Probab. Th. Rel. Fields 71, 293–305 (1986)

Probability Theory and Related Fields

A Nearest Neighbour-Estimator for the Score Function

M. Csörgő*[1] and P. Révész[2]

1 Department of Mathematics and Statistics, Carleton University, Ottawa, Canada K1S 5B6
2 Mathematical Institute of the Hungarian Academy of Sciences, Réaltanoda u. 13–15, Budapest 1053, Hungary

Summary. Let $f(\cdot)$ be a strictly positive density function defined on $(a, b) \subseteq R^1$ with a continuous derivative $f'(\cdot)$ and let $F(x) = \int_a^x f(t)\,dt$, $-\infty \leqq a < x < +\infty$ be the corresponding distribution function. Define the quantile function Q of F by $Q(y) = F^{-1}(y) = \inf\{x: F(x) \geqq y\}$, $0 < y < 1$, the score function $(-1)J$ of the density function f by $J(y) = f'(Q(y))/f(Q(y))$, and the Fisher information $I(f)$ of f by $I(f) = \int_0^1 (J(y))^2\,dy$, assumed to be finite. Given some regularity conditions on F, we propose a sequence of nearest neighbour (N.N.) type estimators J_n for J and prove that for all $\varepsilon \in (0, 1/5)$ there exists an estimator $J_{n,\varepsilon}$ of J such that for all $\delta \in (0, (5\varepsilon/18) \wedge (\varepsilon/12 + 1/40))$ we have

$$\sup_{n^{-\delta} \leqq y \leqq 1 - n^{-\delta}} |J_{n,\varepsilon}(y) - J(y)| \overset{\text{a.s.}}{=} O(n^{-1/5+\varepsilon}(\log n)^{1/2}),$$

and $I_n(f) \xrightarrow{\text{a.s.}} I(f)$, where $I_n(f) = \int_0^1 (\tilde{J}_n(y))^2\,dy$, with $\tilde{J}_n(y) = J_{n,\varepsilon}(y)$ if $n^{-\delta} \leqq y \leqq 1 - n^{-\delta}$ and zero otherwise.

1. Introduction

Let $f(\cdot)$ be a strictly positive density function defined on $(a, b) \subseteq R^1$ with a continuous derivative $f'(\cdot)$ and let $F(x) = \int_a^x f(t)\,dt$, $-\infty \leqq a < x < +\infty$, be the corresponding distribution function. Define the *quantile function* Q of F by

$$Q(y) = F^{-1}(y) = \inf\{x: F(x) \geqq y\}, \qquad 0 < y < 1, \tag{1.1}$$

* This research was supported by a NSERC Canada Grant at Carleton University, Ottawa

and the *score function* $(-1)J$ *of the density function* f by

$$J(y)=\frac{f'(F^{-1}(y))}{f(F^{-1}(y))}=\frac{d}{dy}f(F^{-1}(y)), \qquad 0<y<1. \tag{1.2}$$

The latter function plays an important role in nonparametric and robust statistics (cf. e.g., Hájek and Šidák, 1967; and Huber, 1981). Due to its importance, and because of our lack of knowledge of f in most practical situations, it is desirable to estimate J, given a random sample $X_1, \ldots, X_n$, $n \geqq 1$, on F. Estimators of the score function are particularly important in adaptive estimation (cf. Beran, 1974; Stone, 1975). Indeed, already Hájek and Šidák (1967, p. 259) proposed a sequence of estimators $J_n^*(y)$ for $J(y)$ with the following property

$$P\left\{\int_0^1 (J_n^*(y)-J(y))^2\,dy>\varepsilon\right\}\to 0 \quad \text{as} \quad n\to\infty \tag{1.3}$$

for every $\varepsilon>0$, provided that the *Fisher information* $I(f)$ of f is finite, i.e., we have

$$I(f)=\int_{-\infty}^{\infty}\left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx=\int_0^1 (J(y))^2\,dy<\infty. \tag{1.4}$$

Results in Beran (1974) for another sequence of estimators of the score function J imply rates of convergence for an in probability statement like (1.3) (cf. Remark 3.3).

The aim of the present exposition is to introduce and study another, closely related (cf. Remark 3.2) sequence of estimators J_n for J, based on $X_{1:n}<X_{2:n}<\ldots<X_{n:n}$, the order statistics of our random samples $X_1, \ldots, X_n$, $n \geqq 1$, on F. Our proposed sequence of estimators J_n is

$$J_n(y)=J_{n,\alpha}(y)=\frac{1}{k_n}\sum_{i=\left[\frac{k_n}{2}\right]}^{n-\left[\frac{k_n}{2}\right]}\lambda'\left(\frac{y-\frac{i}{n}}{a_n}\right)\left(X_{i+\left[\frac{k_n}{2}\right]:n}-X_{i-\left[\frac{k_n}{n}\right]:n}\right)^{-1}, \tag{1.5}$$

where $k_n=[n^\alpha]$ ($[\cdot]$ stands for integer part), $1/2<\alpha<1$, $a_n=k_n/n$, and λ is a density function (the window-density), satisfying the following conditions:

(b.i) λ is vanishing outside of $(-1, +1)$,
(b.ii) $\lambda(-x)=\lambda(x)$, $-1\leqq x\leqq 1$,
(b.iii) $|\lambda''(x)|\leqq C$, $-1\leqq x\leqq 1$,

with some $C>0$.

We are going to show (cf. (3.31)) that, under some regularity conditions on F, $J_{n,\alpha}$ is a strongly consistent sequence of estimators for J, namely for all $\varepsilon\in(0, 1/5)$ there exists an estimator $J_{n,\alpha}$ of J with $\alpha=(12-10\varepsilon)/15$ such that for all $\delta\in(0, (5\varepsilon/18)\wedge(\varepsilon/12+1/40))$ we have

$$\sup_{n^{-\delta}\leqq y\leqq 1-n^{-\delta}}|J_{n,\alpha}(y)-J(y)|\overset{\text{a.s.}}{=}O(n^{-1/5+\varepsilon}(\log n)^{1/2}). \tag{1.6}$$

2. Preliminaries on N.N.-Estimation of Densities

In a previous paper (cf. Csörgő and Révész, 1982) we studied the problem of estimating density functions by the nearest neighbour (N.N.) method. Our results herewith are based on those of the latter. Hence we describe their setup here. Let $X_1, X_2, \dots$ be a sequence of i.i.d. r.v. with density function $f(x)=F'(x) = \frac{d}{dx} P\{X_i \leqq x\}$ $(i=1,2,\dots)$. In terms of order statistics $\{X_{k:n};\ 1\leqq k\leqq n\}$ of a random sample on F, for any $1/2<\alpha<\beta<1$, define the sequences

$$A_n = X_{[n^\beta]:n}, \qquad B_n = X_{n-[n^\beta]:n}.$$

On the interval $[A_n, B_n]$ let the (k_n, λ)-N.N. empirical density function of the sample $X_1, X_2, \dots, X_n$ be

$$f_n(x) = \frac{1}{nR_n(x)} \sum_{k=1}^{n} \lambda\left(\frac{x-x_k}{R_n(x)}\right) = \frac{1}{R_n(x)} \int_{-\infty}^{+\infty} \lambda\left(\frac{x-y}{R_n(x)}\right) dF_n(y), \qquad x\in R^1, \tag{2.1}$$

where

$$F_n(y) = \frac{1}{n} \sum_{k=1}^{n} I_{(-\infty,\, y]}(X_k)$$

is the empirical distribution function based on the sample $X_1, X_2, \dots, X_n$, $R_n(x)$ is the smallest possible number for which the interval $\left[x - \frac{R_n(x)}{2},\ x + \frac{R_n(x)}{2}\right]$ contains $k_n = [n^\alpha]$ elements of the sample $X_1, X_2, \dots, X_n$, and λ is an arbitrary density function.

In our above mentioned paper we studied the process

$$\theta_n(x) = k_n^{\frac{1}{2}}(f_n(x) - f(x))/f(x), \qquad x\in[A_n, B_n], \tag{2.2}$$

under some regularity conditions. These conditions form three groups:

(a) regularity conditions of the underlying distribution function $F(x)$;

(b) regularity conditions of the window (i.e., those of the density function λ);

(c) regularity conditions of the sequences $\{k_n\}$, $\{A_n\}$, $\{B_n\}$, i.e., conditions on α and β. We list these conditions as follows:

(a.i) $F(x)$ is twice differentiable on (a, b), where

$$-\infty \leqq a = \sup\{x: F(x)=0\}, \qquad +\infty \geqq b = \inf\{x: F(x)=1\},$$

(a.ii) $F' = f > 0$ on (a, b),

(a.iii) for some $\gamma > 0$ we have

$$\sup_{a<x<b} F(x)(1-F(x)) \frac{|f'(x)|}{f^2(x)} \leqq \gamma,$$

(a.iv) $A = \lim_{x\downarrow a} f(x) < \infty$, $B = \lim_{x\uparrow b} f(x) < \infty$,

(a.v) one of the following conditions hold
(a.v.α) $\min(A, B)>0$,
(a.v.β) if $A=0$ (resp. $B=0$) then f is nondecreasing (resp. nonincreasing) on an interval to the right of a (resp. to the left of b),

(a.vi) $\sup_{a<x<b} \frac{(F(x)(1-F(x)))^2}{f(x)} \leqq C$

with some $C>0$,

(a.vii) $\sup_{a<x<b} |f''(x)| \leqq C$

with some $C>0$,

(b.i) and (b.ii) as before,
(b.iii)* $|\lambda''(x)| \leqq C$, $-1<x<+1$,

with some $C \geqq 0$,

(c) let $\alpha=(4/5-2\varepsilon/3)<\beta=1-\delta$ with $\varepsilon \in (0, 1/5)$ and

$$0<\delta<(5\varepsilon/18) \wedge (\varepsilon/12+1/40).$$

Remark 2.1. We note that f_n of (2.1) with

$$\lambda(x)=\begin{cases}1, & \text{if } -1/2 \leqq x \leqq 1/2,\\ 0, & \text{otherwise,}\end{cases} \tag{2.3}$$

reduces to

$$f_n(x)=k_n/(nR_n(x))=a_n/R_n(x). \tag{2.4}$$

Remark 2.2. Let $x \in [A_n, B_n]$ be such that $F_n(x)=i/n$ ($i \in [n^\beta, n-n^\beta]$). Then in definition (2.1) $R_n(x)$ may be replaced by

$$\tilde{R}_n(x)=F_n^{-1}\left(\frac{i+[k_n/2]}{n}\right)-F_n^{-1}\left(\frac{i-[k_n/2]}{n}\right), \tag{2.5}$$

where $F_n^{-1}(y)=\inf\{x: F_n(x) \geqq y\}$, $0<y<1$, the empirical quantile function of $X_1, \ldots, X_n$, and we have also

$$\tilde{R}_n(x) \geqq R_n(x)/2. \tag{2.6}$$

Remark 2.3. The conditions (a.i)–(a.v) were used in Csörgő and Révész (1978), where we also alluded to how wide a class of distributions satisfied them (cf. also Parzen, 1979). Condition (a.vi) also seems to be a weak one. On condition (c) we note that about α it says that $2/3<\alpha<4/5$.

3. Consistency of our Proposed Sequence of Estimators for the Score Function

First a few definitions. Let

$$J_n^{(1)}(y)=\frac{1}{a_n}\int_{y-a_n}^{y+a_n} \lambda\left(\frac{y-u}{a_n}\right) df(F^{-1}(u)), \tag{3.1}$$

$$J_n^{(2)}(y)=\frac{-1}{a_n}\int_{y-a_n}^{y+a_n} f(F_n^{-1}(u))\, d_u\left(\frac{y-u}{a_n}\right) \tag{3.2}$$

$$J_n^{(3)}(y) = -\int_{y-a_n}^{y+a_n} \frac{1}{R_n(F_n^{-1}(u))}\, d_u \lambda\left(\frac{y-u}{a_n}\right), \tag{3.3}$$

where $y \in [n^\beta/n,\ 1-n^\beta/n]$.

Lemma 3.1. *Assume conditions* (b.i)–(b.iii). *Then*

$$\sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} |J(y) - J_n^{(1)}(y)|$$
$$\leqq \begin{cases} C \sup\limits_{n^\beta/n \leqq y \leqq 1-n^\beta/n}\ \sup\limits_{0 \leqq s \leqq n^\alpha/n} |J(y+s)-J(y)| = C_{1n}(J), \\ C a_n \sup\limits_{n^\beta/n - a_n \leqq y \leqq 1-n^\beta/n + a_n} |J'(y)| = C_{2n}(J'), \\ C a_n^2 \sup\limits_{n^\beta/n - a_n \leqq y \leqq 1-n^\beta/n + a_n} |J''(y)| = C_{3n}(J''), \end{cases} \tag{3.4}$$

where C is a positive constant depending only on $\lambda(\cdot)$.

Remark 3.1. Depending on the continuity and/or the indicated differentiabilities of the score function J over $(0, 1)$, the above inequalities are meant to be used accordingly appropriately.

Proof of Lemma 3.1. Assuming that J is twice differentiable on $(0, 1)$, we prove here the third inequality of (3.4). The other two indicated inequalities are proved similarly. We have

$$\begin{aligned} J_n^{(1)}(y) &= \frac{1}{a_n} \int_{y-a_n}^{y+a_n} J(u)\, \lambda\left(\frac{y-u}{a_n}\right) du \\ &= \int_{-1}^{1} J(a_n t + y)\, \lambda(t)\, dt \\ &= \int_{-1}^{1} J(y)\, \lambda(t)\, dt + \int_{-1}^{1} a_n t J'(y)\, \lambda(t)\, dt + \int_{-1}^{1} \frac{a_n^2 t^2}{2} J''(x)\, \lambda(t)\, dt \\ &= J(y) + \frac{a_n^2}{2} \int_{-1}^{1} t^2 J''(x)\, \lambda(t)\, dt, \qquad \text{(cf. (b.ii))}, \end{aligned}$$

where $x = x(y, t, n)$ and $|x-y| \leqq a_n |t| \leqq a_n$. Hence the third inequality of (3.4).

Let $u_n(y) = n^{\frac{1}{2}}(E_n^{-1}(y) - y)$, $0 \leqq y \leqq 1$, where E_n is the empirical distribution function of n independent uniform $-(0, 1)$ r.v., and E_n^{-1} is the empirical quantile function of the latter random sample.

Lemma A (Csörgő-Révész, 1978; Theorem 2; or Csörgő-Révész, 1981; Theorem 4.5.5). *With* $\delta_n = 25 n^{-1}$ loglog n *we have*

$$\overline{\lim_{n\to\infty}} \sup_{\delta_n \leqq y \leqq 1-\delta_n} (y(1-y) \log\log n)^{-\frac{1}{2}} |u_n(y)| \leqq 4 \quad \text{a.s.} \tag{3.5}$$

Lemma B (Csörgő-Révész, 1978; Lemma 1, or Csörgő-Révész, 1981; Lemma 4.5.2). *Under the conditions* (a.i)–(a.iii) *we have*

$$\frac{f(F^{-1}(y_1))}{f(F^{-1}(y_2))} \leqq \left\{\frac{y_1 \vee y_2}{y_1 \wedge y_2} \, \frac{1-(y_1 \wedge y_2)}{1-(y_1 \vee y_2)}\right\}^{\gamma} \tag{3.6}$$

for every pair $y_1, y_2 \in (0, 1)$ *and* γ *as in* (a.iii).

Lemma 3.2. *Under the conditions* (a.i)–(a.iii) *we have*

$$\overline{\lim_{n\to\infty}} \left(\frac{n^{\beta}}{\log\log n}\right)^{\frac{1}{2}} \sup_{n^{\beta}/n \leqq y \leqq 1-n^{\beta}/n} \left|\frac{f(F^{-1}(y))}{f(F_n^{-1}(y))} - 1\right| \leqq C \quad \text{a.s.} \tag{3.7}$$

where $C = C(\gamma)$ *is a positive constant.*

Proof. By Lemma B we have

$$\frac{f(F^{-1}(y))}{f(F_n^{-1}(y))} = \frac{f(F^{-1}(y))}{f(F^{-1}(F(F_n^{-1}(y))))} \leqq \left\{\frac{y \vee F(F_n^{-1}(y))}{y \wedge F(F_n^{-1}(y))} \, \frac{1-(y \wedge F(F_n^{-1}(y)))}{1-(y \vee F(F_n^{-1}(y)))}\right\}^{\gamma}. \tag{3.8}$$

Consider

$$\sup_{n^{\beta}/n \leqq y \leqq 1-n^{\beta}/n} \left(\left(\frac{F(F_n^{-1}(y))}{y} - 1\right)\right) = \max_{n^{\beta} \leqq i \leqq n-n^{\beta}} \frac{n}{i-1} \left(F(X_{i:n}) - \frac{i-1}{n}\right)$$

$$= \max_{n^{\beta} \leqq i \leqq n-n^{\beta}} \left(\frac{i}{i-1} \left(\frac{F(X_{i:n}) - \frac{i}{n}}{i/n}\right) + \frac{1}{i-1}\right)$$

$$= \max_{n^{\beta} \leqq i \leqq n-n^{\beta}} \left(\frac{i}{i-1} \, \frac{n^{\frac{1}{2}}\left(F(X_{i:n}) - \frac{i}{n}\right)}{\left(\frac{i}{n} \log\log n\right)^{\frac{1}{2}}} \left(\frac{\log\log n}{i}\right)^{\frac{1}{2}} + \frac{1}{(i-1)}\right).$$

By Lemma A

$$\overline{\lim_{n\to\infty}} \sup_{n^{\beta}/n \leqq y \leqq 1-n^{\beta}/n} \left(\frac{n^{\beta}}{\log\log n}\right)^{\frac{1}{2}} \left(\frac{F(F_n^{-1}(y))}{y} - 1\right) \leqq C \quad \text{a.s.} \tag{3.9}$$

for some positive constant C. Hence

$$\sup_{n^{\beta}/n \leqq y \leqq 1-n^{\beta}/n} \left(\frac{F(F_n^{-1}(y))}{y}\right)^{\gamma} \leqq \left(1 + \frac{C_n(\omega)\log\log n)^{\frac{1}{2}}}{n^{\beta/2}}\right)^{\gamma}$$

$$\leqq \left(1 + \frac{2\gamma\, C_n(\omega)(\log\log n)^{\frac{1}{2}}}{n^{\beta/2}}\right),$$

provided n is large enough, where with C as in (3.9)

$$\overline{\lim_{n}}\, C_n(\omega) \leqq C \quad \text{a.s.},$$

and the latter combined with (3.9) now gives

$$\overline{\lim_{n\to\infty}} \sup_{n^{\beta}/n \leqq y \leqq 1-n\beta/n} \left(\frac{n^{\beta}}{\log\log n}\right)^{\frac{1}{2}} \left(\left(\frac{F(F_n^{-1}(y))}{y}\right)^{\gamma} - 1\right) \leqq 2\gamma C \quad \text{a.s.} \tag{3.10}$$

Replacing $F(F_n^{-1}(y))/y$ by any of

$$y/F(F_n^{-1}(y)), \quad (1-y)/(1-F(F_n^{-1}(y))), \quad (1-F(F_n^{-1}(y)))/(1-y),$$

(3.10) remains true, and hence (3.7) is now proved.

Lemma 3.3. *Assume conditions* (a.i)–(a.iv), *also* (b.i) *and* (b.iii), *and that* $2/3<\alpha<\beta<1$. *Then*

$$\sup_{n^\beta/n\leq y\leq 1-n^\beta/n} |J_n^{(1)}(y)-J_n^{(2)}(y)|\leq n\frac{(\log\log n)^{\frac{1}{2}}}{n^{\beta/2}\,n^\alpha}\,C_n(\omega)\leq\frac{(\log\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}\,C_n(\omega), \tag{3.11}$$

where $C_n(\omega)$ *is a sequence of positive r.v. such that*

$$\overline{\lim_{n\to\infty}}\, C_n(\omega)\leq K \quad \text{a.s.} \tag{3.12}$$

with some positive constant K.

Proof. By (3.1) and integration by parts

$$\begin{aligned} J_n^{(1)}(y) &= \frac{-1}{a_n}\int_{y-a_n}^{y+a_n} f(F^{-1}(u))\,d_u\lambda\left(\frac{y-u}{a_n}\right)\\ &= \frac{-1}{a_n}\int_{y-a_n}^{y+a_n} (C_n(\omega,u)+1)\,f(F_n^{-1}(u))\,d_u\lambda\left(\frac{y-u}{a_n}\right),\end{aligned}$$

where

$$C_n(\omega,u)=\left(\frac{f(F^{-1}(u))}{f(F_n^{-1}(u))}-1\right).$$

Hence, by definition of $J_n^{(2)}$ we get

$$\begin{aligned} &\sup_{n^\beta/n\leq y\leq 1-n^\beta/n} |J_n^{(1)}(y)-J_n^{(2)}(y)|\\ &\quad=\sup_{n^\beta/n\leq y\leq 1-n^\beta/n}\left|\frac{1}{a_n}\int_{-1}^{1} C_n(\omega,y+a_n v)\,f(F_n^{-1}(y+a_n v)\,d\lambda(v)\right|\\ &\quad\leq\sup_{n^\beta/n\leq y\leq 1-n^\beta/n}\frac{1}{a_n}\int_{-1}^{1}|C_n(\omega,y+a_n v)|\cdot|f(F_n^{-1}(y+a_n v)|\cdot|\lambda'(v)|\,dv,\end{aligned}$$

and by Lemma 3.2

$$\overline{\lim_{n\to\infty}}\left(\frac{n^\beta}{\log\log n}\right)^{\frac{1}{2}}\sup_{n^\beta/n\leq y\leq 1-n^\beta/n}|C_n(\omega,u)|\leq C,$$

with $C=C(\gamma)>0$. This also completes the proof of Lemma 3.3, upon observing also that $f(F_n^{-1})$ and λ' are bounded by our assumptions.

Lemma 3.4. *Assume conditions* (a.i)–(a.vii), (bi)–(b.iii) *and* (c). *Then*

$$\sup_{n^\beta/n\leq y\leq 1-n\beta/n}|J_n^{(2)}(y)-J_n^{(3)}(y)|\leq\frac{(\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}\,C_n(\omega), \tag{3.13}$$

where $C_n(\omega)$ *is a sequence of positive r.v. such that*

$$\overline{\lim_{n\to\infty}}\, C_n(\omega) \leqq K \quad \text{a.s.} \tag{3.14}$$

with some positive constant K.

Proof. We have

$$\begin{aligned} J_n^{(2)}(y) &= \frac{-1}{a_n} \int_{y-a_n}^{y+a_n} f(F_n^{-1}(u))\, d_u \lambda\left(\frac{y-u}{a_n}\right) \\ &= \frac{-1}{a_n} \int_{y-a_n}^{y+a_n} (\tilde{C}_n(\omega, u)+1) f_n(F_n^{-1}(u)))\, d_u \lambda\left(\frac{y-u}{a_n}\right), \end{aligned} \tag{3.15}$$

where $\tilde{C}_n(\omega, u) = \dfrac{f(F_n^{-1}(u))}{f_n(F_n^{-1}(u))} - 1$. Applying now Consequence in Sect. 5 of [4] in terms of f_n defined in (2.4), we get

$$\overline{\lim_{n\to\infty}} \left(\frac{n a_n}{\log n}\right)^{\frac{1}{2}} \sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} \left|\frac{f_n(F_n^{-1}(y))}{f(F_n^{-1}(y))} - 1\right| \overset{\text{a.s.}}{=} (2(1-\alpha))^{\frac{1}{2}}, \tag{3.16}$$

and hence also

$$\overline{\lim_{n\to\infty}} \left(\frac{n a_n}{\log n}\right)^{\frac{1}{2}} \sup_{n^\beta/n \leqq u \leqq 1-n^\beta/n} |\tilde{C}_n(\omega, u)| \overset{\text{a.s.}}{=} (2(1-\alpha))^{\frac{1}{2}}. \tag{3.17}$$

Letting

$$\left(\frac{n a_n}{\log n}\right)^{\frac{1}{2}} \sup_{n^\beta/n \leqq u \leqq 1-n^\beta/n} \tilde{C}_n(\omega, u) = C_n(\omega), \tag{3.18}$$

we have

$$\overline{\lim_{n\to\infty}} |C_n(\omega)| \leqq K, \quad \text{a.s.}, \tag{3.19}$$

for some constant $K>0$.

Since f is bounded, (3.16) also implies

$$\overline{\lim_{n\to\infty}} \sup_{n^\beta/n \leqq u \leqq 1-n^\beta/n} |f_n(F_n^{-1}(u))| \leqq C, \quad \text{a.s.}, \tag{3.20}$$

with some positive constant C.

Thus on recalling again the definition of f_n in (2.4),

$$\begin{aligned} &\sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} |J_n^{(2)}(y) - J_n^{(3)}(y)| \\ &= \sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} \frac{1}{a_n} \left| \int_{-1}^{1} \tilde{C}_n(\omega, y+a_n v) f_n(F_n^{-1}(y+a_n v))\, \lambda'(v)\, dv \right| \\ &\leqq \frac{1}{a_n} \left(\frac{\log n}{n a_n}\right)^{\frac{1}{2}} L = L \frac{(\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}, \quad \text{a.s.}, \end{aligned} \tag{3.21}$$

and (3.13) is now proved.

Lemma 3.5. *Assume conditions* (a.i)–(a.vii), (b.i)–(b.iii) *and* (c). *Then*

$$\sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} |J_n^{(3)}(y) - J_n(y)| \leqq C_n(\omega)/n^{(3\alpha-2)}, \tag{3.22}$$

where $C_n(\omega)$ *is a sequence of positive r.v. such that*

$$\overline{\lim_{n\to\infty}}\, C_n(\omega) \leqq K, \quad \text{a.s.,} \tag{3.23}$$

with some positive constant K.

Proof. We have

$$J_n^{(3)}(y) = \int_{y-a_n}^{y+a_n} \frac{-1}{R_n(F_n^{-1}(u))}\, d_u \lambda\left(\frac{y-u}{a_n}\right) = \frac{1}{a_n} \int_{y-a_n}^{y+a_n} \frac{1}{R_n(F_n^{-1}(u))}\, \lambda'\left(\frac{y-u}{a_n}\right) du, \tag{3.24}$$

and (cf. (1.5) and (2.5))

$$J_n(y) = \frac{1}{a_n} \sum_{i=\left[\frac{k_n}{2}\right]}^{n-\left[\frac{k_n}{2}\right]} \lambda'\left(\frac{y-\frac{i}{n}}{a_n}\right) \frac{1}{nR_n\left(F_n^{-1}\left(\frac{i}{n}\right)\right)}. \tag{3.25}$$

Hence

$$\begin{aligned}
&\sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} |J_n^{(3)}(y) - J_n(y)| \\
&= \frac{1}{a_n} \sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} \left| \int_{y-a_n}^{y+a_n} \frac{1}{R_n(F_n^{-1}(u))}\, \lambda'\left(\frac{y-u}{a_n}\right) du \right. \\
&\qquad \left. - \sum_{i=\left[\frac{k_n}{2}\right]}^{n-\left[\frac{k_n}{2}\right]} \lambda'\left(\frac{y-\frac{i}{n}}{a_n}\right) \frac{1}{nR_n\left(F_n^{-1}\left(\frac{i}{n}\right)\right)} \right| \\
&\leqq \frac{1}{a_n} \sup_{n^\beta/n \leqq y \leqq 1-n^\beta/n} \sum_{i=[n^\beta]}^{n-[n^\beta]} \frac{1}{nR_n\left(F_n^{-1}\left(\frac{i}{n}\right)\right)} \sup_{\frac{i-1}{n} \leqq u \leqq \frac{i}{n}} \left| \lambda'\left(\frac{y-u}{a_n}\right) - \lambda'\left(\frac{y-\frac{i}{n}}{a_n}\right) \right| \\
&\leqq \frac{1}{a_n} \sup_{-1 \leqq v \leqq 1} |\lambda''(v)| \frac{1}{na_n} \sum_{i=[n^\beta]}^{n-[n^\beta]} \frac{1}{n\tilde{R}_n\left(F_n^{-1}\left(\frac{i}{n}\right)\right)} \\
&\leqq \frac{1}{na_n^2} \frac{n}{k_n} C \sum_{i=[n^\beta]}^{n-[n^\beta]} \frac{1}{n} f_n\left(F_n^{-1}\left(\frac{i}{n}\right)\right) \\
&= \frac{n^2}{k_n^3} C \int_{n^\beta}^{1-n^\beta/n} f_n(F_n^{-1}(u))\, du = C_n(\omega)/n^{3\alpha-2},
\end{aligned} \tag{3.26}$$

where, by (3.20), $\overline{\lim_{n\to\infty}}\, C_n(\omega) \leqq C$ a.s., and where f_n is defined as in (2.4) and the last inequality of (3.26) is by formulas (2.3) through (2.6). The proof of Lemma 3.5 is now complete.

A combination of Lemmas 3.3, 3.4 and 3.5 yields

Lemma 3.6. *Under the conditions* (a.i)–(a.vii), (b.i)–(b.iii) *and* (c), *we have*

$$\varlimsup_{n\to\infty} \frac{n^{3\alpha/2-1}}{(\log n)^{\frac{1}{2}}} \sup_{n^\beta/n \leq y \leq 1-n^\beta/n} |J_n^{(1)}(y)-J_n(y)| \leq C \quad \text{a.s.,} \tag{3.27}$$

where C is a positive constant, and J_n *is as in* (1.5).

The latter Lemma and (3.4) combined result in the following strong consistency for our sequence of estimators J_n of the score function J:

Theorem 3.1. *Given conditions* (a.i)–(a.vii), (b.i)–(b.iii) *and* (c), *we have*

$$\sup_{n^\beta/n \leq y \leq 1-n^\beta/n} |J(y)-J_n(y)| \overset{\text{a.s.}}{=} \begin{cases} O\left(C_{1n}(J) \vee \dfrac{(\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}\right), \\ O\left(C_{2n}(J') \vee \dfrac{(\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}\right), \\ O\left(C_{3n}(J'') \vee \dfrac{(\log n)^{\frac{1}{2}}}{n^{3\alpha/2-1}}\right), \end{cases} \tag{3.28}$$

where $C_{1n}(J)$, $C_{2n}(J')$ *and* $C_{3n}(J'')$ *are defined in* (3.4).

In order to rationalize somewhat the meaning of Theorem 3.1, we introduce the following condition

$$\sup_{0<u<1} u(1-u)|J'(u)| \leq K \tag{3.29}$$

with some positive constant K.

It appears to be true that most frequently used distributions in statistics satisfy the latter condition. Indeed, when they do, the second statement of (3.28) reduces to the following

Consequence 3.1. *Given conditions* (a.i)–(a.vii), (b.i)–(b.iii), (c) *and* (3.29), *we have*

$$\sup_{n^\beta/n \leq y \leq 1-n^\beta/n} |J_n(y)-J(y)| \overset{\text{a.s.}}{=} O((\log n)^{1/2}/n^{3\alpha/2-1}), \tag{3.30}$$

i.e., for all $\varepsilon \in (0, 1/5)$ *there exists an estimator* $J_{n,\alpha}$ *as in* (1.5) *for* J *with* $\alpha = 4/5 - 2\varepsilon/3$ *such that for all* $\delta \in (0, (5\varepsilon/18) \wedge (\varepsilon/12 + 1/40))$ *we have*

$$\sup_{n^{-\delta} \leq y \leq 1-n^{-\delta}} |J_{n,\alpha}(y)-J(y)| \overset{\text{a.s.}}{=} O(n^{-1/5+\varepsilon}(\log n)^{1/2}). \tag{3.31}$$

Proof. Since by condition (3.29), $C_{2n}(J') = O(n^{\alpha-\beta})$, and by condition (c), $\beta - \alpha > 3\alpha/2 - 1$, we have (3.30), and hence also (3.31).

As a corollary to our Consequence 3.1, we now prove that

$$I_n(f) = \int_0^1 (\bar{J}_n(y))^2\, dy \tag{3.32}$$

is a strongly consistent estimator of the *Fisher information* $I(f)$ of (1.4), where

$$\bar{J}_n(y)=\begin{cases} J_n(y) & \text{if } n^\beta/n \leqq y \leqq 1-n^\beta/n \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

Namely, we have

Consequence 3.2. *Given the conditions of Consequence* 3.1 *and* (1.4), *we get*

$$I_n(f) \xrightarrow{\text{a.s.}} I(f). \tag{3.34}$$

Proof. Consider

$$\begin{aligned} |I_n(f)-I(f)| &= \left|\int_0^1 ((\bar{J}_n(y))^2-(J(y))^2)\,dy\right| \\ &\leqq \int_0^{n^\beta/n} J^2 + \int_{n^\beta/n}^{1-n^\beta/n} |J_n^2-J^2| + \int_{1-n^\beta/n}^1 J^2. \end{aligned}$$

By (1.4) $I(f)<\infty$, hence $\int_0^{n^\beta/n} J^2$ and $\int_{1-n^\beta/n}^1 J^2 \to 0$ as $n\to\infty$. By Consequence 3.1 we have

$$\int_{n^{\beta-1}}^{1-n^{\beta-1}} |J_n^2-J^2| = \int_{n^{\beta-1}}^{1-n^{\beta-1}} |(J_n-J)^2+2(J_n-J)\,J| \overset{\text{a.s.}}{=} O((\log n)^{1/2}\, n^{1-3\alpha/2}) \int_0^1 |J|$$

and (3.34) is proven, because of $\int |J| \leqq (I(f))^{1/2}<\infty$.

Assuming a bit more than in (3.29), we have also

Consequence 3.3. *Given conditions* (a.i)–(a.vii), (b.i)–(b.iii), (c) *and, instead of* (3.29),

$$\sup_{0<u<1} |J'(u)| \leqq K \tag{3.35}$$

with some positive constant K, *we have*

$$\int_0^1 (\hat{J}_n(u)-J(u))^2\,du \overset{\text{a.s.}}{=} O(n^{-2/5+2\varepsilon}\log n + n^{-3\delta}), \tag{3.36}$$

where ε, $\delta>0$ *are as in* (3.31) *and*

$$\hat{J}_n(u)=\begin{cases} J_{n,\alpha}(u) & \text{if } n^\beta/n \leqq u \leqq 1-n^\beta/n \\ J_{n,\alpha}(n^\beta/n) & \text{if } 0<u\leqq n^\beta/n \\ J_{n,\alpha}(1-n^\beta/n) & \text{if } 1-n^\beta/n \leqq u<1. \end{cases} \tag{3.37}$$

Proof. By (3.30) and (3.31)

$$\int_{n^\beta/n}^{1-n^\beta/n} (\hat{J}_n(u)-J(u))^2\,du \overset{\text{a.s.}}{=} O\left(\frac{\log n}{n^{3\alpha-2}}\right) = O(n^{-2/5+2\varepsilon}\log n). \tag{3.38}$$

Next consider

$$\begin{aligned}
&\sup_{0<u<n^\beta/n} |\hat{J}_n(u)-J(u)| \\
&\leq \sup_{0<u<n^\beta/n} |\hat{J}_n(u)-\hat{J}_n(n^\beta/n)| + \sup_{0<u<n^\beta/n} |\hat{J}_n(n^\beta/n)-J(n^\beta/n)| \\
&\quad + \sup_{0<u<n^\beta/n} |J(n^\beta/n)-J(u)| \\
&\leq O((\log n)^{1/2}/n^{3\alpha/2-1}) + \sup_{0<u<1} |J'(u)| \frac{n^\beta}{n} \quad \text{a.s.}
\end{aligned} \tag{3.39}$$

The latter upper bound then implies

$$\begin{aligned}
\int_0^{n^\beta/n} (\hat{J}_n(u)-J(u))^2\, du &\leq \frac{n^\beta}{n}\left(O\left(\frac{(\log n)^{1/2}}{n^{3\alpha/2-1}}\right)+K\frac{n^\beta}{n}\right)^2 \quad \text{a.s.} \\
&= n^{-\delta}(O(n^{-1/5+\varepsilon}(\log n)^{1/2}+n^{-\delta}))^2,
\end{aligned} \tag{3.40}$$

and a similar statement holds when integrating from $1-n^\beta/n$ to 1. Combining now (3.38) with (3.40), we get (3.36).

Remark 3.2. It is equally natural to use the sequence of estimators $J_n^{(3)}$ instead of J_n for J. Clearly, our results hold true also in terms of $J_n^{(3)}$ as well. The latter form of estimators makes sense also if we assume condition (b.iii)* only instead of (b.iii). However, when replacing (b.iii) by (b.iii)*, our present proofs do not imply the validity of our results any more. In particular, if λ is the density of the uniform $(-1/2, 1/2)$-law, then we cannot say anything about the behaviour of $J_n^{(3)}$. We note also that in the latter case $J_n^{(3)}$ is similar to the estimator of J, investigated by Hájek and Šidák (1967, p. 259), who proved (1.3) for their sequence of estimators.

Remark 3.3. The referee has pointed out to us that under $\int_0^1 |J'(u)|\, du<\infty$ formulae (2.43) and (2.44) of Beran (1974) with $M^4=N=n$ prove

$$c_n^{-1} n^{1/4} \int_0^1 (J_n(u)-J(u))^2\, du \xrightarrow{P} 0, \tag{3.41}$$

for every sequence $c_n \to \infty$ with the therin considered sequence of estimators J_n of J. The result of (3.36), by the optimal choices $\varepsilon=13/90$ and $\delta=1/27-\eta/3$, reduces to $O(n^{-1/9+\eta})$, $\eta>0$. While the latter is an almost sure rate, it is not as good as that of (3.41). We should note also that the condition (3.35) is also stronger than requiring $\int_0^1 |J'|<\infty$.

Acknowledgement. The authors wish to thank the referee for the careful reading of their manuscript and for a number of valuable remarks and suggestions, which have much improved the presentation of results.

References

1. Beran, R.: Asymptotically efficient adaptive rank estimates in location models. Ann. Stat. **2**, 63–74 (1974)

2. Csörgő, M., Révész, P.: Strong approximation of the quantile process. Ann. Statist. **6**, 882–894 (1978)
3. Csörgő, M., Révész, P.: Strong Approximations in Probability and Statistics. New York: Academic Press 1981
4. Csörgő, M., Révész, P.: An invariance principle for N.N. empirical density functions. In: Gnedenko, B.V., Puri, M.L., Vincze, I. (eds.). Coll. Math. Soc. J. Bolyai, **32**. Non-parametric Statistical Inference, pp. 151–170. Amsterdam: North-Holland 1982
5. Hájek, J., Šidák, Z.: Theory of Rank Tests. New York: Academic Press 1967
6. Huber, P.J.: Robust Statistics. New York: Wiley 1981
7. Parzen, E.: Nonparametric statistical data modeling. J. Am. Stat. Assoc. **74**, 105–131 (1979)
8. Stone, C.J.: Adaptive maximum likelihood estimators of a location parameter. Ann. Statist. **3**, 267–284 (1975)

Received August 25, 1982; in revised form June 2, 1985