Probab. Th. Rel. Fields 78, 249–252 (1988)

Probability Theory and Related Fields

A Short Proof of Motoo's Combinatorial Central Limit Theorem Using Stein's Method

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Summary. A new proof of Motoo's combinatorial central limit theorem (see Motoo 1957) is given using a method of Stein (1972) and a combinatorial method of Bolthausen (1984). This proof is shorter than Motoo's and other wellknown proofs (see e.g. Hájek 1961).

1. Introduction

Let $A_n=(a_{nij})_{1\leqq i,j\leqq n}$ be an $n\times n$-matrix of real numbers such that

$$\sum_{i=1}^{n} a_{nij}=0 \quad \text{for all } j; \qquad \sum_{j=1}^{n} a_{nij}=0 \quad \text{for all } i$$

and

$$\sum_{i,j=1}^{n} a_{nij}^2=n-1.$$

Let further σ_n be uniformly distributed on the set of permutations of $\{1,\ldots,n\}$ and

$$T_{A_n}=\sum_{i=1}^{n} a_{ni\sigma_n(i)}.$$

The following central limit theorem was proved first by Motoo (1957).

Theorem. *Assume that the Lindeberg type condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{|a_{nij}|>\varepsilon} a_{nij}^2=0 \tag{1}$$

holds for all $\varepsilon>0$, *then* T_{A_n} *is asymptotically standard normally distributed.*

We mention that under some appropriate conditions (1) is also necessary (see Hajek (1961), Theorem 4.1 for the special case $a_{nij}=e_{ni}\,d_{nj}$ and Chen (1978), Corollary 5.1 for the general case). Further proofs of the above theorem are due to e.g. Hájek (1961) for the special case $a_{nij}=e_{ni}d_{nj}$ and Ho and Chen (1978), Chen (1978), who considered more general problems.

The purpose of this paper is to give a new proof of the above theorem which is shorter than Motoo's and the above mentioned proofs.

We use a method of Stein (1972) and a combinatorial method of Bolthausen (1984). Stein originally introduced his method for obtaining rates of convergence in a central limit theorem for sums of nearly independent random variables. Since, roughly spoken, this method is a refined kind of the method of moments, it is not surprising that it is also suitable for our theorem, whose earlier, weaker versions were proved by the method of moments (see e.g. Hoeffding (1951)). The shortness of our proof compared to the approaches of Ho and Chen (1978) and Chen (1978), who also use Stein's method, is on the one side due to a simpler and more direct application of this method and on the other side due to the additional use of Bolthausen's combinatorial method.

2. Proof of the Theorem

It suffices to show

$$E(h(T_{A_n}))\to\Phi(h)\qquad\text{as}\quad n\to\infty \tag{2}$$

for a fixed continuous function h which can be extended continuously to $\bar{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$. Here $\Phi(h)$ denotes the standard normal expectation of h.

In the following we fix $n\geqq 2$. Therefore, for simplicity we drop the index n of A_n, a_{nij} etc.

In order to apply Stein's method we define the function

$$f(x)=\psi(x)^{-1}\int_{-\infty}^{x}(h(y)-\Phi(h))\,\psi(y)\,dy,$$

where ψ denotes the density of the standard normal distribution. This function has the following properties:

$$f'(x)-xf(x)=h(x)-\Phi(h)\qquad\text{for all}\quad x\in\mathbb{R}, \tag{3}$$

$$\lim_{x\to\pm\infty}f'(x)=0\quad\text{and}\quad f'\text{ is uniformly continuous.} \tag{4}$$

For (4) we used (3) and applied the rule of l'Hospital to $xf(x)$.

Next we need Bolthausen's combinatorial method (see Bolthausen (1984), pp. 383–384), which we repeat here for the convenience of the reader.

We define a random element (I_1, I_2, J_1, J_2) in N^4, where $N=\{1,\dots,n\}$, in the following way: (I_1, I_2, J_1) is uniformly distributed on N^3, and given this, one has $J_2=J_1$ on $\{I_1=I_2\}$ and J_2 is uniformly distributed on $N-\{J_1\}$

on $\{I_1 \neq I_2\}$. Let π_1 be a random permutation, which is uniformly distributed on the permutations of N and independent of (I_1, I_2, J_1, J_2). Define

$$I_3 = \pi_1^{-1}(J_1), \qquad I_4 = \pi_1^{-1}(J_2),$$
$$J_3 = \pi_1(I_1), \qquad J_4 = \pi_1(I_2),$$

and

$$\underline{I} = (I_1, I_2, I_3, I_4).$$

Of course, $I_1 = I_2$ holds if and only if $I_3 = I_4$. For each fixed $\underline{i} = (i_1, i_2, i_3, i_4) \in N^4$ which satisfies the condition $i_1 = i_2 \Leftrightarrow i_3 = i_4$, we fix once for all a permutation $t(\underline{i})$ of N, which maps i_1 to i_4 and i_2 to i_3 and which leaves the numbers outside $\{i_1, i_2, i_3, i_4\}$ fixed. Let further $s(i_1, i_2)$ be the transposition of i_1 and i_2. Then we put

$$\pi_2 = \pi_1 \circ t(\underline{I}), \qquad \pi_3 = \pi_2 \circ s(I_1, I_2).$$

We note the following simple results:

(5) $\pi_2(I_1) = J_2, \quad \pi_2(I_2) = J_1, \quad \pi_3(I_1) = J_1, \quad \pi_3(I_2) = J_2.$

(6) π_1, π_2, π_3 are independent of $\underline{I}$ and have the same law.

(7) π_2 and (I_1, J_1) are independent.

(8) $(I_k, \pi_l(I_k))$ is uniformly distributed on N^2 for all $1 \leqq k \leqq 4$, $1 \leqq 1 \leqq 3$.

For (6) and (7) one may consult Bolthausen (1984), Lemma, and (8) follows from (6) and the fact that all I_k are uniformly distributed on N.

Furthermore let

$$T_i = \sum_j a_{j\pi_i(j)}; \quad i = 1, 2, 3 \quad \text{and} \quad \Delta T_i = T_{i+1} - T_i; \quad i = 1, 2.$$

Then (5) gives that ΔT_2 depends on (I_1, I_2, J_1, J_2). With this notations we obtain using (5) and (6)

$$\begin{aligned} E(T_A f(T_A)) &= E(T_3 f(T_3)) = nE(a_{I_1 J_1} f(T_3)) \\ &= nE(a_{I_1 J_1} f(T_2)) + nE(a_{I_1 J_1} \Delta T_2 f'(T_1)) \\ &\quad + nE\left(a_{I_1 J_1} \Delta T_2 \int_0^1 (f'(T_1 + \Delta T_1 + t \Delta T_2) - f'(T_1))\, dt\right). \end{aligned}$$

From (7) we see that the first summand is zero and from the independence of π_1 and (I_1, I_2, J_1, J_2) we get that the second summand equals $nE(a_{I_1 J_1} \Delta T_2) E(f'(T_1)) = E(f'(T_A))$.

Therefore, using (3) we get

$$\begin{aligned} |E(h(T_A)) - \Phi(h)| &= |E(f'(T_A)) - E(T_A f(T_A))| \\ &\leqq nE\left(|a_{I_1 J_1} \Delta T_2| \int_0^1 |f'(T_1 + \Delta T_1 + t \Delta T_2) - f'(T_1)|\, dt\right). \end{aligned}$$

Now we fix $\theta>0$. Then (4) gives $\delta>0$ and $0\leqq K<\infty$ such that $|f'(x)|\leqq K$ for all $x\in\mathbb{R}$ and $|f'(x)-f'(y)|\leqq\theta$ whenever $|x-y|\leqq\delta$. Thus we obtain further

$$\begin{aligned}&\leqq 2K\, nE(|a_{I_1J_1}\, \Delta T_2|\, 1_{\{|\Delta T_1|+|\Delta T_2|>\delta\}})\\ &\quad+\theta nE(|a_{I_1J_1}\, \Delta T_2|\, 1_{\{|\Delta T_1|+|\Delta T_2|\leqq\delta\}})=A_1+A_2, \quad \text{say.}\end{aligned}$$

In order to estimate A_1 we remark that ΔT_1 is the sum of 8 summands of the form $\pm a_{\alpha\beta}$ and ΔT_2 is the sum of 4 summands of the form $\pm a_{\mu\nu}$, where α, $\mu\in\{I_1, \ldots, I_4\}$ and β, $\nu\in\{\pi_l(I_k)$: $1\leqq l\leqq 3$, $1\leqq k\leqq 4)$. Thus, defining $\varepsilon=\delta/12$, we have to estimate

$$\begin{aligned}nE(|a_{I_1J_1}||a_{\mu\nu}|\, 1_{\{|a_{\alpha\beta}|>\varepsilon\}})&\leqq n[\varepsilon^2 E(1_{\{|a_{\alpha\beta}|>\varepsilon\}})+\varepsilon E(|a_{I_1J_1}|\, 1_{\{|a_{I_1J_1}|>\varepsilon\}}\, 1_{\{|a_{\alpha\beta}|>\varepsilon\}})\\ &\quad+\varepsilon E(|a_{\mu\nu}|\, 1_{\{|a_{\mu\nu}|>\varepsilon\}}\, 1_{\{|a_{\alpha\beta}|>\varepsilon\}})\\ &\quad+E(|a_{I_1J_1}|\, 1_{\{|a_{I_1J_1}|>\varepsilon\}}|a_{\mu\nu}|\, 1_{\{|a_{\mu\nu}|>\varepsilon\}})]\\ &\leqq 4nE(|a_{I_1J_1}|^2\, 1_{\{|a_{I_1J_1}|>\varepsilon\}})=4n^{-1}\sum_{|a_{ij}|>\varepsilon} a_{ij}^2\to 0.\end{aligned}$$

For the last inequality we used the inequality of Schwarz and (8).

Finally we look at A_2:

$$\begin{aligned}A_2&\leqq\theta nE(|a_{I_1J_1}\, \Delta T_2|)\\ &\leqq 4\theta\, nE(|a_{I_1J_1}|^2)=4\theta\, n^{-1}\sum_{i,j} a_{ij}^2\leqq 4\theta.\end{aligned}$$

Since $\theta>0$ was arbitrary, this proves the theorem.

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Received June 11, 1987; received in revised form January 13, 1988