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Probability Theory and Related Fields

Uniform Convergence of Sums of Order Statistics to Stable Laws

A. Janssen

Universitäts-GH-Siegen, Fachbereich Mathematik,
Hölderlinstrasse 3, D-5900 Siegen, Federal Republic of Germany

Summary. Let $X_1, X_2, \ldots$ denote an i.i.d. sequence of real valued random variables which ly in the domain of attraction of a stable law Q with index $0<\alpha<1$. Under a von Mises condition we show that the sum of order statistics

$$a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n}+\sum_{i=n+1-r(n)}^{n} X_{i:n}\right)$$

converges to Q with respect to the norm of total variation if for instance $\min(k(n), r(n)) \to \infty$.

1. Introduction

During the last years there was much interest concerning the asymptotic behaviour of sums of order statistics

$$W_n=a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n}+\sum_{i=n+1-r(n)}^{n} X_{i:n}\right) \tag{1.1}$$

where $X_{1:n} \leqq X_{2:n} \leqq \ldots \leqq X_{n:n}$ denote the order statistics of an i.i.d. sample $X_1, \ldots, X_n$. Assume that X_1 belongs to the domain of attraction of a stable law with the index α of stability, $0<\alpha<2$. Weak limit laws for W_n can be found in Csörgő et al. [1, 2]. Janssen [9] showed, among other results, that for $1<\alpha<2$ there exists a sequence of random variables Z_n with the same distribution as Y_n such that Z_n converges in L^1.

In the present paper we use the technique of Janssen [9] to establish a limit theorem with respect to the uniform convergence over all Borel sets W_n in case $0<\alpha<1$.

Before we present the results we briefly give a motivation for the examination of the convergence of W_n in the variational distance. The norm of total variation is an essential tool used in connection with the asymptotic approximation of statistical models by much simpler models, for instance see Reiss [11] in connection with extreme value models. Below we see that in our situation it is enough to consider the $k(n)$ smallest and the $r(n)$ largest order statistics of $X_1, \ldots, X_n$ where $\min(k(n), r(n)) \to \infty$ may converge to infinity as slow as we want. As an application consider for instance any statistic T on $\mathbb{R}$. Then we obtain as a consequence of our results always the convergence of the distributions

$$\mathscr{L}(T(W_n)) \to \mathscr{L}(T(W))$$

in the variational distance as $n \to \infty$. Here W denotes the limit of the sequence W_n. In order to give a second application consider the normalized sum of the whole sample namely $\tilde{W}_n = a_n^{-1} \sum_{i=1}^{n} X_i$, which often appears as a sequence of test statistics. For any sequence $\varphi_n: \mathbb{R} \to [0, 1]$ of tests we then obtain

$$E(\varphi_n(W_n)) - E(\varphi_n(\tilde{W}_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

The investigation of convergence results for distributions with respect to the variational distance has a long history as well as for extreme order statistics as for sums of i.i.d. random variables. In the literature also results appear which are concerned with the slightly stronger pointwise convergence of the underlying densities often labeled as local limit theorems. In connection with the uniform convergence of extreme order statistics we refer to de Haan and Resnick [8], Sweeting [12], Falk [3], and further references therein. For local limit theorems of sums of i.i.d. variables the reader should consult the references of Zolotarev [13], p. 274 ff.

In order to explain the result we will introduce further notations. Assume always that X_1 belongs to the domain of attraction of a stable law with index $0 < \alpha < 1$. Then there exists a function L varying slowly at infinity such that

$$G(y) = P(|X_1| > y) = y^{-\alpha} L(y) \tag{1.2}$$

and

$$(1 - F(y))/G(y) \to p \quad \text{and} \quad F(-y)/G(y) \to q \quad \text{as} \quad y \uparrow \infty \tag{1.3}$$

for some $p \in [0, 1]$, $p + q = 1$, where F denotes the distribution function of X_1. Let for $s \in (0, 1)$

$$G^{-1}(s) = \inf\{t: G(t) \leqq s\}$$

denote the inverse of G. Then we always choose

$$a_n = G^{-1}(1/n). \tag{1.4}$$

Note that the normalizing constants a_n^{-1} are the same as in Corollary 3.1 of [1] which are denoted by A_n^* in [2].

Assume that $Y_1, Y_2, \ldots, \tilde{Y}_1, \tilde{Y}_2, \ldots$ are two i.i.d. sequences of exponential distributed random variables. Set

$$\Gamma_k = \sum_{i=1}^{k} Y_i \quad \text{and} \quad \tilde{\Gamma}_k = \sum_{i=1}^{k} \tilde{Y}_i. \tag{1.5}$$

Then well-known results from extreme value theory show that

$$a_n^{-1} X_{k:n} \to -q^{1/\alpha} \Gamma_k^{-1/\alpha} \quad \text{and} \quad a_n^{-1} X_{n+1-k:n} \to p^{1/\alpha} \tilde{\Gamma}_k^{-1/\alpha} \tag{1.6}$$

in distribution as $n \to \infty$. As pointed out in [1, 2] and [9] the asymptotic behaviour of W_n (1.1) only depends approximately on a finite number of extreme order statistics for $0 < \alpha < 1$, i.e.:

$$W_n \to W = -q^{-1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} + p^{1/\alpha} \sum_{i=1}^{\infty} \tilde{\Gamma}_i^{-1/\alpha} \tag{1.7}$$

in distribution as $n \to \infty$, whenever $0 \leqq k(n) \leqq n - r(n) \leqq n$ and

$$q/(k(n)+1) + p/(r(n)+1) \to 0 \tag{1.8}$$

as $n \to \infty$. Assume now that F is absolutely continuous with a Lebesgue density f such that in case $q \neq 0$ the von Mises condition

$$\lim_{x \to -\infty} -x f(x)/F(x) = \alpha \tag{1.9}$$

and in case $p \neq 0$

$$\lim_{x \to \infty} x f(x)/(1 - F(x)) = \alpha \tag{1.10}$$

is fulfilled. Then it is well-known that under (1.9) a finite number of extreme order statistics is convergent uniformly over all Borel sets

$$\sup_{A \in \mathscr{B}^k} |\mathscr{L}(a_n^{-1}(X_{1:n}, \ldots, X_{k:n}))(A) - \mathscr{L}(-q^{-1/\alpha}(\Gamma_1^{-1/\alpha}, \ldots, \Gamma_k^{-1/\alpha}))(A)| \to 0 \tag{1.11}$$

as $n \to \infty$ for each $k \in \mathbb{N}$, cf. Falk [3] or Sweeting [12]. Thus, motivated by (1.11), the assertion (1.6) suggests that under a von Mises condition uniform convergence over all Borel sets also holds in (1.7). Recall that by Scheffé's theorem the convergence uniformly over all Borel sets is equivalent to the λ-stochastic convergence of the corresponding Lebesgue densities of the Lebesgue measure λ.

Finally, let us mention that the von Mises condition (1.9) is a natural condition in connection with uniform limit theorems in extreme value theory. It is pointed out by Sweeting [12] that the von Mises condition (1.9) is equivalent to the convergence of the densities of $a_n^{-1} X_{1:n}$ uniformly on compact sets, see also de Haan and Resnick [8].

2. Main Result

Let $\mathscr{L}(X)$ denote the distribution of a random variable X. For two probability measures P and Q on the Borel sets $\mathscr{B}^n$ on $\mathbb{R}^n$ let

$$\|P-Q\| = \sup_{A \in \mathscr{B}^n} |P(A)-Q(A)| \tag{2.1}$$

denote the variational distance.

(2.1) **Theorem.** *Let X_1 be a random variable belonging to the domain of attraction of a stable distribution with index $0<\alpha<1$, i.e.* (1.2) *and* (1.3) *hold. Choose W_n, W and a_n as in* (1.1), (1.8), (1.7) *and* (1.4) *respectively. Assume that $\mathscr{L}(X_1)$ is absolutely continuous and that the von Mises condition* (1.9) *is satisfied whenever $q \neq 0$ and let* (1.10) *be satisfied whenever $p \neq 0$. Then we obtain*

$$\|\mathscr{L}(W_n)-\mathscr{L}(W)\| \to 0 \tag{2.2}$$

as $n \to \infty$.

Note that the assertion (2.2) can not be formulated for $1 \leqq \alpha < 2$ in this form since then the sum defining W in (1.7) is no longer convergent. It is an open problem whether the sequence $W_n - b_n$ is convergent in the norm of total variation in case $1 \leqq \alpha < 2$ for some centering constants b_n.

3. Proofs

Let ε_x denote the point mass at x, 1_A the indicator function of a set A and let $*$ indicate that convolution operation.

Subsequently let as always assume that $q \neq 0$. First we mention a well-known consequence of the von Mises condition (1.9).

(3.1) **Lemma.**

$$b_n := \sup_{-y \leqq -a_n^{-1/2}} \int_{-\infty}^{-1} a_n |y|\, f(a_n |y|\, w)/F(a_n y) - \alpha |w|^{-\alpha-1}|\, dw \to 0 \tag{3.1}$$

as $n \to \infty$.

The proof is an easy application of the theory of regularly varying functions and Scheffé's lemma. □

First consider $p \neq 0$. According to (3.1) and (1.11) we can find a sequence of integers $s(n) \leqq \min(k(n)), r(n)) - 1$ with $0 \leqq s(n) \to \infty$ such that

$$s(n)\, b(n) \to 0, \qquad s(n)/a_n^{\alpha/2} \to 0 \tag{3.2}$$

and

$$\|\mathscr{L}(a_n^{-1}(X_{1:n}, \ldots, X_{s(n)+1:n})) - \mathscr{L}(-q^{1/\alpha}(\Gamma_1^{-1/\alpha}, \ldots, \Gamma_{s(n)+1}^{-1/\alpha}))\| \to 0 \tag{3.3}$$

and

$$\|\mathscr{L}(a_n^{-1}(X_{n:n}, \ldots, X_{n-s(n):n})) - \mathscr{L}(p^{-1/\alpha}(\tilde{\Gamma}_1^{-1/\alpha}, \ldots, \tilde{\Gamma}_{s(n)+1}^{-1/\alpha}))\| \to 0. \tag{3.4}$$

In addition assume that $s(n)$ increases so slow that

$$s(n)^{1/\alpha} a_n^{-1} X_{s(n)+1:n} \to -q^{1/\alpha} \tag{3.5}$$

and

$$s(n)^{1/\alpha} a_n^{-1} X_{n-s(n):n} \to p^{1/\alpha} \tag{3.6}$$

in probability as $n \to \infty$. Note that the existence of $s(n)$ with (3.5) and (3.6) is a consequence of (1.6) and the law of large numbers. In the case $p=0$ we may choose $0 \leqq s(n) \to \infty$ such that

$$s(n) \leqq k(n) - 1 \tag{3.7}$$

and the assertions (3.2), (3.3) and (3.5) are satisfied. Under these assumptions we prove that the extreme order statistics have a dominating influence.

(3.2) **Lemma.** *(a) For* $p \neq 0$ *we obtain*

$$\left\| \mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n} + \sum_{i=n+1-r(n)}^{n} X_{i:n}\right)\right) - \mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{s(n)+1} X_{i:n} + \sum_{i=n-s(n)}^{n} X_{i:n}\right)\right) \right\| \to 0 \tag{3.8}$$

as $n \to \infty$.

(b) Assume $p=0$. *Then*

$$\left\| \mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n} + \sum_{i=n+1-r(n)}^{n} X_{i:n}\right)\right) - \mathscr{L}\left(a_n^{-1} \sum_{i=1}^{s(n)+1} X_{i:n}\right) \right\| \to 0 \tag{3.9}$$

as $n \to \infty$.

Proof. In a first step let us give the proof of (3.8) in case $p \neq 0$. For $w_1 < w_2$ denote by

$${}^{w_1}X_1, \ldots, {}^{w_1}X_{s(n)}, X_1^{(w_1, w_2)}, \ldots, X_{n-2s(n)-2}^{(w_1, w_2)}, X_1^{w_2}, \ldots, X_{s(n)}^{w_2}$$

independent random variables such that

$${}^{w_1}X_i, [X_i^{(w_1, w_2)}, X_i^{w_2}, \text{respectively}],$$

have the joint density

$$f 1_{(-\infty, w_1]}/F(w_1), [f 1_{[w_1, w_2]}/(F(w_2)-F(w_1)), f 1_{[w_2, \infty)}/(1-F(w_2)), \text{respectively}].$$

We recall from Reiss [11] that the conditional distribution of

$$(X_{1:n}, \ldots, X_{s(n):n}, X_{s(n)+2:n}, \ldots, X_{n-1-s(n):n}, X_{n+1-s(n):n}, \ldots, X_{n:n})$$

given $(X_{s(n)+1:n}, X_{n-s(n):n}) = (w_1, w_2)$ is equal in distribution to

$$({}^{w_1}X_{1:s(n)}, \ldots, {}^{w_1}X_{s(n):s(n)}, X_{1:n-2s(n)-2}^{(w_1, w_2)}, \ldots,$$
$$X_{n-2s(n)-2:n-2s(n)-2}^{(w_1, w_2)}, X_{1:s(n)}^{w_2}, \ldots, X_{s(n):s(n)}^{w_2}).$$

For $y_1 < y_2$ we introduce the random variable

$$Z_n^{(y_1,y_2)} = a_n^{-1}\left[\sum_{i=1}^{k(n)-s(n)-1} X_{i:n-2s(n)-2}^{a_n(y_1,y_2)} + \sum_{i=n-r(n)-s(n)}^{n-2s(n)-2} X_{i:n-2s(n)-2}^{a_n(y_1,y_2)}\right]. \tag{3.10}$$

Let now A be a Borel subset of $\mathbb{R}$. Then we obtain

$$\begin{aligned}
&\left|\mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n} + \sum_{i=n+1-r(n)}^{n} X_{i:n}\right)\right)(A) - \mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{s(n)+1} X_{i:n} + \sum_{i=n-s(n)}^{n} X_{i:n}\right)\right)(A)\right| \\
&\leqq \iint \left|\mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} [{}^{a_n y_1}X_{i:s(n)} + X_{i:s(n)}^{a_n y_2}] + y_1 + y_2 + z\right)(A) - \mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} [{}^{a_n y_1}X_{i:s(n)} + X_{i:s(n)}^{a_n y_2}] + y_1 + y_2\right)(A)\right| \\
&\quad \cdot \mathscr{L}(Z_n^{(y_1,y_2)})(dz)\, \mathscr{L}(a_n^{-1}(X_{s(n)+1:n}, X_{n-s(n):n}))\, d(y_1, y_2) \\
&\leqq \iint g_n(z, y_1)\, \mathscr{L}(Z_n^{(y_1,y_2)})(dz)\, \mathscr{L}(a_n^{-1}(X_{s(n)+1:n}, X_{n-s(n):n}))\, d(y_1, y_2),
\end{aligned} \tag{3.11}$$

where g_n denotes the integrand

$$g_n(z, y_1) = \left\|\mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}X_{i:s(n)} + z\right) - \mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}X_{i:s(n)}\right)\right\|. \tag{3.12}$$

For $w < 0$ let now ${}^wZ_1, {}^wZ_2, \ldots$ be an i.i.d. sequence of Pareto distributed random variables with the density

$$(\alpha |x|^{-\alpha-1} |w|^{\alpha})\, 1_{(-\infty, w]}(x).$$

Thus for $y_1 < 0$

$$\begin{aligned}
g_n(z, y_1) &\leqq 2\left\|\mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}X_i\right) - \mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}Z_i\right)\right\| \\
&\quad + \left\|\mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}Z_i + z\right) - \mathscr{L}\left(a_n^{-1}\sum_{i=1}^{s(n)} {}^{a_n y_1}Z_i\right)\right\| \\
&=: p_n(y_1) + q_n(z, y_1).
\end{aligned} \tag{3.13}$$

First we deal with p_n for $y_1<0$. Notice that

$$\begin{aligned} p_n(y_1) &\leqq 2s(n)\,\|\mathscr{L}({}^{a_n y_1}X_1)-\mathscr{L}({}^{a_n y_1}Z_1)\| \\ &= s(n)\int_{-\infty}^{-1} |a_n|y_1|f(a_n|y_1|\,w)/F(a_n y_1)-\alpha|w|^{-\alpha-1}|\,dw. \end{aligned} \tag{3.14}$$

Thus Lemma (3.1) together with the conditions (3.2) and (3.5) yield

$$\int_{-\infty}^{0} p_n(x/s(n)^{1/\alpha})\,\mathscr{L}(s(n)^{1/\alpha}\,a_n^{-1}\,X_{s(n)+1:n})\,dx\to 0. \tag{3.15}$$

Next we will handle with the second term $q_n(z,y_1)$. Recall that the Pareto random variable ${}^{-1}Z_1$ belongs to the domain of attraction of some stable law Q and

$$\left\|\mathscr{L}\left(n^{-1/\alpha}\sum_{i=1}^{n}{}^{-1}Z_i\right)-Q\right\|\to 0 \tag{3.16}$$

which is a consequence of a well-known local limit theorem, cf. Petrov [10], p. 213 and references therein. Thus we arrive at the following upper bound for $q_n(z,y)$ in case $y<0$:

$$\begin{aligned} q_n(z,y) &= \left\|\mathscr{L}\left(s(n)^{-1/\alpha}\sum_{i=1}^{s(n)}{}^{-1}Z_i\right)+z/(|y|\,s(n)^{1/\alpha}) \right. \\ &\quad \left. -\mathscr{L}\left(s(n)^{-1/\alpha}\sum_{i=1}^{s(n)}{}^{-1}Z_i\right)\right\| \\ &\leqq 2\left\|\mathscr{L}\left(s(n)^{-1/\alpha}\sum_{i=1}^{s(n)}{}^{-1}Z_i\right)-Q\right\|+\left\|Q-Q*\varepsilon_{z/(|y|\,s(n)^{1/\alpha})}\right\|. \end{aligned} \tag{3.17}$$

By (3.16) the first term of (3.17) tends to zero. Define now

$$h(t)=\|Q-Q*\varepsilon_t\| \tag{3.18}$$

which is a bounded continuous function. In order to prove that (3.11) converges to zero it is now by (3.13), (3.15) and (3.17) enough to show that

$$\begin{aligned} &\iint_{\{w_1<0\}} h(z/w_1)\,\mathscr{L}(Z_n^{s(n)^{-1/\alpha}(w_1,w_2)})\,dz\,\mathscr{L}(s(n)^{1/\alpha}\,a_n^{-1}(X_{s(n)+1:n},X_{n-s(n):n})) \\ &\cdot d(w_1,w_2)\to 0 \end{aligned} \tag{3.19}$$

as $n\to\infty$. In view of (3.5) and (3.6) we may restrict ourselves to the domain $(w_1,w_2)\in B:=[-K_1,-K_2]\times[0,K_1]$ with $0<K_2<q^{1/\alpha}$, $K_1>1$, where the integration (3.19) is carried out. For these pairs (w_1,w_2) we will calculate the inner integral of (3.19). Choose $\delta>0$ such that $h(t)\leqq\varepsilon$ if $|t|\leqq\delta$. Then Markov's inequality yields

$$\int h(z/w_1)\,\mathscr{L}(Z_n^{s(n)^{-1/\alpha}(w_1,w_2)})\,dz\leqq\varepsilon+E|Z_n^{s(n)^{-1/\alpha}(w_1,w_2)}|/(\delta w_1). \tag{3.20}$$

From the definition of Z_n (cf. (3.10)) we obtain

$$E|Z^{s(n)^{-1/\alpha}(w_1,w_2)}| \leqq n a_n^{-1} \int_{[-a_n K_1 s(n)^{-1/\alpha}, a_n K_1 s(n)^{-1/\alpha}]} |x| \, dF(x)/(F(a_n w_2 s(n)^{-1/\alpha}) - F(a_n w_1 s(n)^{-1/\alpha})). \tag{3.21}$$

Note that the denominator of (3.21) is for large n uniformly bounded below by some constant $c>0$ for all $(w_1, w_2) \in B$. From Feller [6], p. 283 we recall that for some $K>0$

$$\int_{[-z,z]} |x| \, dF(x) \sim K z G(z) \tag{3.22}$$

as $z \to \infty$. Combining (1.4), (3.4) and (3.22) we see that (3.21) converges uniformly to zero as $n \to \infty$ for all $(w_1, w_2) \in B$ and thus the assertion is proved for $pq \neq 0$. In case $p=0$ we need a modification of the proof above. Assume first that $r(n) \neq 0$ or $k(n)=n$. As in (3.11) we obtain

$$\left|\mathscr{L}\left(a_n^{-1}\left(\sum_{i=1}^{k(n)} X_{i:n} + \sum_{i=n+1-r(n)}^{n} X_{i:n}\right)\right)(A) - \mathscr{L}\left(a_n^{-1} \sum_{i=1}^{s(n)+1} X_{i:n}\right)(A)\right|$$
$$\leqq \iint g_n(z+y_2, y_1) \, \mathscr{L}(Z_n^{(y_1,y_2)}) \, dz \, \mathscr{L}(a_n^{-1}(X_{s(n)+1:n}, X_{n:n})) \, d(y_1, y_2) \tag{3.23}$$

where now

$$Z_n^{(y_1,y_2)} = a_n^{-1}\left[\sum_{i=1}^{k(n)-s(n)-2} X_{i:n-s(n)-2}^{a_n(y_1,y_2)} + \sum_{i=n-s(n)-r(n)}^{n-s(n)-2} X_{i:n-s(n)-2}^{a_n(y_1,y_2)}\right].$$

As in (3.13) we split g_n in various parts, namely

$$g_n(z+y_2, y_1) \leqq p_n(y_1) + q_n(z, y_1) + q_n(y_2, y_1). \tag{3.24}$$

The first term of (3.24) is treated in (3.15). Note that similar to (3.17)–(3.19) we obtain

$$\int q_n(y_2, y_1) \, \mathscr{L}(a_n^{-1}(X_{s(n)+1:n}, X_{n:n})) \, d(y_1, y_2) \to 0 \tag{3.25}$$

since

$$\int_{\{w_1<0\}} h(w_2/w_1) \, \mathscr{L}(s(n)^{1/\alpha} a_n^{-1} X_{s(n)+1:n}, a_n^{-1} X_{n:n}) \, d(w_1, w_2) \to 0 \tag{3.26}$$

because of $a_n^{-1} X_{n:n} \to 0$.

Next we prove similarly as in (3.19)–(3.22) that

$$\iint h(z/w_1) \, \mathscr{L}(Z_n^{(s(n)^{-1/\alpha} w_1, w_2)}) \, dz \, \mathscr{L}(s(n)^{1/\alpha} a_n^{-1} X_{s(n)+1:n}, a_n^{-1} X_{n:n}) \cdot d(w_1, w_2) \to 0 \tag{3.27}$$

as $n \to \infty$. Assume first that $F(0)<1$ which implies

$$P(a_n^{-1} X_{n:n} > 0) \to 1.$$

Thus we may restrict ourselves to the domain B as above. For $(w_1, w_2) \in B$ we see that

$$E|Z_n^{(s(n)^{-1/\alpha}w_1, w_2)}| \leqq n a_n^{-1} \int_{[-a_n K_1 s(n)^{-1/\alpha}, a_n K_1]} |x| \cdot dF(x)/(F(a_n w_2) - F(a_n w_1 s(n)^{-1/\alpha})) \tag{3.28}$$

Note that for $p=0$

$$n a_n^{-1} \int_{[0, a_n K_1]} |x| \, dF(x) \to 0 \tag{3.29}$$

which is a consequence of (1.3) and (3.22). (Consider measures $F_t(x) = (F(x) + tF(-x))/(1+t)$ and choose $t \downarrow 0$.)

As in (3.20) and (3.21) we see by (3.29) that (3.28) converges to zero for $n \to \infty$ which now yields the assertion (3.27).

In the case $F(0)=1$ the largest order statistic $X_{n:n}$ converges to the upper endpoint $d \leqq 0$ of the distribution. Thus we may restrict ourselves to the domain $(w_1, w_2) \in B_n = [-K_1, -K_2] \times \{w_2 : w_2 \geqq (d-1)/a_n\}$. Observe now that (3.28) converges uniformly for all sequences $(w_1, w_2) = (w_1, w_2(n)) \in B_n$ to zero. Thus (3.27) follows. If we now take (3.24), (3.25) and (3.27) into account we see that (3.9) is proved. In the case $r(n)=0$ and $k(n)<n$ the integrand $g_n(z+y_2, y_1)$ of (3.23) must be substituted by $g_n(z, y_1)$. Then the proof obviously carries over. □

(3.3) **Lemma.** *Assume that T_n, S_n, T and S are real-valued random variables such that*

$$\|\mathscr{L}(T_n) - \mathscr{L}(T)\| \to 0 \tag{3.30}$$

and

$$\|\mathscr{L}(T_n, S_n) - \mathscr{L}(T_n) \otimes \mathscr{L}(S_n)\| \to 0 \tag{3.31}$$

holds as $n \to \infty$.

(a) If $\|\mathscr{L}(S_n) - \mathscr{L}(S)\| \to 0$ then

$$\|\mathscr{L}(T_n + S_n) - \mathscr{L}(T+S)\| \to 0 \tag{3.32}$$

as $n \to \infty$.

(b) Assume that $\mathscr{L}(T)$ is absolutely continuous and $S_n \to 0$ in probability. Then we obtain

$$\|\mathscr{L}(T_n + S_n) - \mathscr{L}(T)\| \to 0 \tag{3.33}$$

as $n \to \infty$.

Proof. (a) Notice that (3.32) is trivial whenever T_n and S_n are independent. The assumption (3.31) yields the desired result.

(b) Let $\tilde{S}_n$ be a copy of S_n which is independent with respect to T_n. Thus

$$\begin{aligned} \|\mathscr{L}(T_n + \tilde{S}_n) - \mathscr{L}(T)\| &\leqq \int \|\mathscr{L}(T_n + s) - \mathscr{L}(T)\| \, \mathscr{L}(\tilde{S}_n)\, ds \\ &\leqq \|\mathscr{L}(T_n) - \mathscr{L}(T)\| + \int \|\mathscr{L}(T+s) - \mathscr{L}(T)\| \, \mathscr{L}(\tilde{S}_n)\, ds \to 0 \end{aligned}$$

since $s \mapsto \|\mathscr{L}(T+s) - \mathscr{L}(T)\|$ is continuous. Now the assertion (3.33) follows from (3.31). □

Lemma (3.1) has an immediate consequence.

(3.4) **Lemma.** *Assume that* $0<\alpha<1$. *Then we obtain*

$$\left\|\mathscr{L}(W)-\mathscr{L}\left(-q^{-1/\alpha}\sum_{i=1}^{n}\Gamma_i^{-1/\alpha}+p^{1/\alpha}\sum_{i=1}^{n}\tilde{\Gamma}_i^{-1/\alpha}\right)\right\|\to 0 \tag{3.34}$$

as $n\to\infty$.

Proof. Assume first that $pq\neq 0$. Then we may choose $X_1=W$ and $k(n)=n-r(n)$ with $\min(r(n),k(n))\to\infty$. There exists a sequence $s(n)$ such that the assumptions of Lemma (3.1) are satisfied with $\{s(n)\colon n\in\mathbb{N}\}=\mathbb{N}$. It is well-known that the density f of the stable distribution satisfies the von Mises condition. For instance note that f is increasing on some interval $(-\infty,y]$, which is by de Haan [7] sufficient for the validity of (1.9). It is also possible to prove (1.9) using the exact decay of the density f as well as the decay of F at $-\infty$, which is well-known.

In addition we choose $s(n)$ small enough such that

$$\begin{aligned}&\|\mathscr{L}(X_{1:n},\ldots,X_{s(n)+1:n},X_{n-s(n):n},\ldots,X_{n:n})\\&-\mathscr{L}(X_{1:n},\ldots,X_{s(n)+1:n})\otimes\mathscr{L}(X_{n-s(n):n},\ldots X_{n:n})\|\to 0\end{aligned} \tag{3.35}$$

as $n\to\infty$. The validity of (3.35) was for instance proved by Falk and Kohne [4] or Falk and Reiss [5]. Note that $a_n^{-1}\sum_{i=1}^{n}X_i$ has a stable distribution and obviously

$$\left\|\mathscr{L}\left(a_n^{-1}\sum_{i=1}^{n}X_i\right)-\mathscr{L}(W)\right\|\to 0 \tag{3.36}$$

holds as $n\to\infty$. Consequently (3.36) and (3.8) yield

$$\left\|\mathscr{L}(W)-\mathscr{L}\left(a_n^{-1}\left[\sum_{i=1}^{s(n)+1}X_{i:n}+\sum_{i=n-s(n)}^{n}X_{i:n}\right]\right)\right\|\to 0. \tag{3.37}$$

On the other hand we obtain from (3.3), (3.4) and Lemma (3.3)

$$\begin{aligned}&\left\|\mathscr{L}\left(a_n^{-1}\left[\sum_{i=1}^{s(n)+1}X_{i:n}+\sum_{i=n-s(n)}^{n}X_{i:n}\right]\right)\right.\\&\left.-\mathscr{L}\left(-q^{1/\alpha}\sum_{i=1}^{s(n)+1}\Gamma_i^{-1/\alpha}+p^{1/\alpha}\sum_{i=1}^{s(n)+1}\tilde{\Gamma}_i^{-1/\alpha}\right)\right\|\to 0.\end{aligned} \tag{3.38}$$

Thus the desired result is a consequence of (3.37) and (3.38). The case $p=0$ can be treated similarly if we choose $k(n)=n$ and consider (3.9) instead of (3.8). □

Proof of Theorem (2.1). Assume first that $pq\neq 0$. Then we choose a sequence of non-negative integers $s(n)\to\infty$ which increases slow enough such that (3.2)–

(3.6) and (3.35) hold. If we now combine (3.8), (3.34) and (3.38) then the assertion (2.2) follows. In the case $p=0$ we use (3.9), (3.34) and the subsequent assertion

$$\left\| \mathscr{L}\left(a_n^{-1} \sum_{i=1}^{s(n)+1} X_{i:n}\right) - \mathscr{L}\left(-\sum_{i=1}^{s(n)+1} \Gamma_i^{-1/\alpha}\right) \right\| \to 0, \tag{3.39}$$

which is a consequence of (3.3). All together we see that then the desired assertion (2.2) follows. □

References

1. Csörgő, M., Csörgő, S., Horváth, L., Mason, D.M.: Normal and stable convergence of integral functions of the empirical distribution function. Ann. Probab. **14**, 86–118 (1986)
2. Csörgő, S., Horváth, L., Mason, D.M.: What portion of the sample makes a partial sum asymptotically stable or normal? Probab. Th. Rel. Fields **72**, 1–16 (1986)
3. Falk, M.: Uniform convergence of extreme order statistics. Habilitation thesis. University of Siegen (1985)
4. Falk, M., Kohne, W.: On the rate at which the sample extremes become independent. Ann. Probab. **14**, 1339–1346 (1986)
5. Falk, M., Reiss, R.-D.: Independence of order statistics. Ann. Probab. (to appear in 1988)
6. Feller, W.: An introduction to probability theory and its applications, vol. II, 2nd edn. New York London: Wiley (1971)
7. De Haan, L.: On regular variation and its application to the weak convergence of sample extremes, 3rd edn. (Mathematical Centre Tracts, vol. 32) Amsterdam 1975
8. De Haan, L., Resnick, S.I.: Local limit theorems for sample extremes. Ann. Probab. **10**, 396–413 (1982)
9. Janssen, A.: The domain of attraction of stable laws and extreme order statistics. Prob. Math. Statist. (1989) (to appear)
10. Petrov, V.V.: Sums of independent random variables (Ergebn. Math. Grenzgeb. vol. 82) Berlin Heidelberg New York: Springer 1975
11. Reiss, R.-D.: Approximate distributions of order statistics (with applications to nonparametric statistics), forthcoming monograph. To appear in: Springer series in Statistics (1988)
12. Sweeting, T.J.: On domains of uniform local attraction in extreme value theory. Ann. Probab. **13**, 196–205 (1985)
13. Zolotarev, V.M.: One dimensional stable distributions. Transl. Mathem. Monogr., Am. Math. Soc., Providence (1986)

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