_{\beta \in I^3} 1(\mu L^{-1} \leqq |\beta(i)| \leqq \mu L \text{ for } i = 1, 2, 3,\ \beta(i)_0 \neq \beta(1)_0$$
$$\text{for } i = 2, 3,\ |N^{\beta(2)} - N^{\beta(3)}| \leqq 2a)\, P(|N^{\beta(2)} - N^{\beta(1)}| \leqq a \,|\, \mathscr{G}(\{\beta(2), \beta(3)\}))\Big).$$

In the above $N^{\beta(1)}$ is independent of $\mathscr{G}(\{\beta(2), \beta(3)\})$ because $\beta(1)_0 \neq \beta(i)_0$ $(i = 2, 3)$. The conditional probability in the above is therefore equal to (use Lemma 2.1(b))

$$2^{-|\beta(1)|}\, P_0^{x_{\beta(1)_0}}(B_{|\beta(1)|/\mu} \in \bar{B}(N^{\beta(2)}(\omega); a)) \leqq 2^{-|\beta(1)|}\, c_3\, a^d L^{d/2}$$

(recall $\beta(1)/\mu \geqq L^{-1}$). Substitute this into the above bound on E_2 to see that

$$E_2 \leqq 4c^{-2}\psi_d(a)^{-2} c_3\, a^d L^{d/2}\, {}^\circ{}^*E^{m_\mu}(M_{L^{-1},L}({}^*\mathbb{R}^d))$$
$$\cdot {}^\circ{}^*E^{m_\mu}\Big(\mu^{-4}\sum_{j=0}^{K}\sum_{\beta\in I_j^2} 1(\max|\beta(i)|\leqq \mu L, |N^{\beta(1)}-N^{\beta(2)}|\leqq 2a)$$
$$+\mu^{-4}\sum_{\beta\in I^2} 1(\beta(1)_0 \neq \beta(2)_0, \max|\beta(i)|\leqq \mu L, |N^{\beta(2)}-N^{\beta(1)}|\leqq a)\Big)$$
$$\leqq 4c^{-2}c_3 L^{1+d/2} m(\mathbb{R}^d)\, a^d \psi_d(a)^{-2}(m(\mathbb{R}^d)^\circ E_1(V_2(2a,L,\mu)) + P_0^0(|B_{2/L}|\leqq a)$$
$$\cdot {}^\circ{}^*E^{m_\mu}(M_{0,L}({}^*\mathbb{R}^d))^2)$$
$$\leqq 4c^{-2}c_3 L^{1+d/2} m(\mathbb{R}^d)\, a^d \psi_d(a)^{-2}(m(\mathbb{R}^d)\, L\, 12 c_2(d)\, \psi_d^0(a) + c_3\, a^d L^{d/2+2} m(\mathbb{R}^d)^2)$$
(Lemma 5.10 and (5.38))
$$= c_4(L, m(\mathbb{R}^d), d)\, a^d \psi_d(a)^{-2}(\psi_d^0(a) + a^d).$$

Therefore we get

$$(5.40)\qquad E_2 \leqq \begin{cases} c_5(L, m(\mathbb{R}^d), d)\, a^{d-4}(\log^+\log^+(1/a))^{-2} & \text{if } d>4 \\ c_5(L, m(\mathbb{R}^d), d)\,(\log^+ 1/a)^{-1}(\log^+\log^+ 1/a)^{-2} & \text{if } d=4. \end{cases}$$

Set $a_n = e^{-n}$ and note that the right sides of (5.40) and (5.39) are summable over $n \geqq 2$. Hence the same is true of the left side of (5.35) and by Borel-Cantelli for P-a.a. ω for every $L \in \mathbb{N}$ and for $Y_{L^{-1},L}$-a.a. x there is an $N(\omega, x, L) \in \mathbb{N}$ such that

$$Y_{L^{-1},L}(B(x; a_n)) \leqq c(d)\, \psi_d(a_n) \text{ if } n \geqq N(\omega, x, L).$$

Therefore there is a $c_6(d)$ such that for P-a.a. ω and each $L \in \mathbb{N}$,

$$\limsup_{a\downarrow 0} Y_{L^{-1},L}(B(x; a))/\psi_d(a) \leqq c_6(d) \qquad \text{for } Y_{L^{-1},L}\text{-a.a. } x.$$

By Lemma 3 of Rogers and Taylor (1961) (or more precisely the trivial refinement stated as Theorem 1.4(a) in Perkins (1988a)) we obtain

$$Y_{L^{-1},L}(A) \leqq c_6(d)\, \psi_d - m(A) \qquad \forall A \in \mathscr{B}(\mathbb{R}^d)\ P\text{-a.s.}$$

Let $L \uparrow \infty$ to obtain the required result on our Loeb space. It is then easy to transfer this result over to the canonical space of continuous measure-valued paths, for example, by noting that the above upper bound on (5.35) transfers immediately. □

Theorem 1.4 is immediate from Theorems 5.7, 5.8 and 5.11.

The above result and its proof extends without change to super-symmetric stable process of index α. These measure-valued processes, $\{X_t: t \geqq 0\}$ may be constructed as in Theorem 2.2, but the original collection of d-dimensional Brownian motions $\{B^\beta: \beta \in I\}$ is replaced by a collection of d-dimensional symmetric stable processes of index $\alpha \in (0, 2]$, $\{Y^\beta: \beta \in I\}$ scaled so that

$$E(e^{i\langle\theta, Y_t\rangle}) = e^{-t|\theta|^\alpha}\ \theta \in \mathbb{R}^d.$$

(This is an easy modification of Theorem 4.3 in Chapter 8 of Ether and Kurtz (1986)). Define Y_t and $Y_{r,s}$ as before.

Notation.

$$\psi_{d,\alpha}(x)=\begin{cases} x^{2\alpha}\log^+\log^+ 1/x & \text{if } d>2\alpha \\ x^{2\alpha}(\log^+ 1/x)\log^+\log^+ 1/x & \text{if } d=2\alpha. \end{cases}$$

Theorem 5.12. *Let Q_α^m denote the law on $C([0,\infty), M_F(\mathbb{R}^d))$ of the super symmetric stable process of index $\alpha\in(0,2]$ and dimension d which starts at $m\in M_F(\mathbb{R}^d)$. Assume $d\geqq 2\alpha$. There are constants $c_{5.10}(d,\alpha)$ and $c_{5.11}(d,\alpha)$ such that for each $m\in M_F(\mathbb{R}^d)$ and Q^m-a.a. ω*

$$Y_\infty(A)\leqq c_{5.10}\,\psi_{d,\alpha}\text{-}m(A) \quad \forall A\in\mathscr{B}(\mathbb{R}^d), \tag{5.41}$$

$$\limsup_{a\downarrow 0} Y_{L^{-1},L}(B(x,a))/\psi_{d,\alpha}(a)\leqq c_{5.11} \tag{5.42}$$

for $Y_{L^{-1},L}$-a.a. x *and all* $L\in\mathbb{N}$.

6. Upper Bounds for the Multiple Points of X

In this section the results of Sects. 3 and 5 are used to obtain upper bounds on the Hausdorff measure of $A\cap\bar{R}_k$, where $A\subset\mathbb{R}^d$, and $\bar{R}_k$ is the set of k-multiple points of X (see Sect. 1). We assume $d\geqq 4$ throughout this section and work with respect to Q^m($m\in M_F$, $m\neq 0$ is fixed) on the canonical space of continuous M_F-valued paths, which we now denote by Ω. To avoid measurability difficulties we will also use Q^m to denote the associated outer measure, defined on all subsets of Ω.

Notation.

$$\psi_2(x)=\begin{cases} x^{d-4} & \text{if } d\geqq 5 \\ (\log^+ 1/x)^{-1} & \text{if } d=4 \end{cases}$$

$$T(\lambda)=\inf\{t\geqq 0: X_t(\mathbb{R}^d)\geqq\lambda\}.$$

Recall the definitions of $\mathscr{H}_0$ and $\mathscr{H}$ from Sect. 1.

Lemma 6.1. *Let $\phi\in\mathscr{H}_0$, $k\in\mathbb{N}$ and assume $\psi(x)=\phi(x)\,\psi_2(x)^{-k}\in\mathscr{H}$. Let $I_i=[r_i,s_i]$, $i=1,\ldots,k$ be disjoint, and suppose $0<\delta\leqq\min\{r_i-s_{i-1}: 1\leqq i\leqq k\}\wedge 1$, where $s_0=0$. There is a constant $c_{6.1}(d)$ such that for any $A\subset\mathbb{R}^d$ and $\lambda>0$*

$$Q^m\left(\psi\text{-}m\left(A\cap\left(\bigcap_{i=1}^k \bar{R}(I_i)\right)\right)\geqq\lambda,\, s_k<T(\delta^{-1})\right)\leqq c_{6.1}^k\,\delta^{-kd/2}\,\lambda^{-1}\,\phi\text{-}m(A).$$

Proof. Let $\{I_i\}$, δ, and A be as above. Let $A \subset \bigcup_{j=1}^{\infty} B_j$, where $B_j = B(x_j; d_j/2)$ and $\max_j d_j \leqq \varepsilon(1, \delta, \delta^{-1})$. (Here ε is as in Theorem 3.2). Then

$$Q^m\left(B_j \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right) \neq \phi, s_k < T(\delta^{-1})\right)$$

$$\leqq E^m\left(\mathbf{1}\left(B_j \cap \left(\bigcap_{i=1}^{k-1} \bar{R}(I_i)\right) \neq \phi, r_k - \delta < T(\delta^{-1})\right) Q^{X(r_k-\delta)}(Y_{\delta, s_k - r_k + \delta}(B_j) > 0,\right.$$

$$\left. s_k - r_k + \delta < T(\delta^{-1}))\right) \quad \text{(by the Markov property at } t = r_k - \delta)$$

$$\leqq c_1(d)(\delta^{1-d/2}\delta^{-1} + 2^{-1})\,\psi_2(d_j)\, Q^m\left(B_j \cap \left(\bigcap_{i=1}^{k-1} \bar{R}(I_i)\right) \neq \phi, s_{k-1} < T(\delta^{-1})\right)$$

(Theorem 3.2 and (3.1.1))

$$\leqq c_{6.1}(d)\,\delta^{-d/2}\psi_2(d_j)\, Q^m\left(B_j \cap \left(\bigcap_{i=1}^{k-1} \bar{R}(I_i)\right) \neq \phi, s_{k-1} < T(\delta^{-1})\right).$$

Proceeding inductively, we obtain

$$Q^m\left(B_j \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right) \neq \phi, s_k < T(\delta^{-1})\right) \leqq c_{6.1}(d)^k \delta^{-dk/2} \psi_2(d_j)^k. \tag{6.1}$$

Choose a sequence of covers of A, $\{B_j^n : j \in \mathbb{N}\}$, $(n \in \mathbb{N})$ as above such that

$$\phi - m(A) = \lim_{n \to \infty} \sum_{j=1}^{\infty} \phi(d_j^n), \qquad (d_j^n \text{ is the diameter of } B_j^n).$$

Then

$$Q^m\left(\psi - m\left(A \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right)\right) \geqq \lambda, s_k < T(\delta^{-1})\right)$$

$$\leqq Q^m\left(\liminf_{n\to\infty} \sum_{j=1}^{\infty} \psi(d_j^n)\,\mathbf{1}\left(B_j^n \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right) \neq \phi\right) \geqq \lambda, s_k < T(\delta^{-1})\right)$$

$$\leqq \lambda^{-1} c_{6.1}(d)^k \delta^{-dk/2} \liminf_{n\to\infty} \sum_{j=1}^{\infty} \psi(d_j^n)\,\psi_2(d_j^n)^k$$

(by (6.1) and Fatou's Lemma)

$$= \lambda^{-1} c_{6.1}^k\, \delta^{-dk/2} \phi - m(A). \quad \square$$

Theorem 6.2. *Let $\phi \in \mathscr{H}_0$, $k \in \mathbb{N}$ and assume $\psi(x) = \phi(x)\,\psi_2(x)^{-k} \in \mathscr{H}$. Let $A \subset \mathbb{R}^d$.*

(a) *If $\phi - m(A) = 0$, then $\psi - m(A \cap \bar{R}_k) = 0$* Q^m-a.s.

(b) *If A has σ-finite $\phi - m$, then $A \cap \bar{R}_k$ has σ-finite $\psi - m$* Q^m-a.s.

Proof. (a) If $\phi - m(A)=0$ then Lemma 6.1 implies that $\psi - m\left(A \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right)\right)=0$ Q^m-a.s. (here $\{I_i\}$ are as in Lemma 6.1). $\bar{R}_k$ is a countable union of sets of the form $\bigcap_{i=1}^{k} \bar{R}(I_i)$. (a) follows from the countable sub-additivity of the outer measure $\psi - m$.

(b) Assume without loss of generality that $\phi - m(A) < \infty$. Lemma 6.1 implies that $\psi - m\left(A \cap \left(\bigcap_{i=1}^{k} \bar{R}(I_i)\right)\right) < \infty$ Q^m-a.s. Here $\{I_i\}$ are as in Lemma 6.1 and we have used the fact that $T(\delta^{-1})=\infty$ for small enough $\delta>0$ a.s. Since $\bar{R}_k$ is a countable union of sets of the form $\bigcap_{i=1}^{k} \bar{R}(I_i)$, we are done. □

In the above theorem recall that for $\psi \in \mathscr{H}_f$, $\psi - m(A)=0$ iff $A=\phi$ and A has σ-finite $\psi - m$ iff A is countable. Therefore setting $\phi=(\psi_2)^k$ in the above, we obtain Theorem 1.5(a), (b). Theorem 1.5(c) is immediate from the above result.

Taking $A=\mathbb{R}^4$ and $\phi(x)=x^4$ in Theorem 6.2(b) we see that for $d=4$, $\bar{R}_k$ has σ-finite $x^4\left(\log \frac{1}{x}\right)^k - m$, i.e., we get Theorem 1.6(b). To obtain the slightly more precise result for $d>4$ in Theorem 1.6(a) and, in particular, to show $\bar{R}_k=\phi$, if $k=d/(d-4)$, we will use the results of Sect. 5 together with Lemma 6.1. A technical measurability lemma is required.

Lemma 6.3. *Let* $\psi \in \mathscr{H}$, $k \in \mathbb{N}$ *and* K *be a compact subset of* $\mathbb{R}^d$. *Then* $H(v_1, \ldots, v_k)=\psi - m\left(K \cap \left(\bigcap_{i=1}^{k} S(v_i)\right)\right)$ *is a Borel measurable mapping on* $M_F(\mathbb{R}^d)^k$.

Proof. Let $B_{\mathbb{Q}}=\{B(x;r): x \in \mathbb{Q}^d, r \in \mathbb{Q}^{>0}\}$. Let $\mathscr{B}_F$ denote the (countable) set of finite subsets of $B_{\mathbb{Q}}$. Fix $\lambda>0$, and let $A=\{v \in M_F^k: H(v)<\lambda\}$. We claim that

$$A=\bigcup_{R=1}^{\infty} \bigcap_{L=1}^{\infty} \cup' \left\{(v_1, \ldots, v_k): \sum_{i=1}^{N'} \min_{1 \leq j \leq k} v_j(B_i')=0\right\}, \tag{6.2}$$

where $\cup'$ indicates the union is over all $\{B_i: i \leq N\}$ and $\{B_i': i \leq N'\}$ in $\mathscr{B}_F$ such that (recall dB = diameter of B)

$$K \subset \bigcup_{i=1}^{N} B_i \cup \left(\bigcup_{i=1}^{N'} B_i'\right), \quad dB_i < L^{-1} \quad \text{for all } i \leq N, \tag{6.3}$$

$$\text{and} \quad \sum_{i=1}^{N} \psi(dB_i) < \lambda - R^{-1}.$$

Assume first $H(v_1, \ldots, v_k)<\lambda$. Choose R such that $H(v_1, \ldots, v_k)<\lambda - R^{-1}$ and for a given $L \in \mathbb{N}$ we choose $\{B_i: i \leq N\} \in \mathscr{B}_F$ such that

$$K \cap \left(\bigcap_{i=1}^{k} S(v_i)\right) \subset \bigcup_{i=1}^{N} B_i, \quad dB_i < L^{-1} \quad \text{and} \quad \sum_{i=1}^{N} \psi(dB_i) < \lambda - R^{-1}. \tag{6.4}$$

Here we have used the compactness of the set being covered, the definition of $\psi-m$, and the continuity of ψ (the last, to obtain balls in $B_{\mathbb{Q}}$). Note that if $\psi\in\mathscr{H}_\infty\cup\mathscr{H}_f$ it may be necessary to take $\{B_i\colon i\leqq N\}=\phi$. (6.4) implies

$$K'\equiv\left(\bigcup_{i=1}^{N} B_i\right)^c\cap K\subset\bigcup_{i=1}^{k} S(v_i)^c.$$

Hence for each x in K', there is a $B'_x\in B_{\mathbb{Q}}$ containing x and an $i\leqq k$ such that $v_i(B'_x)=0$. By compactness of K' there is a finite subcover of K', $\{B'_i\colon i\leqq N'\}\in\mathscr{B}_F$ such that

$$\sum_{i=1}^{N'}\min_{j\leqq k} v_j(B'_i)=0.$$

This and (6.4) prove (6.3) and therefore shows $(v_1,\ldots,v_k)$ belongs to the set on the right side of (6.2).

It is easy to reverse the above argument and establish the opposite inclusion. This gives (6.2), and the result follows because the Borel measurability of the set on the right side of (6.2) is evident. □

Proof of Theorem 1.6(a). The second assertion of Theorem 1.6(a) is immediate from the first. Fix $d>4$, $k\in\mathbb{N}$ and let $\psi(x)=x^{d-k(d-4)}\log^+\log^+\frac{1}{x}$. Let $I_i=[r_i,s_i]$, $1\leqq i\leqq k$ be disjoint compact intervals such that

$$0<\delta=\min\{r_i-s_{i-1}\colon i\leqq k\}\wedge 1\qquad (s_0=0).$$

It suffices to prove that if K is compact ($K=\overline{B(0,N)}$ for large N will do), then

$$\psi-m\left(\bigcap_{j=1}^{k}\bar{R}(I_j)\cap K\right)<\infty\ Q^m\text{-a.s.}\tag{6.5}$$

Let

$$A_1=\left\{(v_1,\ldots,v_k)\in(M_F)^k\colon \psi-m\left(\bigcap_{j=1}^{k}S(v_j)\cap K\right)<\infty\right\}.$$

Then A_1 is a Borel subset of $(M_F)^k$ by Lemma 6.3. Therefore

$$\begin{aligned}A_2&=\left\{(\omega,\omega')\in\Omega^2\colon \psi-m(\bar{R}(I_1)(\omega)\cap\left(\bigcap_{j=2}^{k}\bar{R}([r_j-s_1,s_j-s_1])(\omega'))\cap K\right)<\infty\right\}\\&=\{(\omega,\omega')\in\Omega^2\colon (Y_{r_1,s_1}(\omega),Y_{r_2-s_1,s_2-s_1}(\omega'),\ldots,Y_{r_k-s_1,s_k-s_1}(\omega'))\in A_1\}\in\mathscr{F}^0_{s_1}\times\mathscr{F}^0.\end{aligned}$$

Here $\{\mathscr{F}_t^0\}$ denotes the canonical filtration for X, completed in the usual way, and $\mathscr{F}^0=\mathscr{F}^0_\infty$.

The Markov property implies that if $A\in\mathscr{F}^0_{s_1}\times\mathscr{F}^0$ then

$$Q^m(\{\omega'\colon(\omega',\theta_{s_1}\omega')\in A\}\mid\mathscr{F}^0_{s_1})(\omega)=Q^{X_{s_1}(\omega)}(\{\omega'\colon(\omega,\omega')\in A\})\quad\text{a.s.}$$

Apply this result with $A=A_2$ to see that

$$(6.6)\qquad \begin{aligned} &Q^m\left(\psi-m\left(\bigcap_{j=1}^{k}\bar{R}(I_j)\cap K\right)<\infty\,|\,\mathscr{F}^0_{s_1}\right)(\omega)\\ &=Q^m(\{\omega':(\omega',\theta_{s_1}\omega')\in A_2\}\,|\,\mathscr{F}^0_{s_1})(\omega)\\ &=Q^{X_{s_1}(\omega)}\left(\left\{\omega':\psi-m\left(\bar{R}(I_1)(\omega)\right.\right.\right.\\ &\left.\left.\left.\cap\left(\bigcap_{j=2}^{k}\bar{R}([r_j-s_1,s_j-s_1])(\omega')\right)\cap K\right)<\infty\right\}\right)\quad \text{a.s.}\end{aligned}$$

Theorem 5.7 shows that $\psi_1-m(\bar{R}(I_1)(\omega))<\infty$ Q^m-a.s. Fix such an ω and apply Lemma 6.1 with $X_{s_1}(\omega)$ in place of m, $k-1$ in place of k, ψ_1 in place of ϕ and $\bar{R}(I_1)(\omega)\cap K$ in place of A. That result shows that the probability on the right side of (6.6) is 1 (note that $\psi_1(x)\,\psi_2(x)^{-(k-1)}=\psi$). This gives us (6.5) and completes the proof. □

7. The Closed Support of X_t

As was mentioned in the Introduction, it is known (Dawson-Hochberg (1979), Perkins (1988a)) that if $d>2$ then w.p. 1 for all $t>0$, X_t is supported on a random Borel set with finite ϕ_1-m, where

$$\phi_1(x)=x^2\log^+\log^+ 1/x.$$

It is not difficult to use the Levy modulus from Sect. 4 and the methods of Sect. 5 to show that the *closed* support of X_t has finite ϕ_1-m a.s. for each $t>0$.

Theorem 7.1. *Let $d>2$. There are $0<c_{7.1}(d)\leqq c_{7.2}(d)<\infty$ such that for any $m\in M_F(\mathbb{R}^d)$ and $t>0$.*

$$(7.1)\qquad \begin{aligned}&c_{7.1}\,\phi_1-m(A\cap S(X_t))\leqq X_t(A)\leqq c_{7.2}\,\phi_1-m(A\cap S(X_t))\\ &\textit{for all } A\in\mathscr{B}(\mathbb{R}^d)\qquad Q^m\text{-a.s.}\end{aligned}$$

The upper bound on X_t follows immediately from Theorem A of Perkins (1988a). In fact that result shows that the upper bound holds for all $t>0$ simultaneously w.p. 1. We will not give a proof of the lower bound on X_t here. Stronger results, including a proof of (7.1) for all $t>0$ a.s., and a slightly less precise result if $d=2$, are given in Perkins (1989).

The estimate on $Q^m(X_t(B(x,\varepsilon))>0)$ in Theorem 3.1 ($d\geqq 3$) leads to upper bounds on the Hausdorff measure of $A\cap\left(\bigcap_{i=1}^{k}S(X_{t_i})\right)$ and sufficient conditions for this set to be empty, just as for $A\cap\bar{R}_k$ in the previous section. The proofs will be omitted.

Theorem 7.2. *Let* $d>2, k\in\mathbb{N}$ *and* $0<t_1<\ldots<t_k$. *Assume* $\phi\in\mathscr{H}_0$ *and* $\psi(x)=\phi(x)\,x^{-k(d-2)}\in\mathscr{H}$. *Let* $A\subset\mathbb{R}^d$. *If* $\psi-m(A)=0$ *(respectively, is finite), then* $\psi-m\left(A\cap\left(\bigcap_{i=1}^{k} S(X_{t_i})\right)\right)=0$ *(respectively, is finite)* Q^m-a.s.

Corollary 7.3. *Let* $d>2$, $k\in\mathbb{N}$, $0<t_1<\ldots<t_k$ *and* $A\subset\mathbb{R}^d$.

(a) *If* $x^{k(d-2)}-m(A)=0$ *(respectively, is finite) then* $A\cap\left(\bigcap_{i=1}^{k} S(X_{t_i})\right)=\phi$ *(respectively is finite)* Q^m-a.s.

(b) $\dim\left(A\cap\left(\bigcap_{i=1}^{k} S(X_{t_i})\right)\right)\leqq\dim A-k(d-2)$ Q^m-a.s., *where a negative dimension implies the set is empty.*

Results in Perkins (1988b) show this Corollary is essentially sharp. Taking $A=\mathbb{R}^d$ in the above we see that

$$\dim\bigcap_{i=1}^{k} S(X_{t_i})\leqq d-k(d-2)\quad\text{a.s.}\quad d>2, \tag{7.2}$$

and hence $\bigcap_{i=1}^{k} S(X_{t_i})=\phi$ if $k>d(d-2)^{-1}$. To handle the critical case $k=d(d-2)^{-1}$, argue exactly as in Sect. 6 but using Theorem 7.1 in place of the results in Sect. 5.

Theorem 7.4. *Let* $d>2, k\in\mathbb{N}$ *and* $0<t_1<\ldots<t_k$. *Then* $\bigcap_{i=1}^{k} S(X_{t_i})$ *has finite* $x^{d-k(d-2)}\log\log\frac{1}{x}-m$ Q^m-a.s. *In particular this set is* a.s. *empty if* $k\geqq d(d-2)^{-1}$; i.e., *if* $k>1$ *and* $d>3$, *or* $k>2$ *and* $d=3$.

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