Probab. Th. Rel. Fields 78, 223–230 (1988)

Probability Theory and Related Fields

Comparison of Moments for Tangent Sequences of Random Variables

Paweł Hitczenko
Department of Statistics, Academy of Physical Education, Warsaw, Poland

Summary. It is shown that for all tangent sequences (d_n) and (e_n) of nonnegative or conditionally symmetric random variables and for every function Φ satisfying the growth condition $\Phi(2x) \leqq \alpha\Phi(x)$ the following inequality holds: $E\Phi\left(\sup_n \left|\sum_{k=1}^n d_k\right|\right) \leqq cE\Phi\left(\sup_n \left|\sum_{k=1}^n e_k\right|\right)$. This generalizes results of J. Zinn and proves a conjecture of S. Kwapień and W.A. Woyczyński.

The aim of this paper is to prove the following conjecture of S. Kwapień and W.A. Woyczyński [2, Conjecture 2.1].

Let $\Phi: R_+ \to R_+$ be an increasing continuous function satisfying the Δ_2-condition (i.e., $\Phi(2x) \leqq \alpha\Phi(x)$, $x \geqq 0$) and such that $\Phi(0)=0$. Then, there exists a constant c (depending only on Φ) such that for all sequences (d_n) and (e_n) of adapted random variables with identical conditional distributions which are either non-negative or conditionally symmetric we have:

$$E\Phi\left(\sup_n \left|\sum_{k=1}^n d_k\right|\right) \leqq cE\Phi\left(\sup_n \left|\sum_{k=1}^n e_k\right|\right). \tag{1}$$

Such "decoupling" inequalities have been introduced in [3] as a tool in the study of multiple stochastic integrals and have been extended by several authors [4]. For example for $\Phi(x)=|x|^p$, $0<p<1$ inequality (1) was obtained by J. Zinn [4] or [5].

Our proof is based on the techniques of D.L. Burkholder [1]. We shall use the following notation. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \dots$ be an ascending sequence of sub-σ-fields of $\mathscr{F}$. A sequence (d_n) of random variables on $(\Omega, \mathscr{F}, P)$ is adapted if d_n is $\mathscr{F}_n$-measurable $n=1, 2, \dots$. As usual d^* stands for $\sup_n |d_n|$, $d_n^* = \max_{k \leqq n} |d_k|$ and if $f_n = d_1 + \dots + d_n$ then $S(f) = \left(\sum_{k=1}^\infty d^2{}_k\right)^{\frac{1}{2}}$. If (d_n) and (e_n) are adapted sequences of random variables, then (e_n) is said to be tangent to (d_n) if for each real number x we have that $P(d_n$

$>x\,|\,\mathscr{F}_{n-1})=P(e_n>x\,|\,\mathscr{F}_{n-1})$ a.s., $n=1,2,\ldots$. A sequence (d_n) is conditionally symmetric if $(-d_n)$ is tangent to (d_n). The letter c is used to denote a positive real number, not necessarily the same from one use to the next.

We begin with the following lemma (for a different proof with constant 6 instead of 2 see [2, Corollary 2.1]).

Lemma 1. *Assume that* $A_n, B_n \in \mathscr{F}_n$, $n=1,2,\ldots$ *and that there exists a positive number* c such that $P(A_n\,|\,\mathscr{F}_{n-1}) \leqq cP(B_n\,|\,\mathscr{F}_{n-1})$, $n=1,2,\ldots$. *Then*

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq (c+1)\, P\left(\bigcup_{n=1}^{\infty} B_n\right).$$

In particular, if $P(A_n\,|\,\mathscr{F}_{n-1})=P(B_n\,|\,\mathscr{F}_{n-1})$ $n=1,2,\ldots$ *then*

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq 2P\left(\bigcup_{n=1}^{\infty} B_n\right).$$

Proof. Let $C_1=A_1$, $D_1=B_1$, $C_n=B_1^c \cap \ldots \cap B_{n-1}^c \cap A_n$, $D_n=B_1^c \cap \ldots \cap B_{n-1}^c \cap B_n$, $n=2,3,\ldots$ where A^c denotes the complement of A. Then D_n, $n=1,2,\ldots$ are disjoint and $\bigcup_{n=1}^{\infty} D_n=\bigcup_{n=1}^{\infty} B_n$.

$$\begin{aligned} P\left(\bigcup_{n=1}^{\infty} A_n\right) &= P\left(\bigcup_{n=1}^{\infty} A_n \cap \bigcup_{n=1}^{\infty} B_n\right)+P\left(\bigcup_{n=1}^{\infty} A_n \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\leqq P\left(\bigcup_{n=1}^{\infty} B_n\right)+P\left(\bigcup_{n=1}^{\infty} A_n \cap \bigcap_{n=1}^{\infty} B_n^c\right). \end{aligned}$$

The second term on the right-hand side can be estimated from above as follows:

$$\begin{aligned} P\left(\bigcup_{n=1}^{\infty} A_n \cap \bigcup_{n=1}^{\infty} B_n^c\right) &\leqq P\left(\bigcup_{n=1}^{\infty}\left(A_n \cap \bigcap_{k=1}^{n-1} B_k^c\right)\right)=P\left(\bigcup_{n=1}^{\infty} C_n\right) \\ &= P\left(\sum_{n=1}^{\infty} I(C_n) \geqq 1\right) \leqq E\sum_{n=1}^{\infty} I(C_n)=E\sum_{n=1}^{\infty} E(I(C_n)\,|\,\mathscr{F}_{n-1}) \\ &= E\sum_{n=1}^{\infty} EI(B_1^c \cap \ldots \cap B_{n-1}^c)\cdot P(A_n\,|\,\mathscr{F}_{n-1}) \\ &\leqq cE\sum_{n=1}^{\infty} EI(B_1^c \cap \ldots \cap B_{n-1}^c)\cdot P(B_n\,|\,\mathscr{F}_{n-1}) \\ &= cE\sum_{n=1}^{\infty} I(D_n)=c\sum_{n=1}^{\infty} P(D_n)=cP\left(\bigcup_{n=1}^{\infty} D_n\right)=cP\left(\bigcup_{n=1}^{\infty} B_n\right). \end{aligned}$$

Thus

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq (1+c)\cdot P\left(\bigcup_{n=1}^{\infty} B_n\right). \quad \text{Q.E.D.}$$

Now, we are going to prove inequality (1) for nonnegative random variables. The case of conditionally symmetric random variables will be obtained as a corollary to the nonnegative case.

Theorem 2. *Let Φ be as above. Then there exists a positive constant c (depending only on Φ) such that for all tangent sequences (d_n) and (e_n) of nonnegative random variables one has:*

$$E\Phi\left(\sum_{n=1}^{\infty} d_n\right) \leqq c E\Phi\left(\sum_{n=1}^{\infty} e_n\right).$$

Proof. Let Φ, (d_n) and (e_n) satisfy the required conditions. Put

$$\begin{aligned} z_n &= d_{n-1}^* \vee e_{n-1}^* \\ a_n &= d_n I(d_n \leqq 2 z_n) \\ b_n &= d_n I(d_n > 2 z_n) \\ x_n &= e_n I(e_n \leqq 2 z_n) \qquad n = 1, 2, \ldots. \end{aligned}$$

Then

$$E\Phi\left(\sum_{n=1}^{\infty} d_n\right) \leqq c E\Phi\left(\sum_{n=1}^{\infty} a_n\right) + c E\Phi\left(\sum_{n=1}^{\infty} b_n\right).$$

To estimate the second term on the right-hand side observe that on the set $\{d_n > 2 z_n\}$

$$d_n + 2 z_n \leqq 2 d_n \leqq 2 z_{n+1},$$

hence

$$b_n \leqq 2(z_{n+1} - z_n)$$

and

$$\sum_{n=1}^{\infty} b_n \leqq 2 z^* \leqq 2(d^* \vee e^*).$$

Consequently,

$$E\Phi\left(\sum_{n=1}^{\infty} b_n\right) \leqq c E\Phi(d^* \vee e^*) \leqq c E\Phi(d^*) + c E\Phi(e^*).$$

The quantity $E\Phi\left(\sum_{n=1}^{\infty} a_n\right)$ can be estimated in the following way: let $\delta > 0$, $\beta > 1 + \delta$. For a positive number λ define the stopping times μ, ν and τ as follows:

$$\mu = \inf\left\{n : \sum_{k=1}^{n} a_k > \lambda\right\},$$

$$\nu = \inf\left\{n : \sum_{k=1}^{n} a_k > \beta \lambda\right\},$$

$$\tau = \inf\left\{n : \sum_{k=1}^{n} x_k > \delta \lambda \text{ or } z_{n+1} > \delta \lambda\right\}.$$

If $v_k = I(\mu < k \leqq \nu \wedge \tau)$ then v_k is $\mathscr{F}_{k-1}$-measurable, $k = 1, 2, \ldots$. Note that

$$\{\tau \geqq k\} \subset \left\{z_k \leqq \delta\lambda, \sum_{j=1}^{k-1} x_j \leqq \delta\lambda\right\}.$$

Since $z_k \leqq \delta\lambda \Rightarrow x_k \leqq 2\delta\lambda$, we have

$$\sum_{n=1}^{\infty} v_n x_n \leqq 3\delta\lambda I(\mu < \infty) = 3\delta\lambda I\left(\sum_{n=1}^{\infty} a_n > \lambda\right\}.$$

On $\{\nu < \infty, \tau = \infty\}$, since $a_\mu \leqq d_\mu \leqq \delta\lambda$ we have

$$\sum_{k=1}^{\nu} a_k > \beta\lambda \quad \text{and} \quad \sum_{k=1}^{\infty} v_k a_k = \sum_{k=1}^{\nu} a_k - \sum_{k=1}^{\mu-1} a_k - a_\mu > \beta\lambda - \lambda - \delta\lambda.$$

Therefore

$$\begin{aligned} &P\left(\sum_{n=1}^{\infty} a_n > \beta\lambda, \left(\sum_{n=1}^{\infty} x_n\right) \vee z^* \leqq \delta\lambda\right) \\ &\leqq P\left(\sum_{n=1}^{\infty} v_n a_n > (\beta - 1 - \delta)\lambda\right) \\ &\leqq \frac{1}{(\beta-1-\delta)\lambda} E \sum_{n=1}^{\infty} v_n a_n = \frac{1}{(\beta-1-\delta)\lambda} E \sum_{n=1}^{\infty} v_n E(a_n | \mathscr{F}_{n-1}) \\ &= \frac{1}{(\beta-1-\delta)\lambda} E \sum_{n=1}^{\infty} v_n E(x_n | \mathscr{F}_{n-1}) \\ &= \frac{1}{(\beta-1-\delta)\lambda} E \sum_{n=1}^{\infty} v_n x_n \leqq \frac{3\delta}{\beta-1-\delta} P\left(\sum_{n=1}^{\infty} a_n > \lambda\right). \end{aligned}$$

By Lemma 7.1 of [1] we obtain

$$\begin{aligned} E\Phi\left(\sum_{n=1}^{\infty} a_n\right) &\leqq cE\Phi\left(\sum_{n=1}^{\infty} x_n\right) + cE\Phi(z^*) \\ &\leqq cE\Phi\left(\sum_{n=1}^{\infty} e_n\right) + cE\Phi(d^*) + cE\Phi(e^*). \end{aligned}$$

Consequently

$$E\Phi\left(\sum_{n=1}^{\infty} d_n\right) \leqq cE\Phi\left(\sum_{n=1}^{\infty} e_n\right) + cE\Phi(d^*) + cE\Phi(e^*).$$

By Lemma 1 applied to the sets $A_n = \{d_n > t\}$, $B_n = \{e_n > t\}$, $n = 1, 2, \ldots$ we infer that for each positive number t

$$P(d^* > t) \leqq 2P(e^* > t),$$

hence

$$E\Phi(d^*) \leqq cE\Phi(e^*)$$

and finally

$$E\Phi\left(\sum_{n=1}^{\infty} d_n\right) \leqq cE\Phi\left(\sum_{n=1}^{\infty} e_n\right) + cE\Phi(e^*) \leqq cE\Phi\left(\sum_{n=1}^{\infty} e_n\right)$$

which completes the proof.

Remark 3. For functions $\Phi(x) = |x|^p$, $0 < p < 1$ Theorem 2 has been obtained by J. Zinn [5, Theorem 1.4(ii)] under the assumption that

$$\forall t > 0 \qquad \sum_{n=1}^{\infty} E(d_n \wedge t \mid \mathscr{F}_{n-1}) \leqq \sum_{n=1}^{\infty} E(e_n \wedge t \mid \mathscr{F}_{n-1})$$

which is weaker than ours. We would like to indicate another proof of this result briefly; which also allows us to replace concave powers by any concave function Φ. As was observed by D.L. Burkholder [1, the proof of Theorem 20.1], in order to prove that

$$E\Phi(X) \leqq cE\Phi(Y)$$

for concave function Φ and nonnegative random variables X and Y, it suffices to show that for each positive number t, one has

$$EX \wedge t \leqq cEY \wedge t.$$

Let (x_n) and (y_n) be adapted sequences of nonnegative random variables such that $y_n \leqq 1$, $n = 1, 2, \ldots$ and

$$\sum_{n=1}^{\infty} E(x_n \mid \mathscr{F}_{n-1}) \leqq \sum_{n=1}^{\infty} E(y_n \mid \mathscr{F}_{n-1}).$$

Then, by [1, Theorem 20.1] and our assumption

$$E\left(\sum_{n=1}^{\infty} x_n\right) \wedge 1 \leqq 2E\left(\sum_{n=1}^{\infty} E(x_n \mid \mathscr{F}_{n-1})\right) \wedge 1$$

$$\leqq 2E\left(\sum_{n=1}^{\infty} E(y_n \mid \mathscr{F}_{n-1})\right) \wedge 1,$$

and if $\tau=\inf\left\{n\colon \sum_{k=1}^{n} y_k>1\right\}$, then

$$E\left(\sum_{n=1}^{\infty} E(y_n|\mathscr{F}_{n-1})\right)\wedge 1$$

$$\leqq E\sum_{n=1}^{\tau} E(y_n|\mathscr{F}_{n-1})+P(\tau<\infty)=E\sum_{n=1}^{\tau} y_n+P(\tau<\infty)$$

$$\leqq E\left(\sum_{n=1}^{\infty} y_n\right)\wedge 1+2P(\tau<\infty)\leqq 3E\left(\sum_{n=1}^{\infty} y_n\right)\wedge 1.$$

Hence

$$E\left(\sum_{n=1}^{\infty} x_n\right)\wedge 1\leqq 6E\left(\sum_{n=1}^{\infty} y_n\right)\wedge 1.$$

Now, an application of the above inequality to the sequences $x_n=\frac{d_n}{t}\wedge 1$ and $y_n=\frac{e_n}{t}\wedge 1$ yields

$$E\Phi\left(\sum_{n=1}^{\infty} d_n\right)\leqq 6E\Phi\left(\sum_{n=1}^{\infty} e_n\right)$$

as required.

Now we turn our attention to the case of conditionally symmetric random variables. The following theorem is a simple corollary to the proof of the two-sided convex function inequality between the square function and the maximal function of a martingale [1, the proof of Theorem 15.1]. We include the proof for the sake of completeness. Recall that if Φ is an increasing, continuous function satisfying Δ_2-condition and $\Phi(0)=0$, then there exists a constant c such that for every martingale (f_n) with difference sequence (d_n) such that $|d_n|\leqq w_n$, where w_n is $\mathscr{F}_{n-1}$-measurable, $n=1, 2, \ldots$ the following inequalities hold:

$$E\Phi(f^*)\leqq cE\Phi(S(f))+cE\Phi(w^*) \tag{2}$$

and

$$E\Phi(S(f))\leqq cE\Phi(f^*)+cE\Phi(w^*) \tag{3}$$

(see Sect. 12 of [1]).

Theorem 4. *Let Φ be as in Theorem 2. Then there exists a constant c (depending only on Φ) such that for every adapted and conditionally symmetric sequence (d_n) the following inequality holds:*

$$c^{-1}E\Phi(S(f))\leqq E\Phi(f^*)\leqq cE\Phi(S(f)),$$

where $f_n=d_1+\ldots+d_n$, $n=1, 2, \ldots$.

Proof. Let (d_n) be a conditionally symmetric sequence and $f_n = d_1 + \ldots + d_n$, $n = 1, 2, \ldots$. Following Burkholder [1, the proof of Theorem 15.1] write

$$f_n = g_n + h_n$$

where

$$g_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} d_k I(|d_k| \leqq 2 d_{k-1}^*),$$

$$h_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} d_k I(|d_k| > 2 d_{k-1}^*).$$

Then

$$f^* \leqq g^* + h^* \leqq g^* + \sum_{n=1}^{\infty} |b_n|, \tag{4}$$

$$S(g) \leqq S(f) + S(h) \leqq S(f) + \sum_{n=1}^{\infty} |b_n| \tag{5}$$

and

$$E\Phi(f^*) \leqq cE\Phi(g^*) + cE\Phi\left(\sum_{n=1}^{\infty} |b_n|\right).$$

Applying inequality (2) to the martingale (g_n) and using (5) we get

$$\begin{aligned} E\Phi(g^*) &\leqq cE\Phi(S(g)) + cE\Phi(2d^*) \\ &\leqq cE\Phi(S(f)) + cE\Phi\left(\sum_{n=1}^{\infty} |b_n|\right) + cE\Phi(d^*). \end{aligned}$$

Since $d^* \leqq S(f)$ and $\sum_{n=1}^{\infty} |b_n| \leqq 2d^* \leqq 2S(f)$ we conclude that

$$E\Phi(f^*) \leqq cE\Phi(S(f)).$$

The proof of the reverse inequality is the same: use $S(f) \leqq S(g) + \sum_{n=1}^{\infty} |b_n|$ instead of (4), $g^* \leqq f^* + \sum_{n=1}^{\infty} |b_n|$ instead of (5) and (3) instead of (2). Q.E.D.

Corollary 5. *Let Φ be as in Theorem 4. Then there exists a positive number c (depending only on Φ) such that for all tangent and conditionally symmetric sequences (d_n) and (e_n) we have that*

$$E\Phi(f^*) \leqq cE\Phi(g^*),$$

where $f_n = d_1 + \ldots + d_n$ and $g_n = e_1 + \ldots + e_n$, $n = 1, 2, \ldots$.

Proof. By Theorem 4 we have

$$E\Phi(f^*) \leqq cE\Phi(S(f)),$$
$$E\Phi(S(g)) \leqq cE\Phi(g^*),$$

and by Theorem 2 applied to the function $\Psi(t) = \Phi(t^{\frac{1}{2}})$, and sequences (d_n^2) and (e_n^2) we get

$$E\Phi(S(f)) \leqq cE\Phi(S(g)). \quad \text{Q.E.D.}$$

References

1. Burkholder, D.L.: Distribution function inequalities for martingales. Ann. Probab. **1**, 19–42 (1973)
2. Kwapień, S., Woyczyński, W.A.: Semimartingale integrals via decoupling inequalities and tangent processes. Case Western Reserve University, preprint, (1986)
3. McConnell, T., Taqqu, M.: Double integration with respect to symmetric stable processes. Preprint
4. Seminar notes on multiple stochastic integration, polynomial chaos and their applications. Case Western Reserve University, preprint (1985)
5. Zinn, J.: Comparison of martingale difference sequences. In: Beck, A., Dudley, R., Hahn, M., Kuelbs, J., Marcus, M. (eds.). Probability in Banach spaces V. (Lect. Notes Math., vol. 1153, pp. 453–457.) Berlin Heidelberg New York: Springer 1985

Received November 3, 1986; received in revised form December 9, 1987