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Probability Theory and Related Fields

Shocks in the Asymmetric Exclusion Process

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Summary. In this paper, we consider limit theorems for the asymmetric nearest neighbor exclusion process on the integers. The initial distribution is a product measure with asymptotic density λ at $-\infty$ and ρ at $+\infty$. Earlier results described the limiting behavior in all cases except for $0<\lambda<1/2$, $\lambda+\rho=1$. Here we treat the exceptional case, which is more delicate. It corresponds to the one in which a shock wave occurs in an associated partial differential equation. In the cases treated earlier, the limit was an extremal invariant measure. By contrast, in the present case the limit is a mixture of two invariant measures. Our theorem resolves a conjecture made by the third author in 1975 [7]. The convergence proof is based on coupling and symmetry considerations.

1. The Theorem

The asymmetric nearest neighbor exclusion process on the integers Z is the Markov process η_t on $X=\{0,1\}^Z$ which evolves in the following way: Take $1/2<p\leqq 1$, and set $q=1-p$. Sites $x\in Z$ for which $\eta(x)=1$ are considered to be occupied, and sites for which $\eta(x)=0$ are vacant. At independent exponential times with parameter one, a particle at x attempts to move to $y=x+1$ with probability p and to $y=x-1$ with probability q. The transition takes place if y is vacant and is suppressed if y is occupied. A more formal description of this process, together with an exposition of many of the results known about it, can be found in Chap. VIII of [7].

The transition semigroup for η_t is given by $S(t)f(\eta)=E^\eta f(\eta_t)$, with $f\in C(X)$. The distribution of the process at time t when the initial distribution is μ will be denoted by $\mu S(t)$. For $0\leqq\alpha\leqq 1$, ν_α will denote the product measure on X with $\nu_\alpha\{\eta\colon \eta(x)=1\}=\alpha$. These product measures are important in this context

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because they are exactly the extremal invariant measures which are also shift invariant. The purpose of this paper is to complete the proof of the following convergence theorem.

(1.1) **Theorem.** *Suppose μ is a product measure on X for which the following limits exist:*

$$\lambda = \lim_{x \to -\infty} \mu\{\eta: \eta(x)=1\} \quad \text{and} \quad \rho = \lim_{x \to +\infty} \mu\{\eta: \eta(x)=1\}. \tag{1.2}$$

Then

(a) $$\lim_{t \to \infty} \mu S(t) = \nu_{1/2} \quad \text{if } \lambda \geqq 1/2 \text{ and } \rho \leqq 1/2,$$

(b) $$\lim_{t \to \infty} \mu S(t) = \nu_{\rho} \quad \text{if } \rho \geqq 1/2 \text{ and } \lambda + \rho > 1,$$

(c) $$\lim_{t \to \infty} \mu S(t) = \nu_{\lambda} \quad \text{if } \lambda \leqq 1/2 \text{ and } \lambda + \rho < 1,$$

and

(d) *if $\lambda + \rho = 1, 0 < \lambda < 1/2$, and (1.2) is strengthened to*

$$\sum_{x \leqq 0} |\mu\{\eta: \eta(x)=1\} - \lambda| < \infty \quad \text{and} \quad \sum_{x \geqq 0} |\mu\{\eta: \eta(x)=1\} - \rho| < \infty, \tag{1.3}$$

then $\lim_{t \to \infty} \mu S(t) = \frac{1}{2}\nu_{\lambda} + \frac{1}{2}\nu_{\rho}$.

Parts (a), (b) and (c) of this theorem were proved in [5]. An improved proof, together with an extension of this result to nonnearest neighbor systems, was given in [6]. Part (d) is more delicate, as can be seen from the fact that (1.3) is required in place of (1.2), and that the limit is not an extremal invariant measure. This makes part (d) more interesting as well. Without (1.3), the limit of $\mu S(t)$ may not exist at all, as was shown in [5]. The purpose of this paper is to prove part (d). It gives an affirmative answer to Conjecture 1.6 of [5]. A weaker form of (d) was obtained recently by the first author in [1].

The following informal comments are intended to explain why a mixture of product measures might appear in the limit in part (d) of Theorem (1.1). The idea is that the configuration at any time should consist of a region which is not in equilibrium, surrounded on the left and right by infinite regions which are roughly in the equilibria ν_λ and ν_ρ respectively. The location of this central "disturbance" presumably moves somewhat like a symmetric random walk, while its size is of smaller order than its distance from the origin. Therefore, if one looks near the origin, one sees ν_λ if the disturbance is far to the right, and ν_ρ if it is far to the left. By symmetry, each of these situations should occur with probability 1/2.

Section 4 explains this heuristic argument in somewhat greater detail. These ideas are then used to suggest several open problems, which would generalize part (d) of Theorem (1.1). At the same time, we will explain the connection between these problems and a particular partial differential equation. Part (d) above corresponds to the case in which a shock occurs at zero in the solution of this partial differential equation, thus explaining the word "shock" in the title of the paper.

We next state two results, Theorems (1.4) and (1.6), which contain the main ingredients of the proof of part (d) of Theorem (1.1). After that, we will show how (d) follows from them and some earlier results about the exclusion process. Theorem (1.4) is proved in Sect. 2, while Theorem (1.6) is proved in Sect. 3. The proofs which we give use some coupling ideas which are not entirely standard. They also use the available symmetries in an essential way, and require the nearest neighbor assumption in crucial places. From now on, we will always take $\lambda+\rho=1$ and $0<\lambda<1/2$.

(1.4) **Theorem.** *Suppose μ is the product measure on X with*

$$\mu\{\eta:\eta(x)=1\}=\begin{cases}\lambda & \text{if } x<0\\ 1-\lambda & \text{if } x\geqq 0.\end{cases} \tag{1.5}$$

If $t_n\uparrow\infty$ and $\nu=\lim_{n\to\infty}\mu S(t_n)$ exists, then ν is translation invariant.

(1.6) **Theorem.** *Let μ be as in the statement of Theorem* (1.4). *Then* [(1.7) $p\mu S(t)\{\eta:\eta(x)=1,\eta(x+1)=0\}-q\mu S(t)\{\eta:\eta(x)=0,\eta(x+1)=1\}\leqq(p-q)\lambda(1-\lambda)$] *for all $t\geqq 0$ and all $x\in Z$.*

The above statement may seem somewhat mysterious. It is shown in Sect. 3 that the difference between the right and left sides of (1.7) is

$$\lim_{y\to-\infty}\frac{d}{dt}\sum_{u=y}^{x}\mu S(t)\{\eta:\ \eta(u)=1\}.$$

Note that the rate at which particles flow to the right across any given point in each equilibrium ν_λ and $\nu_{1-\lambda}$ is $(p-q)\lambda(1-\lambda)$. Thus (1.7) asserts that the flow is smaller at any point and any time than it is in these equilibria.

Proof of part (d) of Theorem (1.1). First consider the case in which μ has the marginals given in (1.5). Suppose $t_n\uparrow\infty$ and $\nu=\lim_{n\to\infty}\mu S(t_n)$. By Theorem (1.4), ν is translation invariant. Therefore by Proposition (3.2) of [1], there is a probability measure γ on $[\lambda,1-\lambda]$ such that

$$\nu=\int\nu_\alpha\,\gamma(d\alpha).$$

The outline of the argument which leads to this conclusion is as follows: Since ν is translation invariant and is the limit of the distribution of the process along a sequence of times which tend to infinity, ν must be an equilibrium measure for the process. But all translation invariant equilibria are exchangeable by Theorem (3.9) of Chap. VIII of [7]. By De Finetti's theorem and the fact that μ lies stochastically between ν_λ and $\nu_{1-\lambda}$, ν must therefore be a mixture of product measures with densities between λ and $1-\lambda$.

Using the above representation for ν,

$$\nu\{\eta:\eta(x)=1,\ \eta(x+1)=0\}=\nu\{\eta:\eta(x)=0,\ \eta(x+1)=1\}=\int\alpha(1-\alpha)\,\gamma(d\alpha)$$

for all x. Setting $t=t_n$ in (1.7), passing to the limit as $n\to\infty$, and using the assumption $p>q$ then gives

$$\int\alpha(1-\alpha)\,\gamma(d\alpha)\leqq\lambda(1-\lambda).$$

Therefore γ can put mass only at λ and $1-\lambda$. To determine how much mass is put at each point, a symmetry argument is needed. A basic property of the exclusion process is that the set of zero sites evolves according to an exclusion process with p and q reversed. Therefore,

$$\mu S(t)\{\eta:\eta(x)=1\}+\mu S(t)\{\eta:\eta(-x-1)=1\}=1$$

for all t and all x. It follows that $\nu\{\eta:\ \eta(x)=1\}=1/2$, so that γ must put mass $1/2$ at each of λ and $1-\lambda$, and hence

$$\nu=(1/2)\,\nu_\lambda+(1/2)\,\nu_{1-\lambda}.$$

All weak limits of $\mu S(t)$ are thus the same. Since X is compact, we conclude that

$$\lim_{t\to\infty}\mu S(t)=\tfrac{1}{2}\nu_\lambda+\tfrac{1}{2}\nu_{1-\lambda} \tag{1.8}$$

as asserted.

It now remains to extend this result to product measures satisfying (1.3). The first observation is that if μ is a product measure satisfying (1.3), then there is another product measure $\tilde{\mu}$ so that $\tilde{\mu}\{\eta:\ \eta(x)=1\}=\lambda$ for all but finitely many negative x, and $\tilde{\mu}\{\eta:\ \eta(x)=1\}=1-\lambda$ for all but finitely many positive x, so that the total variation distance between μ and $\tilde{\mu}$ is arbitrarily small. Since the total variation distance between $\mu S(t)$ and $\tilde{\mu}S(t)$ is decreasing in t, it is enough to prove (1.8) for all such product measures. For

$$m:\{-n,\ -n+1,\ \dots,\ n-1,\ n\}\to[0,1],$$

let μ_m be the product measure with marginals given by

$$\mu_m\{\eta:\eta(x)=1\}=\begin{cases}\lambda & \text{if } x<-n\\ m(x) & \text{if } |x|\leqq n\\ 1-\lambda & \text{if } x>n.\end{cases}$$

If $\lambda\leqq m(x)\leqq 1-\lambda$ for all x, then μ_m lies stochastically between two translates of the product measure with marginals given by (1.5). According to Proposition (2.12) of Chap. VIII of [7], these stochastic inequalities are preserved by the evolution, so that (1.8) is satisfied by $\mu=\mu_m$ in this case. To remove the constraints on $m(x)$, we proceed as in the proof of Theorem (1.8) of [5]. Set $N=2^{2n+1}$, and let $m_1,\dots,m_N$ be the N choices of m's which satisfy $m(x)=0$

or 1 for each $-n \leqq x \leqq n$. Then any μ_m can be written as a convex combination as follows:

$$\mu_m = \sum_{k=1}^{N} \mu_{m_k} \prod_{x=-n}^{n} \{m(x)^{m_k(x)} [1-m(x)]^{1-m_k(x)}\}.$$

Applying $S(t)$ to this identity, and using the fact that we already know that (1.8) holds for $\mu = \mu_m$ whenever $m(x)$ is close to 1/2 for all $-n \leqq x \leqq n$, it then follows that (1.8) holds for each $\mu = \mu_m$ with m unrestricted. □

2. Proof of Theorem (1.4)

The basic idea behind the proof of Theorem (1.4) is to couple μ together with a translate of μ so that discrepancies between the two configurations occur in a certain way, and then to show that these discrepancies disappear eventually with probability one. We begin with two lemmas which play important roles in the proofs of both Theorems (1.4) and (1.6). They assert monotonicity and symmetry properties of an associated process with a special particle whose behavior is different than that of the other particles.

To describe the associated process, suppose first that η_t and ζ_t are copies of the exclusion process which are coupled using the basic coupling (see Sect. 2 of Chap. VIII of [7]). In this joint realization of η_t and ζ_t, particles at the same site in the two configurations choose the same exponential times at which to try to move, and choose the same neighbor to which to try to move. Whether the attempted transition occurs or not depends, of course, on the rest of the configuration of each process separately.

Suppose now that initially, $\eta_0 \leqq \zeta_0$, and $\eta_0(x)=0$ and $\zeta_0(x)=1$ at exactly one site $x \in Z$. Then this situation persists at later times. Let X_t denote the location of this discrepancy between η_t and ζ_t. Then (η_t, X_t) is the associated Markov process on $\{(\eta, x) \in X \times Z\colon \eta(x)=0\}$ which we will study. Note that η_t is a version of the exclusion process, and that X_t can be regarded as the location of a special type of particle over which all particles in η_t have priority in the following sense: If a particle in η_t tries to move to X_t, then that particle and the special particle exchange positions.

(2.1) **Lemma.** *Consider two copies (η_t, X_t) and (ζ_t, Y_t) of the process described above. If $\eta_0 \leqq \zeta_0$ and $X_0 \geqq Y_0$, then the two processes can be coupled together in such a way that $\eta_t \leqq \zeta_t$ and $X_t \geqq Y_t$ for all t. Therefore, if f is an increasing function on Z which grows at most polynomially, then $E^{(\eta,x)} f(X_t)$ is a decreasing function of η for fixed $x \in Z$ and $t \geqq 0$.*

Proof. The second statement follows immediately from the first. The polynomial growth assumption on f is made simply to guarantee that the expectation is well defined. In order to construct the required coupling, first apply the basic coupling to η_t and ζ_t. Note that $\eta_t \leqq \zeta_t$ for all t. Couple X_t and Y_t together so that the attempted movement of X_t and Y_t is always the same, and is independent of the movement of the other particles, except in the following two cases:

(i) $X_t = Y_t$ with $\eta_t(X_t+1)=0$ and $\zeta_t(Y_t+1)=1$ when movement is to the right, and (ii) $X_t = Y_t$ with $\eta_t(X_t-1)=0$ and $\zeta_t(Y_t-1)=1$ when movement is to the left. Clearly $X_t \geqq Y_t$ is preserved, except possibly in cases (i) and (ii). In case (i), note that X_t moves to the right at rate p and Y_t moves to the right at rate q. Since $p>q$, these transitions can be coupled so that X_t and Y_t move together at rate q and X_t moves to the right alone at rate $p-q$, thus preserving the inequality $X_t \geqq Y_t$. Case (ii) is analogous, with transitions to the left replacing transitions to the right. □

(2.2) **Lemma.** *Consider two copies (η_t, X_t) and (ζ_t, Y_t) of the process described above. If $X_0 = Y_0 = 0$ and $\eta_0(x) = 1 - \zeta_0(-x)$ for all $x \neq 0$, then the two processes can be coupled together so that $Y_t = -X_t$ for all t. In particular, if $X_0 = 0$ and η_0 is random and satisfies*

$$\{\eta_0(x), x \neq 0\} \stackrel{d}{=} \{1 - \eta_0(-x), x \neq 0\},$$

then X_t has a distribution which is symmetric about 0 for any $t \geqq 0$. (The above equality denotes equality in distribution.)

Proof. First construct (η_t, X_t), and then define (ζ_t, Y_t) by $Y_t = -X_t$ and $\zeta_t(Y_t + x) = 1 - \eta_t(X_t - x)$ for all $x \neq 0$. It is then a simple matter to check that (ζ_t, Y_t) has the correct transition rates. □

(2.3) **Corollary.** *Consider two copies (η_t, X_t) and (ζ_t, Y_t) of the above process. Set $X_0 = Y_0 = 0$ and $\eta_0(1) = \zeta_0(-1) = 0$. If $\{\eta_0(x), x \neq 0, 1\}$ are independent Bernoulli random variables with*

$$P[\eta_0(x)=1] = \begin{cases} \lambda & \text{if } x<0 \\ 1-\lambda & \text{if } x>1, \end{cases}$$

and $\{\zeta_0(x), x \neq 0, -1\}$ are independent random variables with

$$P[\zeta_0(x)=1] = \begin{cases} \lambda & \text{if } x<-1 \\ 1-\lambda & \text{if } x>0, \end{cases}$$

then $EX_t \geqq EY_t$ for all t.

Proof. Write

$$EX_t = \lambda E(X_t \mid \eta_0(-1)=1) + (1-\lambda) E(X_t \mid \eta_0(-1)=0)$$

and

$$EY_t = \lambda E(Y_t \mid \zeta_0(1)=0) + (1-\lambda) E(Y_t \mid \zeta_0(1)=1).$$

By Lemma (2.2),

$$E(X_t \mid \eta_0(-1)=1) = 0 \quad \text{and} \quad E(Y_t \mid \zeta_0(1)=1) = 0.$$

On the other hand, the conditional distributions

$$\{\eta_0(x), x \neq 0 \mid \eta_0(-1)=0\} \quad \text{and} \quad \{\zeta_0(x), x \neq 0 \mid \zeta_0(1)=0\}$$

agree, and lie stochastically below the conditional distributions

$$\{\eta_0(x), x \neq 0 \mid \eta_0(-1)=1\} \quad \text{and} \quad \{\zeta_0(x), x \neq 0 \mid \zeta_0(1)=1\}.$$

Therefore, by Lemma (2.1),

$$E(X_t \mid \eta_0(-1)=0) = E(Y_t \mid \zeta_0(1)=0) \geqq 0.$$

Since $\lambda < 1/2$, it follows that $EX_t \geqq EY_t$. □

(2.4) **Lemma.** *There exists a joint distribution of (η, ζ) with the following properties:*

(a) *$\{\eta(x), x \in Z\}$ are independent Bernoulli random variables with*

$$P[\eta(x)=1] = \begin{cases} \lambda & \text{if } x \leqq -1 \\ 1-\lambda & \text{if } x \geqq 0, \end{cases}$$

(b) *$\{\zeta(x), x \in Z\}$ are independent Bernoulli random variables with*

$$P[\zeta(x)=1] = \begin{cases} \lambda & \text{if } x \leqq 0 \\ 1-\lambda & \text{if } x \geqq 1, \end{cases}$$

(c) *with probability one, there exists an $n \geqq 0$ and integers $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ so that $x_i < y_i$, $\eta(x_i)=1$, $\zeta(x_i)=0$, $\eta(y_i)=0$, $\zeta(y_i)=1$ for each $1 \leqq i \leqq n$, and $\eta(u) = \zeta(u)$ for all $u \notin \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and*

(d) *conditionally on $n, x_1, \ldots, x_n, y_1, \ldots, y_n$, $n \geqq 1$, and the values of $\eta(u)$ for $x_1 \leqq u \leqq y_n$, $\{\eta(u), u < x_1$ or $u > y_n\}$ are independent Bernoulli random variables, with parameter λ for $u < x_1$ and $1-\lambda$ for $u > y_n$.*

Proof. Start by choosing $\eta(x) = \zeta(x)$ for $x \leqq -1$, with the appropriate distribution. Next choose $\eta(0) = \zeta(0) = 1$ with probability λ, $\eta(0) = \zeta(0) = 0$ with probability λ, and $\eta(0) = 1$, $\zeta(0) = 0$ with probability $1 - 2\lambda$. If $\eta(0) = \zeta(0)$, continue choosing $\eta(x) = \zeta(x)$ for $x \geqq 1$ with the appropriate distribution. If $\eta(0) = 1$, $\zeta(0) = 0$, complete the choice of $\eta(x)$, $\zeta(x)$ for $x \geqq 1$ in the following way. Suppose $\eta(x)$, $\zeta(x)$ have been chosen for $x < y$. If

$$\sum_{x=0}^{y-1} \eta(x) > \sum_{x=0}^{y-1} \zeta(x),$$

let $\eta(y)$ and $\zeta(y)$ be independent Bernoulli random variables with parameter $1-\lambda$, while if

$$\sum_{x=0}^{y-1} \eta(x) = \sum_{x=0}^{y-1} \zeta(x), \tag{2.5}$$

let $\eta(y)$ and $\zeta(y)$ be the same Bernoulli random variable with parameter $1-\lambda$. Note that (2.5) will happen eventually, because the random walk on z with increments $-1, +1$ with probability $\lambda(1-\lambda)$ each and 0 with probability $\lambda^2 + (1-\lambda)^2$ is recurrent. Once (2.5) occurs for one y, it will hold for all larger y's. □

Proof of Theorem (1.4). Assign to (η_0, ζ_0) the distribution constructed in Lemma (2.4). Then η_0 is distributed according to the measure μ in Theorem (1.4), and ζ_0 is distributed according to a translate μ_1 of μ. So if $f \in C(X)$,

$$\int f \, d\mu S(t) - \int f \, d\mu_1 S(t) = E f(\eta_t) - E f(\zeta_t). \tag{2.6}$$

In order to prove the theorem, it therefore suffices to show that if (η_t, ζ_t) evolves according to the basic coupling, then

$$\lim_{t \to \infty} P[\eta_t \equiv \zeta_t] = 1. \tag{2.7}$$

To see this, note that (2.7) implies that the right side of (2.6), and hence its left side, converges to zero as $t \to \infty$.

We need to show that the $2n$ discrepancies identified in (c) of Lemma (2.4) eventually disappear with probability one. By (d) of Lemma (2.4), it is enough to prove (2.7) when the initial distribution (η_0, ζ_0) is of the following type: (a) for some N, $\{\eta_0(u) = \zeta_0(u), |u| > N\}$ are independent Bernoulli random variables, with parameter λ for $u < -N$ and $1 - \lambda$ for $u > N$, and (b) there are $2n$ integers $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ in $[-N, N]$ with $x_i < y_i$ so that $\eta_0(u) = \zeta_0(u)$ for $u \in [-N, N] \setminus \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, $\eta_0(x_i) = 1$, $\zeta_0(x_i) = 0$, $\eta_0(y_i) = 0$, and $\zeta_0(y_i) = 1$ for $1 \leqq i \leqq n$. The configurations in $[-N, N]$ may be assumed to be nonrandom. Such a pair (η_0, ζ_0) will be said to be good with n pairs of discrepancies.

We next note that it suffices to take $n = 1$. To see this, suppose that (η_0, ζ_0) is good with n pairs of discrepancies, and define $\eta^0, \ldots, \eta^n$ via $\eta^k(x_i) = 1$ and $\eta^k(y_i) = 0$ if $i \leqq k$, $\eta^k(x_i) = 0$ and $\eta^k(y_i) = 1$ if $i > k$, and $\eta^k(u) = \eta_0(u) = \zeta_0(u)$ for all $u \notin \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Then $\eta^0 = \zeta_0$, $\eta^n = \eta_0$, and for each $0 \leqq k < n$, (η^{k+1}, η^k) is good with one pair of discrepancies. Of course, (2.7) for the original good pair (η_0, ζ_0) with n pairs of discrepancies follows from

$$\lim_{t \to \infty} P^{(\eta^{k+1}, \eta^k)}[\eta_t \equiv \zeta_t] = 1$$

for each k.

Finally consider the coupled process which has the following initial distribution: $\{\eta_0(u) = \zeta_0(u), u \neq -1, 0\}$ are independent Bernoulli, with parameter λ for $u < -1$ and parameter $1 - \lambda$ for $u > 0$, $\eta_0(-1) = 1$, $\eta_0(0) = 0$, $\zeta_0(-1) = 0$, $\zeta_0(0) = 1$. It is not hard to see that the distribution of the process at any time $t > 0$ has the property that any distribution which is good with one pair of discrepancies is absolutely continuous with respect to it. Hence it suffices to prove (2.7) for the above initial distribution.

Now regard the extra particles in the η and ζ configurations respectively as being the special particles in the associated processes which were discussed at the beginning of this section, and denote their locations by X_t and Y_t. The key observation is that on the event $\{\eta_t \neq \zeta_t\}$, $\eta_t(X_t) = 1$, $\eta_t(Y_t) = 0$, $\zeta_t(X_t) = 0$, and $\zeta_t(Y_t) = 1$, with $X_t < Y_t$, while on the event $\{\eta_t = \zeta_t\}$, $X_t = Y_t$. By Corollary (2.3), $EY_t \leqq EX_t + 1$. So, the distribution of $Y_t - X_t$ remains tight. Since $\{X_t = Y_t\}$ is an increasing event, it is a simple matter to check that $\lim_{t \to \infty} P[X_t < Y_t] = 0$. The idea is that if X_t and Y_t are a certain distance apart, then it is easy to give lower bounds, which depend only on that distance, on the probability that $Y_{t+1}-X_{t+1}\leqq Y_t-X_t$ with η_{t+1} having no particles between X_{t+1} and Y_{t+1}, and therefore also on the probability that $X_{t+2}=Y_{t+2}$. Consequently, $\lim_{t\to\infty} P[\eta_t\neq\zeta_t]=0$, which completes the proof of Theorem (1.4) □

3. Proof of Theorem (1.6)

The first step in the proof of Theorem (1.6) is to identify the difference between the right and left sides of (1.7) as a limit of time derivatives. Then we will evaluate these time derivatives by coupling the product measure μ having marginals given by (1.5) with $\mu S(t)$ for infinitesimally small t.

To carry out this program, we will need to use the generator Ω of the exclusion process:

$$\Omega f(\eta)=p\sum_{\eta(x)=1,\eta(x+1)=0}[f(\eta_{x,x+1})-f(\eta)]+q\sum_{\eta(x)=1,\eta(x-1)=0}[f(\eta_{x,x-1})-f(\eta)].$$

Here $\eta_{x,y}$ is obtained from η by interchanging the x and y coordinates. For more on the generator and its relation to the evolution, see Chaps. I and VIII of [7].

(3.1) **Lemma.** *For any $x\in Z$ and $t\geqq 0$,*

$$(p-q)\lambda(1-\lambda)-p\mu S(t)\{\eta:\eta(x)=1,\eta(x+1)=0\}+q\mu S(t)\{\eta:\eta(x)=0,\eta(x+1)=1\}$$
$$=\lim_{y\to-\infty}\frac{d}{dt}\sum_{u=y}^{x}\mu S(t)\{\eta:\eta(u)=1\}.$$

Proof. For fixed x and y in Z, let

$$f(\eta)=\sum_{u=y}^{x}\eta(u).$$

By semigroup theory,

$$\frac{d}{dt}\sum_{u=y}^{x}\mu S(t)\{\eta:\eta(u)=1\}=\frac{d}{dt}\int S(t)f\,d\mu=\int\Omega f\,d\mu S(t).$$

Therefore, the evaluation of Ωf yields

$$\frac{d}{dt}\sum_{u=y}^{x}\mu S(t)\{\eta:\eta(u)=1\}=p\mu S(t)\{\eta:\eta(y-1)=1,\eta(y)=0\}$$
$$-q\mu S(t)\{\eta:\eta(y-1)=0,\eta(y)=1\}-p\mu S(t)\{\eta:\eta(x+1)=0,\eta(x)=1\}$$
$$+q\mu S(t)\{\eta:\eta(x+1)=1,\eta(x)=0\}.$$

Since ν_λ is invariant for the process by Theorem (2.1) of Chap. VIII of [7],

$$\lim_{y\to-\infty} \mu S(t)\{\eta:\eta(y-1)=1,\eta(y)=0\}=\nu_\lambda\{\eta:\eta(u)=1,\eta(u+1)=0\}=\lambda(1-\lambda),$$

and

$$\lim_{y\to-\infty} \mu S(t)\{\eta:\eta(y-1)=0,\eta(y)=1\}=\nu_\lambda\{\eta:\eta(u)=1,\eta(u-1)=0\}=\lambda(1-\lambda\}.$$

The required result now follows by passing to the limit as $y\to-\infty$. □

Theorem (1.6) will follow once we show that the limit in Lemma (3.1) is nonnegative. The idea is that since ν_λ and $\nu_{1-\lambda}$ are invariant for the exclusion process, it should be possible to couple μ and $\mu S(t)$ together in such a way that discrepancies occur only at -1 and 0, at least to first order in t. It should then be possible to study the derivative in Lemma (3.1) by following the discrepancies.

To make this idea precise, consider the basic coupling $(\eta_{t,s},\zeta_{t,s})$ of two copies of the exclusion process. Time is denoted by t as usual. The parameter s is assumed to satisfy

$$0\leqq s\leqq \frac{(1-\lambda)^2}{(1-2\lambda)[q+(p-q)(2\lambda-\lambda^2)]}. \tag{3.2}$$

The distribution of $\zeta_{0,s}$ is chosen so that for small s, it is approximately equal to $\mu S(s)$. Specifically, the initial distribution $(\eta_{0,s},\zeta_{0,s})$ is chosen as follows: $\{\eta_{0,s}(x)=\zeta_{0,s}(x),\ x\neq-1,0\}$ are independent Bernoulli random variables, with parameter λ if $x<-1$ and parameter $1-\lambda$ if $x>0$. Independently of these variables,

$$\begin{pmatrix}\zeta_{0,s}(-1) & \zeta_{0,s}(0)\\ \eta_{0,s}(-1) & \eta_{0,s}(0)\end{pmatrix}$$

takes the following values, with the probabilities shown below:

Value	Probability
$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	λ^2
$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\lambda(1-\lambda)$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$s(1-2\lambda)[q+(p-q)\lambda^2]$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$s\lambda(1-\lambda)(1-2\lambda)(p-q)$
$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\lambda(1-\lambda)$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad s\lambda(1-\lambda)(1-2\lambda)(p-q)$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad (1-\lambda)^2 - s(1-2\lambda)[q+(p-q)(2\lambda-\lambda^2)]$$

The last probability is nonnegative by (3.2). Note that $\eta_{0,s}$ has distribution μ for each s, and that $(\zeta_{0,s}(-1), \zeta_{0,s}(0))$ takes the following values, with the probabilities shown below:

Value	Probability
(1, 1)	$\lambda(1-\lambda)+s\lambda(1-\lambda)(1-2\lambda)(p-q)$
(1, 0)	$\lambda^2+s(1-2\lambda)[q+(p-q)\lambda^2]$
(0, 1)	$(1-\lambda)^2-s(1-2\lambda)[q+(p-q)(2\lambda-\lambda^2)]$
(0, 0)	$\lambda(1-\lambda)+s\lambda(1-\lambda)(1-2\lambda)(p-q)$

Up to first order in s, $(\zeta_{0,s}(-1), \zeta_{0,s}(0))$ has the same distribution as does $(\eta(-1), \eta(0))$ under $\mu S(s)$. This is what lies behind the next lemma, whose proof is basically a generator computation. As in Chap. I of [7], let

$$D(X) = \{f \in C(X) : \sum_x \sup_\eta |f(\eta_x) - f(\eta)| < \infty\},$$

where η_x is obtained from η by flipping the coordinate $\eta(x)$. This class of functions is important because it is a core for the generator Ω and has the property that $f \in D(X)$ implies $S(t) f \in D(X)$.

(3.3) **Lemma.** *For any* $f \in D(X)$,

$$\lim_{s \downarrow 0} \frac{E f(\zeta_{0,s}) - \int f \, d\mu S(s)}{s} = 0.$$

Proof. By the definition of the generator,

$$\lim_{s \downarrow 0} \frac{\int f \, d\mu S(s) - \int f \, d\mu}{s} = \int \Omega f \, d\mu.$$

Using the explicit expression for the distribution of $\zeta_{0,s}$, we also see that

$$\lim_{s \downarrow 0} \frac{E f(\zeta_{0,s}) - \int f \, d\mu}{s} = \int f h \, d\mu,$$

where h is the following function:

$$h(\eta) = \begin{cases} (1-2\lambda)(p-q) & \text{if } \eta(-1)=1, \ \eta(0)=1 \\ (1-2\lambda)[q\lambda^{-2}+(p-q)] & \text{if } \eta(-1)=1, \ \eta(0)=0 \\ -(1-2\lambda)[q+(p-q)(2\lambda-\lambda^2)](1-\lambda)^{-2} & \text{if } \eta(-1)=0, \ \eta(0)=1 \\ (1-2\lambda)(p-q) & \text{if } \eta(-1)=0, \ \eta(0)=0. \end{cases}$$

We therefore need to show that

$$\int \Omega f \, d\mu = \int f h \, d\mu \qquad \text{for all } f \in D(X).$$

For $x \leqq -2$ or $x \geqq 0$, μ is invariant under the permutation of the coordinates $\eta(x)$ and $\eta(x+1)$. So for these values of x,

$$\begin{aligned} &\int f(\eta_{x,x+1})\{p\eta(x)[1-\eta(x+1)]+q\eta(x+1)[1-\eta(x)]\}\,d\mu \\ &\quad = \int f(\eta)\{p\eta(x+1)[1-\eta(x)]+q\eta(x)[1-\eta(x+1)]\}\,d\mu. \end{aligned}$$

We therefore obtain

$$\begin{aligned} &\int [f(\eta_{x,x+1})-f(\eta)]\{p\eta(x)[1-\eta(x+1)]+q\eta(x+1)[1-\eta(x)]\}\,d\mu \\ &\quad = (p-q)\int f(\eta)[\eta(x+1)-\eta(x)]\,d\mu. \end{aligned}$$

On the other hand, for $x=-1$,

$$\begin{aligned} &\int f(\eta_{-1,0})\{p\eta(-1)[1-\eta(0)]+q\eta(0)[1-\eta(-1)]\}\,d\mu \\ &\quad = \int f(\eta)\{p\eta(0)[1-\eta(-1)]\,\lambda^2(1-\lambda)^{-2}+q\eta(-1)[1-\eta(0)](1-\lambda)^2\,\lambda^{-2}\}\,d\mu. \end{aligned}$$

Consequently,

$$\begin{aligned} \int [f(\eta_{-1,0})-f(\eta)]\{p\eta(-1)[1-\eta(0)]+q\eta(0)[1-\eta(-1)]\}\,d\mu \\ = (p-q)\int f(\eta)[\eta(0)-\eta(-1)]\,d\mu \\ + \int f(\eta)\{q(1-2\lambda)\,\lambda^{-2}\,\eta(-1)[1-\eta(0)] \\ + p(2\lambda-1)(1-\lambda)^{-2}\eta(0)[1-\eta(-1)]\}\,d\mu. \end{aligned}$$

It follows that

$$\begin{aligned} \int \Omega f\,d\mu &= \int \sum_x [f(\eta_{x,x+1})-f(\eta)]\{p\eta(x)[1-\eta(x+1)]+q\eta(x+1)[1-\eta(x)]\}\,d\mu \\ &= \lim_{N\to\infty}(p-q)\int f(\eta)[\eta(N)-\eta(-N)]\,d\mu \\ &\quad + \int f(\eta)\{q(1-2\lambda)\,\lambda^{-2}\,\eta(-1)[1-\eta(0)] \\ &\quad + p(2\lambda-1)(1-\lambda)^{-2}\,\eta(0)[1-\eta(-1)]\}\,d\mu. \end{aligned}$$

For $f \in D(X)$,

$$\lim_{N\to\infty}\int f(\eta)\,\eta(N)\,d\mu = (1-\lambda)\int f\,d\mu,$$

and

$$\lim_{N\to-\infty}\int f(\eta)\,\eta(-N)\,d\mu = \lambda\int f\,d\mu.$$

Substituting this above, we obtain $\int \Omega f\,d\mu = \int f h\,d\mu$ as desired. □

Define $(\eta_{t,1}, \zeta_{t,1}), (\eta_{t,2}, \zeta_{t,2})$, and $(\eta_{t,3}, \zeta_{t,3})$ by conditioning $(\eta_{t,s}, \zeta_{t,s})$ on the events

$$\begin{pmatrix} \zeta_{0,s}(-1) & \zeta_{0,s}(0) \\ \eta_{0,s}(-1) & \eta_{0,s}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

respectively. (After conditioning, the distributions no longer depend on s.) Note that these initial configurations are the only ones in which $\eta_{0,s}$ and $\zeta_{0,s}$ differ.

(3.4) **Lemma.**

$$\begin{aligned} \frac{d}{dt} \sum_{u=y}^{x} \mu S(t)\{\eta : \eta(u) = 1\} &= (1-2\lambda)[q + (p-q)\lambda^2] \, E \sum_{u=y}^{x} [\zeta_{t,1}(u) - \eta_{t,1}(u)] \\ &\quad + \lambda(1-\lambda)(1-2\lambda)(p-q) \, E \sum_{u=y}^{x} [\zeta_{t,2}(u) - \eta_{t,2}(u)] \\ &\quad + \lambda(1-\lambda)(1-2\lambda)(p-q) \, E \sum_{u=y}^{x} [\zeta_{t,3}(u) - \eta_{t,3}(u)]. \end{aligned}$$

Proof. Let

$$g(\eta) = \sum_{u=y}^{x} \eta(u).$$

Then $g \in D(X)$, so $f = S(t) g \in D(X)$ by Theorem (3.9) of Chap. I of [7]. Therefore, letting $\mu_t = \mu S(t)$, we can compute

$$\begin{aligned} \frac{d}{dt} \sum_{u=y}^{x} \mu_t\{\eta : \eta(u) = 1\} &= \lim_{s \downarrow 0} \frac{1}{s} \sum_{u=y}^{x} [\mu_{t+s}\{\eta : \eta(u) = 1\} - \mu_t\{\eta : \eta(u) = 1\}] \\ &= \lim_{s \downarrow 0} \frac{1}{s} [\int g \, d\mu_{t+s} - \int g \, d\mu_t] \\ &= \lim_{s \downarrow 0} \frac{1}{s} [\int f \, d\mu_s - \int f \, d\mu] \\ &= \lim_{s \downarrow 0} \frac{E f(\zeta_{0,s}) - \int f \, d\mu}{s}, \end{aligned}$$

where the last equality follows from Lemma (3.3). This in turn equals

$$\begin{aligned} \lim_{s \downarrow 0} \frac{E f(\zeta_{0,s}) - E f(\eta_{0,s})}{s} &= \lim_{s \downarrow 0} \frac{E g(\zeta_{t,s}) - E g(\eta_{t,s})}{s} \\ &= \lim_{s \downarrow 0} \frac{1}{s} E \sum_{u=y}^{x} [\zeta_{t,s}(u) - \eta_{t,s}(u)]. \end{aligned}$$

The assertion of the lemma now follows from the specified distribution of (η_0, ζ_0). $\square$

Proof of Theorem (1.6). Combining Lemmas (3.1) and (3.4), it now suffices to show that

(3.5) $$\lim_{y\to-\infty} E\sum_{u=y}^{x}[\zeta_{t,1}(u)-\eta_{t,1}(u)]\geqq 0$$

and

(3.6) $$\lim_{y\to-\infty} E\sum_{u=y}^{x}[\zeta_{t,2}(u)-\eta_{t,2}(u)+\zeta_{t,3}(u)-\eta_{t,3}(u)]\geqq 0.$$

For the first inequality, note that either $\zeta_{t,1}=\eta_{t,1}$ or for some $u<v$, $\zeta_{t,1}(u)=1$, $\zeta_{t,1}(v)=0$, $\eta_{t,1}(u)=0$, $\eta_{t,1}(v)=1$. Therefore

$$\sum_{u=-\infty}^{x}[\zeta_{t,1}(u)-\eta_{t,1}(u)]$$

equals 0 or 1 with probability one for each $x\in Z$ and $t\geqq 0$.

To prove (3.6), we need to return to Lemmas (2.1) and (2.2). Let (η_t, X_t) and (ζ_t, Y_t) be defined by $\eta_t=\eta_{t,2}$, $\zeta_t=\zeta_{t,3}$,

$$X_t = \text{that } z \text{ for which } \zeta_{t,2}(z)=1, \quad \eta_{t,2}(z)=0,$$

and

$$Y_t = \text{that } z \text{ for which } \zeta_{t,3}(z)=0, \quad \eta_{t,3}(z)=1.$$

Then (η_t, X_t) and (ζ_t, Y_t) evolve according to the mechanism described at the beginning of Sect. 2. Furthermore,

$$E\sum_{u=y}^{x}[\zeta_{t,2}(u)-\eta_{t,2}(u)]=P[y\leqq X_t\leqq x],$$

and

$$E\sum_{u=y}^{x}[\zeta_{t,3}(u)-\eta_{t,3}(u)]=-P[y\leqq Y_t\leqq x].$$

It therefore suffices to show that

$$P(X_t\leqq x)\geqq P(Y_t\leqq x).$$

Now, Lemma (2.2) implies that

$$P(X_1\leqq x)=P(Y_t\geqq -1-x).$$

It is therefore enough to show that $P(Y_t\geqq -x)-P(Y_t\leqq x)\geqq 0$. To check this, apply Lemma (2.1) to the increasing function

$$f(u)=1_{\{u\geqq -x\}}-1_{\{u\leqq x\}}$$

to conclude that $Ef(Y_t)$ (with $\zeta_0(-1)=0$) is greater than it would be if $\zeta_0(-1)$ were instead Bernoulli with parameter λ. But in that case, since f is odd, $Ef(Y_t)$

would be zero by Lemma (2.2). It follows that $Ef(Y_t) \geqq 0$ in the present case, and so

$$P(Y_t \geqq -x) - P(Y_t \leqq x) \geqq 0$$

as desired. □

4. Open Problems

In this section we introduce several open problems which, if solved, could be viewed as generalizations of part (d) of Theorem (1.1). We begin by defining τ_x: $X \to X$ by $\tau_x(\eta)(y) = \eta(y-x)$. The corresponding mapping on the set of all probability measures on X will also be denoted by τ_x. Suppose that μ is the product probability measure on X with marginals given by

$$\mu\{\eta : \eta(x) = 1\} = \begin{cases} \lambda & \text{if } x < 0 \\ \rho & \text{if } x \geqq 0. \end{cases}$$

where $0 \leqq \lambda, \rho \leqq 1$. In [2] and [3], it has been shown that

$$\text{(4.1)} \qquad \lim_{\varepsilon \downarrow 0} \mu S(t/\varepsilon)\, \tau_{[x/\varepsilon]} = v_{u(x,t)} \qquad \text{if } \lambda \geqq \rho \text{ or } x \neq (1-\lambda-\rho)(p-q)\, t,$$

when $[\cdot]$ denotes the integer part. The parameter $u(x, t)$ of the limiting product measure is the weak entropy solution of the partial differential equation

$$\text{(4.2)} \qquad \frac{\partial u}{\partial t} + (p-q)\frac{\partial}{\partial x}[u(1-u)] = 0$$

with initial condition

$$u(x, 0) = \begin{cases} \lambda & \text{if } x < 0 \\ \rho & \text{if } x \geqq 0. \end{cases}$$

The solution to this equation exhibits a shock wave if $\lambda < \rho$. For this fact and relevant definitions and results concerning weak solutions, the reader is referred to [9]. In the present case, the shock wave travels at the velocity $(1-\lambda-\rho)(p-q)$. Hence, it should not be surprising that the limit in (4.1) is not known only when $\lambda < \rho$ and $x = (1-\lambda-\rho)(p-q)\, t$. Our first open problem is therefore to determine the limit in this case. We have settled the special case $\rho = 1-\lambda > 1/2$ and $x = 0$ in this paper. Some of the techniques used in [1] and the present paper are likely to be helpful in the general case. New ideas will probably be required as well. The first result concerning the limit in (4.1) was obtained in [8], and is described and somewhat generalized in Sect. 5 of Chap. VIII of [7].

Our second problem is also related to the behavior of the process in the presence of a shock wave. We will present it in its simplest context. Let μ

be the above product measue with $\rho=1-\lambda$. Is it the case that for some increasing function $f(t)$

$$\lim_{t\to\infty} \mu S(t)\, \tau_{[cf(t)]} = \alpha(c)\, \nu_\lambda + (1-\alpha(c))\, \nu_{1-\lambda} \tag{4.3}$$

for all $-\infty<c<\infty$, where $\alpha(\cdot)$ is a strictly increasing continuous function? Something like this is true in the symmetric case, when $p=1/2$ (see Theorem 5.2 of Chap. VIII of [7]). This result in the symmetric case, together with the corollary to Theorem 1 in [10], suggest that $f(t)=\sqrt{t}$ is the right choice. Showing the existence of the limit in (4.3), and then identifying the function $\alpha(c)$ are problems which appear to require a new approach. The case $c=0$ in (4.3) is just part (d) of Theorem (1.1).

There is a heuristic argument based on coupling which explains why one might expect (4.3) to hold for some choice of $f(t)$. Consider the basic coupling of three copies of the exclusion process η_t, γ_t and ζ_t in which η_0 has distribution ν_λ, ζ_0 has distribution $\nu_{1-\lambda}$, γ_0 is distributed according to the product measure μ with marginals given by (1.5), and $\eta_t(x)\leqq\gamma_t(x)\leqq\zeta_t(x)$ for all $x\in Z$ and $t\geqq 0$ (see Sect. 2 of Chap. VIII of [7] for a description of the basic coupling). Then η_t has distribution ν_λ for all t, ζ_t has distribution $\nu_{1-\lambda}$ for all t, and γ_t has distribution $\mu S(t)$. Consider the leftmost x so that $\gamma_t(x)=1$ and $\eta_t(x)=0$, and call its position X_t. Similarly, let Y_t be the rightmost x so that $\zeta_t(x)=1$ and $\gamma_t(x)=0$. If $p=1$, then $Y_t\leqq X_t$ for all t. One might expect that even if $1/2<p<1$, the drift to the right of the particles would be strong enough so that $(Y_t-X_t)^+$ would not be too large. So, it may be that X_t and Y_t move more or less together approximately like a Brownian motion. By definition, $\gamma_t(x)=\eta_t(x)$ for $x<X_t$ and $\gamma_t(x)=\zeta_t(x)$ for $x>Y_t$. Therefore, if g is a function on X which depends only on the coordinates $\eta(x)$ for $a\leqq x\leqq b$, it follows that

$$\begin{aligned}\int g\, d\mu S(t) = E g(\gamma_t) = E[g(\eta_t), X_t>b] + E[g(\zeta_t), X_t\leqq b, Y_t<a] \\ + E[g(\gamma_t), X_t\leqq b, Y_t\geqq a].\end{aligned}$$

While it appears difficult to construct a rigorous argument based on these ideas, they do suggest that (4.3) is probably true.

Since the first three parts of Theorem (1.1) and (4.1) have been proved for systems in which the nearest neighbor assumption is replaced by a first moment assumption on $p(x,y)$ (see [6] and [2], respectively), our third open problem is to remove the nearest neighbor hypothesis in part (d) of Theorem (1.1) as well. Again, a new technique seems to be needed.

It is of course possible to combine two or three of the above problems, or to state them for other processes, such as those studied in [4]. For many of these processes, the limit in (4.1) is known, except in the presence of a shock wave (see [2]). These shock waves occur when the initial condition is "increasing". As far as we know, the only progress on these problems that has been made is in [10], which deals with the zero range process.

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