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Probability Theory and Related Fields

Comparison of Location Models of Weibull Type Samples and Extreme Value Processes

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Summary. Four different location parameter models are compared within the sufficiency and deficiency concept. The starting is a location model of a Weibull type sample with shape parameter $-1<a<1$. Here our basic inequality concerns the approximate sufficiency of the k lower extremes. In addition, the lower extremes are approximately equal, in distribution, to $(S_m^{1/(1+a)}+t)_{m\leq k}$ where S_m is the sum of m i.i.d. standard exponential random variables and t is the location parameter. The final step leads us to the model of extreme value processes $(S_m^{1/(1+a)}+t)_{m=1,2,3\ldots}$

1. Introduction

Consider a location family P_t, $t\in\mathbb{R}$, with Lebesgue densities f_t given by $f_t(x)=f(x-t)$ where f is of Weibull type; that is, for some known value $a>-1$ we have

$$(1.1)\qquad f(x)=\begin{cases}x^a\,r(x) & \text{if } x>0\\ 0 & \text{if } x\leq 0\end{cases}$$

where r is a sufficiently regular function. Notice that we get Weibull densities if $r(x)=(1+a)\exp(-x^{1+a})$, and we get certain generalized Pareto densities if $r(x)=(1+a)\,1_{(0,1)}(x)$.

It is well known that the Fisher information is finite for Weibull densities if and only if $a>1$. In this case the sequence of product experiments $(\mathbb{R}^n, \mathbb{B}^n, (P^n_{n^{-1/2}t})_{t\in\mathbb{R}})$ is locally asymptotically normal (LAN). This result remains to hold for $a=1$ if $n^{-1/2}$ is replaced by $(n\log n)^{-1/2}$. Thus it is clear that for $a\geqq 1$ a fixed number of extreme order statistics asymptotically does not contain any information about the given experiments. The situation changes completely if $a=0$: It is well known that the sample minimum $X_{1:n}$ is sufficient under exponential distributions with unknown location parameter.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with common distribution P_t and let $X_{1:n}\leqq\ldots\leqq X_{n:n}$ denote the corresponding order statistics. It is well known that the order statistic $(X_{1:n}, \ldots, X_{n:n})$ is sufficient. In the present article we will reduce the number of order statistics to the $k(n)$ lower extremes $X_{1:n}, \ldots, X_{k(n):n}$ and will calculate bounds for the loss of information if $-1<a<1$.

These calculations will be carried out within the framework of deficiency of statistical experiments. Before giving a detailed outline of our central ideas we make a short comment about specific statistical procedures treated in literature.

It is necessary to distinguish between a parametric set-up (e.g., Polfeldt (1970) and the present paper) and a nonparametric one (e.g., Weiss (1971) and Hall (1982)). In the first case one evaluates the location parameter of the distribution. We refer to Polfeldt (1970) and Janssen (1988) for the treatment of estimation and testing procedures. Further references can be found in Hall (1982). In the second case one is only interested in the tail of the distribution which is characterized by functional parameters like the endpoint and the tail index a. If $a=0$ then the sample minimum properly centered is an asymptotically efficient estimator of the left endpoint. If $a \neq 0$ then the sample minimum still attains the optimal rate, however it is inefficient. The performance of estimators can be improved when several extremes are taken into account. Asymptotically efficient estimators are obtained if $k \equiv k(n)$ goes to infinity as the sample size n goes to infinity. In this context Weiss (1971) uses a quick estimator and Hall (1982) a "maximum likelihood estimator" of the endpoint. The variance of estimators decreases when $k(n)$ increases; our present results give some in sight in this relation. However, in the nonparametric set-up one also has to take into account the bias of the estimator leading to a different kind of problems.

We introduce the rescaled experiment

$$E_n = (\mathbb{R}^n, \mathbb{B}^n, (P^n_{\delta_n t})_{t \in \mathbb{R}}) \tag{1.2}$$

where δ_n will be specified below; we have $\delta_n = o(n^{-1/2})$. The second statistical experiment is

$$E_{n,k} = (\mathbb{R}^k, \mathbb{B}^k, (V_{t,k,n})_{t \in \mathbb{R}}) \tag{1.3}$$

for some $k \leq n$ where $V_{t,k,n}$ is the distribution of $\delta_n^{-1}(X_{1:n}, \ldots, X_{k:n})$ under $P^n_{\delta_n t}$; obviously,

$$V_{t,k,n} = \mathscr{L}(\delta_n^{-1}(X_{1:n}, \ldots, X_{k:n}) + t \mid P^n_0). \tag{1.4}$$

Observe that $E_{n,k}$ is less informative than E_n.

Finally we introduce the statistical experiments G_k and G which arise out of approximations to $E_{n,k}$. Let Y_i, $i \in \mathbb{N}$, be an i.i.d. sequence of standard exponential random variables and put

$$S_m = \sum_{i=1}^{m} Y_i. \tag{1.5}$$

Define

$$Q_{t,k} = \mathscr{L}((S_m^{1/(1+a)} + t)_{m \leq k}) \tag{1.6}$$

and

(1.7) $$Q_t = \mathscr{L}((S_m^{1/(1+a)} + t)_{m \in \mathbb{N}}).$$

Then,

(1.8) $$G_k = (\mathbb{R}^k, \mathbb{B}^k, (Q_{t,k})_{t \in \mathbb{R}})$$

and

(1.9) $$G = (\mathbb{R}^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}}, (Q_t)_{t \in \mathbb{R}}).$$

The comparison between the four different statistical experiments will be carried out according to the following diagram:

$$\begin{array}{ccc} E_n & \overset{(4.7)}{\longleftrightarrow} & E_{n,k} \\ (5.15)\updownarrow & & \updownarrow(5.8) \\ G & \underset{(5.4)}{\longleftrightarrow} & G_k \end{array}$$

Denote by $\| \ \|$ the variational distance between probability measures; that is $\|Q_0 - Q_1\| = \sup_A |Q_0(A) - Q_1(A)|$ where A varies over the measurable sets. Given two dominated statistical experiments $H_i = (\Omega_i, \mathscr{A}_i, (Q_{i,\theta})_{\theta \in \Theta})$ with Polish Borel spaces $(\Omega_i, \mathscr{A}_i)$ for $i = 0, 1$ define the deficiency

(1.10) $$\delta(H_1, H_0) = \inf_K \sup_{\theta \in \Theta} \|Q_{0,\theta} - K Q_{1,\theta}\|$$

where K ranges over all Markov kernels from $\mathscr{A}_1$ to $\mathscr{A}_0$ and $K Q$ is defined by

(1.11) $$K Q(A) = \int K(A, \cdot)\, dQ.$$

The symmetric deficiency of H_0 and H_1 is defined by

(1.12) $$\Delta(H_0, H_1) = \max\{\delta(H_0, H_1), \delta(H_1, H_0)\}.$$

For arbitrary experiments the definition of $\Delta(\cdot, \cdot)$ can be found in Strasser (1985), p. 296.

If

(1.13) $$\|V_{0,k,n} - Q_{0,k}\| \to 0 \quad \text{as} \quad n \to \infty$$

then it is clear that

(1.14) $$\Delta(E_{n,k}, G_k) \to 0 \quad \text{as} \quad n \to \infty.$$

It is well known that a von Mises condition is sufficient for (1.13) (see Falk (1985) or Sweeting (1985)). The von Mises condition is equivalent to the condition that r is slowly varying at zero; i.e.

$$r(x\,t)/r(t) \to 1 \quad \text{as} \quad t \downarrow 0$$

for each $x > 0$.

Next we examine restrictions of experiments to compact sets. W.l.g. we can assume that the compact set is equal to the interval $[0, s]$. We will write Δ_s to indicate that $\Theta = [0, s]$.

It was pointed out in Janssen (1988) that

$$\Delta_s(E_n, G) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.15}$$

Remember that $E_{n,k}$ is less informative than E_n. However, from Janssen (1988) we know that in addition to (1.15)

$$\Delta_s(E_{n,k(n)}, G) \to 0 \quad \text{as} \quad n \to \infty \tag{1.16}$$

whenever $k(n) \leq n$ and $k(n) \to \infty$ as $n \to \infty$. Combining (1.15) and (1.16) we also get

$$\Delta_s(E_{n,k(n)}, E_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.17}$$

Thus the lower extremes are asymptotical sufficient if $k(n) \to \infty$ as $n \to \infty$. Rates in (1.17) can be deduced from our basic Theorem (2.8).

Next we give some further remarks concerning the literature. Under strong regularity conditions the limiting behaviour of likelihood processes in the case of densities with singularities was investigated in Ibragimov and Has'minskii (1981), Chap. V and VI (compare these results with Janssen (1988)). In connection with nonparametric models the approximate sufficiency of sparse order statistics was studied for instance by Weiss (1980) and Reiss (1986). For the background concerning the comparison of statistical experiments we refer to Le Cam (1986), Strasser (1985) and Torgerson (1976).

The paper is organized as follows. In Sect. 2 we establish an upper bound of $\Delta_s(E_{n,k}, E_n)$. A careful study of the asymptotic behaviour of the upper bound reveals that the bound essentially consists of two parts. The first part heavily depends on the number k of order statistics under consideration. The second part expresses the dependence on the sample sizes n. This suggests to examine these terms separately in the Sect. 3 and 4. The comparison to the models G_k and G is carried out in Sect. 5.

2. The Basic Inequality

Let $-1 < a < 1$ and let P_t again be defined via a density f of the form (1.1) and the location parameter t. Moreover, $X_{1:n}, \ldots, X_{n:n}$ are the corresponding order statistics of a sample of size n. Denote by

$$K_t^{(n,k)}(\cdot \mid x) \tag{2.1}$$

the conditional distribution of $(X_{1:n}, \ldots, X_{n:n})$ given $(X_{1:n}, \ldots, X_{k:n})=x$ $:=(x_1, \ldots, x_k)$ under the parameter t. It seems to us that the Markov kernel $K_0^{(n,k)}$ is appropriate to calculate an upper bound for the deficiency of E_n and $E_{n,k}$ as defined in Sect. 1. A moments reflection shows that it suffices to establish an upper bound of

$$\|\mathscr{L}((X_{1:n}, \ldots, X_{n:n})|P_t^n)-K_0^{(n,k)}\mathscr{L}((X_{1:n}, \ldots, X_{k:n})|P_t^n)\|$$

for some fixed parameter t. The upper bound will depend on three auxiliary functions h, g and ψ: First define the L^2-function (w.r.t. Lebesgue measure) h by

$$h(x)=(x-1)^{a/2}\,1_{(1,\infty)}(x)-x^{a/2}\,1_{(0,\infty)}(x). \tag{2.2}$$

Then for every $t>0$ with $r(t)>0$ define

$$g(t)=\left\|\left(\frac{f^{1/2}(t(\cdot-1))-f^{1/2}(t\cdot)}{r^{1/2}(t)\,t^{a/2}}-h(\cdot)\right)1_{(1,\infty)}(\cdot)\right\|_2 \tag{2.3}$$

where $\|\cdot\|_2$ denotes the L^2-norm. Moreover,

$$\psi(y)=\left(\int_{1+y}^{\infty} h^2(x)\,dx\right)^{1/2} \quad \text{for } y\geqq 0. \tag{2.4}$$

It is immediate that ψ is a decreasing, continuous function.

Before formulating our basic inequality we indicate the relation of the auxiliary function g to conditions and arguments as previously given in Janssen (1988) and Janssen et al. (1988), § 10. Assume that the function r is positive on the interval $(0, x_0)$ and slowly varying at zero. Obviously,

$$\frac{f^{1/2}(t(x-1))-f^{1/2}(tx)}{r^{1/2}(t)\,t^{a/2}}\to h(x) \quad \text{as} \quad t\downarrow 0 \tag{2.5}$$

pointwise for $x>0$. If the L^2-convergence in (2.5) is valid then $g(t)\to 0$ as $t\downarrow 0$ and vice versa since

$$[f^{1/2}(tx)/(r^{1/2}(t)\,t^{a/2})-x^{a/2}]\,1_{(0,1)}(x)\to 0 \quad \text{as} \quad t\downarrow 0$$

in L^2. Subsequently we will find mild conditions which ensure that the L^2-convergence in (2.5) holds. Assume that

$$\int_{\varepsilon}^{\infty}(f^{1/2}(x-t)-f^{1/2}(x))^2\,dx=o(t^{1+a}\,r(t)) \quad \text{as} \quad t\downarrow 0 \tag{2.6)(i}$$

for each $\varepsilon>0$. Secondly, the function $r(x)$ splits in two non-negative functions $r(x)=r_1(x)\;l(x)$, where r_1 is absolutely continuous on $(0, x_0)$, slowly varying at zero with

$$x(\log r_1(x))'\to 0 \quad \text{as} \quad x\downarrow 0. \tag{2.6)(ii}$$

In addition, let $l(x)$ denote a continuous function on $[0, x_0]$ with $l(0)>0$ and

$$\text{(2.6)(iii)} \qquad \int_0^{x_0/2} (l^{1/2}(x+t)-l^{1/2}(x))^2\, r_1(x)\, x^a\, dx = o(t^{1+a}\, r(t))$$

as $t \downarrow 0$.
We remark that (2.6) (i) and (2.6) (iii) are satisfied whenever f and l are absolutely continuous on $(0, \infty)$ and $[0, x_0]$ respectively, and

$$\text{(2.7)} \qquad \int_\varepsilon^\infty \frac{|f'(x)|^\lambda}{f(x)^{\lambda-1}}\, dx + \int_0^{x_0} \left|\frac{l'(x)}{l(x)}\right|^\lambda f(x)\, dx < \infty$$

for some $\lambda \in (1+a, 2] \cap [1, 2]$ and each $\varepsilon > 0$. These implications can easily be checked by making use of arguments of Ibragimov and Has'minskii (1981), p. 282.

(2.8) **Theorem.** *For $k \in \{1, \ldots, n\}$, $t \geqq 0$ and $\delta > 0$ such that $k/n \leqq F(\delta) < F(\delta+t) < 1$ the following inequality holds:*

$$\text{(2.9)} \qquad \|\mathscr{L}((X_{1:n}, \ldots, X_{n:n})|P_t^n) - K_0^{(n,k)}\, \mathscr{L}(X_{1:n}, \ldots, X_{k:n})|P_t^n)\|$$
$$\leqq (n-k)^{1/2} \{t^{\frac{1+a}{2}}\, r^{1/2}(t)(1-F(\delta+t))^{-1/2} \left[g(t) + E_{P_0^n}\, \psi\left(\frac{1}{t} X_{k:n}\right)\right]$$
$$+\sqrt{2} \exp\left(-n\left(F(\delta)-\frac{k}{n}\right)^2/3\right)\Big\}.$$

Before turning to the proof of Theorem (2.8) we give an upper and lower bound for the term involving the function ψ. If $a=0$ then $\psi=0$ whence the term $E_{P_0^n}\, \psi\left(\frac{1}{t} X_{k:n}\right)$ is equal to zero. In general, this term can be expressed by moments of $X_{k:n}$. We have

$$\text{(2.10)} \qquad E_{P_0^n}\, \psi\left(\frac{1}{t} X_{k:n}\right) \begin{cases} \geqq \left(|a|\, t^{\frac{1-a}{2}}/(2(1-a)^{1/2})\right) E_{P_0^n}\left([1+X_{k:n}]^{\frac{a-1}{2}}\right) \\ \leqq \left(|a|\, t^{\frac{1-a}{2}}/(2(1-a)^{1/2})\right) E_{P_0^n}\left(X_{k:n}^{\frac{a-1}{2}}\right). \end{cases}$$

To prove (2.10) for $a \neq 0$ apply the mean value theorem and verify that

$$\psi(y) \leqq \left(\int_{1+y}^\infty \left(\frac{a}{2}\right)^2 (x-1)^{a-2}\, dx\right)^{1/2} = \frac{|a|}{2(1-a)^{1/2}}\, y^{\frac{a-1}{2}}.$$

Calculate the lower bound in an analogous way.

The proof of Theorem (2.8) is split up into several steps.

The first auxiliary lemma concerns an arbitrary family P_t, $t \geqq 0$, of probability measures with distribution functions F_t and Lebesgue densities f_t. The conditional distribution $K_t^{(n,k)}$ is defined as in (2.1). We know from Reiss (1986) that

$K_t^{(n,k)}(\cdot\,|\,x)$ is the product of the k Dirac measures at $x_1, \ldots, x_k$ and of the distribution of the order statistic $(X_{1:n-k}, \ldots, X_{n-k:n-k})$ of a sample of size $n-k$ under P_{t,x_k} which is P_t truncated on the left at x_k. W.l.g. we may assume that $F_t(x_k)<1$. Then P_{t,x_k} has the Lebesgue density

$$f_{t,x_k} = f_t/(1-F_t(x_k))\, 1_{[x_k,\infty)}. \tag{2.11}$$

Denote by $d(P,Q)$ the Hellinger distance between the probability measures P and Q; that is

$$d(P,Q) = \left(\frac{1}{2}\int\left[\left(\frac{dP}{d(P+Q)}\right)^{1/2} - \left(\frac{dQ}{d(P+Q)}\right)^{1/2}\right]^2 d(P+Q)\right)^{1/2}. \tag{2.12}$$

Remember that

$$\|P-Q\| \leqq \sqrt{2}\, d(P,Q) \leqq \sqrt{2}. \tag{2.13}$$

The following lemma is in the spirit of the Theorem in Reiss (1986).

(2.14) **Lemma.**

$$\begin{aligned} &\|\mathscr{L}((X_{1:n}, \ldots, X_{n:n})\,|\,P_t^n) - K_0^{(n,k)}\, \mathscr{L}((X_{1:n}, \ldots, X_{k:n})\,|\,P_t^n)\| \\ &\leqq \sqrt{2}(n-k)^{1/2} \int H(P_{t,x_k}, P_{0,x_k})\, d\mathscr{L}(X_{k:n}\,|\,P_t^n)(x_k) \end{aligned} \tag{2.15}$$

where by definition

$$H(P_{t,x_k}, P_{0,x_k}) = \begin{cases} 1 & \text{if} \quad F_t(x_k)=1 \\ d(P_{t,x_k}, P_{0,x_k}) & \quad\;\; \text{otherwise.} \end{cases}$$

Proof. Notice that the left-hand side of (2.15) equals

$$\begin{aligned} &\sup_{B\in\mathbb{B}^n} \left|\int [K_t^{(n,k)}(B\,|\,x) - K_0^{(n,k)}(B\,|\,x)]\, d\mathscr{L}((X_{1:n}, \ldots, X_{k:n})\,|\,P_t^n)(x)\right| \\ &\leqq \int \sup_{B\in\mathbb{B}^n} |K_t^{(n,k)}(B\,|\,x) - K_0^{(n,k)}(B\,|\,x)|\, d\mathscr{L}((X_{1:n}, \ldots, X_{k:n})\,|\,P_t^n)(x) \\ &= \int \|P_{t,x_k}^{n-k} - P_{0,x_k}^{n-k}\|\, d\mathscr{L}((X_{1:n}, \ldots, X_{k:n})\,|\,P_t^n)(x). \end{aligned} \tag{2.16}$$

It is well known that for two probability measures P and Q the subsequent inequality holds for each positive integer $m\in\mathbb{N}$.

$$\|P^m - Q^m\| \leqq \sqrt{2m}\, d(P,Q). \tag{2.17}$$

Thus the inequality (2.15) is a consequence of (2.16) and (2.17).

Next we formulate a large deviation result for order statistics.

(2.18) **Lemma.** *Let F be the underlying distribution function and δ such that $F(\delta) \geqq k/n$. Then*

$$P\{X_{k:n} > \delta\} \leqq \exp\left(-n\left(F(\delta) - \frac{k}{n}\right)^2 / 3\right]. \tag{2.19}$$

Proof. Lemma 3.3 in Reiss (1981 a) and the quantile transformation yield

$$P\{X_{k:n}>\delta\}\leqq\exp\left[\frac{-n(F(\delta)-\mu)^2}{3(\sigma^2+(F(\delta)-\mu))}\right]$$

where $\mu=k/(n+1)$ and $\sigma^2=\mu(1-\mu)$. Now (2.19) is immediate.

We are in the proper position to establish the

Proof of Theorem (2.8). Since $(P_t)_{t\in\mathbb{R}}$ is a location family Lemma (2.14) implies that the left-hand side of (2.9) is less than or equal to

$$\begin{aligned}(2.20)\quad &(n-k)^{1/2}\sqrt{2}\int H(P_{t,t+x_k},P_{0,t+x_k})\,d\mathscr{L}(X_{k:n}|P_0^n)(x_k)\\ &\leqq(n-k)^{1/2}\left\{\int_0^\delta\left[\int(f_{t,t+x_k}^{1/2}(x)-f_{0,t+x_k}^{1/2}(x))^2\,dx\right]^{1/2}d\mathscr{L}(X_{k:n}|P_0^n)(x_k)\right.\\ &\quad\left.+\sqrt{2}\,P(\{X_{k:n}>\delta\})\right\}\end{aligned}$$

where the second step is achieved by integration over $[0,\delta]$ and (δ,∞). Lemma (2.18) gives an upper bound of $P\{X_{k:n}>\delta\}$.

Since $x+y\geqq2\sqrt{xy}$ for $x,y\geqq0$ we obtain for $x_k<\delta$

$$\begin{aligned}(2.21)\quad &\int(f_{t,t+x_k}^{1/2}(x)-f_{0,t+x_k}^{1/2}(x))^2\,dx\\ &=2\left[1-\langle(1-F(x_k+t))(1-F(x_k))\rangle^{-1/2}\int_{x_k+t}^{\infty}f^{1/2}(x-t)f^{1/2}(x)\,dx\right]\\ &=2\left[1-\langle(1-F(x_k+t))(1-F(x_k))\rangle^{-1/2}\{(1-F(x_k+t)+1-F(x_k))/2\right.\\ &\quad\left.-\frac{1}{2}\int_{x_k+t}^{\infty}(f^{1/2}(x-t)-f^{1/2}(x))^2\,dx\}\right]\\ &\leqq(1-F(\delta+t))^{-1}\int_{x_k+t}^{\infty}(f^{1/2}(x-t)-f^{1/2}(x))^2\,dx\\ &=(1-F(\delta+t))^{-1}\,t\int_{1+x_k/t}^{\infty}(f^{1/2}(t(x-1))-f^{1/2}(tx))^2\,dx.\end{aligned}$$

According to the definition of g and h

$$\begin{aligned}(2.22)\quad &\|(f^{1/2}(t(\cdot-1))-f^{1/2}(t\cdot))1_{(1+x_k/t,\infty)}\|_2\\ &\leqq r^{1/2}(t)\,t^{a/2}[\|h\,1_{(1+x_k/t,\infty)}\|_2+g(t)].\end{aligned}$$

Combining (2.20)–(2.22) and taking into account the definition of ψ the proof is completed.

Further insight into the right-hand side of our basic inequality (2.9) will be achieved by the asymptotic considerations made in the next section.

3. The Asymptotic Information Contained in k Smallest Order Statistics

Theorem (2.8) will be applied to sequences $\delta_n t$ of location parameters as $n \to \infty$ where $\delta_n = F^{-1}\left(\frac{1}{n}\right)$ and $t \geqq 0$. The results of the present section will be proved under particular mild conditions on the underlying density f which is of the form (1.1) with $-1 < a < 1$. Hereafter we assume that the function r is positive on some interval $(0, x_0)$ and slowly varying at zero. Moreover assume that in (2.5) the L^2-convergence holds.

Note that

$$\delta_n = n^{-1/(1+a)} L\left(\frac{1}{n}\right) \tag{3.1}$$

where L is a further function which is slowly varying at zero. The left-hand side in (2.9) will be denoted by $\rho(n, k, t)$; thus we have

$$\rho(n, k, \delta_n t) = \| \mathscr{L}((X_{1:n}, \ldots, X_{n:n}) | P^n_{\delta_n t}) - K_0^{(n,k)} \mathscr{L}((X_{1:n}, \ldots, X_{k:n}) | P^n_{\delta_n t}) \|. \tag{3.2}$$

Notice that $\sup\limits_{0 \leqq t \leqq s} \rho(n, k, \delta_n t)$ is an upper bound of the symmetric deficiency $\Delta_s(E_n, E_{n,k})$.

(3.3) **Theorem.** *The following two inequalities hold:*

$$\lim_{n \in \mathbb{N}} \sup_{0 \leqq t \leqq s} \rho(n, k, \delta_n t) \leqq s^{(1+a)/2} (1+a)^{1/2} E\psi\left(\frac{1}{s} S_k^{1/(1+a)}\right) \tag{3.4}$$

and

$$E\psi\left(\frac{1}{s} S_k^{1/(1+a)}\right) \leqq \frac{|a| \, s^{(1-a)/2}}{2(1-a)^{1/2}} E(S_k^{(a-1)/(2(1+a))}). \tag{3.5}$$

(3.4) and (3.5) will be combined with the inequality

$$k^{-p} < E(S_k^{-p}) \leqq (k-1-k_0)^{-p} \tag{3.6}$$

for $k > p > 0$ where k_0 is defined by

$$k_0 = \begin{cases} [p] & \text{if } p \notin \mathbb{N} \\ p-1 & \text{if } p \in \mathbb{N} \end{cases}.$$

Concerning the proof of (3.6) we refer to Janssen et al. (1988), Lemma (5.2). Now it is immediate that

$$\lim_{n \in \mathbb{N}} \sup_{0 \leqq t \leqq s} \rho(n, k, \delta_n t) = O(k^{(a-1)/(2(1+a))}) \tag{3.7}$$

and, since ρ is monotone decreasing in k, we have

$$\lim_{n\in\mathbb{N}} \lim_{0\leq t\leq s} \rho(n, k(n), \delta_n t) = 0 \tag{3.8}$$

if $k(n) \leq n$ satisfies $k(n) \to \infty$ as $n \to \infty$.

Theorem (3.3) will be the decisive tool to establish an upper bound of $\Delta_s(G, G_k)$.

Proof of Theorem (3.3). Fix $\delta > 0$ and note that

$$(1+a)\, F(x) \sim x\, f(x) = x^{a+1}\, r(x) \tag{3.9}$$

as $x \downarrow 0$. Since $F(F^{-1}(x) = x$ we get

$$(\delta_n t)^{1+a}\, r(\delta_n t) \sim (1+a)\, t^{1+a}/n \tag{3.10}$$

as $n \to \infty$. Recall from Seneta (1976) that

$$r(\delta_n t)/r(\delta_n) \to 1 \quad \text{as} \quad n \to \infty \tag{3.11}$$

uniformly in t on compact sets $D \subset (0, \infty)$. Moreover we make use of

$$t_n^{1+a}\, r(\delta_n t_n)/r(\delta_n) \to 0 \quad \text{as} \quad t_n \downarrow 0 \tag{3.12}$$

which is a well-known property of slowly varying functions. Combining (3.10)–(3.12),

$$\lim_{n\in\mathbb{N}} \sup_{0\leq t\leq s} (n-k)^{1/2} (\delta_n t)^{(1+a)/2}\, r^{1/2}(\delta_n t) = s^{(1+a)/2} (1+a)^{1/2}. \tag{3.13}$$

From (1.13) we conclude that for fixed k

$$E_{P_0^n}\, \psi\left(\frac{1}{\delta_n t} X_{k:n}\right) \to E\psi\left(\frac{1}{t} S_k^{1/(1+a)}\right) \quad \text{as} \quad n \to \infty. \tag{3.14}$$

From the discussion in Sect. 2, (2.5), we know that

$$g(t) \to 0 \quad \text{as} \quad t \downarrow 0. \tag{3.15}$$

Inequality (3.4) is immediate from (3.13)–(3.15) and the basic inequality (2.9).

The proof of (3.5) is analogous to that of (2.10).

Notice that the right-hand side of (3.4) is equal to zero for $k=1$ if $a=0$. Thus the sample minimum is asymptotically sufficient under our present conditions if the shape parameter a is equal to zero.

Under the present conditions we are also able to prove the following inequality which will further be pursued in Sect. 4.

(3.16) **Lemma.** *Let* $0 < \lambda < 1$ *and* $\varepsilon > 0$. *There exists a constant* $C > 0$ *such that*

$$\sup_{0\leq t\leq s} \rho(n, k, \delta_n t) \leq C\left[k^{(a-1)/(2(1+a))+\varepsilon} + \sup_{0\leq t\leq s} g(t\delta_n)\right] \tag{3.17}$$

for all $n \in \mathbb{N}$ *and* $k \leqq n\lambda$. *Moreover*, (3.17) *holds with* $\varepsilon = 0$ *if, in addition, the slowly varying function r satisfies the condition*

$$r(x) \to c \in (0, \infty) \quad \text{as} \quad x \downarrow 0. \tag{3.18}$$

Proof. Throughout $C > 0$ denotes a generic constant which does not depend on n and $k \leqq n\lambda$. According to (3.13),

$$\sup_{n \in \mathbb{N}} \sup_{0 \leqq t \leqq s} n^{1/2} (\delta_n t)^{(1+a)/2} r^{1/2}(\delta_n t) \leqq C. \tag{3.19}$$

Inequality (2.10) yields,

$$\sup_{0 < t \leqq s} E_{P_0^n} \psi\left(\frac{1}{\delta_n t} X_{k:n}\right) \leqq C\, \delta_n^{(1-a)/2} E_{P_0^n}(X_{k:n}^{(a-1)/2}). \tag{3.20}$$

Write $X_{k:n} = F^{-1}(U_{k:n})$ where $U_{k:n}$ is the kth order statistic of i.i.d. (0, 1)-uniformly distributed random variables. Note that

$$H(\eta) = F^{-1}(\eta)^{(a-1)/2} \tag{3.21}$$

is regular varying at zero with the index of variation $(a-1)/(2(1+a))$. From Janssen et al. (1988), Lemma (5.2), we recall that

$$E\left(H(U_{k:n})/H\left(\frac{1}{n}\right)\right) \leqq C\, k^{(a-1)/(2(1+a))+\varepsilon} \tag{3.22}$$

for every $n \in \mathbb{N}$ and $k_0 \leqq k \leqq n$ with k_0 being fixed. In addition, the term ε can be omitted if $H(\eta) \sim c\,\eta^{(a-1)/(2(1+a))}$ as $\eta \downarrow 0$.

Since $\delta_n^{(1-a)/2} = 1/H(1/n)$ the assertions (3.20) and (3.22) yield

$$\sup_{0 \leqq t \leqq s} E_{P_0^n} \psi\left(\frac{1}{\delta_n t} X_{k:n}\right) \leqq C\, k^{(a-1)/(2(1+a))+\varepsilon}. \tag{3.23}$$

Moreover notice that for each $k \leqq \lambda n$

$$\exp(-n(F(\delta) - k/n)^2/3) \leqq \exp(-k(F(\delta) - \lambda)^2/3\lambda). \tag{3.24}$$

Next we choose $\delta > 0$ such that $\lambda < F(\delta) < 1$. If n is large enough we may apply the inequality (2.9). If we take now (3.19), (3.23) and (3.24) into account we obtain (3.17) where C is a constant depending only on ε and λ. In addition we see that under (3.18) we may choose $\varepsilon = 0$.

(3.25) *Remark.* If $a = 0$ then we find a constant $d > 0$ such that Lemma (3.16) holds with (3.17) replaced by

$$\sup_{0 \leqq t \leqq s} \rho(n, k, \delta_n t) \leqq C[\exp(-dn) + \sup_{0 \leqq t \leqq s} g(t\delta_n)]. \tag{3.26}$$

The proof of (3.26) runs along the lines of the preceding proof if we observe that $\psi \equiv 0$ if $a = 0$.

Our present method also enables us to treat the following problem. Let f be a density of the form (1.1) with support $[0, b]$ having a second singularity at the upper endpoint of the distribution such that f is absolutely continuous inside (x_1, b), $0<x_1<b$, and

$$\text{(3.27) (i)} \qquad \lim_{x\downarrow 0} \frac{f(b-x)}{f(x)} = c\in(0,\infty),$$

$$\text{(3.27) (ii)} \qquad x(\log r(x))' + (b-x)(\log(f(b-x)(b-x)^{-a}))' \to 0 \quad \text{as} \quad x\downarrow 0.$$

Replace condition (2.6) (i) by

$$\text{(3.27) (iii)} \qquad \int_{\varepsilon}^{b-\varepsilon} (f^{1/2}(x-t) - f^{1/2}(x))^2\, dx = o(t^{1+a}\, r(t))$$

for each $\varepsilon>0$. Then the lower and upper extremes $X_{1:n}, \ldots, X_{k(n,1):n}$, $X_{n-k(n,2):n}, \ldots, X_{n:n}$ are asymptotically sufficient in the sense of this section whenever $\min\{k(n,1), k(n,2)\} \to \infty$ as $n\to\infty$ whenever $-1<a<1$. Thus we are able to extend and to strengthen results of Weiss (1979) who only considered the case $a=0$. The details are omitted in order not to overload the present paper.

4. The Rate of Convergence When n Tends to Infinity

The upper bound (3.17) of $\sup_{0\le t\le s} \rho(n, k, \delta_n t)$ involves the term $\sup_{0\le t\le s} g(t\delta_n)$ which converges to zero as $n\to\infty$. In the present section we will establish the rate of convergence of $\sup_{0\le t\le s} g(t\delta_n)$.

Throughout this section we assume that the function r in (1.1) is bounded and can be written in the form

$$\text{(4.1)} \qquad r(x) = c\, e^{\tilde{h}(x)} \quad \text{for } 0<x<x_0$$

where $c\in(0,\infty)$ and $\tilde{h}$ satisfies the condition

$$\text{(4.2)} \qquad |\tilde{h}(x)| \leqq L x^{\gamma}$$

for some constant $L>0$ and $\gamma>0$.

Notice that in the case of the Weibull density with shape parameter a we have $\tilde{h}(x) = -x^{1+a}$ and thus (4.2) is satisfied with $\gamma = 1+a$.

(4.1) implies that

$$\text{(4.3)} \qquad r(x)\to c \quad \text{as} \quad x\downarrow 0$$

Under condition (4.3) it is convenient to replace the rate of local alternatives δ_n by

(4.4) $$\tilde{\delta}_n = n^{-1/(1+a)}.$$

We remark that $\tilde{\delta}_n \sim (c/(1+a))^{1/(1+a)}\,\delta_n$.

(4.5) **Theorem.** *Let* $-1<a<1$ *and* $a \neq 0$. *Assume in addition to condition* (4.1) *that* f *is absolutely continuous inside* $(0, \infty)$, *and that*

(4.6) $$\int_0^\infty \frac{(r'(x))^2}{r(x)}\, x^a\, dx < \infty.$$

Then, for every $\lambda \in (0, 1)$ *there exists a constant* $C>0$ *such that for every positive integer* n *and* $k \leqq n\lambda$ *the following inequality holds:*

(4.7) $$\sup_{0 \leqq t \leqq s} \rho(n, k, t\, n^{-1/(1+a)}) \leqq C\,[k^{(a-1)/(2(1+a))} + n^{\max\{a-1,\,-2\gamma\}/(2(1+a))}].$$

Proof. Define

(4.8) $$\tilde{g}(t) = \left\| \left(\frac{f^{1/2}(t(\cdot-1)) - f^{1/2}(t\cdot)}{t^{a/2}} - c^{1/2}\, h \right) 1_{(1,\infty)} \right\|_2$$

with h as in (2.2). We remark that (3.17) holds with g and δ_n replaced by $\tilde{g}$ and $\tilde{\delta}_n$. Thus (4.7) is valid if

(4.9) $$\tilde{g}(t) = O(t^{\min\{1-a,\, 2\gamma\}/2}).$$

Assume first that $a>0$. Then,

(4.10)
$$\begin{aligned}
\tilde{g}(t) &= \left(\int_1^\infty [(x-1)^{a/2}\, r^{1/2}(t(x-1)) - x^{a/2}\, r^{1/2}\, t x) - c^{1/2}\{(x-1)^{a/2} - x^{a/2}\}]^2\, dx \right)^{1/2} \\
&\leqq \left(\int_1^\infty [(x-1)^{a/2} (r^{1/2}(t(x-1)) - r^{1/2}(t x))]^2\, dx \right)^{1/2} \\
&\quad + \left(\int_1^\infty [(r^{1/2}(t x) - c^{1/2})((x-1)^{a/2} - x^{a/2})]^2\, dx \right)^{1/2} \\
&=: g_1(t) + g_2(t).
\end{aligned}$$

Note that by Fubini's theorem

(4.11)
$$\begin{aligned}
g_1^2(t) &= \int_0^\infty x^a [r^{1/2}(t x) - r^{1/2}(t x + t)]^2\, dx \\
&= t^{-1-a} \int_0^\infty y^a \left(\int_y^{y+t} \frac{r'(\tau)}{2 r^{1/2}(\tau)}\, d\tau \right)^2 dy \\
&\leqq \frac{t^{-1-a}}{4} \int_0^\infty y^a\, t \int_0^\infty \frac{(r'(\tau))^2}{r(\tau)}\, 1_{[y, y+t]}(\tau)\, d\tau\, dy \\
&\leqq \frac{t^{1-a}}{4} \int_0^\infty \frac{(r'(\tau))^2}{r(\tau)}\, t^a\, d\tau
\end{aligned}$$

since a is non-negative. Condition (4.2) yields

$$|r^{1/2}(x)-c^{1/2}|\leqq L' x^{\gamma} \tag{4.12}$$

for some $L'>0$ whenever $x>0$. On the other hand observe that

$$[(x-1)^{a/2}-x^{a/2}]^2 \, 1_{(1,\infty)}(x) \leqq (x-1)^a \, 1_{\left[1,1+\frac{|a|}{2}\right]}(x)+\left(\frac{a}{2}\right)^2 \cdot (x-1)^{a-2} \, 1_{\left(1+\frac{|a|}{2},\infty\right)}(x). \tag{4.13}$$

Let us introduce

$$h(x,t)=[(r^{1/2}(tx)-c^{1/2})((x-1)^{a/2}-x^{a/2})]^2.$$

If we take (4.12) and (4.13) into account we obtain for $a>0$

$$\int_1^{1+\frac{|a|}{2}} h(x,t)\,dx=O(t^{2\gamma}) \tag{4.14}$$

and

$$\int_{1+\frac{|a|}{2}}^{1+1/t} h(x,t)\,dx=t^{2\gamma} K' \int_{1+\frac{|a|}{2}}^{1+1/t} (x-1)^{a-2+2\gamma}\,dx = O(t^{\min(2\gamma,1-a)}). \tag{4.15}$$

The boundedness of $r^{1/2}(x)$ yields

$$\int_{1+1/t}^{\infty} h(x,t)\,dx \leqq C\,\psi^2\left(\frac{1}{t}\right)=O(t^{1-a}). \tag{4.16}$$

Thus, (4.9) is proved for $a>0$. The case $a<0$ can be treated similarly. Instead of (4.10) we use the identity

$$\begin{aligned}&(x-1)^{a/2}\,r^{1/2}(t(x-1))-x^{a/2}\,r^{1/2}(tx)-c^{1/2}\{(x-1)^{a/2}-x^{a/2}\}\\&=x^{a/2}[r^{1/2}(t(x-1))-r^{1/2}(tx)]\\&+[r^{1/2}(t(x-1))-c][(x-1)^{a/2}-x^{a/2}].\end{aligned}$$

Then the proof carries over and can easily be completed.

We remark that for $a<0$ condition (4.6) implies that r is bounded since (4.6) implies that $\int_0^{\infty} |r'(x)|\,x^a\,dx<\infty$.

The case of $a=0$ will be treated in Theorem (4.22).

(4.17) *Example.* Let $f(x)=(1+a)\, x^a \exp(-x^{1+a})\, 1_{(0,\infty)}(x)$ be the Weibull density for $a\in(-1,0)\cup(0,1)$. Then for some $C>0$

$$\sup_{0\leqq t\leqq s} \rho(n,k,t\,n^{-1/(1+a)}) \leqq \begin{cases} C\,[k^{(a-1)/(2(1+a))}+n^{(a-1)/(2(1+a))}] & \text{if } a \geqq -1/3 \\ C\,[k^{(a-1)/(2(1+a))}+n^{-1}] & \text{if } a < -1/3 \end{cases} \tag{4.18}$$

whenever $k\leqq \lambda n$ with $\lambda<1$.

The second example concerns the generalized Pareto densities for $a<0$. Notice that for $a\geqq 0$ there is another jump at the left-hand side of the range of the distribution so that we have to include the upper extremes into our considerations.

(4.19) *Example.* Consider the generalized Pareto distribution with the density

$$f(x)=(1+a)\, x^a\, 1_{(0,1)}(x) \tag{4.20}$$

for $-1<a<0$. Then f is not absolutely continuous inside $(0,\infty)$ but (2.5) still holds. We obtain

$$\sup_{0\leqq t\leqq s} \rho(n,k,t\,n^{-1/(1+a)}) \leqq C(k^{(a-1)/(2(1+a))}+n^{a/(2(1+a))}) \tag{4.21}$$

whenever $k\leqq \lambda\, n$ with $\lambda<1$. Notice that $\tilde{g}(t)\sim d\, t^{-a/2}$ as $t\downarrow 0$ for some $d>0$ where $\tilde{g}$ is defined in (4.8).

(4.22) **Theorem.** *Assume that for $a=0$ f is an arbitrary density of the form* (1.1) *such that* (4.3) *holds for some $c>0$. In addition let f be absolutely continuous inside* $(0,\infty)$ *and*

$$\int_0^\infty (|f'(x)|^\eta / f(x)^{\eta-1})\, dx<\infty \tag{4.23}$$

for some $\eta\in(1,2]$. Then, for every $\lambda\in(0,1)$ there exists a constant $C>0$ such that for every positive integer n and $k\leqq n\,\lambda$ the following inequality holds:

$$\sup_{0\leqq t\leqq s} \rho(n,k,t\,n^{-1}) \leqq C\, n^{(1-\eta)/2}. \tag{4.24}$$

Proof. We recall from Ibragimov and Has'minskii (1981), p. 284, that

$$\int_t^\infty (f^{1/2}(x-t)-f^{1/2}(x))^2\, dx \leqq \int_t^\infty (f^{1/\eta}(x-t)-f^{1/\eta}(x))^\eta\, dx \leqq \left(\frac{t}{\eta}\right)^\eta \int_0^\infty |f'(y)|^\eta / f(y)^{\eta-1}\, dy \tag{4.25}$$

which yields

(4.26) $$\tilde{g}(t)=O(t^{(\eta-1)/2}).$$

At this stage we may apply the same arguments used in the proof of Theorem (4.5) which yield the inequality (4.24).

5. Comparison of G, G_k, E_n and $E_{n,k}$

Remember that a bound of the symmetric deficiency $\Delta_s(E_n, E_{n,k})$ was established in (4.7). In the present section we will include the statistical experiments G_k and G into our considerations. It is easy to see that for $a \neq 0$,

(5.1) $$\Delta_s(G, G_k) > 0$$

since G and G_k have different likelihood processes. Moreover, if $a=0$, then

(5.2) $$\Delta_s(G, G_1)=0.$$

Theorem (3.3) enables us to calculate a bound in (5.1).

(5.3) **Theorem.** *For $-1<a<1$ and $s>0$:*

(5.4) $$\begin{aligned}\Delta_s(G, G_k) &\leqq s^{(1+a)/2}(1+a)^{1/2}\, E\psi\left(\frac{1}{s} S_k^{1/(1+a)}\right)\\ &= O(|a|\, k^{(a-1)/(2(1+a))}).\end{aligned}$$

Proof. The triangle inequality yields for arbitrary $n \geqq k$

$$\Delta_s(G, G_k) \leqq \Delta_s(G, E_n) + \Delta_s(E_n, E_{n,k}) + \Delta(E_{n,k}, G_k).$$

According to (1.15), $\Delta_s(G, E_n) \to 0$ as $n \to \infty$. In addition, $\Delta(E_{n,k}, G_k) \to 0$ as $n \to \infty$ under conditions such that (1.13) holds. Thus an application of Theorem (3.3) and (3.6) immediately leads to (5.4).

The link between the statistical experiments $E_{n,k}$ and G_k will be established by means of the following lemma which is a modification of Corollary 2.48 in Falk (1986). A detailed proof of Lemma (5.5) is given in Reiss (1988).

(5.5) **Lemma.** *Assume that for some $a>-1$ condition* (4.1) *holds with $c=1+a$. Then there exists $C>0$ such that for every n and $k \leqq n$:*

(5.6) $$\begin{aligned}&\| \mathscr{L}((n^{1/(1+a)}(X_{1:n}, \ldots, X_{k:n}))\,|\,P_0^n) - \mathscr{L}((S_j^{1/(1+a)})_{j\leqq k})\|\\ &\leqq C[(k/n)^{\gamma/(1+a)}\, k^{1/2} + k/n].\end{aligned}$$

We remark that the right-hand side of (5.6) can be replaced by k/n if f is a generalized Pareto density (see Reiss (1981 b)). It was proved in Weiss (1971) that the left-hand side of (5.6) converges to zero if $k(n)=O(n^\kappa)$ for every $\kappa>0$.
As an immediate consequence of Lemma (5.5) we obtain

(5.7) **Theorem.** *If* (4.1) *holds then for* $a>-1$ *there exists* $C>0$ *such that for every* n *and* $k \leqq n$ *the following inequality holds:*

$$\Delta(G_k, E_{n,k}) \leqq C\left[\left(\frac{k}{n}\right)^{\gamma/(1+a)} k^{1/2} + \frac{k}{n}\right]. \tag{5.8}$$

For Weibull densities we have $\gamma=(1+a)$ whence (5.8) holds with the upper bound $C\,k^{3/2}/n$.

(5.9) *Remark.* Let $X_{1:n}, \ldots, X_{n:n}$ be the order statistic of standard Weibull random variables with shape parameter $a>-1$. Denote by d again the Hellinger distance. Put

$$P_{k,n} = \mathscr{L}(n^{1/(1+a)}(X_{1:n}, \ldots, X_{k:n}))$$

and let $Q_{0,k}$ be the distribution as defined in (1.6). Direct computations show that

$$1-d^2(P_{k,n}, Q_{0,k}) = \left(\frac{n!}{(n-k)!\, n^k}\right)^{1/2} \prod_{i=1}^{k} \left(\frac{1}{2}\left(1+\frac{n-k}{n}\right)+\frac{i}{2n}\right)^{-1}. \tag{5.10}$$

By straightforward computations we obtain for $k \leqq n/3$

$$\exp\left(-\frac{5(k-2)^3}{192 n^2}\right) \leqq 1-d^2(P_{k,n}, Q_{0,k}) \leqq \exp\left(-\frac{k^3}{96 n^2}\right). \tag{5.11}$$

This yields

$$\|P_{k(n),n} - Q_{0,k(n)}\| \to 0 \quad \text{iff } k(n)^{3/2}/n \to 0 \tag{5.12}$$

as $n \to \infty$. Moreover, if $k \leqq n/3$ then

$$\Delta(G_k, E_{n,k}) \leqq \tfrac{1}{4}\left(\tfrac{5}{6}\right)^{1/2} k^{3/2}/n \tag{5.13}$$

in accordance with the result of Theorem (5.7).

At the present stage of our investigations we already have a chain of relations at our disposal which leads from E_n over $E_{n,k}$ and G_k to G. Thus we also can evaluate an upper bound of $\Delta_s(G, E_n)$.

(5.14) **Theorem.** *Assume that the density* f *satisfies the conditions* (1.1), (2.5), (4.1), (4.2) *and* (4.6) *for* $a \in (-1, 1)$. *Consider the corresponding sequence of experiments* $(E_n)_{n \in \mathbb{N}}$ *given in* (1.2) *where* δ_n *is specified in* (3.1). *Then we obtain*

$$\Delta_s(G, E_n) = O(n^{\beta(\gamma, a)}) \tag{5.15}$$

where

$$\beta(\gamma, a)=\begin{cases}\dfrac{a-1}{3+a} & \gamma \geqq \dfrac{2(1+a)}{1-a}, a \neq 0,\\ \dfrac{\gamma(a-1)}{2(1+a)(1+\gamma)} & \text{if } 0<\gamma<\dfrac{2(1+a)}{1-a}, a \neq 0,\\ \max\left(-\gamma, -\dfrac{1}{2}\right) & a=0.\end{cases} \tag{5.16}$$

Proof. Without restrictions we may substitute δ_n by $n^{-1/(1+a)}$ since we can enlarge the interval $[0, s]$, see (4.4). Assume first $a \neq 0$.

Combining (4.7), (5.4) and (5.8) we obtain

$$\begin{aligned}\Delta_s(G, E_n) &\leqq \Delta_s(G, G_k)+\Delta_s(G_k, E_{n,k})+\Delta_s(E_{n,k}, E_n)\\ &O\left[k^{(a-1)/(2(1+a))}+\left(\frac{k}{n}\right)^{\gamma/(1+a)} k^{1/2}+\frac{k}{n}+n^{\max\{a-1, -2\gamma\}/(2(1+a))}\right]\end{aligned} \tag{5.17}$$

uniformly over all $k \leqq \lambda n$ if $\lambda<1$.
Now choose

$$k(n)=[n^{\gamma/(1+\gamma)}]. \tag{5.18}$$

Then we see that for $0<\gamma \leqq \gamma_0 := \dfrac{2(1+a)}{1-a}$ the following inequality holds:

$$k(n)^{(a-1)/(2(1+a))}+\left(\frac{k(n)}{n}\right)^{\gamma/(1+a)} k(n)^{1/2}+\frac{k(n)}{n}+n^{-\gamma/(1+a)}=O(n^{\beta(\gamma, a)}) \tag{5.19}$$

which yields the desired bound (5.15). Note that in the case $\gamma>\gamma_0$ the assumption (4.2) also holds for γ_0 instead of γ. Thus we obtain a bound of the order $O(n^{\beta(\gamma_0, a)})$ whenever $-a \leqq \gamma_0$.

For $a=0$ and $k=1$ we deduce from (4.24), (5.4) and (5.8) the upper bound

$$K[n^{-\gamma}+n^{-1}+n^{-1/2}] \tag{5.20}$$

in (5.17). Thus (5.15) is proved.
Finally, we will give a proof of (1.15) which uses the arguments developed in this paper.

(5.21) **Lemma.** *Assume that condition* (1.1) *holds where r is positive on* $(0, x_0)$ *and slowly varying at zero such that $g(t) \to 0$ as $t \downarrow 0$. Then,*

$$\Delta_s(E_n, G) \to 0. \tag{5.22}$$

Proof. First we prove that $(E_n)_{n \in \mathbb{N}}$ is a Cauchy sequence w.r.t. Δ. For $\varepsilon>0$ choose k and n_0 such that

$$\Delta_s(E_n, E_{n,k}) \leqq \varepsilon$$

for all $n \geqq n_0$. In addition, choose $n_1 \geqq n_0$ such that for all $n, m \geqq n_1$

$$\Delta_s(E_{n,k}, E_{m,k}) \leqq \alpha_k(n) + \alpha_k(m) \leqq \varepsilon$$

where $\alpha_k(n) = \| V_{0,k,n} - Q_{0,k} \|$. Hence

$$\Delta_s(E_n, E_m) \leqq 3\varepsilon$$

whenever $n, m \geqq n_1$.

By Le Cam (1986) the sequence E_n is convergent to some experiment in view of the completeness of the distance Δ_s. On the other hand one can choose $k(n) \leqq n$, $k(n) \to \infty$ such that

(5.23) $$\Delta_s(G_{k(n)}, E_{n,k(n)}) \leqq \alpha_{k(n)}(n) \to 0.$$

Note that also

(5.24) $$\Delta_s(E_n, E_{n,k(n)}) \to 0$$

as $n \to \infty$. It is easy to see by a martingale argument, compare for instance with Janssen (1988), Lemma 6.4, that

$$G_k \to G$$

weakly as $k \to \infty$ in the topology of the weak convergence for experiments. Thus (5.23) and (5.24) show that

$$E_n \to G$$

weakly and the assertion is proved.

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