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Probability Theory and Related Fields

Markov Type Properties for Mixtures of Probability Measures*

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Summary. We show that, given a general Markov type property **M**, and a finite dimensional set of probability measures $\mathscr{H}$, the set of elements of $\mathscr{H}$ having **M** can be described by finitely many quadratic equations. We apply the result to the problem of the global Markov property for nonextremal Gibbs states.

0. Introduction

We begin with two results from the theory of lattice spin systems, which have been motivating for our work. We are interested in the set $\mathscr{G}$ of so-called Gibbs measures (Gibbs states); these are measures that already have a certain Markov type property. The "interesting" states from the physical point of view are those Gibbs states that have a stronger version of that property, the so-called global Markov property (GMP). Often, GMP is shown of states that are extremal in the convex set $\mathscr{G}$; the two results below are dealing with nonextremal ones.

(i) Higuchi [1] showed that for certain Gibbs states P_0, P_1, both having GMP, automatically all convex combinations of the P_i have GMP.

(ii) Miyamoto [2] treats one dimensional spin systems, and implicitly (see Sect. 2) shows that $\mathscr{G}$ is a zero-, one-, or three dimensional simplex whose vertices all have GMP; moreover, the nonextremal states in $\mathscr{G}$ with GMP are exactly those on a (possibly degenerate) hyperbolic paraboloid in $\mathscr{G}$.

Here now, we are in the case of an arbitrary measurable space, and we are given a general notion of Markov property. We show that inside an n-dimensional set $\mathscr{H}$ of probability measures, those having the given Markov property are exactly those that lie on the intersection of a certain finite set of quadrics.

The proof is quite easy. Given probability measures $P_0, \ldots, P_n$ whose affine hull contains $\mathscr{H}$, every "instance" of the Markov property for $P_\alpha = \sum \alpha_i P_i$ (where $\sum \alpha_i = 1$, $P_\alpha \in \mathscr{H}$) gives rise to a quadratic equation in α with stochastic coefficients that is to hold a.s. Buth we are able to disregard of "unimportant" events

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thus transforming the equation into a system of deterministic equations. We then only have to run through all instances of the Markov property; the number of equations can be made finite using a dimensionality argument.

This paper contains: Sect. 1: Definitions; Sect. 2: Example (The Gibbs states with GMP in the case of a one dimensional spin system); Sect. 3: Proof of the general result; Sect. 4: Application (Conditions for the GMP to hold for all Gibbs states).

1. Definitions

For the general result of this paper (Sect. 3) we only need Definitions 1 and 3. Definitions 2 and 4, dealing with the special case of lattice systems and Gibbs measures, are needed for the examples in Sect. 2 and 4.

Definition 1. Let a measurable space $(\Omega, \mathscr{A})$ be given. A *general Markov property* is given by a system $\mathscr{T}$ of triples $(\mathscr{B}, \mathscr{C}, \mathscr{B}')$ of sub-σ-algebras of $\mathscr{A}$. Call any triple $\mathscr{S}=(B, \mathscr{C}, B')$ with $B\in\mathscr{B}$, $B'\in\mathscr{B}'$ with $(\mathscr{B}, \mathscr{C}, \mathscr{B}')\in\mathscr{T}$, a *standard situation from* $\mathscr{T}$. A probability measure P on $(\Omega, \mathscr{A})$ is said to *fulfil* $\mathbf{M}(\mathscr{S})$ (the "instance" of the Markov property given by $\mathscr{S}$) if $\mathscr{S}=(B, \mathscr{C}, B')$, and B and B' are conditionally independent under P, given $\mathscr{C}$. P is said to have the *Markov property given by* $\mathscr{T}$ if for all $\mathscr{S}$ from $\mathscr{T}$, P has $M(\mathscr{S})$.

Definition 2 (Lattice systems). Let $\Omega=S^\Gamma$ where S is a finite set (the "state space"), and $\Gamma\subseteq\mathbb{R}^d$ is a discrete locally finite set. Let $\mathscr{V}$ be a set of finite subsets of Γ. We call $\gamma, \gamma'\in\Gamma$ neighbours if they belong to the same $V\in\mathscr{V}$. Let $\mathscr{A}$ be the product-σ-algebra on Ω. For $\Lambda\subseteq\Gamma$, put $\Omega_\Lambda=S^\Lambda$. $\mathscr{A}_\Lambda$ = product-σ-algebra on Ω_Λ, and let $\mathscr{B}_\Lambda=\mathscr{A}_\Lambda\times\Omega_{\Gamma\setminus\Lambda}$ be the σ-algebra of sets "measurable from within Λ". Put

$$\mathscr{T}_{\mathrm{LMP}}=\{(\mathscr{B}_\Lambda, \mathscr{B}_C, \mathscr{B}_{\Lambda'}):\ \Lambda\subseteq\Gamma \text{ finite}, \Lambda'=\Gamma\setminus(\Lambda\cup C), \text{and } C \text{ shields } \Lambda \text{ from } \Lambda'\};$$

here C is said to shield Λ from Λ' iff for all $V\in\mathscr{V}$, $V\cap\Lambda=\emptyset$ if $V\cap\Lambda'\neq\emptyset$. We call the Markov property given by $\mathscr{T}_{\mathrm{LMP}}$ the *local Markov property* (LMP). Put

$$\mathscr{T}_{\mathrm{GMP}}=\{(\mathscr{B}_\Lambda, \mathscr{B}_C, \mathscr{B}_{\Lambda'}):\ \Lambda\subseteq\Gamma, \Lambda'=\Gamma\setminus(\Lambda\cup C), \text{and } C \text{ shields } \Lambda \text{ from } \Lambda'\};$$

$\mathscr{T}_{\mathrm{GMP}}$ gives rise to the so-called *global Markov property*.

Definition 3. Let $\mathscr{G}\subseteq\mathscr{M}^1(\Omega, \mathscr{A})$ be given and $P_0, \ldots, P_n$ be affinely independent elements of $\mathscr{G}$. Put $\mathscr{H}=\mathscr{A}(P_0, \ldots, P_n)\cap\mathscr{G}$, where $\mathscr{A}$ ... denotes the affine hull. Given $\alpha\in\mathbb{R}^{n+1}$, put $P_\alpha=\sum\alpha_i P_i$. There exists an obvious isomorphism between $\mathscr{H}$ and $W_{\mathscr{H}}=\{\alpha\in\mathbb{R}^{n+1}: P_\alpha\in\mathscr{H}\}$.

Definition 4 (Gibbs measures). Given a lattice system, we briefly describe the so-called Gibbs measures (more information on Gibbs measures is contained, e.g., in [3]). Let $\Phi=(\Phi_V)_{V\in\mathscr{V}}$ be an interaction, i.e., a system of functions $\Phi_V: \Omega_V\to\mathbb{R}\cup\{\infty\}$. Given $\Lambda\subseteq\Gamma$ finite, and a configuration τ "outside Λ" ($\tau\in\Omega_{\Gamma\setminus\Lambda}$) there is a canonical way of defining a probability measure $\Pi_{\Lambda,\tau}(\cdot)$

on $(\Omega_\Lambda, \mathscr{A}_\Lambda)$ by choosing its density w.r.t. counting measure proportional to $\exp(-\sum_{V \cap \Lambda \neq \emptyset} \Phi_V(\omega^\frown\tau))$, $^\frown$ denoting concatenation. $\Pi_{\Lambda,\tau}$ can be ragarded as a probability on $(\Omega, \mathscr{B}_\Lambda)$ as well. $P \in \mathscr{M}^1(\Omega, \mathscr{A})$ now is called Gibbs iff for all finite $\Lambda \subseteq \Gamma$ and $B \in \mathscr{B}_\Lambda$,

$$P(B|\tau) = \Pi_{\Lambda,\tau}(B) \qquad P|\mathscr{B}_{\Gamma\setminus\Lambda}-\text{a.s.}$$

It is easy to see that all Gibbs measures have LMP, and that $\{P : P \text{ Gibbs}\}$ forms a convex subset of $\mathscr{M}^1(\Omega, \mathscr{A})$. In our examples (Sect. 2 and 4) we let $\mathscr{G} = \{P : P \text{ Gibbs}\}$.

2. The Gibbs Measures with the Global Markov Property in the case of a One Dimensional Lattice Spin System

As an example, let us briefly describe the set of Gibbs measures with GMP in the case of a one dimensional lattice spin system (i.e., $\Gamma = \mathbb{Z}$, $S = \{0, 1\}$) with nearest neighbour interaction (i.e., $\mathscr{V} = \{(i, i+1) : i \in \mathbb{Z}\}$).

Assume that we have to check GMP for a given Gibbs state P. By induction on the number of "connective components" of C inside Γ, and use of LMP, we may restrict ourselves to standard situations $\mathscr{S} = (B, \mathscr{B}_C, B')$ where C has only one "connective component", i.e., $C = \{i, \ldots, j\}$ for certain $i, j \in \mathbb{Z}$. Since we assumed a nearest neighbour interaction, we may further restrict to singletons C, $C = \{i\}$, $i \in \mathbb{Z}$. But P fulfils the Markov property given by $\mathscr{T} = \{(\mathscr{B}_\Lambda, \mathscr{C}, \mathscr{B}_{\Lambda'}) : \mathscr{C} = \mathscr{B}_C$ for $C = \{i\}$, $i \in \mathbb{Z}$, C shields Λ and $\Lambda'\}$ iff P represents a symmetrical Markov chain; this is just the case treated Miyamoto [2], whose results we are now going to apply ([2] does not consider the most general notion of interaction, but it is easy to generalize).

From [2] we see that the extremal Gibbs states can be labelled P_σ (where $\sigma = (\sigma_-, \sigma_+)$, $\sigma_-, \sigma_+ \in \{0, 1\}$, think of σ_- as of the boundary configuration to the "far left", σ_+ being that to the "far right"), and exactly one of the following cases holds: 1) all P_σ agree; 2) $P_{0,0} = P_{0,1} \neq P_{1,0} = P_{1,1}$; 3) $P_{0,0} = P_{1,0} \neq P_{0,1} = P_{1,1}$; 4) all P_σ differ. So, G is a zero-, one-, or three dimensional simplex. Put $\mathscr{H} = \mathscr{A}(P_{0,0}, P_{0,1}, P_{1,0}, P_{1,1}) \cap \mathscr{G} = \mathscr{G}$.

In [2] it is shown that $\{P \in \mathscr{H} : P$ represents a Markov chain$\}$ $(= \{P \in \mathscr{G} : P$ has GMP$\}) \cong \{P_\alpha : \sum \alpha_\sigma = 1, \alpha^t F \alpha = 0\}$, where for certain $a, b > 0$,

$$F = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -b & 0 \\ 0 & -b & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

if we enumerate the σ in lexicographical order.

So, in the case of a one dimensional lattice spin system, we find that the set of Gibbs states in $\mathscr{G}$ having GMP is isomorphic to a (possibly degenerate) hyperbolic paraboloid in $\mathscr{G}$.

3. Proof of the General Result

Let $\mathscr{M}^n$ be the set of symmetric $n \times n$ matrices, and let $\mathscr{M}_0^n = \{M \in \mathscr{M}^n$: M has vanishing diagonal elements$\}$. $\mathscr{M}^n$ and $\mathscr{M}_0^n$ are vector spaces over $\mathbb{R}$ of dimension $\binom{n+1}{2}$, resp. $\binom{n}{2}$.

Given $F \in \mathscr{M}^n$, let $Q(F) = \{\alpha \in \mathbb{R}^n: \sum \alpha_i = 1, \alpha^t F \alpha = 0\}$ be the associated quadric. For $\mathscr{F} \subseteq \mathscr{M}^n$, put $Q(\mathscr{F}) = \bigcap_{F \in \mathscr{F}} Q(F)$.

Lemma 1. *a) Given $\mathscr{F} \subseteq \mathscr{M}^n$, there exists $\mathscr{F}_0 \subseteq \mathscr{F}$ such that $\# \mathscr{F}_0 \leqq \binom{n+1}{2}$, and $Q(\mathscr{F}) = Q(\mathscr{F}_0)$.*

b) Let Δ be a system of sets of matrices from $\mathscr{M}^n$ that is closed under finite intersections. Then there exists $\mathscr{F}_1 \in \Delta$ such that $Q(\mathscr{F}_1) = \bigcup_{\mathscr{F} \in \Delta} Q(\mathscr{F})$ (by (a), a finite $\mathscr{F}_0 \subseteq \mathscr{F}_1$ has the same Q).

Proof. (a) $Q(\mathscr{F})$ is the set of solutions of the following system of equations:

$$\sum \alpha_i = 1; \tag{1}$$

$$\sum_{i \leqq j} \eta_{ij} f_{ij} = 0 \qquad (F \in \mathscr{F}), \tag{2}$$

where $\eta_{ii} = \alpha_{ii}^2$, $\eta_{ij} = 2\alpha_i a_j$ $(i<j)$. But (2) can be replaced by a maximal linearly independent subsystem

$$\sum_{i \leqq j} \eta_{ij} f_{ij} = 0 \qquad (F \in \mathscr{F}_0), \tag{2'}$$

where $\# \mathscr{F}_0 \leqq \dim \mathscr{M}^n = \binom{n+1}{2}$.

(b) Assume the contrary. Then, since Δ is closed under finite intersections, and since $Q(\mathscr{F} \cap \mathscr{F}') \supseteq Q(\mathscr{F}) \cup Q(\mathscr{F}')$, there exists a tower

$$Q(\mathscr{F}_0) \subsetneqq Q(\mathscr{F}_1) \subsetneqq \dots \subsetneqq Q(\mathscr{F}_{\binom{n+1}{2}+1}),$$

where $\mathscr{F}_i \supsetneqq \mathscr{F}_{i+1}$, $i = 0, \dots, \binom{n+1}{2}$. But this would yield more than $\binom{n+1}{2}$ many linearly independent matrices $F \in \mathscr{M}^n$.

For the following lemma and theorem, let $(\Omega, \mathscr{A})$ and $\mathscr{G} \subseteq \mathscr{M}^1(\Omega, \mathscr{A})$ be arbitrary. Let $P_0, \dots, P_{n+1} \in \mathscr{G}$ be affinely independent, put $\mathscr{H} = \mathscr{G} \cap \mathscr{A}(P_0, \dots, P_{n+1})$, and assume the existence of $\alpha_0 \in W_{\mathscr{H}}$ with all components positive. Let $\mathbf{M}$ denote a general Markov property, given a certain $\mathscr{T}$. For any standard situation $\mathscr{S}$ from $\mathscr{T}$, let $\mathbf{M}(\mathscr{S})$ denote the corresponding "instance" of the Markov property.

Lemma 2. *Given a standard situation $\mathscr{S}$ from $\mathscr{T}$, there exists $\mathscr{F} \subseteq \mathscr{M}^{n+1}$ such that $\# \mathscr{F} \leqq \binom{n+2}{2}$, and*

$$\{\alpha \in W_{\mathscr{H}}: P_\alpha \quad has \quad \mathbf{M}(\mathscr{S})\} = Q(\mathscr{F}) \cap W_{\mathscr{H}}.$$

In case $\mathbf{M}(\mathscr{S})$ already holds for the P_i, we may choose $\mathscr{F} \subseteq \mathscr{M}_0^{n+1}$, hence $\# \mathscr{F} \leqq \binom{n+1}{2}$.

Proof. Put $P = P_{\alpha_0}$. Given any measure Q, we denote its restriction $Q|\mathscr{C}$ to $\mathscr{C}$ by $Q^{\mathscr{C}}$.

Claim 1. There exists a $\mathscr{C}$-measurable function F with values in $\mathscr{M}^{n+1}$ such that for all $\alpha \in W_{\mathscr{H}}$:

$$P_\alpha \quad \text{has} \quad \mathbf{M}(\mathscr{S}) \quad \text{iff} \quad \alpha^t \cdot F \cdot \alpha = 0 \qquad P^{\mathscr{C}} - \text{a.s.};$$

F has values in $\mathscr{M}_0^{n+1}$ if the P_i already have $\mathbf{M}(\mathscr{S})$.

Proof of Claim 1. Obviously, $P_\alpha \ll P$, $P_i \ll P$ (all i), hence $Q \ll P$ for $Q = P_\alpha$, P_i as well as $Q = (1_D P_\alpha)$, $(1_D P_i)$ where $D = B$, B' or $B \cap B'$. Now, for these D,

$$\begin{aligned} P_\alpha(D|\mathscr{C}) &= d(1_D P_\alpha)^{\mathscr{C}}/dP_\alpha^{\mathscr{C}} \\ &= \frac{d(1_D P_\alpha)^{\mathscr{C}}/dP^{\mathscr{C}}}{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}}} \qquad P_\alpha^{\mathscr{C}} - \text{a.s.} \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbf{M}(\mathscr{S}) \qquad \text{holds for } P_\alpha \\ \Leftrightarrow\ & \frac{d(1_{B\cap B'} P_\alpha)^{\mathscr{C}}/dP^{\mathscr{C}}}{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}}} = \frac{d(1_B P_\alpha)^{\mathscr{C}}/dP^{\mathscr{C}}}{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}}} \cdot \frac{d(1_{B'} P_\alpha)^{\mathscr{C}}/dP^{\mathscr{C}}}{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}}} \ P_\alpha^{\mathscr{C}} - \text{a.s.} &(1)\\ \Leftrightarrow\ & \frac{d(1_{B\cap B'} P_\alpha)^{\mathscr{C}}}{dP^{\mathscr{C}}} \cdot \frac{dP_\alpha^{\mathscr{C}}}{dP^{\mathscr{C}}} - \frac{d(1_B P_\alpha)^{\mathscr{C}}}{dP^{\mathscr{C}}} \cdot \frac{d(1_{B'} P_\alpha)^{\mathscr{C}}}{dP^{\mathscr{C}}} = 0 \ P^{\mathscr{C}} - \text{a.s.} &(2)\\ \Leftrightarrow\ & \alpha^t \cdot \tilde{F} \cdot \alpha = 0 \qquad P^{\mathscr{C}} - \text{a.s.}, &(3)\\ & \text{where } \tilde{f}_{ij} = \frac{d(1_{B\cap B'} P_i)^{\mathscr{C}}}{dP^{\mathscr{C}}} \cdot \frac{dP_j^{\mathscr{C}}}{dP^{\mathscr{C}}} - \frac{d(1_B P_i)^{\mathscr{C}}}{dP^{\mathscr{C}}} \cdot \frac{d(1_{B'} P_j)^{\mathscr{C}}}{dP^{\mathscr{C}}}, \\ \Leftrightarrow\ & \alpha^t \cdot F \cdot \alpha = 0 \qquad P^{\mathscr{C}} - \text{a.s.}, &(4)\\ & \text{where} \quad f_{ij} = (1/2)(\tilde{f}_{ij} + \tilde{f}_{ji}). \end{aligned}$$

Obviously, F has values in $\mathscr{M}^{n+1}$, and F is $\mathscr{C}$-measurable. We have to clearify the equivalence (1)$\Leftrightarrow$(2). Notice that

if the equation in (1) holds outside N where $P_\alpha^{\mathscr{C}}(N) = 0$, then the equation in (2) holds outside $N \setminus \{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}} = 0\}$, which is a $P^{\mathscr{C}}$-zero-set, and

if the equation in (2) holds outside N where $P^{\mathscr{C}}(N) = 0$, then the equation in (1) holds outside $N \cup \{dP_\alpha^{\mathscr{C}}/dP^{\mathscr{C}} = 0\}$, which is a $P_\alpha^{\mathscr{C}}$-zero-set.

These statements yield the desired equivalence.

If $\mathbf{M}(\mathscr{S})$ already holds for the P_i then the diagonal elements of F vanish P – a.s. Thus w.l.o.g., always $F \in \mathscr{M}_0^{n+1}$. This proves Claim 1.

Now, put $\Delta = \{\{F(\omega): \omega \in N^c\}: P^{\mathscr{C}}(N)=0,\ N \in \mathscr{C}\}$. The system Δ is stable under (even countably infinite) intersections, hence, by Lemma 1, there exist $\mathscr{F}_0 \subseteq \mathscr{F}_1 \in \Delta$, $\mathscr{F}_0$ finite, such that $Q(\mathscr{F}_0) = Q(\mathscr{F}_1) = \bigcup_{\mathscr{F} \in \Delta} Q(\mathscr{F})$.

Claim 2. P_α has $\mathbf{M}(\mathscr{S})$ iff $\alpha \in Q(\mathscr{F}_0)$.

Proof of Claim 2.

$$\begin{aligned}
& P_\alpha \text{ has } \mathbf{M}(\mathscr{S}) && (1)\\
\Leftrightarrow\ & \text{there exists } N_\alpha \in \mathscr{C} \text{ such that } P(N_\alpha)=0, \quad \text{and} &&\\
& \alpha^t \cdot F(\omega) \cdot \alpha = 0 \qquad \text{for all } \omega \neq N_\alpha && (2)\\
\Leftrightarrow\ & \alpha \in Q(\mathscr{F}_1) && (3)\\
\Leftrightarrow\ & \alpha \in Q(\mathscr{F}_0). && (4)
\end{aligned}$$

Here, in the equivalence (2)$\Leftrightarrow$(3), "$\Rightarrow$" is obvious since $Q(\mathscr{F}_1) = \bigcup_{\mathscr{F} \in \Delta} Q(\mathscr{F})$; "$\Leftarrow$" is obvious since also $\mathscr{F}_1 \in \Delta$, which means that there exists $N \in \mathscr{S}$ with $P(N)=0$, and $\alpha^t F(\omega)\, \alpha = 0$ for all $\omega \neq N$. Put $N_\alpha = N$. This proves the claim.

The last remark also implies that N_α can be chosen independently of $\alpha \in W_{\mathscr{H}}$. In case the P_i already have $\mathbf{M}(\mathscr{S})$, everything happens inside $\mathscr{M}_0^{n+1}$. This proves Lemma 2.

Now the proof of the main theorem is very easy.

Theorem. *Under the assumptions made just before Lemma 2, there exists $\mathscr{F}_0 \subseteq \mathscr{M}^{n+1}$ with cardinality $\leqq \binom{n+2}{2}$ ($\mathscr{F}_0 \subseteq \mathscr{M}_0^{n+1}$ with cardinality $\leqq \binom{n+1}{2}$ in case the P_i already have the Markov property) such that*

$$\{\alpha\colon\ P_0 \textit{ has the Markov property}) \cap W_{\mathscr{H}} = Q(\mathscr{F}_0) \cap W_{\mathscr{H}}.$$

Proof. Given $\mathscr{S}$, let $\mathscr{F}_0(\mathscr{S})$ be the finite system found in Lemma 2. Put $\mathscr{F} = \cup\{\mathscr{F}_0(\mathscr{S})\colon \mathscr{S}$ is a standard situation from $\mathscr{T}\}$. Choose a finite $\mathscr{F}_0 \subseteq \mathscr{F}$ ($\mathscr{F}_0$ has the appropriate cardinality by Lemma 1) such that $Q(\mathscr{F}_0) = Q(\mathscr{F})$. Then, for any $\alpha \in W_{\mathscr{H}}$,

$$\begin{aligned}
& P_\alpha \text{ has the Markov property}\\
\Leftrightarrow\ & \text{for all } \mathscr{S} \text{ from } \mathscr{T}, P_\alpha \text{ has } \mathbf{M}(\mathscr{S})\\
\Leftrightarrow\ & \text{for all } \mathscr{S} \text{ from } \mathscr{T}, \alpha \in Q(\mathscr{F}_0(\mathscr{S}))\\
\Leftrightarrow\ & \alpha \in Q(\mathscr{F})\\
\Leftrightarrow\ & \alpha \in Q(\mathscr{F}_0).
\end{aligned}$$

This shows the theorem.

4. Application: Conditions for the Global Markov Property to hold for all Gibbs Measures

We have been motivated by the random field setting. Here we have affinely independent Gibbs states $P_0, \ldots, P_n$, and $\mathscr{G}$ is the set of all Gibbs states (for a given lattice system). $\mathscr{H}=\mathscr{G}\cap\mathscr{A}(P_0, \ldots, P_n)$ then is a convex set; α_0 as needed in Lemma 2 exists. In the case of extremal P_i, or if $\mathscr{G}$ itself is finite dimensional, $\mathscr{H}$ is an n-dimensional simplex; $W_{\mathscr{H}}$ then is the standard n-simplex.

4.1. Now, let $n=1$, and P_0, P_1 already have GMP. Then there are exactly two possibilities:

(i) $\mathscr{H}\cap\{P\in\mathscr{G}: P \text{ has GMP}\}=\mathscr{H}$ or
(ii) $\mathscr{H}\cap\{P\in\mathscr{G}: P \text{ has GMP}\}=\{P_0, P_1\}$.

This is obvious since any quadratic equation in one variable with more than two solutions is trivial.

Remark. Case (i) applies, e.g., in the case where for each standard situation $\mathscr{S}$, $P_0^{\mathscr{C}}\perp P_1^{\mathscr{C}}$ (which implies that P_0, P_1 are extremal). In particular, this is the case where P_0, P_1 are the minimal, resp. maximal Gibbs states for a translation invariant attractive interaction, see Higuchi [1]. In the setting of Lemma 2, we have (with $\alpha_0=(1/2, 1/2)$) for P-almost all ω, either $dP_0^{\mathscr{C}}/dP^{\mathscr{C}}=0$ (hence $d(1_D P_0)^{\mathscr{C}}/dP^{\mathscr{C}}=0$ for $D=B, B', B\cap B'$), or $dP_1^{\mathscr{C}}/dP^{\mathscr{C}}=0$ (hence $d(1_D P_1)^{\mathscr{C}}/dP^{\mathscr{C}}=0$ for $D=B, B', B\cap B'$), i.e., $F\equiv 0$ and $Q(F)=\mathscr{H}$.

4.2. By a straightforward generalisation to the case of $n>1$, we find the following condition for the GMP to hold for all elements of $\mathscr{G}$:

(+) P_i has GMP ($i=0, \ldots, n$), and for all pairs (i, j) from $\{0, \ldots, n\}$ there exists $\alpha\in(0, 1)$ such that $\alpha P_i+(1-\alpha) P_j$ has GMP.

It is not necessary to check GMP for arbitrary convex combinations of the P_i. This is clear: in view of the theorem, (+) enforces all $F\in\mathscr{F}_0$ to be zero since any quadric in $W_{\mathscr{H}}$ that contains all e_i, and $\alpha(i, j)\, e_i+(1-\alpha(i, j))\, e_j$ for certain $\alpha(i, j)\in(0, 1)$ has to be trivial.

In particular we have GMP for all elements from $\mathscr{G}$ if $\dim\mathscr{G}=n$, and we have GMP for all extremal states, and for nontrivial convex combinations of any pair of extremal states. It would be nice to show this even in the case of an infinite dimensional $\mathscr{G}$. From the theorem, it would be possible to show GMP for all states that are finite convex combinations of extremal states, and the set of such states in dense in $\mathscr{G}$ w.r.t. weak convergence. But we do not know how the "Markov equations" GMP($\mathscr{S}$) behave under weak limits.

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