Probab. Th. Rel. Fields 79, 249–269 (1988)

Probability Theory and Related Fields

The Itô Formula for Anticipative Processes with Nonmonotonous Time Scale via the Malliavin Calculus

A.S. Ustunel

2, Bd. Auguste Blanqui, F-75013 Paris, France

Summary. This work is devoted to derive Itô-type formulae for anticipative stochastic processes with nonmonotonous time using the Malliavin Calculus techniques and the fundamental theorem of the differential calculus. The same method is applied also to give an Itô-Ventcell type formula in the anticipative case.

Introduction

Suppose that F is a smooth function on $\mathbb{R}^n$ and (W_t) is a standard $\mathbb{R}^n$-valued Wiener process. The Itô formula says that $(F(W_t); t\in[0,1])$ is a semimartingale (more precisely an Itô process) with the following decomposition:

$$F(W_t)=F(W_0)+\int_0^t \partial_i F(W_s)\,dW_s^i+(1/2)\int_0^t \triangle F(W_s)\,ds.$$

On the other hand, if φ is a smooth Wiener functional with zero expectation, the Clark-Itô representation theorem gives φ as

$$\varphi=\int_0^1 E[\dot{\nabla}\varphi(s)\,|\,\mathscr{F}_s]\,dW_s$$

where $\nabla\varphi$ is the Sobolev derivative of φ on the canonical Wiener space (cf. [18]), so it is a random variable with values in the Cameron-Martin space of absolutely continuous functions with square integrable density with respect to the Lebesgue measure on [0, 1] denoted by H, and the Lebesgue density of $\nabla\varphi$ is denoted by $\dot{\nabla}\varphi$. $E[\cdot|\mathscr{F}_s]$ denotes the conditional expectation with respect to the data of the Wiener paths upto the time s. Now, if we look at $F(W_t)$ as an element of L^2, denoting by $\langle\cdot,\cdot\rangle$ the duality form, we have

$$\langle F(W_t),\varphi\rangle=\langle F(W_0),\varphi\rangle+\int_0^t \langle\partial_i F(W_s),\dot{\nabla}\varphi(s)\rangle\,ds$$

$$+(1/2)\int_0^t \langle\triangle F(W_s),\varphi\rangle\,ds.$$

This means that the mapping $t \to \langle F(W_t), \varphi \rangle$ is absolutely continuous with respect to the Lebesgue measure and the corresponding density is

$$\sum_i \langle \partial_i F(W_s), \dot{\nabla}_i \varphi(s) \rangle + (1/2) \langle \triangle F(W_s), \varphi \rangle$$

which is defined ds-almost everywhere. There is no reason for not proceeding in the reverse order to prove the Itô formula: What we need is to calculate the Lebesgue density of $t \to \langle F(W_t), \varphi \rangle$ for any test function φ, then to use the identity

$$\langle F(W_t), \varphi \rangle = \langle F(W_0), \varphi \rangle + \int_0^t \frac{d}{ds} \langle F(W_s), \varphi \rangle \, ds$$

and identify the integral with the help of the Clark-Itô representation and the Fubini theorems. This work is devoted to the applications of the idea explained above with the following generalizations: Instead of the Wiener process we take an Itô process. When we make the calculations explained above, since the Clark-Itô Representation Theorem is a consequence of the integration by parts formula on the Wiener space, for which the adaptedness condition is superfluous (cf. [5, 18, 21]), we realize that we can supress it. In this case the natural extension of the Itô integral which is most compatible with the Malliavin calculus formalism is the divergence operator and it was shown in [3] that it coincides with the Skorohod integral on a reasonable domain. Consequently, we take an Itô process with a Skorohod integral part, an absolutely continuous part and an initial condition, neither being adapted.

If one can achieve the project explained above, since he would be working in a frame in which the orientation of time is not important, he may ask himself if the ordinary time scale can be replaced such that each coordinate of the above process depends on a clock $\theta_i(t)$, where θ_i is a C^1-function from $[0, 1]$ into itself. These kinds of problems are encountered in random differential geometry (cf. [1, 19]) when one constructs stochastic integrals indexed by the chains. This problem is solved in Sect. III of this work.

In all these situations, in order to calculate the Lebesgue densities, we have proceeded by calculating the pointwise derivatives, however, as the reader will realize, this is not sufficient to use the Fundamental Theorem of the Differential Calculus, in fact we also need the derivative to be continuous. In order to circumvent this difficulty we use the nice technique of P. Malliavin called redefinition, which consists of taking the conditional expectations on a smooth basis of the Cameron-Martin space and applying to them the finite dimensional Sobolev injection theorems (cf. [7]). For the same reason, we use a class of test functionals which is smaller than that of S. Watanabe, which we have constructed with the redefinition technique.

Let us explain the order of the sections. Having recalled some elementary results about the Malliavin calculus and the distributions on the classical Wiener space, we prove the Itô formula for the anticipative Itô processes with the ordinary time scale in the Sect. II. Section III is devoted to the proof of the Itô formula for anticipative Itô processes with a nonmonotonous, differentiable time scale. In Sect. IV we prove an Itô type formula for a random field $F(x, \omega)$ when we replace x with an anticipating Itô process and, as shown in the example of application, this result covers the results of Hitsuda also (cf. [4]).

We think that the method used here to prove all these formulae will be useful for studying the higher dimensional random fields, since one can now profit from the tools of the classical differential geometry thanks to the redefinition technique. Let us note that we already have applied this method and the results of Sects. III and IV to study the anticipative stochastic differential equations and the filtering of the diffusion processes (cf. [20]).

The results of the Sect. II have been announced as a note in CRAS (cf. [19]) and at that time we learned that a similar result had also been found by Nualart-Pardoux (cf. [9]). In fact A. Badrikian has let us know that the formula (II.2) was going back to the work of Sevljakov in 1981 (cf. [13]) which is reproved in [12]. We also acknowledge that this work has profited from the constructive remarks of an anonymous referee.

I. Preliminaries and Notations

Ω denotes the classical Wiener space $C([0, 1], \mathbb{R}^d)$, H is the Cameron-Martin space, i.e., the set of absolutely continuous functions on $[0, 1]$ with values in $\mathbb{R}^d$, having square integrable densities with respect to the Lebesgue measure on $[0, 1]$ and μ is the standard Wiener measure on Ω for which H is the reproducing kernel Hilbert space. We denote by $(\mathscr{F}_t; t\in[0, 1])$ the canonical increasing family of the sigma-algebras on Ω, completed with respect to the Wiener measure μ. The infinitesimal generator of the Ω-valued Ornstein-Uhlenbeck process is denoted by A (cf. [21]). If M is a separable Hilbert space, $D_{p,k}(M)$, $p\in(1, \infty)$, $k\in\mathbb{Z}$, denotes the Banach space which is the completion of the M-valued polynomials on Ω with respect to the following norm:

$$\|\eta\|_{D_{p,k}(M)} = \|(I-A)^{k/2}\eta\|_{L^p(\mu; M)}$$

where $(I-A)^n\eta$ is defined coordinatewise. $D(M)$ is defined as the projective limit of the spaces $(D_{p,k}(M); p\in(1, \infty), k\in\mathbb{Z})$ and its continuous dual is denoted by $D'(M)$. Let us note that if $M=\mathbb{R}$ we shall simply write $D_{p,k}$, D, D' instead of $D_{p,k}(\mathbb{R})$, $D(\mathbb{R})$, $D'(\mathbb{R})$.

If $K\in D(H)$, we denote by $\delta_t K$ the divergence operator applied to the H-valued random variable whose density with respect to the Lebesgue measure is defined by $1_{[0,t]}(s)\dot{K}_s$ where $\dot{K}_s$ is the density of K. Recall that in the contexte of the Malliavin Calculus the divergence is defined as the adjoint of the Sobolev derivative in the direction of H (cf. [5, 21]) and if the above densities are adopted to the filtration $(\mathscr{F}_t; t\in[0, 1])$ then it coincides with the classical Itô integral of K and in the general case Gaveau-Trauber have proved that (cf. [3]) it coincides with the Skorohod integral on $D_{2,1}(H)$ (cf. [15] for a general definition of this integral). Consequently $\delta_t K$ will also be denoted as $\int_0^t \dot{K}_s \delta W_s$ with the integral notation. For the extension of the above results to the space of the distributions D' we refer reader [16–18].

II. A Short Proof of the Itô Formula for Anticipative Processes

We call an anticipative Itô process a stochastic process (X_t) with values in $\mathbb{R}^m$ such that

$$X_t = (X_t^1, \dots, X_t^m),$$

where

$$X_t^i = X_0^i + \int_0^t \dot{K}_s^i \delta W_s + \int_0^t \xi_s^i ds$$

and K^i denotes the H-valued random variable whose density with respect to the Lebesgue measure on $[0, 1]$ is $\dot{K}_s^i$ (it takes its values in $\mathbb{R}^d$). First we shall suppose that K^i are in $D(H)$, X_0^i are in D and ξ^i are in $D(H([0, 1], \mathbb{R}))$ for $i = 1, \dots, m$.

Let (h_n) be a complete, orthonormal basis in H whose elements are (the restrictions of) smooth functions on $[0, 1]$, and (k_n) be a basis of $H([0, 1], \mathbb{R})$ consisting also of smooth functions. Denote by V_n the sigma-algebra generated by $(h_1, \dots, h_n)$, $n \geqq 1$, where h_i is regarded as the Gaussian random variable

$$\int_0^1 \dot{h}_i(s) \cdot dW_s(x) = h_i(x) = (h_i, x), \qquad x \in \Omega.$$

Look at $E[X_0 | V_n]$: From Doob's Lemma, it can be written as $f_n((h_1, x), \dots, (h_n, x))$ where f_n is a Borel measurable function from $\mathbb{R}^n$ into $\mathbb{R}^m$, and, since X_0^i is in D for all $i = 1, \dots, m$, using the finite dimensional Sobolev injection theorems, we can take f_n in $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ by modifying it on a set of Lebesgue measure zero. This procedure will be denoted by $E[X_0 | V_n\tilde{]}$ and it is called redefinition by P. Malliavin (cf. [7]). For the notational convenience we will denote it by X_0^n. We shall also redefine K^i and ξ^i in the following manner:

$$K_i^{n,i} = \sum_{j=1}^n E[(K^i, h_j) | V_n\tilde{]} h_j,$$

$$\xi^{n,i} = \sum_{j=1}^n E[(\xi^i, k_j) | V_n\tilde{]} k_j$$

and let us define X_t^n as

$$X_t^{n,i} = X_0^{n,i} + \int_0^t \dot{K}_s^{n,i} \delta W_s + \int_0^t \xi_s^{n,i} ds, \qquad i = 1, \dots, m.$$

Proposition II.1. *Let F be a C^2-function on $\mathbb{R}^m$ with bounded derivatives. We then have*

$$F(X_t^n)-F(X_0^n)=\int_0^t \partial_i F(X_s^n)\dot{K}_s^{n,i}\,\delta W_s+\int_0^t \partial_i F(X_s^n)\xi_s^{n,i}\,ds$$

$$+\int_0^t \partial_{ij}F(X_s^n)\Bigg[(1/2)(\dot{K}_s^{n,i},\dot{K}_s^{n,j})+(\dot{K}_s^{n,i},\dot{\nabla}X_0^{n,j}(s))$$

$$+\left(\dot{K}_s^{n,i},\int_0^s \dot{\nabla}\xi_r^{n,j}(s)\,dr\right)+\left(\dot{K}_s^{n,i},\int_0^s \dot{\nabla}\dot{K}_r^{n,j}(s)\,\delta W_r\right)\Bigg]ds \qquad \text{(II.1)}$$

almost surely, for any $t\in[0,1]$, where we used the usual summation convention.

Proof. We shall give a proof for $d=1$ for the notational simplicity. In this case X_t^n is written as

$$X_t^n=X_0^n+\int_0^t \dot{K}_s^n\,\delta W_s+\int_0^t \xi_s^n\,ds$$

and the orthonormal sequences (k_j) and (h_i) can be taken as identical. In the sequel of the proof, for typographical reasons, we shall drop the index n.

Using the Taylor formula, we have

$$F(X_{t+h})-F(X_t)=F'(X_t)(X_{t+h}-X_t)+(1/2)F''(X_t+a(t,t+h)X_{t+h})(X_{t+h}-X_t)^2$$

where a comes from Rolle's Theorem.

Let us denote by D_0 the algebra generated by the following set:

$$\{E[\varphi\,|\,V_k]\tilde{\ };\ \varphi\in D,\ k\in\mathbb{N}\}$$

where $E[\cdot\,|\,V_k]\tilde{\ }$ denotes as before the redefined conditional expectation with respect to the sigma-algebra V_k. From the convergence of the martingales and the commutation relations between $E[\cdot|V_k]$ and the Ornstein-Uhlenbeck operator A, it is easy to see that D_0 is a dense subset of D. Consequently, it is sufficient to prove the formula (II.1) on the test functions belonging to D_0 since the both sides of it are already in $L^2(\mu)$. To achieve this we shall calculate

$$\frac{d}{dt}\langle F(X_t),\varphi\rangle,\qquad \varphi\in D_0$$

using the second order development that we have written above:

$$\left\langle\frac{1}{h}F'(X_t)(X_{t+h}-X_t),\varphi\right\rangle=\frac{1}{h}E(F'(X_t)(\delta_{t+h}K-\delta_t K+\xi_{t+h}-\xi_t)\varphi)$$

where $\xi_t = \int_0^t \dot{\xi}_s\,ds$. Using the very definition of the divergence operator, we obtain:

$$\frac{1}{h} E(F'(X_t)(\delta_{t+h}K - \delta_t K)\varphi) = \frac{1}{h} E \int_t^{t+h} \dot{K}_s \dot{\nabla}[F'(X_t)\varphi](s)\,ds$$

$$= \frac{1}{h} E \int_t^{t+h} \dot{K}_s [F''(X_t)\varphi(\dot{\nabla} X_0(s) + \dot{\nabla}\delta_t K(s) + \dot{\nabla}\xi_t(s))$$

$$+ F'(X_t)\dot{\nabla}\varphi(s)]\,ds,$$

using the commutation relations between the divergence operator and the Sobolev derivative (cf. [10, 18]) we find that the above expression is equal to

$$\frac{1}{h} E \int_t^{t+h} \varphi \dot{K}_s F''(X_t)\left[\dot{\nabla} X_0(s) + \int_0^t \dot{\nabla}\dot{K}_r(s)\,\delta W_r + \int_0^t \dot{\nabla}\dot{\xi}_r(s)\,dr\right] ds$$

$$+ \frac{1}{h} E \int_t^{t+h} \dot{K}_s F'(X_t)\dot{\nabla}\varphi(s)\,ds,$$

whose limit is obvious when h goes to zero. Let us look at the terms of the second order: First, since F is C^3, then the term $F''(X_t + aX_{t+h})$ is strongly Sobolev differentiable,

$$(1/h)\dot{E}[F''(X_t + aX_{t+h})(X_{t+h} - X_t)^2\varphi]$$

$$\cong (1/h)\,\mathrm{E}[F''(X_t + aX_{t+h})(\delta_{t+h}K - \delta_t K)^2\varphi]$$

$$= (1/h)\langle \nabla[F''(X_t + aX_{t+h})\varphi\,\delta^t_{t+h}K],\, K^t_{t+h}\rangle$$

$$= (1/h)\langle \varphi\,\delta^t_{t+h}K\,\nabla F''(X_t + aX_{t+h}) + F''(X_t + aX_{t+h})\nabla(\varphi\,\delta^t_{t+h}K),\, K^t_{t+h}\rangle$$

where $\delta^t_{t+h}K$ denotes $\delta_{t+h}K - \delta_t K$, K^t_{t+h} is the element of $D(H)$ whose density is equal to $1_{[t,t+h]}(s)\dot{K}_s$, "$\cong$" means equality when h goes to zero and $\langle\cdot,\cdot\rangle$ denotes the duality between $D'(H)$ and $D(H)$ (we shall use the same duality bracket for the case $H = R$ but this does not create any confusion). It is evident that the term

$$(1/h)\langle F''(X_t + aX_{t+h})\nabla(\varphi\,\delta^t_{t+h}K),\, K^t_{t+h}\rangle \to \langle F''(X_t)\dot{K}^2_t,\, \varphi\rangle$$

when h goes to zero and we have also

$$(1/h)\langle \varphi\,\delta^t_{t+h}K\,\nabla F''(X_t + aX_{t+h}),\, K^t_{t+h}\rangle \to 0$$

as h goes to zero.

We have finally proved that

$$\frac{d}{dt}\langle F(X_t), \varphi\rangle = \langle \dot K_t F'(X_t), \dot\nabla\varphi(t)\rangle + \langle \dot\xi_t F'(X_t), \varphi\rangle$$

$$+\left\langle F''(X_t)\dot K_t\left[\dot\nabla X_0(t) + (1/2)\dot K_t + \int_0^t \dot\nabla \dot K_r(t)\,\delta W_r\right.\right.$$

$$\left.\left. + \int_0^t \dot\nabla \dot\xi_r(t)\,dr\right], \varphi\right\rangle$$

for any $\varphi \in D_0$. Now let us recall that, thanks to the redefinitions made above, this derivative is continuous with respect to the parameter t (it would have been C^∞ if we had chosen F to be C^∞), hence we can apply the fundamental theorem of the differential calculus:

$$\langle F(X_t), \varphi\rangle = \langle F(X_0), \varphi\rangle + \int_0^t \frac{d}{ds}\langle F(X_s), \varphi\rangle\, ds$$

and the only term to calculate is the first term in the derivative:

$$\int_0^t \langle \dot K_s F'(X_s), \dot\nabla\varphi(s)\rangle\, ds = E\int_0^t \dot K_s F'(X_s)\dot\nabla\varphi(s)\, ds$$

$$= \left\langle \int_0^\bullet 1_{[0,t]}(s)\dot K_s F'(X_s)\, ds, \nabla\varphi\right\rangle$$

$$= \left\langle \delta\left(\int_0^\bullet 1_{[0,t]}(s)\dot K_s F'(X_s)\, ds\right), \varphi\right\rangle$$

$$= \left\langle \int_0^t \dot K_s F'(X_s)\,\delta W_s, \varphi\right\rangle$$

since δ is the adjoint of ∇. For the other terms the calculations are evident and a limit procedure to pass from C^3 to C^2 completes the proof. Q.E.D.

Remark. With the notations of the above proof and under the same hypothesis, we have

$$E[\sup_t |X_t|^p] < +\infty$$

for any $p < \infty$, consequently, even if F'' is not bounded but of polynomial growth the argument that we have used to show that the term of the Taylor formula corresponding to $\dot\nabla[F''(X_t + aX_{t+h})](s)$ does not contribute to the limit still holds and this fact will be used in the proof of the Theorem II.1.

Proposition II.2. *Under the hypothesis of the Proposition II.1, we have, for any* $t \in [0, 1]$,

$$F(X_t) = F(X_0) + \int_0^t \partial_i F(X_s) \dot{K}_s^i \delta W_s + \int_0^t \partial_i F(X_s) \dot{\xi}_s^i \, ds$$

$$+ \int_0^t \partial_{ij} F(X_s) \Bigg[(\dot{K}_s^i, (1/2) K_s^j + \nabla X_0^j(s) + \int_0^s \dot{\nabla} \dot{\xi}_r^j(s) \, dr$$

$$+ \int_0^s \dot{\nabla} \dot{K}_r^j(s) \, \delta W_r \Big) \Bigg] ds \tag{II.2}$$

μ-almost surely.

Proof. As in the preceeding proposition we shall work in the one dimensional case: First let us note that each term of the formula (II.2) is well defined. For example:

$$E \left| \int_0^t F''(X_s) \dot{K}_s \left(\int_0^s \dot{\nabla} \dot{K}_r(s) \, \delta W_r \right) ds \right|^2$$

$$\leqq c \left(E \int_0^1 |\dot{K}_s|^2 \, ds \right) \left(E \int_0^1 \left(\int_0^s \dot{\nabla} \dot{K}_r(s) \, \delta W_r \right)^2 ds \right)$$

$$\leqq c \, \|K\|_{D_{2,0}(H)}^2 \left(E \int_0^1 \left(\int_0^1 |\dot{\nabla} \dot{K}_r(s)|^2 \, dr + \int_0^s \int_0^s \ddot{\nabla}^2 \dot{K}_r(s, u) \ddot{\nabla}^2 \dot{K}_u(s, r) \, dr \, du \right) ds \right)$$

$$\leqq c' \, \|K\|_{D_{2,0}(H)}^2 [\|K\|_{D_{2,1}(H)}^2 + \|K\|_{D_{2,2}(H)}^2]$$

using the equivalence of the norms defined by $(-A)^{1/2}$ and ∇ (i.e., the inequalities of Meyer, cf. [17, 21]), where c and c' are two constants independent of X, K and ξ. Furthermore, by hypothesis and the continuity of the divergence and Lebesgue integral as mappings from $D(H)$ into D we see that X_t^n converge to X_t in D for any t hence $F(X_t^n)$, $F'(X_t^n)$, $F''(X_t^n)$ converge respectively to $F(X_t)$, $F'(X_t)$, $F''(X_t)$ in all the L^p spaces for $p < \infty$ and $t \in [0, 1]$. Hence all the terms can be controlled with the following quantities:

$$\|K\|_{D_{2,2}(H)}, \|\xi\|_{D_{2,1}(H)}, \|X_0\|_{D_{2,1}}, \qquad \|F\|_\infty, \|F'\|_\infty, \|F''\|_\infty,$$

consequently we can use the usual limiting procedure. Q.E.D.

In fact, in the proof of the above proposition we have made a little bit better:

Proposition II.3. *Suppose that F is a twice differentiable function with bounded derivatives and that* $X_0^i \in D_{2,1}$, $K^i \in D_{2,2}(H)$, $\xi^i \in D_{2,1}(H([0, 1], \mathbb{R}^m))$, *then the formula* (II.2) *still holds and its terms are in* $D_{2,0} = L^2(\mu)$.

We have proved the validity of the formula (II.2) for the twice differentiable functions having bounded first and second derivatives. In fact this restriction can be removed if we impose some regularity conditions on the integrands. To do this we need the following

Lemma II.1. *Suppose that Z is a process with the following decomposition:*

$$Z_t = Z_0 + \delta_t H + \int_0^t \dot{\eta}_s \, ds = Z_0 + \delta_t H + \eta_t$$

for any $t \in [0, 1]$. Then, for any $p(1, \infty)$, $k \in \mathbb{Z}$, one has the following relation:

$$\sup_{t \in [0,1]} \|Z_t\|_{D_{p,k}} \leqq c_{p,k} \|H\|_{D_{p,k+1}(H)} + \|\eta\|_{D_{p,k}(H)} + \|Z_0\|_{D_{p,k}}$$

where $c_{p,k}$ is a constant depending only on p and k.

Proof. From the hypothesis, we have

$$\|Z_t\|_{D_{p,k}} \leqq \|\delta_t H\|_{D_{p,k}} + \|\eta_t\|_{D_{p,k}} + \|Z_0\|_{D_{p,k}},$$

moreover, from the continuity properties of the divergence operator on the Sobolev spaces (cf. [5]), one has

$$\|\delta_t H\|_{D_{p,k}} \leqq c_{p,k} \left\| \int_0^{\cdot} 1_{[0,t]}(s) \dot{H}_s \, ds \right\|_{D_{p,k+1}(H)}$$
$$\leqq c_{p,k} \|H\|_{D_{p,k+1}(H)}.$$

For η_t we have a similar but simpler inequality. Q.E.D.

Theorem II.1. *Suppose that F is a C^2-function which is polynomially bounded of order $n > 0$ as well as its first two derivatives and suppose that $K^i \in D_{4n,2}(H)$ with*

$$\sum_i E\left(\int_0^1 |\dot{K}_s^i|^4 \, ds \right)^2 < +\infty$$

and that $X_0^i \in D_{n,1} \cap D_{2,1}$, $\xi \in D_{2n,0}(H([0,1], \mathbb{R}^m)) \cap D_{2,1}([H([0,1], \mathbb{R}^m))$. Then the change of variables formula (II.2) *is still true. In particular, if the above hypothesis holds for any $n \in \mathbb{N}$, then the formula* (II.2) *holds for any C^2 function which is polynomially bounded on R^m.*

Proof. For the notational simplicity we shall proceed in the unidimensional case.

Using the hypothesis and the Lemma II.1, we have the following apriori majoration:

$$\sup_t \left\{ \left\| \int_0^t F'(X_s)\dot{\xi}_s\, ds \right\|_{D_{2,0}}^2 + \|F(X_t)\|_{D_{2,0}}^2 + \left\| \int_0^t F''(X_s)\dot{K}_s \dot{\nabla} X_0(s)\, ds \right\|_{D_{2,0}}^2 \right.$$

$$+ \left\| \int_0^t F''(X_s)\dot{K}_s \left(\int_0^s \dot{\nabla}\dot{K}_r(s)\, \delta W_r \right) ds \right\|_{D_{2,0}}^2$$

$$+ \left\| \int_0^t F''(X_s)\dot{K}_s \left(\int_0^s \dot{\nabla}\dot{\xi}_r(s)\, dr \right) ds \right\|_{D_{2,0}}^2$$

$$+ \left\| \int_0^t F''(X_s)\dot{K}_s^2\, ds \right\|_{D_{2,0}}^2 + \left\| \int_0^t F'(X_s)\dot{K}_s\, \delta W_s \right\|_{D_{2,0}}^2$$

$$\leqq C(n) \left(E \int_0^t |\dot{K}_s|^4\, ds \right)^{1/2} (\|K\|_{D_{4n,2}(H)}^2 + \|\xi\|_{D_{2n,0}(H)}^2 + \|\xi\|_{D_{2,1}(H)}$$

$$+ \|X_0\|_{D_{2,1}}^2 + \|X_0\|_{D_{n,1}}^2),$$

hence we can proceed exactly as in the proof of the Proposition II.1 and the Proposition II.2, i.e., beginning with the smooth coefficients K, X_0 and ξ imposing on K the supplementary condition of the Theorem and taking F as in the hypothesis. The above inequality says that we can pass to the limit and the redefined case is obvious from the proof of the Proposition II.3 and the Remark following it. Q.E.D.

The class of functions with which we have studied till now does not include the exponential function. Because of its importance in probability we shall treat it separately.

Preposition II.4. Suppose that K is in $D_{2,2}(H)$ and that

$$E \int_0^1 \exp(2|\delta_s K|)\dot{K}_s^2\, ds < +\infty$$

then we have

$$\exp(\delta_t K) = 1 + \int_0^t \exp(\delta_s K)\dot{K}_s\, \delta W_s + \int_0^t \exp(\delta_s K)\dot{K}_s((1/2)\dot{K}_s + \int_0^s \dot{\nabla}\dot{K}_r(s)\, \delta W_r)\, ds$$

almost surely, for any t in $[0, t]$.

Proof. Let e_N be the function $\sum_{i<N+1} x^i/(i!)$. Using the previous results, we have the Itô formula for $e_N(\delta_t K)$. Let $\varphi \in D$, then, from the hypothesis

$$E \int_0^t \sum_{i=0}^{\infty} |\delta_s K|^i \frac{1}{(i!)} |\dot{K}_s|\, |\dot{\nabla}\varphi(s)|\, ds$$

$$\leqq \|\varphi\|_{2,1} \left(E \int_0^t (\exp 2|\delta_s K|)\, |\dot{K}_s|^2\, ds \right)^{1/2},$$

hence $\int_0^t e'_N(\delta_s K)\dot{K}_s\,\delta W_s$ converges to $\int_0^t (\exp \delta_s K)\dot{K}_s\,\delta W_s$ in D' in the weak topology. Similarly, we have

$$E\int_0^t \sum_{i=0}^{\infty} |\delta_s K|^i/(i!)\,|\dot{K}_s|\,|\varphi|\,\left|(1/2)\dot{K}_s + \int_0^s \dot{\nabla}\dot{K}_r\left(s\right)\,\delta W_r\right| ds$$

$$\leq \left(E\int_0^1 (\exp 2\,|\delta_s K|)\,|\dot{K}_s|^2\,ds\right)^{1/2}\left(E\left[\varphi^2\int_0^1\left(1/2\,\dot{K}_s + \int_0^s \dot{\nabla}\dot{K}_r(s)\,\delta W_r\right)^2 ds\right]\right)^{1/2}$$

Since K is in $D_{2,2}(H)$, ∇K is in $D_{2,1}(H\otimes_2 H)$ where "$\otimes_2$" denotes the Hilbert-Schmidt tensor product, hence, if we apply δ to its second component, the result will be in $D_{2,0}(H)$ and this implies the finiteness of the right hand side of the inequality. Consequently the second order part of the Itô formula for e_N converges in $\sigma(D', D)$ to the Lebesgue integral in which e_N is replaced with the exponential function. Both sides of the Itô formula are equal as distributions, by the hypothesis they are in $D_{2,0}$ for any $t>0$ and this completes the proof. Q.E.D.

III. Extensions to Nonmonotonous Time

Let v: $[0, 1]\to[0, 1]$ be a C^1-function (in the sense of restriction) which is not necessarily monotonous. If f is a continuous function on $[0, 1]$ then a classical theorem of differential calculus says that

$$\int_{v(0)}^{v(1)} f(s)\,ds = \int_0^1 f(v(s))\,v'(s)\,ds. \tag{III.1}$$

If we take a stochastic integral instead of the Riemann integral and a stochastic process instead of f, can we do something similar? Let us suppose first that $K\in D(H)$ and denote by K^n its finite dimensional redefinition as in the preceeding section. For the sake of simplicity we deal with the one dimensional case. So, if φ is in D_0 (cf. the Sect. II), then we have

$$\left\langle \int_{v(0)}^{v(1)} \dot{K}_s^n\,\delta W_s, \varphi\right\rangle = E\int_{v(0)}^{v(1)} \dot{K}_s^n\,\dot{\nabla}\varphi(s)\,ds$$

since the integrand is infinitely differentiable with respect to s, by (III.1), we can write

$$\left\langle \int_{v(0)}^{v(1)} \dot{K}_s^n\,\delta W_s, \varphi\right\rangle = E\int_0^1 \dot{K}_{\gamma(s)}^n\,\dot{\nabla}\varphi(v(s))\,v'(s)\,ds,$$

if we suppose that

$$\int_0^1 (\operatorname{card}\{u: v(u)=s, v'(u)\neq 0\})^2 \, ds < +\infty, \tag{III.2}$$

then the image of the measure $v'(s)\,ds$ under v is absolutely continuous with respect to the Lebesgue measure (cf. [1, 22]) with the density

$$\varepsilon(s) = \sum_{u \in \bar{v}^1(s)} \operatorname{sign} v'(u)$$

hence we obtain

$$\left\langle \int_{v(0)}^{v(1)} \dot{K}_s^n \, \delta W_s, \varphi \right\rangle = E \int_0^1 \dot{K}_s^n \dot{\nabla} \varphi(s)\, \varepsilon(s)\, ds$$

$$= \left\langle \int_0^1 \dot{K}_s^n \, \varepsilon(s)\, \delta W_s, \varphi \right\rangle, \qquad \varphi \in D_0,$$

consequently

$$\int_{v(0)}^{v(1)} \dot{K}_s^n \, \delta W_s = \int_0^1 \dot{K}_s^n \, \varepsilon(s)\, \delta W_s \quad \text{a.s.},$$

moreover both sides pass well to the limit as n goes to infinity, hence we have

$$\int_{v(0)}^{v(1)} \dot{K}_s \, \delta W_s = \int_0^1 \dot{K}_s \, \varepsilon(s)\, \delta W_s \quad \text{a.s.},$$

now, if K is in $D_{2,1}(H)$ instead of $D(H)$, we can approximate it with $(K^m) \subset D(H)$ and from the continuity of the divergence operator on $D_{2,1}(H)$ we obtain the following

Theorem III.1. *For any $K \in D_{2,1}(H)$, under the condition* (III.2) *we have*

$$\int_{v(0)}^{v(1)} \dot{K}_s \, \delta W_s = \int_0^1 \dot{K}_s \, \varepsilon(s)\, \delta W_s$$

where ε is the degree of the function v.

Remark. If $K \in D_{2,0}(H)$ such that K is adapted then the theorem is still true except that the Skorohod integral should be replaced by the classical Itô integral.

We shall apply the same idea combined with the redefinition technique of the Sect. II to prove the Itô formula for $F(X^1_{\theta_1(t)}, \ldots, X^m_{\theta_m(t)})$ where θ_i: $[0, 1] \to [0, 1]$ are C^1-mappings and X^i are the one-dimensional processes defined by

$$X_T^i = X_0^i + \int_0^t \xi_s^i \, ds + \int_0^t \dot{K}_s^i \, \delta W_s$$

$$= X_0^i + \xi_t^i + \delta_t K^i, \qquad i = 1, \ldots, m.$$

For the typographical reasons, we shall denote $(\theta_1(t), \ldots, \theta_m(t))$ with $\theta(t)$, hence θ defines a C^1-mapping from $[0, 1]$ into $[0, 1]^m$ and $X_\theta(t)$ will denote the vector valued random variable whose components are $X^i_{\theta_i(t)}$, $i=1, \ldots, m$. Let us suppose that K^i, ξ^i are in $D(H)$ and X^i_0 in D for all i. As before we shall denote by $K^{i,n}$, $\xi^{i,n}$, $X^{i,n}$ redefinitions and the processes defined from these redefined processes (cf. Sect. II). We have

Lemma III.1. *Let F: $\mathbb{R}^m \to \mathbb{R}$ be a C^2-function with bounded derivatives, then, for any φ in D_0, the mapping $t \to \langle F(X^{1,n}_{\theta_1(t)}, \ldots, X^{m,n}_{\theta_m(t)}), \varphi\rangle$ is absolutely continuous with respect to the Lebesgue measure on* $[0, 1]$.

Proof. For the notational convenience we shall suppress the index n of the redefinition. If $\varphi \in D_0$, we have, from Taylor's formula:

$$F(X_{\theta(1)}) - F(X_{\theta(0)}) = \sum_k F(X_{\theta(t_{k+1})}) - F(X_{\theta(t_k)});$$

$$\sum_k \langle F(X_{\theta(t_{k+1})}) - F(X_{\theta(t_k)}), \varphi\rangle$$

$$= \sum_k \sum_{i=1}^m \langle \partial_i F(X_{\theta(t_k)})(X^i_{\theta_i(t_{k+1})} - X^i_{\theta_i(t_k)}), \varphi\rangle$$

$$+ \Big\langle (1/2) \sum_k \sum_{i,j=1}^m \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})})(X^i_{\theta_i(t_{k+1})} - X^i_{\theta_i(t_k)})$$

$$\cdot\ (X^j_{\theta_j(t_{k+1})} - X^j_{\theta_j(t_k)}), \varphi\Big\rangle$$

where $(t_0 = 0 < t_1 < \ldots < t_{N+1})$ is a partition of $[0, 1]$. We shall study the terms of the sum as $\sup_i (t_{i+1} - t_i)$ goes to zero. Let us look at first the terms of first order:

$$\sum_{k,i} \langle \partial_i F(X_{\theta(t_k)})(\xi^i_{\theta_i t_{k+1})} - \xi^i_{\theta_i(t_k)} + \delta_{\theta_i(t_{k+1})} K^i - \delta_{\theta_i(t_k)} K^i), \varphi\rangle$$

$$= \sum_{k,i} \Bigg(\int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \langle \partial_i F(X_{\theta(t_k)})\, \dot\xi^i_s, \varphi\rangle\, ds$$

$$+ E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \dot K^i_s \nabla [\varphi\, \partial_i F(X_{\theta(t_k)})](s)\, ds \Bigg)$$

denoting the first sum of the right hand side by I, we see that the above quantity is equal to

$$\mathrm{I} + \sum_{k,i} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} (\dot K^i_s \partial_i F(X_{\theta(t_k)}) \nabla \varphi(s) + \dot K^i_s \varphi \sum_{j=1}^m \partial_{ij} F(X_{\theta(t_k)} \nabla X^j_{\theta_j(t_k)(s)}\, ds$$

$$= \mathrm{I} + \mathrm{II} + \sum_k \sum_{ij} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot K^i_s \partial_{ij} F(X_{\theta(t_k)}) \Bigg[\nabla X^j_0(s) + \int_0^{\theta_j(t_k)} \nabla \dot\xi^j_r(s)\, dr$$

$$+ 1_{[0, \theta_j(t_k)]}(s) \dot K^j_s + \int_0^{\theta_j(t_k)} \nabla K^j_r(s)\, \delta W_r \Bigg] \Bigg)\, ds$$

where II denotes the sum with $\dot{\nabla}\varphi(s)$ and the last equality follows from the commutation relations between ∇ and δ. The only term that we need to study is the one which has the indicator function of the interval $[0, \theta_j(t_k)]$. It can be written as

$$\sum_k 1_{\{\theta_i(t_k)<\theta_i(t_{k+1})\}} 1_{\{\theta_j(t_k)>\theta_i(t_k)\}}$$

$$\cdot E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \dot{K}_s^j 1_{[0,\theta_j(t_k)]}(s)\, \partial_{ij} F(X_{\theta(t_k)})\, ds$$

$$+\sum_k 1_{\{\theta_i(t_k)>\theta_i(t_{k+1})\}} 1_{\{\theta_j(t_k)>\theta_i(t_{k+1})\}}$$

$$\cdot E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \dot{K}_s^i \dot{K}_s^j \varphi\, 1_{[0,\theta_j(t_k)]}(s)\, \partial_{ij} F(X_{\theta(t_k)})\, ds,$$

if $\theta_j(t_k)>\theta_i(t_k)$, from the continuity, we have $\theta_j(t_k)\geqq\theta_i(t_{k+1})$ provided that the distance between t_k and t_{k+1} is sufficiently small, hence the indicator function in the integral of the first sum disappears. Similarly, if $\theta_j(t_k)>\theta_i(t_{k+1})$, we have $\theta_j(t_k)\geqq\theta_i(t_k)$ and the above expression becomes

$$\sum_k 1_{\{\theta'_i(t_k)>0\}} 1_{\{\theta_j(t_k)>\theta_i(t_k)\}} E \int_{t_k}^{t_{k+1}} \varphi \dot{K}_{\theta_i(s)}^i \dot{K}_{\theta_i(s)}^j \partial_{ij} F(X_{\theta(t_k)})\, \theta'_i(s)\, ds$$

$$+\sum_k 1_{\{\theta'_i(t_k)<0\}} 1_{\{\theta_j(t_k)\geqq\theta_i(t_k)\}} E \int_{t_k}^{t_{k+1}} \varphi \dot{K}_{\theta_i(s)}^i \dot{K}_{\theta_i(s)}^j \partial_{ij} F(X_{\theta(t_k)})\, \theta'_i(s)\, ds$$

and the limit of this sum becomes, from the dominated convergence theorem and by writing the sets $\{\theta'_i(s)\lesseqgtr 0\}\cap\{\theta_i(s)>\theta_j(s)\}$ as a countable union of disjoint intervals,

$$E \int_0^1 [1_{\{\theta'_i(s)>0\}} 1_{\{\theta_j>\theta_i\}} + 1_{\{\theta'_i(s)<0\}} 1_{\{\theta_j\geqq\theta_i\}}]$$

$$\cdot \varphi \dot{K}_{\theta_i(s)}^i \dot{K}_{\theta_i(s)}^j \partial_{ij} F(X_{\theta(s)})\, \theta'_i(s)\, ds,$$

let us note also that the first order terms are of finite variation as it follows trivially from these calculations. The only second order term which influences the limit is the following one:

$$(1/2)\sum_k \langle \partial_{ij} F(X_{\theta(t_k)}+\alpha_k X_{\theta(t_{k+1})})(\delta_{\theta_i(t_{k+1})}K^i-\delta_{\theta_i(t_k)}K^i)(\delta_{\theta_j(t_{k+1})}K^j-\delta_{\theta_j(t_k)}K^j), \varphi\rangle$$

$$=(1/2)\sum_k E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \dot{K}_s^i \dot{\nabla}[\varphi(\delta_{\theta_j(t_{k+1})}K^j-\delta_{\theta_j(t_k)}K^j)\, \partial_{ij} F(X_{\theta(t_k)}+\alpha_k X_{\theta(t_{k+1})})](s)\, ds$$

when we develop the Sobolev derivative using exactly the same argument as in the proof of the Proposition II.1 we see, in fact, that the only term having a nonzero limit is

$$(1/2)\sum_k E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})}) 1_{[\theta_j(t_k), \theta_j(t_{k+1})}(s)] \dot{K}_s^j ds$$

$$= (1/2)\sum_k 1_{\{\theta_i(t_{k+1}) > \theta_i(t_k)\}} 1_{\{\theta_j(t_{k+1}) > \theta_i(t_k)\}} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \dot{K}_s^j$$

$$\cdot \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})}) 1_{[0, \theta_j(t_{k+1})]}(s) ds$$

$$-(1/2)\sum_k 1_{\{\theta_i(t_{k+1}) > \theta_i(t_k)\}} 1_{\{\theta_j(t_k) > \theta_i(t_k)\}} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \dot{K}_s^j$$

$$\cdot \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})}) 1_{[0, \theta_j(t_k)]}(s) ds$$

$$+(1/2)\sum_k 1_{\{\theta_i(t_k) > \theta_i(t_{k+1})\}} 1_{\{\theta_j(t_{k+1}) > \theta_i(t_{k+1})\}} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \dot{K}_s^j$$

$$\cdot \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})}) 1_{[0, \theta_j(t_{k+1})]}(s) ds$$

$$-(1/2)\sum_k 1_{\{\theta_i(t_k) > \theta_i(t_{k+1})\}} 1_{\{\theta_j(t_k(> \theta_i(t_{k+1})\}} E \int_{\theta_i(t_k)}^{\theta_i(t_{k+1})} \varphi \dot{K}_s^i \dot{K}_s^j$$

$$\cdot \partial_{ij} F(X_{\theta(t_k)} + a_k X_{\theta(t_{k+1})}) 1_{[0, \theta_j(t_k)]}(s) ds,$$

when $\sup_i(t_{i+1} - t_i)$ goes to zero, the above expression has the following limit:

$$(1/2) E \int_0^1 [1_{\{\theta_i'(s) > 0\}} 1^{(s)}_{\{\theta_j = \theta_i\}} - 1_{\{\theta_i'(s) < 0\}} 1^{(s)}_{\{\theta_i = \theta_j\}}] \varphi \dot{K}^i_{\theta_i(s)} \dot{K}^j_{\theta_i(s)}$$

$$\cdot \partial_{ij} F(X_{\theta(s)}) \theta_i'(s) ds.$$

Hence we have proved that the Radon-Nikodym derivative of $t \mapsto \langle F(X_{\theta(t)}, \varphi \rangle$, for $\varphi \in D_0$, is

$$E[\varphi \partial_i F(X_{\theta(s)}) \dot{\xi}^i_{\theta_i(s)} + \dot{K}^i_{\theta_i(s)} \partial_i F(X_{\theta(s)}) \dot{\nabla} \varphi(\theta_i(s))$$

$$+ \varphi \dot{K}^i_{\theta_i(s)} \partial_{ij} F(X_{\theta(s)} (\dot{\nabla} X_0^j(\theta_i(s)) + \int_0^{\theta_j(s)} \dot{\nabla} \dot{\xi}^i_r(\theta_i(s)) dr$$

$$+ \int_0^{\theta_j(s)} K_r^j(\theta_i(s)) \delta W_r)$$

$$+ \varphi \dot{K}^i_{\theta_i(s)} \dot{K}^j_{\theta_i(s)} \partial_{ij} F(X_{\theta(s)}) (1_{\{\theta_j > \theta_i\}}(s) + 1_{\{\theta_i'(s) < 0\}}(s) + 1_{\{\theta_j = \theta_i\}}(s))$$

$$+ (1/2) \varphi \dot{K}^i_{\theta_i(s)} \dot{K}^j_{\theta_i(s)} \partial_{ij} F(X_{\theta(s)}) (1_{\{\theta_i'(s) > 0\}} - 1_{\{\theta_i'(s) < 0\}})$$

$$\cdot 1^{(s)}_{\{\theta_i = \theta_j\}}] \theta_i'(s). \quad \text{Q.E.D.} \tag{III.1}$$

We have

Proposition III.1. *Suppose that* ξ^i, $K^i \in D(H)$, $X_0^i \in D$, $F \in C_b^2(\mathbb{R}^m)$ *and that* θ_i: $[0, 1] \to [0, 1]$ *is continuously differentiable, for* $i = 1, \ldots, m$, *with*

$$N_\theta = \sum_{i=1}^{m} \int_0^1 (\text{card}\{u: \theta_i(u) = s\})^2 \, ds < +\infty.$$

Then, for any $t \in [0, 1]$, *almost surely we have the following relation:*

$$\begin{aligned} F(X_{\theta(t)}) - F(X_{\theta(0)}) &= \int_0^1 \dot{\xi}_s^i [\sum_{\bar{\theta}_i^{-1}(s) \cap [0,t]} \partial_i F(X_{\theta(u)}) \operatorname{sign} \theta_i'(u)] \, ds \\ &+ \int_0^1 \dot{K}_s^i [\sum_{\bar{\theta}_i^{-1}(s) \cap [0,t]} \partial_i F(X_{\theta(u)}) \operatorname{sign} \theta_i'(u)] \, \delta W_s \\ &+ \int_0^1 K_s^i \dot{\nabla} X_0^j(s) [\sum_{\bar{\theta}_i^{-1}(s) \cap [0,t]} \partial_{ij} F(X_{\theta(u)}) \operatorname{sign} \theta_i'(u)] \, ds \\ &+ \int_0^1 \dot{K}_s^i [\sum_{\bar{\theta}_i^{-1}(s) \cap [0,t]} \partial_{ij} F(X_{\theta(u)}) \left(\int_0^{\theta_j(u)} \nabla \dot{\xi}_r^j(s) \, dr + \int_0^{\theta_j(u)} \nabla \dot{K}_r^j(s) \, \delta W_r \right) \operatorname{sign} \theta_i'(u)] \, ds \\ &+ \int_0^1 \dot{K}_s^i \dot{K}_s^j [\sum_{\bar{\theta}_i^{-1}(s) \cap [0,t]} \partial_{ij} F(X_{\theta(u)}) (1_{\{\theta_j > \theta_i\}}(u) + (1/2) \, 1_{\{\theta_i = \theta_j\}}(u)) \operatorname{sign} \theta_i'(u)] \, ds. \quad \text{(III.2)} \end{aligned}$$

Proof. Using the formula (III.1), we see that (III.2) is true for the finite dimensional redefinitions as we have done in the Sect. II, afterwards we can pass to the limit. Q.E.D.

Theorem III.1. *Suppose that we have:*
$F: \mathbb{R}^m \to \mathbb{R}$ *bounded with bounded derivatives upto second order,*
θ: $[0, 1]^m \to [0, 1]^m$ *is a* C^1*-mapping,*

$$K^i \in D_{2,2}(H), \qquad X_0^i \in D_{2,1}, \qquad \xi^i \in D_{2,1}(H)$$

for $i = 1, \ldots, m$,

$$N_\theta = \sum_{i=1}^{m} \int_0^1 (\text{card}\{u: \theta_i(u) = s\})^2 \, ds < +\infty.$$

Let us denote by X_t^i *the anticipative Itô process defined as*

$$X_t^i = X_0^i + \int_0^t \dot{K}_s^i \, \delta W_s + \xi_t^i, \qquad i = 1, \ldots, m,$$

then, for any $t \in [0, 1]$, *the formula* (III.2) *holds almost everywhere.*

Proof. The proof is again the classical limiting procedure. From the hypothesis, if we approximate X^i, K^i, ξ^i with the smooth objects we know that the formula is valid for these approximations. In order to pass to the limit we see that the left hand side of (III.2) creates no difficulty as well as the terms of the right hand side, except the second term, but, since the limits of all the other terms do exist in $L^2(\mu)$, the limit of the second term does exist and belongs to $L^2(\mu)$. Q.E.D.

Example. Suppose that $(X_t;\, t\in[0,1])$ is the solution of the following stochastic differential equation with smooth coefficients:

$$dX_t^i = A^i(X_t)\cdot dW_t, \qquad i=1,\dots,m,$$

where $A^i\colon \mathbb{R}^m \to \mathbb{R}^d$. Then, for any $F\in C_b^2(\mathbb{R}^m)$, for any θ as in the hypothesis of the theorem, we have

$$F(X_{\theta(t)}) - F(X_{\theta(0)}) = \int_0^1 \Big[\sum_{\bar\theta_i^{-1}(s)\cap[0,t]} \partial_i F(X_{\theta(u)}) \operatorname{sign}\theta_i'(u) \Big] A^i(X_s)\cdot \delta W_s$$

$$+ \sum_{i,j,v,l,n} \int_0^t A_v^i(X_{\theta_i(s)})\, \partial_{ij} F(X_{\theta(s)})$$

$$\cdot \left(\int_0^{\theta_j(s)} \partial_n A_v^j(X_r)\, 1_{[0,r]}(s) [Y_r Y_s^{-1} A_l(X_s)]_n\, \delta W_r^l \right) \theta_i'(s)\, ds$$

$$+ \sum_{ij} \int_0^t (A^i(X_{\theta_i(s)}), A^j(X_{\theta_i(s)}))\, \partial_{ij} F(X_{\theta(s)})$$

$$\cdot\, (1_{\{\theta_j>\theta_i\}}(s) + (1/2)\, 1_{\{\theta_i=\theta_j\}}(s))\, \theta_i'(s)\, ds$$

where Y is the matrix-valued process defined as

$$dY_t = DA_\alpha(X_t)\, Y_t\, dW_t^\alpha$$
$$Y_o = Id$$

$[DA_\sigma]_{ij} = \partial_j A_\alpha^i$, and $\int_0^1 H\,\delta W_t^i$ is the Skorohod integral of He_i, H being scalar valued and e_i is the i-th unit vector of the canonical basis of $\mathbb{R}^d$.

Remarks. (1) With the finiteness hypothesis of N_θ, one can prove the analogous of the Theorem II.1; since the calculations are quite similar to those of above we shall omit it.

(2) Note that the second term the example can be written with the Itô integral as

$$\int_0^1 A_v^i(X_{\theta_i(s)})\, \partial_{ij} F(X_{\theta(s)\}}\, 1_{\{s<\theta_j(s)\}}$$

$$\cdot \left(Y_s^{-1*} \int_0^{\theta_j(t_k)} \mathrm{Y_r}^* \mathrm{DA}^j(X_r)\, dW_r^1, A_1(X_s) \right) \theta_i'(s)\, ds$$

with the usual summation convention.

IV. An Extension to Random Fields

In [4], Hitsuda shows that one can develope $F(W_{t_1},\dots,W_{t_n})$ for the first variable keeping the others fixed for smooth F with the help of the Wiener's chaos decomposition of L^2. In fact what happens can also be understood by using the Malliavin Calculus. First let us give the following

Proposition IV.1. *Suppose that* $F\colon \mathbb{R}^m \times \Omega \to \mathbb{R}$ *is a measurable mapping. Suppose that* F *is in* $\mathcal{S}(\mathbb{R}^n)\tilde{\otimes} D$, *i.e., the completed projective tensor product of the space of rapidly decreasing functions and the space* D *and that* (X_t) *is an n-dimensional Itô process defined as*

$$X_t^i = X_0^i + \int_0^t \dot{K}_s^i \delta W_s + \int_0^t \xi_s^i \, ds, \qquad i = 1, \ldots, n$$

where $X_0^i \in D_{4,1}$, $K^i \in D_{4,2}(H)$, $\xi^i \in D_{4,1}(H([0,1], \mathbb{R}))$. *Then we have the following relation:*

$$\begin{aligned} F(X_t, w) = F(X_0; w) &+ \int_0^t \partial_i F(X_s, \omega)(\dot{K}_s^i \delta W_s + \xi_s^i \, ds) \\ &+ \int_0^t (\dot{\nabla}[\partial_i F](X_s, \omega)(s), \dot{K}_s^i) \, ds \\ &+ \int_0^t \Bigg(\dot{K}_s^i, \frac{1}{2} \dot{K}_s^j + \dot{\nabla} X_0(s) \\ &+ \int_0^s \dot{\nabla} \xi_r^j(s) \, ds + \int_0^s \dot{\nabla} \dot{K}_r^j(s) \, \delta W_r \Bigg) \partial_{ij} F(X_s, \omega) \, ds \end{aligned} \tag{IV.1}$$

for any t *in* $[0, 1]$, μ*-almost surely.*

Proof. It is sufficient to write

$$F(X_t, \omega) = \int_{\mathbb{R}^n} \exp i(y, X_t) \hat{F}(y, \omega) \, dy$$

where "^" denotes the Fourier transformation on $\mathbb{R}^n$ and apply the Itô formula to $\exp i(y, X_t)$ (cf. the Theorem II.1) and use the commutation relation

$$\delta(\varphi K) = \varphi \, \delta K - \nabla_K \varphi$$

for $\varphi \in D$, $K \in D'(H)$ (cf. [18, 21]). By the hypothesis the Lebesgue integral with respect to dy commutes with the Lebesgue integral with respect to ds and with the Skorohod integral. Q.E.D.

The hypothesis that we have imposed on F is very easy to check: Because of the nuclearity of $\mathcal{S}(\mathbb{R}^n)$ (cf. [14]), $F \in \mathcal{S}(\mathbb{R}^n) \tilde{\otimes} D$ if and only if, the equivalence class corresponding to the following random variable

$$\omega \mapsto \langle T, F(\cdot, \omega) \rangle$$

is in D, for any tempered distribution T on $\mathbb{R}^n$. Obviously this formula can be extended to larger classes of functionals. Let us give an extension of it to some Sobolev spaces: The first difficulty is how to define $\partial_x^\alpha F(X_t(\omega), \omega)$ for a multi-index α with $|\alpha| \leqq 2$. Since we have

$$|\partial_X^\alpha F(x, \omega) - \partial_X^\alpha F(Y, \omega)| \leqq c(n) |x - y| \, \| F(., \omega) \|_{W^m}, \; |\alpha| \leqq 2,$$

with $m>(n/2)+2$, where W^m denotes the Sobolev space of order m on $\mathbb{R}^n$, if we suppose that $\|F\|_{W^m}$ is in $D_{p,0}$ with $p>n$, then from the Lemma of Kolmogorov (cf. for instance [8]), $x\mapsto\partial^\alpha F(x,\omega)$ will have an almost surely continuous modification. Therefore $\partial_x^\alpha F(X_t(\omega),\omega)$ will be well defined for $|\alpha|\leqq 2$.

The second difficulty comes from the term

$$\int_0^t ([\dot\nabla\partial_i F](X_s,\omega)(s),\dot K_s^i)\,ds$$

which is the integral of a function of two variables on the diagonal of $[0,1]^2$. We have the following apriori majoration:

$$\left|\int_0^t ([\dot\nabla\partial_i F](X_s,\omega)(s),\dot K_s^i)\,ds\right|$$
$$\leqq c(n,\varepsilon)\int_0^t |\dot K_s^i|\,\|\dot\nabla F(\cdot,\omega)(s)\|_{W^{n/2+\varepsilon+2}}\,ds$$

for an arbitrary $\varepsilon>0$, where the constant $c(n,\mathscr{E})$ depends only on n and ε. Let us denote by $D_{2,1}(W^m)$ the space of Wiener functionals with values in W^m, equipped with the norm

$$\|G\|^2=E\left[\|G\|_{W^m}^2+\int_0^1 \|\dot\nabla G(\omega)(s)\|_{W^m}^2\,ds\right],$$

let us note that this notation is compatible with the Sect. I because of the Meyer inequalities for vector valued Wiener functionals (cf. [17]). Since $\mathscr{S}(R^n)\tilde\otimes D$ is dense in $D_{2,1}(W^m)$, passing to a subsequence, we see that

$$\int_0^t ([\dot\nabla\partial_i F](X_s,\omega)(s),K_s^i)\,ds$$

can be defined as the almost sure limit of the integrals

$$\int_0^t ([\dot\nabla\partial_i F_{n_k}](X_s,\omega)(s),K_s^i)\,ds;\ n_k\in\mathbb{N})$$

where (F_n) converges to F in $D_{2,1}(W^m)$ with $m=(n/2)+2+\varepsilon$. Apriori the definition of this term depends on the choice of the approximating sequence. However, when we look at the things in detail, this is not the case as the following result shows:

Proposition IV.2. *Suppose that $F\in D_{2,1}(W^m)\cap D_{p,0}(W^m)$ with $p>n$, $m>(n/2)+2$, n being the dimension, $X_0^i\in D_{4,1}$, $K^i\in D_{4,2}(H)$ and $\xi^i\in D_{4,1}(H([0,1],\mathbb{R}))$ for $i=1,\dots,n$. Then the formula* (IV.1) *remains valid and the term*

$$\int_0^t ([\dot\nabla\partial_i F](X_s,\omega)(s),K_s^i)\,ds \qquad \text{(IV.2)}$$

is independent of the particular choice of the sequence (F_n) approximating F.

Proof. As we have explained above, both sides of the formula (IV.1) pass to the limit in D' in the weak topology. Furthermore all the terms, except (IV.2) are independent of the choice of (F_n) approximating F, consequently (IV.2) is also independent of (F_n). From the majorations that we have done above all the terms are elements of $D_{2,0}$ hence also the term with the Skorohod integral. Q.E.D.

Let us now apply this result to develop $f(W_{t_1}, W_{t_2})$, for f in $W^4(\mathbb{R}^2)$ to understand what happens in [4]. From (IV.1) we have

$$\begin{aligned} f(W_{t_1}, W_{t_2}) &= f(0, W_{t_2}) + \int_0^{t_1} \partial_x f(W_a, W_{t_2})\,\delta W_a \\ &\quad + \int_0^{t_1} \dot{\nabla}[\partial_x f(W_a, W_{t_2})](a)\,da + (1/2)\int_0^{t_1} \partial_x^2 f(W_a, W_{t_2})\,da \\ &= f(0, W_{t_2}) + \int_0^{t_1} \partial_x f(W_a, W_{t_2})\,\delta W_a + \int_0^{t_1\wedge t_2} \partial_x \partial_y f(W_a, W_{t_2})\,da \\ &\quad + (1/2)\int_0^{t_1} \partial_x^2 f(W_a, W_{t_2})\,da. \end{aligned} \tag{IV.3}$$

Iterating the same formula for each term above with respect to t_2, we obtain

$$\begin{aligned} f(W_{t_1}, W_{t_2}) &= f(0,0) + \int_0^{t_2} \partial_y f(0, W_b)\,dW_b + (1/2)\int_0^{t_2} \partial_y^2 f(0, W_b)\,db \\ &\quad + \int_0^{t_1} \Bigg[\partial_x f(W_a, 0) + \int_0^{t_2} \partial_x \partial_y f(W_a, W_b)\,\delta W_b \\ &\quad + \int_0^{t_2\wedge a} \partial_x^2 \partial_y^2 f(W_a, W_b)\,db + (1/2)\int_0^{t_2} \partial_y^2 \partial_x f(W_a, W_b)\,db\Bigg]\,\delta W_a \\ &\quad + \int_0^{t_2\wedge t_2} da\Bigg[\partial_x \partial_y f(W_a, 0) + \int_0^{t_2} \partial_x \partial_y^2 f(W_a, W_b)\,\delta W_b \\ &\quad + \int_0^{t_2\wedge a} \partial_x^2 \partial_y^2 f(W_a, W_b)\,db + (1/2)\int_0^{t_2} \partial_x \partial_y^3 f(W_a, W_b)\,db\Bigg] \\ &\quad + (1/2)\int_0^{t_1} da\Bigg[\partial_x^2 f(W_a, 0) + \int_0^{t_2} \partial_x^2 \partial_y f(W_a, W_b)\,\delta W_b \\ &\quad + \int_0^{t_2\wedge a} \partial_x^3 \partial_y f(W_a, W_b)\,db + (1/2)\int_0^{t_2} \partial_x^2 \partial_y^2 f(W_a, W_b)\,db\Bigg]. \end{aligned}$$

The same method of iteration can be applied in case which there are more than two variables.

References

1. Bismut, J.-M.: Mécanique aléatoire. (Lect. Notes Math., vol. 866) Berlin Heidelberg New York: Springer 1981
2. Dellacherie, C., Meyer, P.A.: Probabilités et potentiel, Chap. V–VIII. Paris: Hermann 1980
3. Gaveau, B., Trauber, P.: L'Intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. J. Funct. Anal. **46**, 230–238 (1982)
4. Hitsuda, M.: Formula for Brownian partial derivatives. Faculty of Integrated Arts and Sciences, Hiroshima University, vol. 3, pp. 1–15 (1979)
5. Krée, P.: Continuité de la divergence dans les espaces de Sobolev relatifs à l'espace de Wiener. C.R. Acad. Sci. Paris, Ser. I **296**, 833 (1983)
6. Kuo, H.-H.: Gaussian measures on Banach spaces. (Lect. Notes Math., vol. 463) Berlin Heidelberg New York: Springer 1975
7. Malliavin, P.: Implicit functions in finite corank on the Wiener space. In: Ito, K., Watanabe, S. (eds.) Tanaguchi Supposium, Katata 1982. Amsterdam: North Holland 1984
8. Meyer, P.A.: Flot d'une équation differentielle stochastique. In: Azema, J. Yor M. (eds.) Séminaire de probabilités XV 1979/80. (Lect. Notes Math., vol. 850, pp. 103–117) Berlin Heidelberg New York: Springer 1981
9. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands. Preprint (1986)
10. Nualart, D., Zakai, M.: Generalized stochastic integrals and the Malliavin calculus. Preprint (1985)
11. Pardoux, E., Protter, P.: A two-sided stochastic integral and its calculus. Preprint (1985)
12. Sekiguchi, T., Shiota, Y.: L^2-theory of noncausal stochastic integrals. Math. Rep. Toyama Univ. **8**, 119–195 (1985)
13. Sevljakov, A.Ju.: The Itô formula for the extended stochastic integral. Theory Proba. Math. Stat. **22**, 163–174 (1981)
14. Schaefer, H.H.: Topological vector spaces. (Graduate Texts in Mathematics, vol. 3) Berlin Heidelberg New York: Springer 1971
15. Skorohod, A.V.: On a generalization of a stochastic integral. Theory Probab. Appl. **20**, 219–233 (1975)
16. Ustunel, A.S.: Une extension du calcul d'Itô via le calcul des variations stochastiques. C.R. Acad. Sci., Paris, Ser. I **300**, 277–279 (1985)
17. Ustunel, A.S.: Extension of Itô's calculus via Malliavin's calculus. Stochastics 23, 353–375 (1988)
18. Ustunel, A.S.: Representation of the distributions on Wiener space and stochastic calculus of variations. J. Funct. Anal. **70**, 126–139 (1987)
19. Ustunel, A.S.: La formule de changement de variable pour l'intégrale anticipante de Skorohod. C.R. Acad. Sci., Paris, Serie I, **303**, 329–331 (1986)
20. Ustunel, A.S.: Some comments on the filtering of diffusions and the Malliavin Calculus. In: Korezliogli, H., Ustunel, A.S. (eds.) Proceedings, Sihvri. (Lect. Notes Math. vol. 1316, pp. 247–266). Berlin Heidelberg New York: Springer 1988
21. Watanabe. S.: Stochastic differential equations and Malliavin calculus. Bombay: Tata Institute of Fundamental Research 1984
22. Zheng, Wei-An, Meyer, P.A.: Intégrales stochastiques non monotones. In: Azema, J., Yor, M. (eds.) Séminaire de probabilités XVIII, 1982/83. Proceedings, 1984. (Lect. Notes Math., vol. 1059, pp. 154–171) Berlin Heidelberg New York: Springer 1984

Received November 1, 1987; in revised form January 6, 1988