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Probability Theory and Related Fields

Non-Classical Law of the Iterated Logarithm Behaviour for Trimmed Sums

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Summary. We study the law of the iterated logarithm for the partial sum of i.i.d. random variables when the r_n largest summands are excluded, where $r_n = o(\log\log n)$. This complements earlier work in which the case $\log\log n = O(r_n)$ was considered. A law of the iterated logarithm is again seen to prevail for a wide class of distributions, but for reasons quite different from previously.

1. Introduction

Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common distribution function F. For $x > 0$ define

$$G(x) = P(|X| > x), \qquad K(x) = x^{-2} \int_{|y| \leq x} y^2 F(dy)$$

$$Q(x) = G(x) + K(x).$$

If we need to distinguish X from another random variable we will write F_X, G_X, K_X and Q_X.

Let ${}^{(1)}X_n, \ldots, {}^{(n)}X_n$ be an arrangement of $X_1, \ldots, X_n$ in decreasing order of magnitude, i.e. $|{}^{(1)}X_n| \geqq \ldots \geqq |{}^{(n)}X_n|$. We will assume throughout that the distribution function of X is continuous one effect of which is to make the ordering ${}^{(1)}X_n, \ldots, {}^{(n)}X_n$ unique except on a null set. This assumption could be dispensed with but the ensuing technical details would only serve to obscure the main ideas. For $r \geqq 0$ an integer, define ${}^{(r)}S_n = {}^{(r+1)}X_n + \ldots {}^{(n)}X_n$. We write S_n for ${}^{(0)}S_n$. We will refer to ${}^{(r)}S_n$ as a trimmed sum.

The study of trimmed sums is motivated on the one hand by statistical considerations, (although it is perhaps more natural to consider trimming by the order statistics in this context) while on the other hand, probabilistically, by a desire to better understand partial sums of i.i.d. random variables and in particular to understand the role played by the summands of large modulus.

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This in turn leads to a deeper understanding of the classical limit theorems and puts them more sharply into perspective.

The present paper grew out of an attempt to answer some unresolved questions which arose in [3]. One of the main results in [3], Theorem 5.5, states that if the distribution of X satisfies

$$\limsup_{x\to\infty} G(x)/K(x) < \infty \tag{1.1}$$

and r_n is an increasing sequence of integers satisfying

$$\liminf_{n\to\infty} r_n/\log\log n > 0 \tag{1.2}$$

$$\limsup_{n\to\infty} r_n n^{-1} < G(0), \tag{1.3}$$

then

$$0 < \limsup_{n\to\infty} \frac{|{}^{(r_n)}S_n - nEX1(|X| \leq b_n)|}{(n\log\log n\, b_n^2\, K(b_n))^{1/2}} < \infty \tag{1.4}$$

where $G(b_n) = r_n n^{-1}$. Condition (1.1), first introduced by Feller [2], is equivalent to stochastic compactness of S_n and is discussed in detail in [4] where further references can be found. In particular (1.1) holds whenever X is in the domain of attraction of a stable law of index $\alpha \in (0, 2]$. The normalizer in (1.4) is the natural one to use for the Law of the Iterated Logarithm (L.I.L.) in that Pruitt [13] has shown that if X is symmetric, (1.1) holds, $r_n \uparrow \infty$ and $r_n n^{-1} \to 0$ then

$$\frac{{}^{(r_n)}S_n}{(n b_n^2\, K(b_n))^{1/2}} \to N(0, 1) \tag{1.5}$$

where $N(0, 1)$ is normal with mean 0 and variance 1. It is interesting to note that no symmetry assumption is needed for (1.4) to hold, but (1.5) may fail without it.

Results similar to (1.4), for other variants of the trimmed sum, have been discovered recently by several authors, see [5] and [6] for example. In each of these works it is also assumed that r_n satisfies (1.2). In light of (1.5) one might expect that (1.4) holds without this assumption. We will show that this is not the case although an L.I.L. result for ${}^{(r_n)}S_n$ is still available but for entirely different reasons. In (1.4) the large values of ${}^{(r_n)}S_n$ arise due to the cummulative effect of many summands as in the classical LIL, however when $r_n = o(\log\log n)$ the large values of ${}^{(r_n)}S_n$ are determined by a small number of large terms. For example, we will show that if r_n is an increasing sequence of integers tending to ∞ such that

$$r_n(\log\log n)^{-1/2} \to 0 \tag{1.6}$$

and if in addition to (1.1) the distribution of X satisfies

$$\liminf_{x\to\infty} G(x)/K(x) > 0 \tag{1.7}$$

then the large values of ${}^{(r_n-1)}S_n$, after centering, are due entirely to ${}^{(r_n)}X_n$ and further, that ${}^{(r_n)}X_n$ can be normalized to obtain a finite non-zero lim sup. That is, there exist α_n, δ_n such that

$$0<\limsup_{n\to\infty}|{}^{(r_n)}X_n|\,\alpha_n^{-1}=\limsup_{n\to\infty}|{}^{(r_n-1)}S_n-\delta_n|\,\alpha_n^{-1}<\infty \tag{1.8}$$

$$\limsup_{n\to\infty}|{}^{(r_n+1)}X_n|\,\alpha_n^{-1}=\limsup_{n\to\infty}|{}^{(r_n)}S_n-\delta_n|\,\alpha_n^{-1}=0. \tag{1.9}$$

If instead of (1.6) we assume only that

$$r_n(\log\log n)^{-1}\to 0 \tag{1.10}$$

then one can still find α_n such that

$$0<\limsup_{n\to\infty}|{}^{(r_n)}X_n|\,\alpha_n^{-1}<\infty \tag{1.11}$$

but now there may be other summands which are also comparable in size to α_n. Nevertheless by controlling these terms we will show that under (1.1) and (1.7), there exists γ_n and δ_n such that

$$0<\limsup_{n\to\infty}|{}^{(r_n-1)}S_n-\delta_n|\,\gamma_n^{-1}<\infty. \tag{1.12}$$

The normalizer γ_n is given by $N_n\alpha_n$ where $N_n=[r_n^2/\log\log n]+1$ ($[x]$ denotes the integer part of x). The way in which this arises is that roughly speaking, the large values of ${}^{(r_n-1)}S_n$ occur because infinitely often there are N_n terms comparable in size to α_n and these terms all have the same sign. This is quite different from the way the large values arise in the classical LIL, see Sect. 5 for a further discussion.

Condition (1.7) is equivalent, by a famous result of Lévy, to X not being in the domain of partial attraction of the normal law. Thus the class of distributions satisfying (1.1) and (1.7) is still quite large and includes all of those in the domain of attraction of a stable law of index $\alpha\in(0,2)$. We should perhaps point out here that Maller [10], extending earlier work of Kesten [7] in the case $r=0$, has shown that the failure of (1.7) is necessary and sufficient for the existence of an increasing sequence γ_n such that (1.12) holds with $\delta_n=$ median (S_n) and r_n a bounded sequence.

To illustrate the difference between the normalizers in (1.4) and (1.12), assume that X is symmetric stable of index $\alpha\in(0,2)$ and the scale parameter is chosen so that $G(x)\sim x^{-\alpha}$. Then the normalizer in (1.5) is given by $n^{1/\alpha}(\alpha(2-\alpha)^{-1}\,r_n^{1-2/\alpha})^{1/2}$ and so in (1.4) it is $n^{1/\alpha}(\alpha(2-\alpha)^{-1}\,r_n^{1-2/\alpha}\log\log n)^{1/2}$. In (1.12) if we take for example $r_n=[l_p n]$ for $p\geqq 3$, where $l_p n$ is the p^{th} iterate of the logarithm function, then

$$\limsup_{n\to\infty}|{}^{(r_n)}X_n|\,\alpha_n^{-1}=\limsup_{n\to\infty}|{}^{(r_n-1)}S_n|\,\alpha_n^{-1}=e^{2/\alpha}$$

where $\alpha_n=n^{1/\alpha}(l_p n)^{-1/\alpha}\exp((l_2 n+\ldots+l_{p-1} n)(\alpha r_n)^{-1})$.

If (1.7) fails then the non-classical behaviour given by (1.12) need not hold. For example let X have bounded support, then it is easy to see that ${}^{(r_n)}S_n$ satisfies (1.4) no matter how slowly r_n increases to infinity, indeed (1.4) holds for r_n constant. In fact it can be shown that for any random variable X in the domain of attraction of the normal law, there exists an increasing sequence r_n, which depends on X, such that $r_n = o(\log\log n)$ and (1.4) holds, c.f. [8]. On the other hand one can also construct examples of X in the domain of attraction of the normal law for which (1.12) holds provided r_n increases sufficiently slowly, this rate again depending on the distribution. Thus for distributions satisfying (1.1) and (1.7) there is a single level, namely $\log\log n$, which distinguishes between classical and non-classical LIL behaviour, while for distributions attracted to the normal such a cut-off, if it exists, seems to depend on the distribution.

2. Preliminaries

Our basic assumption on the underlying distribution will be

(2.1) $$0 < \liminf_{x\to\infty} G(x)/K(x) \leqq \limsup_{x\to\infty} G(x)/K(x) < \infty$$

Hence for some $\theta > 1$ and all $x > 0$

(2.2) $$G(x) \leqq Q(x) \leqq \theta G(x).$$

By (2.1) and Lemma 2.4 of Pruitt [12], there exists $q > 0$ and $x_0 > 0$ such that for all $x \geqq x_0$

(2.3) $$x^q Q(x) \quad \text{is decreasing.}$$

On the other hand by Lemma 2.1 of [12] $x^2 Q(x)$ is always increasing, thus for any $\xi \in (0,1)$ if $\xi x \geqq x_0$ then

(2.4) $$\xi^2 \theta^{-1} G(\xi x) \leqq G(x) \leqq \theta \xi^q G(\xi x).$$

We will assume that r_n is a sequence of integers such that

(2.5) $$r_n \quad \text{increases to } \infty, \quad r_n (l_2 n)^{-1} \to 0.$$

In order to describe the normalizing sequences α_n and γ_n we must first introduce an auxilliary sequence. Thus let a_n be any sequence of positive reals satisfying the following conditions:

(2.6) $$a_n \text{ is decreasing}$$

(2.7) $$(l n)^{-2} \leqq a_n \leqq (l n)^{-1}$$

(2.8) $$\sum_n a_n n^{-1} e^{\varepsilon r_n} \begin{cases} < \infty & \varepsilon < 0 \\ = \infty & \varepsilon \geqq 0. \end{cases}$$

In the case that r_n satisfies $\liminf r_n(l_p n)^{-1} > 0$ for some $p \geqq 2$, one can easily check that $a_n = ((l n)(l_2 n) \dots (l_{p-1} n))^{-1}$ satisfies (2.6)–(2.8). The proof that such an a_n exists in general is not difficult but will be deferred to the appendix.

It is a simple consequence of the monotonicity of a_n and r_n that if $b > 1$ then

(2.9) $$\sum_k a_{[b^k]} e^{\varepsilon r_{[b^k]}} \begin{cases} < \infty & \text{if } \varepsilon < 0 \\ = \infty & \text{if } \varepsilon \geqq 0 \end{cases}$$

and furthermore, again by monotonicity, if $n_k \geqq b^k$, then for every $\varepsilon < 0$

(2.10) $$\sum_k a_{n_k} e^{\varepsilon r_{n_k}} < \infty.$$

Of course the sequence a_n depends on r_n but note that if a_n satisfies (2.6)–(2.8) then it satisfies (2.6)–(2.8) with r_n replaced by the sequence $r_n + j$ for each fixed j. Also observe that if w_n is any sequence such that $|r_n - w_n| = o(r_n)$ then (2.9) and (2.10) hold with w_n replacing r_n provided we exclude the case $\varepsilon = 0$ in (2.9).

Now let

(2.11) $$\beta_n = \exp((l a_n^{-1} - r_n l r_n + r_n) r_n^{-1}).$$

Thus

(2.12) $$a_n = \exp(r_n - r_n l r_n - r_n l \beta_n).$$

For later reference note that by (2.5) and (2.7) for any $p \in \mathbb{R}$

(2.13) $$r_n^{p+1} (l_2 n)^{-p} \beta_n \geqq (r_n / l_2 n)^p \exp((l_2 n / r_n)) \to \infty.$$

Set

(2.14) $$N_n = [r_n^2 / l_2 n] + 1$$

and for $\lambda > 0$ define

(2.15) $$\alpha_n(\lambda) = \min\{x : G(x) = (\lambda n \beta_n)^{-1}\}$$

and let

(2.16) $$\gamma_n(\lambda) = N_n \alpha_n(\lambda).$$

We will write α_n for $\alpha_n(1)$ and γ_n for $\gamma_n(1)$. The sequence α_n will be used to normalize ${}^{(r_n)}X_n$, while γ_n will be used to normalize ${}^{(r_n - 1)}S_n$. Note that by (2.5) and (2.13) $n \beta_n \to \infty$, so $\alpha_n(\lambda)$ and $\gamma_n(\lambda)$ both tend to infinity for every $\lambda > 0$. Thus by (2.4) if $\lambda_1 < \lambda_2$ and n is sufficiently large

(2.17) $$(\lambda_1 / \theta \lambda_2)^{q^{-1}} \alpha_n(\lambda_2) \leqq \alpha_n(\lambda_1) \leqq (\theta \lambda_1 \lambda_2^{-1})^{1/2} \alpha_n(\lambda_2).$$

In the special case that X is in the domain of attraction of a stable law of index $\alpha \in (0, 2)$, then $G(x)/K(x) \to (2 - \alpha) \alpha^{-1}$. Thus using Lemma 2.4 of [12] instead of (2.4) one can improve (2.17) in this case to

(2.18) $$\alpha_n(\lambda_1) \alpha_n(\lambda_2)^{-1} \to (\lambda_1 \lambda_2^{-1})^{1/\alpha}.$$

In many of our Borel-Cantelli arguments we will be using the same subsequence to sum along, so we will now describe this subsequence and also some of its properties that will be needed.

Let $a>1$ and set $n_1=[(a-1)^{-1}]+1$ and

$$n_{k+1}=\min\{n: r_n > r_{n_k} \quad \text{or} \quad a_n < a_{n_k}/2\} \wedge [a n_k]. \tag{2.19}$$

We first note that for some $b\in(1,a)$

$$n_k \geqq b^k \quad \text{for all } k. \tag{2.20}$$

This is because for each given k, there are $[k/3]$ values of j for which one of the following hold:

$$r_{n_j} > r_{n_{j-1}}, \qquad a_{n_j} < a_{n_{j-1}}/2, \qquad n_j=[a n_{j-1}].$$

In the first case, since r_n is integer valued

$$\begin{aligned}[k/3] &\leqq r_{n_k}\\ &\leqq l_2\, n_k\end{aligned}$$

for large k by (2.5). In the second case

$$\begin{aligned}2^{[k/3]}\, a_{n_1}^{-1} &\leqq a_{n_k}^{-1}\\ &\leqq (l n_k)^2\end{aligned}$$

by (2.7), while in the final case by the definition of n_1, it is not hard to see that for some $c\in(1,a)$, independent of k,

$$n_k \geqq c^{[k/3]}.$$

Consequently (2.20) holds, and also by (2.19)

$$n_{k+1} \leqq a n_k. \tag{2.21}$$

Set $m_k=n_{k+1}-1$. Note that we trivially have

$$r_n \text{ is constant on } [n_k, m_k], \tag{2.22}$$

and since a_n is decreasing we see that

$$\beta_n \quad \text{and} \quad \alpha_n(\lambda) \text{ are increasing on } [n_k, m_k]. \tag{2.23}$$

Furthermore for some constant $c>0$ independent of k

$$\beta_{n_k} \geqq c\,\beta_{m_k}. \tag{2.24}$$

As a consequence of this and (2.21) we have by (2.4) that for some constant $c>0$ independent of k and λ

(2.25) $$\alpha_{n_k}(\lambda) \geqq c\,\alpha_{m_k}(\lambda)$$

(2.26) $$\gamma_{n_k}(\lambda) \geqq c\,\gamma_{m_k}(\lambda).$$

Remark. Throughout we will use the letter c to denote a positive constant whose value may change from one useage to the next.

3. Probability Estimates

For $b>0$ and $d>0$ define

(3.1) $$U_n(d) = \sum_{i=1}^{n} X_i\, 1(|X_i| \leqq d)$$

(3.2) $$J_n(b) = \sum_{i=1}^{n} 1(|X_i| > b).$$

In order to prove our main results we will need probability estimates on the size of $J_n(b)$ and $U_n(d)$. Since we will be working outside the range for which the classical exponential bounds were designed (see p. 266 of [9]) we will use the following estimate which is an immediate consequence of Lemma 3.1 in [12].

Lemma 3.1. *For any* $v>0$, $d>0$, $s>0$ *and all* n

(3.3) $$P(|U_n(d) - EU_n(d)| \geqq 2^{-1} v\, e^{v}\, n\, d\, K(d) + s\, d\, v^{-1}) \leqq 2e^{-s}.$$

Given two sequences s_n and t_n we will write $s_n \approx t_n$ if $s_n t_n^{-1}$ and $s_n^{-1} t_n$ are both bounded as $n \to \infty$.

Lemma 3.2. *There exist positive constants* c_1 *and* c_2 *such that for all* $r \geqq 1$, *all* n *and all* $b \geqq 0$

(3.4) $$c_1 r^{-1/2} \exp(r - r\,l r + r\,l(nG(b)) - 2nG(b)) \leqq P(J_n(b) \geqq r) \leqq c_2 r^{-1/2} \exp(r - r\,l r + r\,l(nG(b)) - (n-r)\,G(b))$$

provided

(3.5) $$n > r^2$$

(3.6) $$nG(b) < r/2.$$

Proof. For any $b \geqq 0$, $r \geqq 1$ and $n \geqq r$

(3.7) $$P(J_n(b) \geqq r) = \sum_{j=r}^{n} \binom{n}{j} G(b)^j (1 - G(b))^{n-j}.$$

Set $u_j=\binom{n}{j} G(b)^j(1-G(b))^{n-j}$. Then for $r\leqq j\leqq n$

$$\frac{u_{j+1}}{u_j}=\frac{(n-j)\,G(b)}{(j+1)(1-G(b))}\leqq\frac{nG(b)}{r(1-G(b))}\leqq\frac{1}{2(1-G(b))}$$

by (3.6). Further since $1\leqq r<n^{1/2}$, we have by (3.6) that $G(b)\leqq(2n^{1/2})^{-1}\leqq 2^{-(3/2)}$ and so $u_{j+1}\,u_j^{-1}\leqq 2^{-1/4}$. Hence

$$\binom{n}{r} G(b)^r(1-G(b))^{n-r}\leqq P(J_n(b)\geqq r)\leqq c\binom{n}{r} G(b)^r(1-G(b))^{n-r} \tag{3.8}$$

where the c is independent of n, r and b. Next by Stirling's formula there exist positive constants c_3 and c_4 such that for all $r\geqq 1$ and all $n>r^2$

$$c_3\, r^{-1/2}\left(\frac{n}{n-r}\right)^{n-r}\left(\frac{n}{r}\right)^r\leqq\binom{n}{r}\leqq c_4\, r^{-1/2}\left(\frac{n}{n-r}\right)^{n-r}\left(\frac{n}{r}\right)^r. \tag{3.9}$$

Now it is a straightforward exercise to check that for all r and n satisfying $1\leqq r^2<n$,

$$e^{r-1}\leqq\left(\frac{n}{n-r}\right)^{n-r}\leqq e^r. \tag{3.10}$$

Also the elementary inequalities $e^{-2x}\leqq 1-x\leqq e^{-x}$ for $0\leqq x\leqq 1/2$, give

$$\exp(-2nG(b))\leqq(1-G(b))^{n-r}\leqq\exp(-(n-r)\,G(b)). \tag{3.11}$$

Thus (3.4) follows from (3.8)–(3.11). □

Corollary 3.3. *For any sequence of integers s_n satisfying $1\leqq s_n^2<n$ and any sequence of real numbers $b_n>0$, if $nG(b_n)\,s_n^{-1}\to 0$ then*

$$P(J_n(b_n)\geqq s_n)\to 0. \tag{3.12}$$

This also follows trivially from Markov's inequality. The following result is an easy consequence of a generalized Borel-Cantelli Lemma.

Lemma 3.4. *Assume B_k, C_k are two sequences of events such that B_k $k=1,2,\ldots$ are independent and for each k, B_k and C_k are independent. If $\Sigma\, P(B_k)=\infty$ and $P(C_k)\to 1$ then $P(B_k\,C_k$ i.o.$)=1$.*

Proof. Let $E_k=B_k C_k$. Then $P(E_k)=P(B_k)\,P(C_k)\sim P(B_k)$ and so $\Sigma P(E_k)=\infty$. If $i<j$ then

$$P(E_i\,E_j)\leqq P(B_i\,B_j)=P(B_i)\,P(B_j)\sim P(E_i)\,P(E_j)$$

as $i\to\infty$. From these two facts, it easily follows that

$$\limsup_{n\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i\,E_j)\right)\left(\sum_{i=1}^{n} P(E_i)\right)^{-2}\leqq 1.$$

The result now follows by $P3$ on page 317 of [14]. □

We conclude this section with a simple Lemma which will prove useful later.

Lemma 3.5. *For any* $x \geqq 0$, $\varepsilon \geqq 0$ *and* $N \geqq 0$

$$[(N+\varepsilon)[x]]+1-[N[x]] \geqq \varepsilon(1+\varepsilon)^{-1} x \tag{3.13}$$

Proof. If $\varepsilon = 0$ the result is trivial, thus we may assume $\varepsilon > 0$. If $x \leqq (1+\varepsilon)\,\varepsilon^{-1}$ then RHS $\leqq 1$ while LHS $\geqq 1$ for all x. If $x > (1+\varepsilon)\,\varepsilon^{-1}$ then $x-1 > (1+\varepsilon)^{-1} x$ and so

$$\begin{aligned} \text{LHS} &\geqq (N+\varepsilon)[x]-N[x] \\ &\geqq \varepsilon(x-1) \\ &\geqq \varepsilon(1+\varepsilon)^{-1} x. \quad \square \end{aligned}$$

4. Main Results

We begin this section by describing the growth of ${}^{(r_n)}X_n$. The only consequences of (2.1) that will be used in this paper are (2.4), (2.17), (2.25) and (2.26). Since these are not needed in the proof of the following result, no restrictions need be placed on the distribution of X.

Theorem 4.1. *Assume that* r_n *satisfies* (2.5), *then*

$$\limsup_{n\to\infty} \frac{|{}^{(r_n)}X_n|}{\alpha_n(\lambda)} \begin{cases} \leqq 1 & \text{if } \lambda > 1 \\ \geqq 1 & \text{if } \lambda < 1. \end{cases} \tag{4.1}$$

Proof. Given $\lambda > 1$, choose $a \in (1, \lambda)$ and let n_k be defined by (2.19). Set $m_k = n_{k+1} - 1$ and observe that by (2.13) and (2.21), $m_k\, G(\alpha_{n_k}(\lambda))\, r_{n_k}^{-1} \to 0$. Thus for large k by (2.21), (2.23) and (3.4)

$$\begin{aligned} &P(|{}^{(r_n)}X_n| > \alpha_n(\lambda) \quad \text{for some } n_k \leqq n \leqq m_k) \\ &= P(J_n(\alpha_n(\lambda)) \geqq r_n \quad \text{for some } n_k \leqq n \leqq m_k) \\ &\leqq P(J_{m_k}(\alpha_{n_k}(\lambda)) \geqq r_{n_k}) \\ &\leqq c_2\, r_{n_k}^{-1/2} \exp(r_{n_k} - r_{n_k}\, l r_{n_k} + r_{n_k}\, l(m_k\, G(\alpha_{n_k}(\lambda)))) \\ &= c_2\, r_{n_k}^{-1/2} \exp(r_{n_k} - r_{n_k}\, l r_{n_k} + r_{n_k}\, l(n_k\, G(\alpha_{n_k}(\lambda))) + r_{n_k}\, l(m_k\, n_k^{-1})) \\ &\leqq c_2\, r_{n_k}^{-1/2} \exp(r_{n_k} - r_{n_k}\, l r_{n_k} - r_{n_k}\, l \beta_{n_k} + r_{n_k}(l a - l \lambda)) \\ &= c_2\, r_{n_k}^{-1/2}\, a_{n_k} \exp((l a - l \lambda)\, r_{n_k}) \end{aligned} \tag{4.2}$$

and this gives rise to a convergent series by (2.10) and (2.20) since $a < \lambda$. The upper bound now follows by the Borel-Cantelli Lemma.

Now fix $\lambda < 1$ and choose D, an integer, large enough that

$$1 - D^{-1} > \lambda \tag{4.3}$$

Set $n_k = D^k$ and

$$A_k = \{J_{n_k}(\alpha_{n_k}(\lambda)) - J_{n_{k-1}}(\alpha_{n_k}(\lambda)) \geqq r_{n_k}\}.$$

Now $J_{n_k}(\alpha_{n_k}(\lambda)) - J_{n_{k-1}}(\alpha_{n_k}(\lambda))$ has the same distributions as $J_{n_k - n_{k-1}}(\alpha_{n_k}(\lambda))$ and one can again easily check that the conditions of Lemma 3.2 are met, so

$$\begin{aligned} P(A_k) &\geqq c_1\, r_{n_k}^{-1/2} \exp(r_{n_k} - r_{n_k}\, l r_{n_k} + r_{n_k}\, l((n_k - n_{k-1})\, G(\alpha_{n_k}(\lambda))) - 2(n_k - n_{k-1})\, G(\alpha_{n_k}(\lambda))) \\ &\geqq c_1\, r_{n_k}^{-1/2} \exp(r_{n_k} - r_{n_k}\, l r_{n_k} - r_{n_k}\, l \beta_{n_k} + r_{n_k}(l(1-D^1) - l\lambda) - 2(\lambda \beta_{n_k})^{-1}) \\ &= c_1\, r_{n_k}^{-1/2}\, a_{n_k} \exp((l(1-D^{-1}) - l\lambda)\, r_{n_k} - 2(\lambda \beta_{n_k})^{-1}) \end{aligned}$$

which gives rise to a divergent series by (2.9) and (4.3) since $r_{n_k} \beta_{n_k} \to \infty$ by (2.13). Since A_k $k = 1, 2, \ldots$ are independent events, $P(A_k \text{ i.o.}) = 1$ from which the result follows. □

Of course in general $\alpha_n(\lambda)$ and α_n need not be comparable, for example when G is slowly varying, however in our case we have

Corollary 4.2. *Assume that r_n satisfies* (2.5);

(a) *if X satisfies* (2.1) *then*

$$0 < \limsup_{n \to \infty} |{}^{(r_n)}X_n|\, \alpha_n^{-1} < \infty \tag{4.4}$$

(b) *if X is in the domain of attraction of a stable law of index $\alpha \in (0, 2)$ then*

$$\limsup_{n \to \infty} |{}^{(r_n)}X_n|\, \alpha_n^{-1} = 1. \tag{4.5}$$

Proof. (a) follows from (2.17) and (4.1) while (b) follows from (2.18) and (4.1). □

Remark. It is easy to see that (4.4) holds more generally than under condition (2.1). What is needed for (4.4) is that $\alpha_n(\lambda_1) \approx \alpha_n(\lambda_2)$ for some $\lambda_1 < 1 < \lambda_2$. This is true for example if there exists $p > 0$ and a non-increasing function f such that $x^p\, G(x) \approx f(x)$ as $x \to \infty$. In particular this is true for many random variables in the domain of attraction of the normal law.

As was pointed out earlier any sequence a_n satisfying (2.6)–(2.8), satisfies (2.6)–(2.8) with r_n replaced by $r_n + j$. Thus defining

$$\begin{aligned} \beta_n(j) &= \exp((l a_n^{-1} - (r_n + j)\, l(r_n + j) + (r_n + j))(r_n + j)^{-1}) \\ \alpha_n(\lambda, j) &= \min\{x : G(x) = (\lambda n \beta_n(j))^{-1}\} \end{aligned} \tag{4.6}$$

we have by Theorem 4.1 that for each $j \geqq 0$

$$\limsup_{n \to \infty} \frac{|{}^{(r_n + j)}X_n|}{\alpha_n(\lambda, j)} \begin{cases} \leqq 1 & \text{if } \lambda > 1 \\ \geqq 1 & \text{if } \lambda < 1. \end{cases} \tag{4.7}$$

In particular for each $j \geqq 0$, $|{}^{(r_n + j)}X_n| \leqq \alpha_n(2, j)$ eventually. Our next aim, Lemma 4.4, is to show that this holds uniformly in j for $0 \leqq j \leqq j_n$, provided $j_n = o(r_n)$. Actually we will not prove quite this much, we will introduce a new sequence

$b_n(j)$, which is more convenient to work with and show that $|^{(r_n+j)}X_n| \leqq b_n(j)$ for all $0 \leqq j \leqq j_n$ eventually. To do this set

(4.8) $$\delta_n = \exp\left(-\frac{l_2 n}{2r_n^2}\right)$$

and for $0 \leqq j \leqq r_n$ define

(4.9) $$b_n(j) = \min\{x: G(x) = (2n\,\delta_n^j \beta_n)^{-1}\}.$$

First note that by (2.5) and (2.7)

$$\begin{aligned} n\delta_n^{r_n}\beta_n &\geqq n \exp\left(-\frac{l_2 n}{2r_n} + \frac{l_2 n - r_n l r_n + r_n}{r_n}\right) \\ &\geqq n r_n^{-1} \\ &\to \infty. \end{aligned}$$

Thus

(4.10) $$b_n(r_n) \to \infty$$

and since $\delta_n < 1$ we trivially have

(4.11) $$b_n(r_n) \leqq b_n(j) < b_n(j-1) \leqq \alpha_n(2)$$

for $1 \leqq j \leqq r_n$. Next observe that for $0 \leqq j \leqq r_n$

$$\begin{aligned} l(\delta_n^j \beta_n \beta_n(j)^{-1}) &= -\frac{j l_2 n}{2r_n^2} + \frac{l a_n^{-1}}{r_n} - l r_n - \frac{l a_n^{-1}}{r_n + j} + l(r_n + j) \\ &\geqq \frac{j l a_n^{-1}}{r_n(r_n + j)} - \frac{j l_2 n}{2r_n^2} \\ &\geqq 0 \end{aligned}$$

by (2.7). Thus

(4.12) $$\delta_n^j \beta_n \geqq \beta_n(j), \quad 0 \leqq j \leqq r_n.$$

Later we will need to compare $b_n(j)$ with $b_n(1)$ and also $b_n(1)$ with α_n. To do this we let

(4.13) $$\alpha = [2l(4\theta)] + 1$$

where θ is given by (2.2). Thus by (2.14)

(4.14) $$\delta_n^{-\alpha N_n} \geqq 4\theta.$$

Lemma 4.3. *If n is sufficiently large, then for all integers $k \geqq 0$ satisfying $\alpha k N_n + 1 \leqq r_n$*

(4.15) $$b_n(\alpha k N_n + 1) \leqq 2^{-k} b_n(1).$$

Proof. If $k = 0$ then the result is trivial, thus we may assume $k \geqq 1$.

By (2.4) if $2^{-k} b_n(1) \geqq x_0$ then

$$G(2^{-k} b_n(1)) \leqq 4^k \theta G(b_n(1)) \leqq (4\theta)^k (2n\delta_n \beta_n)^{-1} \leqq (2n\,\delta^{\alpha k N_n + 1} \beta_n)^{-1} = G(b_n(\alpha k N_n + 1)). \tag{4.16}$$

Thus if we can show that $2^{-k} b_n(1) \geqq x_0$ holds whenever $\alpha k N_n + 1 \leqq r_n$, provided n is sufficiently large, we will be done. Let

$$k_n = \max\{k: 2^{-k} b_n(1) \geqq x_0\}. \tag{4.17}$$

Since $b_n(1) \to \infty$ by (4.10) and (4.11), we must have $k_n \to \infty$. Suppose that $\alpha k_n N_n + 1 \leqq r_n$ infinitely often, then by (4.16) and (4.17) along some subsequence we have both

$$2^{-(k_n+1)} b_n(1) < x_0$$

and

$$2^{-k_n} b_n(1) \geqq b_n(\alpha k_n N_n + 1) \geqq b_n(r_n) \to \infty$$

by (4.10) and (4.11). This is a contradiction and so $\alpha k_n N_n + 1 \geqq r_n$ eventually which completes the proof. □

If $r_n = o((l_2 n)^{1/2})$ then $\delta_n \to 0$ and so for every $\varepsilon > 0$ by (2.4)

$$G(\varepsilon \alpha_n) \leqq \varepsilon^{-2} \theta G(\alpha_n) = \varepsilon^{-2} \theta (n \beta_n)^{-1} \leqq (2n\,\delta_n \beta_n)^{-1} = G(b_n(1))$$

provided n is sufficiently large. Thus if $r_n = o((l_2 n)^{1/2})$ then

$$b_n(1) = o(\alpha_n). \tag{4.18}$$

Lemma 4.4. *Let j_n be any sequence of integers satisfying $j_n r_n^{-1} \to 0$ and set*

$$A_n = \{|^{(r_n + j)} X_n| > b_n(j) \quad \text{for some } 0 \leqq j \leqq j_n\} \tag{4.19}$$

Then

$$P(A_n \text{ i.o.}) = 0. \tag{4.20}$$

Proof. Let $a \in (1, 2)$ and define n_k by (2.19). Set $m_k = n_{k+1} - 1$ and let

$$B_k = \{A_n \text{ for some } n_k \leqq n \leqq m_k\}.$$
$$\hat{b}_k(j) = \min_{n_k \leqq n \leqq m_k} b_n(j).$$

By (2.22), since we may assume j_n is nondecreasing, we have

$$P(B_k) \leqq P(|^{(r_{m_k}+j)}X_{m_k}| > \hat{b}_k(j) \quad \text{for some } 0 \leqq j \leqq j_{m_k}) \leqq j_{m_k} \max_{0 \leqq j \leqq j_{m_k}} P(|^{(r_{m_k}+j)}X_{m_k}| > \hat{b}_k(j)). \tag{4.21}$$

We now wish to apply Lemma 3.2. By (2.5) it is clear that if k is sufficiently large then $\max_{0 \leqq j \leqq j_{m_k}} (r_{m_k}+j)^2 < m_k$. To check (3.6) first observe that by (4.12)

$$\begin{aligned} G(\hat{b}_k(j)) &= \max_{n_k \leqq n \leqq m_k} G(b_n(j)) \\ &= \max_{n_k \leqq n \leqq m_k} (2n\,\delta_n^j\,\beta_n)^{-1} \\ &\leqq \max_{n_k \leqq n \leqq m_k} (2n\,\beta_n(j))^{-1} \\ &\leqq (2n_k\,\beta_{n_k}(j))^{-1} \end{aligned} \tag{4.22}$$

by (2.6) and (2.22). Thus for every $0 \leqq j \leqq j_{m_k}$

$$\begin{aligned} m_k\,G(\hat{b}_k(j)) &\leqq a n_k\,G(\hat{b}_k(j)) \\ &\leqq a(2\beta_{n_k}(j))^{-1} \\ &= (a/2)\exp\left(-\frac{l a_{n_k}^{-1}}{r_{n_k}+j} + l(r_{n_k}+j) - 1\right) \\ &\leqq (a/2)(r_{n_k}+j)\exp\left(-\frac{l a_{n_k}^{-1}}{r_{n_k}+j_{n_k}}\right) \\ &\leqq 1/2(r_{m_k}+j) \end{aligned} \tag{4.23}$$

for large k, independent of j, by (2.5), (2.7) and (2.22). Consequently we can apply Lemma 3.2 to obtain

$$\begin{aligned} P(B_k) \leqq j_{m_k} \max_{0 \leqq j \leqq j_{m_k}} c_2 (r_{m_k}+j)^{-1/2} \exp((r_{m_k}+j) \\ -(r_{m_k}+j)\,l(r_{m_k}+j) + (r_{m_k}+j)\,l(m_k\,G(\hat{b}_k(j))))) \end{aligned} \tag{4.24}$$

Using (2.22) and (4.22) the exponent above can be written as

$$\begin{aligned} &(r_{m_k}+j) - (r_{m_k}+j)\,l(r_{m_k}+j) + (r_{m_k}+j)\,l(n_k\,G(\hat{b}_k(j))) + (r_{m_k}+j)\,l(m_k n_k^{-1}) \\ &\leqq (r_{n_k}+j) - (r_{n_k}+j)\,l(r_{n_k}+j) - (r_{n_k}+j)\,l\beta_{n_k}(j) + (r_{n_k}+j)(la - l2) \\ &= l a_{n_k} + (r_{n_k}+j)(la - l2) \end{aligned} \tag{4.25}$$

by (4.6). Thus, since $a < 2$

$$P(B_k) \leqq c_2\, j_{m_k}\, r_{m_k}^{-1/2}\, a_{n_k}\, e^{(la - l2)r_{n_k}}$$

and this gives rise to a convergent series by (2.10) since $j_{m_k} r_{n_k}^{-1} = j_{m_k} r_{m_k}^{-1} \to 0$. The result now follows from the Borel-Cantelli Lemma. □

In proving the lower bound in Theorem 4.8, we will, roughly speaking, need to ensure that infinitely often the N_n largest terms are all of size α_n and further that all of these terms have the same sign. This will be formulated precisely, together with an additional requirement, in Lemma 4.6. To prove this we must first introduce some further notation.

For $0 \leqq y_1 \leqq y_2$, if $G(y_1) > G(y_2)$ let $X(y_1, y_2)$ be a random variable with distribution function $F_{X(y_1, y_2)}$ given by

$$dF_{X(y_1,y_2)}(x) = 1(y_1 \leqq |x| \leqq y_2)(G(y_1) - G(y_2))^{-1}\, dF_X(x). \tag{4.26}$$

Thus $X(y_1, y_2)$ is X conditioned to have absolute value between y_1 and y_2. Note that

$$G_{X(y_1,y_2)}(x) = \begin{cases} 1 & \text{if } x \leqq y_1 \\ (G(x) - G(y_2))(G(y_1) - G(y_2))^{-1} & \text{if } y_1 \leqq x \leqq y_2 \\ 0 & \text{if } x \geqq y_2 \end{cases} \tag{4.27}$$

We will write $X(y_1)$ for $X(0, y_1)$.

For $r \geqq 2$ and $s \geqq 0$ let $H_{m,r+s,r-1}(y_1, y_2)$ denote the two-dimensional distribution function of $(|^{(r+s)}X_m|, |^{(r-1)}X_m|)$. Observe that this distribution assigns zero probability to the complement of the set $\{(y_1, y_2): 0 \leqq y_1 \leqq y_2 \text{ and } G(y_1) > G(y_2)\}$. The following proposition does not appear to be in the literature, but since variants of it are well known, (see for example Lemma 1.1 of [11]), we will not prove it here.

Proposition 4.5. *Let $X_i(y_1)$, $i = 1, 2, \ldots$, and $X_j(y_1, y_2)$, $j = 1, 2, \ldots$, be sequences of* i.i.d. *random variables with common distributions given by $X(y_1)$ and $X(y_1, y_2)$ respectively. Further assume that these sequences are independent. Then for all $r \geqq 2$, all $0 \leqq s \leqq u$, all bounded Borel functions $\phi_1: R^{u-s} \to \mathbb{R}^1$, $\phi_2: \mathbb{R}^s \to \mathbb{R}^1$, and all Borel sets $B \subseteq [0, \infty) \times [0, \infty)$.*

$$\begin{aligned} E[\phi_1(^{(r+u)}X_m, \ldots, {}^{(r+s+1)}X_m)\, \phi_2(^{(r+s-1)}X_m, \ldots, {}^{(r)}X_m);\\ (|^{(r+s)}X_m|, |^{(r-1)}X_m|) \in B] \\ = \int_B E\phi_1(^{(u-s)}X_{m-r-s}(y_1), \ldots, {}^{(1)}X_{m-r-s}(y_1))\, E\phi_2(^{(s)}X_s(y_1, y_2), \ldots, \\ {}^{(1)}X_s(y_1, y_2))\, dH_{m,r+s,r-1}(y_1, y_2). \end{aligned} \tag{4.28}$$

Remarks. 1. If $s = 0$ or $u = s$ we should explain what is meant by (4.28). If $s = 0$ then $\phi_2 \equiv 1$, while if $u = s$ then $\phi_1 \equiv 1$.

2. The more intuitive way of phrasing (4.28) is that the distribution of $(^{(r+u)}X_m, \ldots, {}^{(r+s+1)}X_m, {}^{(r+s-1)}X_m, \ldots, {}^{(r)}X_m)$ conditioned by $|^{(r+s)}X_m| = y_1$, $|^{(r-1)}X_m| = y_2$ is given by $(^{(u-s)}X_{m-r-s}(y_1), \ldots, {}^{(1)}X_{m-r-s}(y_1), {}^{(s)}X_s(y_1, y_2), \ldots, {}^{(1)}X_s(y_1, y_2))$.

Set

$$p_n = [r_n^2 / l_2 n]. \tag{4.29}$$

Lemma 4.6. *For every integer $N \geqq 1$ there exists $\lambda_2 \in (0,1)$ such that for all $\lambda_1 \in (0, \lambda_2)$ and all $\varepsilon \in (0,1)$*

(4.30) $$P(|^{(r_n+t_n)}X_n| \leqq \alpha_n(\lambda_1), |^{(r_n+s_n)}X_n| \geqq \alpha_n(\lambda_2), E_n \text{ i.o.}) = 1$$

where $s_n = N p_n$, $t_n = [(N+2\varepsilon)\, p_n] + 1$ *and* $E_n = E_n^+ \cup E_n^-$ *where*

(4.31) $$E_n^+ = \left\{\sum_{i=0}^{s_n-1} 1(^{(r_n+i)}X_n > 0) \geqq (1-\varepsilon)\, s_n\right\}$$

(4.32) $$E_n^- = \left\{\sum_{i=0}^{s_n-1} 1(^{(r_n+i)}X_n < 0) \geqq (1-\varepsilon)\, s_n\right\}.$$

Remark. If $s_n = 0$, then E_n is the whole space.

Proof. Fix $N \geqq 1$ and choose $\lambda_2 \in (0,1)$ so that

(4.33) $$l(2\lambda_2) + 2N + 1 < 0$$

Let $n_k = 2^k$, and for notational convenience write $r_k = r_{n_k}$, $s_k = s_{n_k}$, $t_k = t_{n_k}$, $p_k = p_{n_k}$ and $\alpha_k(\lambda) = \alpha_{n_k}(\lambda)$. Define $u_k = [(N+\varepsilon)\ p_k] + 1$ and $v_k = [\varepsilon p_k]$. Note that $u_k > s_k$. Let

$$^{(r)}Z_k = r^{th} \text{ largest random variable in absolute value from amongst } X_{n_{k-1}+1}, \ldots, X_{n_k}$$

$$A_k^+ = \{^{(r_k+i)}Z_k > 0 \quad \text{for all } 0 \leqq i < s_k\}$$

$$A_k^- = \{^{(r_k+i)}Z_k > 0 \quad \text{for all } 0 \leqq i < s_k\}$$

$$A_k = A_k^+ \cup A_k^-$$

$$B_k = \{|^{(r_k+u_k)}Z_k| \leqq \alpha_k(\lambda_1), |^{(r_k+s_k)}Z_k| \geqq \alpha_k(\lambda_2)\}$$

$$C_k = \{J_{n_{k-1}}(\alpha_k(\lambda_1)) \leqq v_k\}.$$

If $s_k = 0$ then A_k^+ and A_k^- are the whole space. Observe that on the event $A_k^+ B_k C_k$ since $u_k + v_k \leqq t_k$ we have

(4.34) $$|^{(r_k+t_k)}X_{n_k}| \leqq \alpha_k(\lambda_1)$$

(4.35) $$|^{(r_k+s_k)}X_{n_k}| \geqq \alpha_k(\lambda_2)$$

and

$$\begin{aligned}\sum_{i=0}^{s_k-1} 1(^{(r_k+i)}X_{n_k} > 0) &\geqq s_k - v_k \\ &\geqq (N-\varepsilon)\, p_k \\ &\geqq (1-\varepsilon)\, s_k.\end{aligned}$$

Similarly on the event $A_k^-\, B_k C_k$ we have (4.34), (4.35) and

(4.36) $$\sum_{i=0}^{s_k-1} 1(^{(r_k+i)}X_{n_k} < 0) \geqq (1-\varepsilon)\, s_k.$$

Hence to prove (4.30), it suffices to show $P(A_k B_k C_k \text{ i.o.})=1$. To do this we will use Lemma 3.4. First observe that $A_k B_k$, $k=1, 2, \ldots$ are independent, and for each k, C_k and $A_k B_k$ are independent. Now setting $v_k'=v_k+1$ we have by (3.12)

$$1-P(C_k)=P(J_{n_{k-1}}(\alpha_k(\lambda_1))\geqq v_k')$$
$$\to 0$$

since

$$\begin{aligned} n_{k-1}\, G(\alpha_k(\lambda_1))(v_k')^{-1} &\leqq (\lambda_1 \beta_k v_k')^{-1} \\ &\leqq (1+\varepsilon)(\varepsilon \lambda_1 \beta_k r_k^2 / l_2 n_k)^{-1} \\ &\to 0 \end{aligned}$$

by (2.13) and (3.13). Next let $m_k=n_k-n_{k-1}=2^{k-1}$, so $\{{}^{(j)}Z_k\colon 1\leqq j\leqq m_k\}\overset{d}{=}\{{}^{(j)}X_{m_k}\colon 1\leqq j\leqq m_k\}$. Thus to compute $P(A_k B_k)$ we can use Proposition 4.5 with

$$\phi_1(x_1, \ldots, x_{u_k-s_k})=1(|x_1|\leqq \alpha_k(\lambda_1))$$

$$\phi_2(z_1, \ldots, z_{s_k})=\begin{cases} \sum_{i=1}^{s_k} 1(z_i>0)+\prod_{i=1}^{s_k} 1(z_i<0) & \text{if } s_k\neq 0 \\ 1 & \text{if } s_k=0 \end{cases}$$

$$B=\{(y_1, y_2)\colon \alpha_k(\lambda_2)\leqq y_1\leqq y_2<\infty\}.$$

Recall that $u_k>s_k$ so there is no need to modify the definition of ϕ_1 to include the case $u_k=s_k$. Observe that if $G(y_1)>G(y_2)$ then

$$\begin{aligned} &E\phi_2({}^{(s_k)}X_{s_k}(y_1, y_2), \ldots, {}^{(1)}X_{s_k}(y_1, y_2)) \\ &\quad =E\phi_2(X_1(y_1, y_2), \ldots, X_{s_k}(y_1, y_2)) \\ &\quad \geqq 2^{-s_k} \end{aligned}$$

while for any $y_1\geqq \alpha_k(\lambda_2)$

$$\begin{aligned} &E\phi_1({}^{(u_k-s_k)}X_{m_k-r_k-s_k}(y_1), \ldots, {}^{(1)}X_{m_k-r_k-s_k}(y_1)) \\ &\quad =1-P(|{}^{(u_k-s_k)}X_{m_k-r_k-s_k}(y_1)|>\alpha_k(\lambda_1)) \\ &\quad \to 1 \end{aligned}$$

uniformly in $y_1\geqq \alpha_k(\lambda_2)$ by (3.4), since by (4.27)

$$\begin{aligned} (m_k-r_k-s_k)\, G_{X(y_1)}(\alpha_k(\lambda_1))(u_k-s_k)^{-1} &\leqq (m_k-r_k-s_k)\, G(\alpha_k(\lambda_1))(u_k-s_k)^{-1} \\ &\leqq (\lambda_1 \beta_k (u_k-s_k))^{-1} \\ &\leqq (1+\varepsilon)(\varepsilon \lambda_1 \beta_k r_k^2 / l_2 n_k)^{-1} \\ &\to 0 \end{aligned}$$

by (2.13) and (3.13). Thus by (4.28), for large k

$$P(A_k B_k)\geqq 2^{-(s_k+1)}\, P(|{}^{(r_k+s_k)}X_{m_k}|\geqq \alpha_k(\lambda_2)),$$

and so to complete the proof we must show that this gives rise to a divergent series. Let $w_n = r_n + s_n$ and write $w_k = w_{n_k}$. Note that $|w_n - r_n| = o(r_n)$ by (2.5) thus it is easy to check that conditions (3.5) and (3.6) are met, so by (3.4)

$$(4.37)\quad P(|^{(w_k)}X_{m_k}| \geqq \alpha_k(\lambda_2)) \geqq c_1 w_k^{-1/2} \exp(w_k - w_k l w_k - w_k l \beta_k - w_k l(2\lambda_2) - (\lambda_2 \beta_k)^{-1}).$$

Now set $\beta'_n = \exp((l a_n^{-1} - w_n l w_n + w_n) w_n^{-1})$ and $\beta'_k = \beta'_{n_k}$. Observe that by (2.5) and (2.7)

$$\begin{aligned} l\beta_k - l\beta'_k &= \frac{s_k l a_k^{-1}}{r_k(r_k + s_k)} + l(1 + s_k r_k^{-1}) \\ &\leqq \frac{2 s_k l_2 n_k}{r_k^2} + o(1) \\ &\leqq 2N + o(1). \end{aligned}$$

Thus for large k, $l\beta_k - l\beta'_k \leqq 2N + 1$ and so

$$\begin{aligned} P(A_k B_k) &\geqq c_1 2^{-(s_k+1)} w_k^{-1/2} \exp(w_k - w_k l w_k - w_k l \beta'_k \\ &\qquad - w_k(l(2\lambda_2) + 2N + 1) - (\lambda_2 \beta_k)^{-1}) \\ &= c_1 2^{-(s_k+1)} w_k^{-1/2} a_{n_k} \exp(-w_k(l(2\lambda_2) + 2N + 1) - (\lambda_2 \beta_k)^{-1}). \end{aligned}$$

Since $|w_n - r_n| = s_n = o(r_n)$ we have $s_k = o(w_k)$, and further by (2.13) that $\beta_k^{-1} = \circ(w_k)$. Thus by (4.33) and the remarks following (2.10), the above give rise to a divergent series and the proof is complete. □

Fix $p > 2$ and let

$$(4.38)\qquad d_n = \min\{x : G(x) = (l_2 n / r_n)^p (n \beta_n)^{-1}\}.$$

One easily checks that $G(d_n) \to 0$ and so $d_n \to \infty$. Let

$$(4.39)\qquad j_n = \left[4p\left(\frac{r_n^2}{l_2 n}\right) l\left(\frac{l_2 n}{r_n}\right)\right] + 1.$$

Note that by (2.5)

$$(4.40)\qquad j_n r_n^{-1} \to 0.$$

Let n_k defined by (2.19) with $a = 2$ and set $m_k = n_{k+1} - 1$. Let

$$(4.41)\qquad \hat{d}_k = \min_{n_k \leqq n \leqq m_k} d_n,$$

then

$$(4.42)\qquad G(\hat{d}_k) \leqq (l_2 m_k / r_{m_k})^p (n_k \beta_{n_k})^{-1}.$$

Observe that

$$\begin{aligned} G(b_{m_k}(j_{m_k})) &= (2m_k\,\delta_{m_k}^{j_{m_k}}\,\beta_{m_k})^{-1} \\ &= (2m_k\,\beta_{m_k})^{-1}\exp(j_{m_k}\,l_2\,m_k(2r_{m_k}^2)^{-1}) \\ &\geqq (l_2\,m_k/r_{m_k})^{2p}(2m_k\,\beta_{m_k})^{-1} \\ &\geqq c(l_2\,m_k/r_{m_k})^{2p}(n_k\,\beta_{n_k})^{-1} \end{aligned}$$

by (2.21) and (2.24) where c is independent of k. Thus by (2.5) for large k, $G(b_{m_k}(j_{m_k}))\geqq G(\hat{d}_k)$ and so

(4.43) $$b_{m_k}(j_{m_k})\leqq \hat{d}_k.$$

Similarly one can show that for large n

(4.44) $$b_n(j_n)\leqq d_n.$$

Now let

(4.45) $$\check{d}_k=\max_{n_k\leqq n\leqq m_k} d_n$$

then by (2.5), (2.21) and (2.24) for some $c>0$, independent of k

$$\begin{aligned} G(\check{d}_k) &\geqq (l_2\,n_k/r_{n_k})^{p}(m_k\,\beta_{m_k})^{-1} \\ &\geqq c(l_2\,n_k/r_{n_k})^{p}(n_k\,\beta_{n_k})^{-1} \\ &= cG(d_{n_k}). \end{aligned}$$

Thus by (2.4) since $d_n\to\infty$

(4.46) $$\check{d}_k\leqq c\,d_{n_k}$$

for large k, where c is independent of k. Also note that by (2.1) for any $\varepsilon>0$, $x^{2+\varepsilon}\,G(x)\to\infty$ as $x\to\infty$, (this actually holds for $\varepsilon=0$ also). Thus by (2.7) and (2.11), $n^{-s}\alpha_n\to\infty$ for all $s<1/2$. In particular $(r_n/l_2\,n)^{p/2}\,\alpha_n\to\infty$ and so by (2.4) for large n

(4.47) $$d_n\leqq\theta^{1/2}(r_n/l_2\,n)^{p/2}\,\alpha_n.$$

From (2.5) it then easily follows that

(4.48) $$d_n\,r_n=o(\gamma_n).$$

Note also that by (2.21), (2.22), (4.42) and (4.46) for some constant c

(4.49) $$\begin{aligned} \check{d}_k\,m_k\,G(\hat{d}_k) &\leqq c\,d_{n_k}(l_2\,n_k/r_{n_k})^{p}\beta_{n_k}^{-1} \\ &= o(\gamma_{n_k}) \end{aligned}$$

by (2.13) and (4.48).

To simplify notation in the next Lemma, it is convenient to define

$$\bar{U}_n(d)=U_n(d)-EU_n(d).$$

Lemma 4.7.

$$\limsup_{n\to\infty} \frac{|U_n(d_n)-EU_n(d_n)|}{\gamma_n}=0 \quad \text{a.s.} \tag{4.50}$$

Proof. Let n_k be defined by (2.19) with $a=2$ and set $m_k=n_{k+1}-1$. By (2.21)–(2.23) it suffices to prove

$$\limsup_{k\to\infty} \max_{n_k\leqq n\leqq m_k} \frac{|\bar{U}_n(d_n)|}{\gamma_{n_k}}=0 \quad \text{a.s.}$$

First observe that for $n_k\leqq n\leqq m_k$

$$\begin{aligned}|\bar{U}_n(d_n)|&=|\bar{U}_n(\hat{d}_k)+\sum_{i=1}^{n} X_i\, 1(\hat{d}_k<|X_i|\leqq d_n)-\sum_{i=1}^{n} EX_i\, 1(\hat{d}_k<|X_i|\leqq d_n)|\\ &\leqq|\bar{U}_n(\hat{d}_k)|+\check{d}_k\sum_{i=1}^{m_k} 1(\hat{d}_k<|X_i|\leqq \check{d}_k)+\sum_{i=1}^{m_k} E|X_i|\, 1(\hat{d}_k<|X_i|\leqq \check{d}_k)\end{aligned}$$

Thus

$$\begin{aligned}\max_{n_k\leqq n\leqq m_k}|\bar{U}_n(d_n)|&\leqq \max_{n_k\leqq n\leqq m_k}|\bar{U}_n(\hat{d}_k)|+\check{d}_k\sum_{1}^{m_k} 1(\hat{d}_k<|X_i|\leqq \check{d}_k)\\ &\quad+\sum_{1}^{m_k} E|X_i|\, 1(\hat{d}_k<|X_i|\leqq \check{d}_k)\\ &=I+II+III.\end{aligned} \tag{4.51}$$

Now by (4.43) for large k

$$\begin{aligned}II&\leqq \check{d}_k\, J_{m_k}(\hat{d}_k)\\ &\leqq \check{d}_k\, J_{m_k}(b_{m_k}(j_{m_k}))\end{aligned}$$

while by (4.19) and (4.40) for large k we have a.s.

$$J_{m_k}(b_{m_k}(j_{m_k}))<r_{m_k}+j_{m_k}.$$

Thus for large k, by (2.22), (4.40) and (4.46)

$$\begin{aligned}II&\leqq 2\check{d}_k\, r_{m_k}\\ &\leqq c\, d_{n_k}\, r_{n_k}\\ &=o(\gamma_{n_k})\end{aligned}$$

by (4.48). Next

$$\begin{aligned}III&\leqq \check{d}_k\, m_k\, G(\hat{d}_k)\\ &=o(\gamma_{n_k})\end{aligned}$$

by (4.49). To deal with I, first observe that for $n_k \leqq n \leqq m_k$ by Chebyshev and (2.2)

$$P(|\bar{U}_{m_k}(\hat{d}_k) - \bar{U}_n(\hat{d}_k)| > \varepsilon \gamma_{n_k}) \leqq (m_k - n)\, \hat{d}_k^2\, K(\hat{d}_k)(\varepsilon \gamma_{n_k})^{-2}$$
$$\leqq \theta \hat{d}_k^2\, m_k\, G(\hat{d}_k)(\varepsilon \gamma_{n_k})^{-2}$$
$$\to 0$$

by (4.48) and (4.49), uniformly in n. Thus by Skorohod's Lemma (Breiman [1] p. 45) for large k

$$P(\max_{n_k \leqq n \leqq m_k} |\bar{U}_n(\hat{d}_k)| > 2\varepsilon \gamma_{n_k}) \leqq 2P(|\bar{U}_{m_k}(\hat{d}_k)| > \varepsilon \gamma_{n_k}).$$

We will now use Lemma 3.1 with $s = 2l_2\, n_k$ and $v = l_2\, n_k (2r_{n_k})^{-1}$. Then

$$s \hat{d}_k\, v^{-1} \leqq 4 d_{n_k}\, r_{n_k} = o(\gamma_{n_k})$$

by (4.48) while

$$\tfrac{1}{2} v e^{v} m_k\, \hat{d}_k\, K(\hat{d}_k) \leqq \theta v e^{v} d_{n_k}\, n_k\, G(\hat{d}_k) = o(\gamma_{n_k})$$

by (4.49). Hence by (3.3) for any $\varepsilon > 0$, if k is sufficiently large

$$P(|\bar{U}_{m_k}(\hat{d}_k)| > \varepsilon \gamma_{n_k}) \leqq 2 \exp(-2l_2\, n_k)$$

which gives rise to a convergent series by (2.20). The result now follows from Borel-Cantelli. □

We now come to our main result describing the L.I.L. behaviour of ${}^{(r_n-1)}S_n$.

Theorem 4.8. *Assume that* r_n *satisfies* (2.5) *and let* γ_n *and* d_n *be given by* (2.16) *and* (4.38) *respectively*

(a) If (2.1) *holds then*

(4.52) $$0 < \limsup_{n\to\infty} |{}^{(r_n-1)}S_n - EU_n(d_n)|\, \gamma_n^{-1} < \infty$$

(b) If (2.1) *holds and in addition* $r_n = o((l_2\, n)^{1/2})$ *then*

(4.53) $$0 < \limsup_{n\to\infty} |{}^{(r_n)}X_n|\, \gamma_n^{-1} = \limsup_{n\to\infty} |{}^{(r_n-1)}S_n - EU_n(d_n)|\, \gamma_n^{-1} < \infty$$

Further

(4.54) $$\limsup_{n\to\infty} |{}^{(r_n+1)}X_n|\, \gamma_n^{-1} = 0$$

(4.55) $$\limsup_{n\to\infty} |{}^{(r_n)}S_n - EU_n(d_n)|\, \gamma_n^{-1} = 0$$

(c) If $r_n = o((l_2\, n)^{1/2})$ *and* X *is in the domain of attraction of a stable law of index* $\alpha \in (0, 2)$, *then* (4.54) *and* (4.55) *hold and* (4.53) *can be strengthened to*

(4.56) $$\limsup_{n\to\infty} |{}^{(r_n)}X_n|\, \gamma_n^{-1} = \limsup_{n\to\infty} |{}^{(r_n-1)}S_n - EU_n(d_n)|\, \gamma_n^{-1} = 1.$$

Proof. First note that to prove (4.54), it suffices by (4.7) to show that $\alpha_n(\lambda, 1)\alpha_n^{-1} \to 0$ for some $\lambda > 1$. To do this it suffices by (2.4) to show that $G(\alpha_n(\lambda, 1))/G(\alpha_n) = \beta_n(\lambda \beta_n(1))^{-1} \to \infty$. But by (2.7)

$$\begin{aligned} l(\beta_n \beta_n(1)^{-1}) &\geqq \frac{l a_n^{-1}}{r_n} - \frac{l a_n^{-1}}{r_n+1} \\ &\geqq \frac{l_2 n}{r_n(r_n+1)} \\ &\to \infty \end{aligned}$$

if $r_n = o((l_2 n)^{1/2})$.

Next observe that (4.56) is an immediate consequence of (4.53) and (4.5). Further (4.53) follows from (4.55) and (4.4). Thus we only have to prove (4.52) and (4.55). We begin with the proof of (4.55) and the upper bound in (4.52), which will be proved simultaneously. Observe that to prove the upper bound in (4.52), it suffices by (4.4) to show that

$$\limsup_{n \to \infty} |{}^{(r_n)}S_n - EU_n(d_n)|\, \gamma_n^{-1} < \infty. \tag{4.57}$$

Fix n; if $|{}^{(r_n+1)}X_n| \leqq d_n$ then

$$|{}^{(r_n)}S_n - U_n(d_n)| \leqq d_n r_n$$

and so by (4.48) and (4.50)

$$|{}^{(r_n)}S_n - EU_n(d_n)| = o(\gamma_n). \tag{4.58}$$

If $|{}^{(r_n+1)}X_n| > d_n$ then

$$\begin{aligned} |{}^{(r_n)}S_n - EU_n(d_n)| &\leqq \Big|\sum_{j=1}^{n-r_n} {}^{(r_n+j)}X_n\, 1(|{}^{(r_n+j)}X_n| > d_n)\Big| \\ &\quad + |U_n(d_n) - EU_n(d_n)| \\ &= I + II. \end{aligned} \tag{4.59}$$

By (4.50), $II = o(\gamma_n)$, thus we have left to estimate I. Let j_n be as in (4.39), then by (4.20), (4.40) and (4.44) we have that $|{}^{(r_n+j_n)}X_n| \leqq d_n$ eventually. Thus for large n, using (4.11), (4.15) and (4.20) we have a.s.

$$\begin{aligned} I &\leqq \sum_{j=1}^{j_n-1} |{}^{(r_n+j)}X_n| \\ &\leqq \sum_{j=1}^{j_n-1} b_n(j) \\ &\leqq \sum_{k=0}^{[j_n/\alpha N_n]} \sum_{j=k\alpha N_n+1}^{(k+1)\alpha N_n} b_n(j) \\ &\leqq \sum_{k=0}^{[j_n/\alpha N_n]} \alpha N_n\, b_n(k\alpha N_n+1) \\ &\leqq \alpha N_n\, b_n(1) \sum_{0}^{[j_n/\alpha N_n]} 2^{-k} \\ &\leqq 2\alpha N_n\, b_n(1). \end{aligned} \tag{4.60}$$

Thus by (4.59) and (4.60)

$$|{}^{(r_n)}S_n - EU_n(d_n)| \leqq \gamma_n(2\alpha\, b_n(1)\,\alpha_n^{-1} + o(1))$$

Now by (2.17) and (4.11), $b_n(1)\,\alpha_n^{-1} = O(1)$ which proves (4.57), while if $r_n = o((l_2 n)^{1/2})$ then $b_n(1)\,\alpha_n^{-1} \to 0$ by (4.18) which proves (4.55).

To prove the lower bound in (4.52) we begin by letting $N=4$ in Lemma 4.6 and choosing $\lambda_2 \in (0,1)$ to satisfy (4.30). By (2.17) we have

$$\alpha_n(\lambda_2) \geqq (\lambda_2/2\theta)^{q^{-1}}\,\alpha_n(2). \tag{4.61}$$

Next choose M an integer, large enough that

$$2^{-M} \leqq (8\alpha)^{-1}(\lambda_2/2\theta)^{q^{-1}} \tag{4.62}$$

and set $\lambda_1 = (16\theta\,\alpha^2 M^2)^{-1}\,\lambda_2$. Observe that $\lambda_1 \in (0, \lambda_2)$ and by (2.17)

$$\alpha_n(\lambda_1) \leqq (4\alpha M)^{-1}\,\alpha_n(\lambda_2). \tag{4.63}$$

Set

$$\varepsilon = (8(1 + (2\theta/\lambda_2)^{q^{-1}}))^{-1} \tag{4.64}$$

in Lemma 4.6 and let

$$D_n = \{|{}^{(r_n+t_n)}X_n| \leqq \alpha_n(\lambda_1),\, |{}^{(r_n+s_n)}X_n| \geqq \alpha_n(\lambda_2),\, E_n\}.$$

Thus $P(D_n \text{ i.o.}) = 1$. If $\omega \in D_n$ and n is sufficiently large then $|{}^{(r_n+s_n)}X_n| > d_n$ since $\alpha_n(\lambda_2) > d_n$ for large n by (2.17) and (4.47). Thus for infinitely many n, D_n occurs and

$$\begin{aligned}|{}^{(r_n-1)}S_n - EU_n(d_n)| &\geqq \left|\sum_{j=0}^{s_n} {}^{(r_n+j)}X_n\right| - \sum_{j=s_n+1}^{n-r_n} |{}^{(r_n+j)}X_n|\, 1(|{}^{(r_n+j)}X_n| > d_n)\\ &\quad - |U_n(d_n) - EU_n(d_n)|\\ &= I - II - III.\end{aligned}$$

Now by (4.20) and (4.44) for large n

$$\begin{aligned}II &\leqq \sum_{j=s_n+1}^{j_n} |{}^{(r_n+j)}X_n|\\ &= \sum_{j=s_n+1}^{t_n-1} |{}^{(r_n+j)}X_n| + \sum_{j=t_n}^{\alpha M N_n} |{}^{(r_n+j)}X_n| + \sum_{\alpha M N_n+1}^{j_n} |{}^{(r_n+j)}X_n|\\ &= II_1 + II_2 + II_3.\end{aligned}$$

Now for large n by (4.1)

$$\begin{aligned} II_1 &\leqq (t_n - s_n - 1)\,|^{(r_n+s_n+1)}X_n| \\ &\leqq 2\varepsilon p_n |^{(r_n)}X_n| \\ &\leqq 2\varepsilon p_n \alpha_n(2) \\ &\leqq (1/4)\, p_n \alpha_n(\lambda_2) \end{aligned}$$

by (4.61) and (4.64). Since $\omega \in D_n$

$$\begin{aligned} II_2 &\leqq \alpha M N_n \alpha_n(\lambda_1) \\ &\leqq (1/4)\, N_n \alpha_n(\lambda_2) \end{aligned}$$

by (4.63). By (4.15) and (4.20)

$$\begin{aligned} II_3 &\leqq \sum_{\alpha M N_n+1}^{j_n} b_n(j) \\ &\leqq \sum_{k=M}^{[j_n/\alpha N_n]} \sum_{k\alpha N_n+1}^{(k+1)\alpha N_n} b_n(j) \\ &\leqq \alpha N_n \sum_{k=M}^{[j_n/\alpha N_n]} b_n(k\alpha N_n+1) \\ &\leqq \alpha N_n \sum_{k=M}^{[j_n/\alpha N_n]} 2^{-k} b_n(1) \\ &\leqq \alpha N_n\, 2^{-M+1}\, b_n(1) \\ &\leqq \alpha N_n\, 2^{-M+1}\, \alpha_n(2) \\ &\leqq (1/4)\, N_n \alpha_n(\lambda_2) \end{aligned}$$

by (4.11), (4.61) and (4.62). Thus for large n, if $\omega \in D_n$

(4.65) $$II \leqq (3/4)\, N_n \alpha_n(\lambda_2).$$

Next for $\omega \in D_n$, if $p_n = 0$ then

(4.66) $$\begin{aligned} I &= |^{(r_n)}X_n| \\ &\geqq \alpha_n(\lambda_2) \\ &= N_n \alpha_n(\lambda_2) \end{aligned}$$

while if $p_n \neq 0$ then $s_n \geqq 4$ and so

$$I = \left| \sum_{j=0}^{s_n-1} {}^{(r_n+j)}X_n\, 1({}^{(r_n+j)}X_n > 0) + \sum_{j=0}^{s_n-1} {}^{(r_n+j)}X_n\, 1({}^{(r_n+j)}X_n < 0) + {}^{(r_n+s_n)}X_n \right|.$$

Now since $\omega \in D_n$ we have that $\omega \in E_n^+ \cup E_n^-$. If $\omega \in E_n^+$ let $j_0 \in [0, s_n - 1]$ be such that ${}^{(r_n+j_0)}X_n > 0$. Then ${}^{(r_n+j_0)}X_n + {}^{(r_n+s_n)}X_n \geqq 0$ and so for large n by (4.1)

$$I \geqq ((1-\varepsilon)\, s_n - 1)\, \alpha_n(\lambda_2) - \varepsilon s_n\, \alpha_n(2). \tag{4.67}$$

If $\omega \in E_n^-$ then the analogous argument shows that (4.67) still holds. Thus by (4.61) and (4.64)

$$I \geqq ((3/4)\, s_n - 1)\, \alpha_n(\lambda_2). \tag{4.68}$$

Now if $p_n \neq 0$ then $(3/4)\, s_n - 1 \geqq N_n$ thus combining this with (4.66) gives that for large n, if $\omega \in D_n$

$$I \geqq N_n\, \alpha_n(\lambda_2). \tag{4.69}$$

Thus by (4.50), (4.65) and (4.69), for infinitely many n

$$\begin{aligned} |{}^{(r_n-1)}S_n - EU_n(d_n)| &\geqq (1/4 + o(1))\, N_n\, \alpha_n(\lambda_2) \\ &\geqq c N_n\, \alpha_n \end{aligned}$$

where $c > 0$ by (2.17) and this completes the proof of the lower bound. □

As an example assume that X is symmetric stable of index $\alpha \in (0, 2)$ with scale parameter chosen so that $G(x) \sim x^{-\alpha}$. If $r_n = o(l_2\, n)$ and $\liminf r_n (l_p\, n)^{-1} > 0$ for some $p \geqq 3$, then as mentioned in section 2 we may take $a_n = ((l n) \ldots (l_{p-1}\, n))^{-1}$ and so

$$\gamma_n = N_n\, n^{1/\alpha} \exp((l_2\, n + \ldots + l_p\, n - r_n\, l r_n + r_n)(\alpha r_n)^{-1}). \tag{4.70}$$

In particular if $r_n = [l_p n]$ for some $p \geqq 3$ then it's easy to see that

$$\gamma_n \sim e^{2/\alpha}\, n^{1/\alpha} (l_p n)^{-1/\alpha} \exp((l_2\, n + \ldots + l_{p-1}\, n)(\alpha r_n)^{-1}) \tag{4.71}$$

As we remarked in the introduction, the assumption of continuity on the distribution of X is not needed. The general case can be dealt with using the techniques described in [3]. In particular take for the definition of ${}^{(r)}S_n$ the one given in Sect. 6 of [3]. Next with $\tilde{G}$ given by (6.1) of [3], let $\tilde{\alpha}_n = \tilde{G}((n\beta_n)^{-1})$ and $\tilde{d}_n = \tilde{G}((l_2\, n/r_n)^p (n\beta_n)^{-1})$ where $p > 2$. Then Theorem 4.8 holds with $\tilde{\alpha}_n$ and $\tilde{d}_n$ replacing α_n and d_n respectively. The proof follows along the lines given here but the technical details are made more complicated.

5. Classical and Non-Classical L.I.L. Behaviour

We would like to explain a little further the remarks made in the introduction about the different ways in which the large values arise in (1.4) and (1.12). For simplicity assume that X is symmetric, else what we are really talking about is fluctuations of ${}^{(r_n)}S_n$ from some centering sequence. We also assume (2.1), so (1.4) and (1.12) both hold without need for centering.

If $r_n(l_2 n)^{-1} \to 0$ let N_n and α_n be given by (2.14) and (2.15) respectively. If $r_n(l_2 n)^{-1} \to \infty$ let $N_n = r_n$ and define α_n by $G(\alpha_n) = r_n n^{-1}$. Then as we have seen, in the first case, the large values of ${}^{(r_n)}X_n$ are comparable to α_n, and the large values of ${}^{(r_n-1)}S_n$ arise because infinitely often there are N_n terms comparable in size to α_n and these terms have the same sign. If $r_n(l_2 n)^{-1} \to \infty$, then by (4.1) of [3], we can again show that the large values of ${}^{(r_n)}X_n$ are comparable to α_n and again there are N_n terms comparable in size to α_n. However the correct normalization for ${}^{(r_n-1)}S_n$ (or ${}^{(r_n)}S_n$) in this case is not $N_n \alpha_n = r_n \alpha_n$, but $(r_n l_2 n)^{1/2} \alpha_n$. There are two things to notice about this. First, the minimal number of summands required to make ${}^{(r_n-1)}S_n$ as large as $(r_n l_2 n)^{1/2} \alpha_n$, is greater than $l_2 n$, more precisely there exists a sequence s_n such that $s_n(l_2 n)^{-1} \to \infty$ and $(|{}^{(r_n)}X_n| + \ldots + |{}^{(r_n+s_n)}X_n|) = o((r_n l_2 n)^{1/2} \alpha_n)$. Secondly, since there are r_n terms of size α_n, there needs to be a lot of cancellation amongst terms in order that $(r_n l_2 n)^{1/2} \alpha_n$ be the correct normalizer for ${}^{(r_n-1)}S_n$. Both of these properties are typical of classical L.I.L. behaviour. For example if $EX^2 < \infty$, one can show that there exists a sequence s_n, depending on X, such that $(l_2 n) = o(s_n)$ and $(|{}^{(1)}X_n| + \ldots + |{}^{(s_n)}X_n|) = o((n l_2 n)^{1/2})$. Furthermore, despite the paradoxical sounding nature of the statement, there has to be a lot of cancellation in order for S_n to take values of order $(n l_2 n)^{1/2}$. One way of expressing this for example, is that if t_n is any sequence such that

$$\limsup_{n\to\infty} \left(\sum_{i=1}^{t_n} {}^{(i)}X_n \right) (n l_2 n)^{-1/2} > 0$$

then

$$\limsup_{n\to\infty} \left(\sum_{i=1}^{t_n} |{}^{(i)}X_n| \right) (n l_2 n)^{-1/2} = \infty.$$

The idea that classical L.I.L. behaviour is due to many moderate summands rather than a few large summands is a common (though often well hidden) theme; see Klass [15] for a nice discussion.

The borderline case $r_n \approx l_2 n$ is not included in (1.12) but is included in (1.4). This might lead one to think of it as giving rise to classical L.I.L. behaviour. However it may be that the techniques used in this paper can be extended to cover this case. Notice that the two definitions of α_n do agree up to constants when $r_n \approx l_2 n$, since we may take $a_n = (ln)^{-1}$. Thus the large values in this case may arise in both ways!

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Appendix

Given a sequence of integers r_n increasing to infinity, we construct a sequence a_n satisfying (2.6)–(2.8). Let

$$\begin{aligned} n_1 &= \min\{n \geqq 10 : r_n = 1\} \\ m_1 &= \max\{n : (ln) \leqq (ln_1)^2\} \\ n_{k+1} &= \min\{n > m_k : r_n \neq r_{m_k}\} \\ m_{k+1} &= \max\{n : (ln) \leqq (ln_{k+1})^2\}. \end{aligned}$$

Clearly $n_k < m_k < n_{k+1}$ and since r_n is integer valued

(A 1) $$r_{n_k} \geqq k.$$

Define

$$a_n = \begin{cases} (l n_k)^{-2} & n_k \leqq n \leqq m_k \\ (l n)^{-2} & m_k < n \leqq n_{k+1}. \end{cases}$$

Clearly a_n satisfies (2.6) and (2.7). To check (2.8) first observe that

$$\begin{aligned} \sum_n a_n n^{-1} &\geqq \sum_k \left(\sum_{n_k}^{m_k} a_n n^{-1} \right) \\ &= \sum_k (l n_k)^{-2} \sum_{n_k}^{m_k} n^{-1} \\ &\geqq \sum_k (l n_k)^{-2} (l(m_k+1) - l n_k) \\ &\geqq \sum_k (l n_k)^{-2} ((l n_k)^2 - (l n_k)) \end{aligned}$$

which diverges. Next let $\varepsilon < 0$, then by (A 1)

$$\begin{aligned} \sum_n a_n n^{-1} e^{\varepsilon r_n} &\leqq \sum_k \left(\sum_{n_k}^{m_k} a_n n^{-1} e^{\varepsilon r_n} + \sum_{m_k+1}^{n_{k+1}} a_n n^{-1} e^{\varepsilon r_n} \right) \\ &\leqq \sum_k e^{\varepsilon k} \sum_{n_k}^{m_k} a_n n^{-1} + \sum_n (n (l n)^2)^{-1}. \end{aligned}$$

Now the latter series converges while

$$\begin{aligned} \sum_{n_k}^{m_k} a_n n^{-1} &= (l n_k)^{-2} \sum_{n_k}^{m_k} n^{-1} \\ &\leqq (l n_k)^{-2} l m_k \\ &\leqq 1, \end{aligned}$$

thus (2.8) holds.

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