Probab. Th. Rel. Fields 79, 175–200 (1988)

Probability Theory and Related Fields

Estimates for the Closeness of Successive Convolutions of Multidimensional Symmetric Distributions

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Summary. Let ξ_1, $\xi_2, \ldots$ be i.i.d random vectors in $\mathbb{R}^k$ with a common distribution $\mathscr{L}(\xi_i)=F$, $i=1,2,\ldots$. Let $S_n=\xi_1+\ldots+\xi_n$. We investigate how small is the difference between $\mathscr{L}(S_n)$ and $\mathscr{L}(S_{n+m})$ in the case when ξ_i have symmetric distributions.

1. Introduction

Let $\xi_1,\ldots,\xi_n,\ldots\in\mathbb{R}^k$ be i.i.d. random vectors with a common distribution $\mathscr{L}(\xi_i)=F$, $i=1,2,\ldots$. Then the sum $\mathbf{S}_n=\xi_1+\ldots+\xi_n$ has the distribution F^n (products and powers of measures will be understood in the convolution sence: $FG=F*G$, $F^n=F^{*n}$). Let m,n be arbitrary natural numbers. We shall study how small is the difference between F^{n+m} and F^n in the sense of uniform distance $\varrho(\cdot,\cdot)$ between distribution functions, i.e. how much may be changed the distribution of $\mathbf{S}_n$ after addition to it of the next summand or of a group of summands.

In author's papers [15–17] it was shown that one can obtain meaningful bounds for $\varrho(F^n,F^{n+m})$ without any moment conditions. Moreover, if the distribution F is centered so that all its marginal distributions have zero as medians, then $\varrho(F^n,F^{n+1})\leq ckn^{-1/2}$ where c is an absolute constant. The proof of this inequality is relatively simple and is based on classical bounds for concentration functions of convolutions.

Essentially more complicated methods are needed to investigate the case of symmetric distributions F. From the above mentioned inequality it follows that in this case $\varrho(F^n,F^{n+1})\leq ckn^{-1/2}$. It can be easily seen that this inequality is optimal with respect to order. But we shall show that it may be essentially improved in the case when the characteristic function $\hat{F}(\mathbf{t})$ is uniformly separated from -1. In particular, we shall prove that $\varrho(F^n,F^{n+1})\leq c(k)n^{-1}$ if $\hat{F}(\mathbf{t})\geq 0$ for all $\mathbf{t}\in\mathbb{R}^k$. Using this fact for the distribution F^2 with symmetric F we obtain a paradoxical statement: for all natural numbers n and for any symmetric distribution F the inequalities

$$\varrho(F^n, F^{n+1}) \leqq ckn^{-1/2}, \quad \varrho(F^n, F^{n+2}) \leqq c(k)n^{-1}$$

are valid and they are both optimal with respect to order.

It is evident that the knowledge about the closeness of F^n and F^{n+1} is useful for studying distributions of the form

$$G = \sum_{s=0}^{\infty} p_s F^s, \quad 0 \leqq p_s \leqq 1, \quad \sum_{s=0}^{\infty} p_s = 1.$$

As an example we obtain Theorem 1.4 which contains a new estimate for the uniform distance between the n-fold convolution of F^n of a symmetric distribution F and the corresponding accompanying law

$$e(nF) = e^{-n} \sum_{s=0}^{\infty} \frac{n^s}{s!} F^s.$$

It will be proved that

$$\varrho(F^n, e(nF)) \leqq c(k)n^{-1/2}.$$

All above mentioned inequalities are especially interesting because they give bounds which are independent of any characteristics of F. Note that these inequalities are multidimentional generalizations of analogous one-dimensional results contained in [16, 17]; [2], §§ 5 and 6 Chap. V.

To prove our results we use a new method of estimating the uniform distance between convolutions. This method is an improved version of the triangular functions method, firstly proposed and applied in one-dimensional case by Arak [1], see also [2]. Another example of the application of this method is the proof of Theorem 1.3 from [19] that will be proved in a separate paper. Now we introduce some necessary notations. Let $\mathfrak{B}_k$ be the σ-field of Borel subsets of the Euclidean space $\mathbb{R}^k$, $\mathfrak{F}_k$ be the set of probability measures on $\mathfrak{B}_k$, $\mathfrak{D}_k$ be the set of infinitely divisible distributions in $\mathfrak{F}_k$, $\mathfrak{F}_k^s \subset \mathfrak{F}_k$ be the set of symmetric distributions in $\mathfrak{F}_k$ i.e. of distributions $\mathscr{L}(\xi)$ for which $\mathscr{L}(\xi) = \mathscr{L}(-\xi)$, $\mathfrak{F}_k^+ \subset \mathfrak{F}_k^s$ be the set of distributions with non-negative for all $\mathbf{t} \in \mathbb{R}^k$ characteristic functions. The notation $c(\cdot)$ will be used for different positive constants depending only on the indicated argument. For example, $c(k)$ depend only on the dimension k, c are absolute constants. The writing $\mathbf{x} \in \mathbb{R}^k$ will further denote that $\mathbf{x} = (x_1, \ldots, x_k)$ where $x_j \in \mathbb{R}^1$, $j = 1, \ldots, k$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ we introduce the usual partial ordering: $\mathbf{x} \leqq \mathbf{y}$ means that $x_j \leqq y_j$ for all $j = 1, \ldots, k$ and we denote $[\mathbf{x}, \mathbf{y}] = \{\mathbf{u} \in \mathbb{R}^k : \mathbf{x} \leqq \mathbf{u} \leqq \mathbf{y}\}$. We shall also write $\mathbf{x} < \mathbf{y}$ if $x_j < y_j$ for all $j = 1, \ldots, k$. Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{Z}$ be the set of all integers, $E_{\mathbf{a}} \in \mathfrak{F}_k$ be the distribution concentrated at a point $\mathbf{a} \in \mathbb{R}^k$, $E = E_{\mathbf{0}}$ where $\mathbf{0}$ is the zero vector in $\mathbb{R}^k$. The symbol θ will be used to denote different quantities for which $|\theta| \leqq 1$. For $F = \mathscr{L}(\xi) \in \mathfrak{F}_k$ we shall denote its distribution function by

$$F(\mathbf{x}) = F\{\{\mathbf{u} \in \mathbb{R}^k : \mathbf{u} < \mathbf{x}\}\}, \quad \mathbf{x} \in \mathbb{R}^k;$$

its characteristic function by

$$\hat{F}(\mathbf{t}) = \int_{\mathbb{R}^k} \exp(i(\mathbf{t}, \mathbf{x})) F\{d\mathbf{x}\}, \quad \mathbf{t} \in \mathbb{R}^k, \quad (\mathbf{t}, \mathbf{x}) = \sum_{j=1}^{k} t_j x_j;$$

its concentration function by

$$Q(F,\mathbf{h}) = \sup_{\mathbf{y}\in\mathbb{R}^k} F\{[\mathbf{y},\mathbf{y}+\mathbf{h}]\}, \quad \mathbf{h}\in\mathbb{R}^k, \quad \mathbf{h}\geqq\mathbf{0};$$

and $F^{(j)}=\mathscr{L}(\xi_j)$, $j=1,\dots,k$. We shall estimate the uniform distance between distribution functions

$$\varrho(F,G) = \sup_{\mathbf{x}\in\mathbb{R}^k} |F(\mathbf{x})-G(\mathbf{x})|$$

where $F,G\in\mathfrak{F}_k$. For $F\in\mathfrak{F}_k$, $\lambda\geqq\mathbf{0}$ introduce a distribution $e(\lambda F)\in\mathfrak{D}_k$ by

$$e(\lambda F) = e^{-\lambda}\sum_{s=0}^{\infty} \lambda^s F^s/s!, \quad F^0=E$$

Its characteristic function is equal to $\exp(\lambda(\hat{F}(\mathbf{t})-1))$, $\mathbf{t}\in\mathbb{R}^k$.

Now we return to the statement of the problem. It is evident that if a distribution $F\in\mathfrak{F}_k$ is concentrated on a hyperplane which does not contain zero and is orthogonal to one of coordinate axes then $\varrho(F^n,F^{n+m})=1$ for any natural numbers n,m. In particular, we can consider the case when $F=E_{\mathbf{a}}$, $\mathbf{a}\in\mathbb{R}^k$, $\mathbf{a}\neq\mathbf{0}$. On the other hand, if all distributions $F^{(j)}\in\mathfrak{F}_1$, $j=1,\dots,k$, are either non-degenerate or equal to $E\in\mathfrak{F}_1$ then, as is shown in [15], $\varrho(F^n,F^{n+1})\to 0$ and, moreover, there exists $c(F)$ such that

$$\varrho(F^n,F^{n+1}) \leqq \frac{c(F)}{\sqrt{n}}, \quad n\in\mathbb{N}. \tag{1.1}$$

A point $\mathbf{a}\in\mathbb{R}^k$ will be called the $\mathbf{q}$-quantile of a distribution $F\in\mathfrak{F}_k$, where $\mathbf{q}\in\mathbb{R}^k$, $\mathbf{0}\leqq\mathbf{q}\leqq\mathbf{1}$, $\mathbf{1}=(1,1,\dots,1)\in\mathbb{R}^k$, if $F^{(j)}\{(-\infty,a_j)\}\leqq q_j$, $F^{(j)}\{(a_j,\infty)\}\leqq 1-q_j$ for all $j=1,\dots,k$. In [15] it was also shown that if $\mathbf{0}$ is the $\mathbf{q}$-quantile of a distribution $F\in\mathfrak{F}_k$ and $\mathbf{0}<\mathbf{q}<\mathbf{1}$ then

$$\varrho(F^n,F^{n+1}) \leqq cn^{-1/2}\sum_{j=1}^{k} (q_j(1-q_j))^{-1/2} \tag{1.2}$$

for all $n\in\mathbb{N}$. If $\mathbf{q}=(1/2,1/2,\dots,1/2)\in\mathbb{R}^k$ then the inequality (1.2) turns into

$$\varrho(F^n,F^{n+1}) \leqq ckn^{-1/2}, \quad n\in\mathbb{N}. \tag{1.3}$$

In particular, the inequality (1.3) is valid for any symmetric distribution $F\in\mathfrak{F}_k^s$.

Let F be a one-dimensional lattice symmetric distribution concentrated on the set of odd numbers. Then the distributions F^n, $n=1,2,\dots$ are concentrated either on the set of odd numbers or on the set of even ones in accordance with evenness of a number n. Therefore, $\varrho(F^n,F^{n+1})\geqq Q(F^n,0)/2$. For many distributions, e.g. for $F=E_{-1}/2+E_1/2$ the concentration function $Q(F^n,0)$ behaves as $c(F)n^{-1/2}$ if $n\to\infty$. This indicates that the order of decreasing with respect to n of the right-hand side of (1.3) cannot be increased without additional conditions.

It is easy to show the distribution $F\in\mathfrak{F}_1^s$ is concentrated on the set of odd numbers if and only if its characteristic function $\hat{F}(t)$ is equal to -1 at the point $t=(2m+1)\pi$, where $m\in\mathbb{Z}$, for example, $\hat{F}(t)=\cos t$ for $F=E_{-1}/2+E_1/2$. The following Theorem 1.1 says that the separation from -1 of the characteristic function of a distribution $F\in\mathfrak{F}_k^s$ leads to more quick decreasing of $\varrho(F^n, F^{n+1})$ than the inequality (1.3) is able to provide. For $0<\alpha<2$ define the classes of probability laws

$$\mathfrak{F}_k^\alpha=\{F\in\mathfrak{F}_k^s: \hat{F}(\mathbf{t})\geqq -1+\alpha \quad \text{for all} \quad \mathbf{t}\in\mathbb{R}^k\}.$$

It is easy to see that $\mathfrak{F}_k^{\alpha_1}\subset\mathfrak{F}_k^{\alpha_2}$ for $\alpha_1>\alpha_2$ and $\mathfrak{F}_k^1=\mathfrak{F}_k^+$.

Theorem 1.1. *Let* $0<\alpha<2$ *and* $F\in\mathfrak{F}_k^\alpha$. *Then*

$$\begin{aligned}\varrho(F^n, F^{n+1}) &\leqq c(k)(n^{-1}+\exp(-n\alpha+ck\ln^3 n))\\ &=c(k)n^{-1}+o(n^{-1}).\end{aligned} \tag{1.4}$$

for any $n\in\mathbb{N}$.

The considerable part of this paper (Sect. 2–6) is devoted to the proof of Theorem 1.1. Now we deduce a number of consequences of this theorem.

Corollary 1.1. *For all* $n\in\mathbb{N}$ *the following inequality holds:*

$$\sup_{F\in\mathfrak{F}_k^+}\varrho(F^n, F^{n+1})\leqq c(k)n^{-1}.$$

Proof. It is sufficient to apply Theorem 1.1 for $\alpha=1$.

Corollary 1.2. *For any* $a\geqq 0$, $b\geqq 0$, $F\in\mathfrak{F}_k^s$ *the inequality*

$$\varrho(e(aF), e((a+b)F))\leqq\min\{b, c(k)ba^{-1}\} \tag{1.5}$$

is valid.

Proof. For $a\leqq 1$ the inequality (1.5) can be easily deduced from the formula $e((a+b)F)=e(aF)e(bF)$ and from the following well known property of the uniform distance: for any $F, G, H\in\mathfrak{F}_k$

$$\varrho(FH, GH)\leqq\varrho(F, G). \tag{1.6}$$

In fact,

$$\varrho(e(aF), e(aF)e(bF))\leqq\varrho(E, e(bF))\leqq 1-e^{-b}\leqq b.$$

Let now $a>1$ and let $m\in\mathbb{Z}$ be the largest integer which is less or equal to a/b. Applying Corollary 1.1 for the distribution $G=e(bF)\in\mathfrak{F}_k^+$ and using the inequality (1.6) we obtain:

$$\begin{aligned}\varrho(e(aF), e((a+b)F)) &\leqq\varrho(e(mbF), e((m+1)bF))\\ &=\varrho(G^m, G^{m+1})\leqq\frac{c(k)}{m+1}\leqq\frac{c(k)b}{a}.\end{aligned}$$

Remember that, according to (1.3),

$$\sup_{F\in\mathfrak{F}_k^s}\varrho(F^n, F^{n+1})\leqq ckn^{-1/2} \tag{1.7}$$

and the order of decreasing of the right-hand side of (1.7) with respect to n cannot be enlarged. Using Corollary 1.1 we shall show that $\sup\limits_{F\in\mathfrak{F}_k^s}\varrho(F^n, F^{n+2})$ decreases essentially more quickly than the very close to it left-hand side of (1.7). At the same time we shall estimate $\varrho(F^n, F^{n+m})$ for each $m\in\mathbb{N}$.

Theorem 1.2. *For any $m, n \in \mathbb{N}$ the following inequalities are valid:*

$$\sup_{F \in \mathfrak{F}_k^s} \varrho(F^n, F^{n+2}) \leqq c(k) n^{-1}, \tag{1.8}$$

$$\sup_{F \in \mathfrak{F}_k^s} \varrho(F^n, F^{n+2m}) \leqq c(k) mn^{-1}, \tag{1.9}$$

$$\sup_{F \in \mathfrak{F}_k^s} \varrho(F^n, F^{n+2m+1}) \leqq ckn^{-1/2} + c(k) mn^{-1}, \tag{1.10}$$

and, consequently,

$$\sup_{1 \leqq m \leqq \sqrt{n}} \sup_{F \in \mathfrak{F}_k^s} \varrho(F^n, F^{n+m}) \leqq c(k) n^{-1/2}.$$

Proof. Let $n = 2l$, $l \in \mathbb{N}$. If $F \in \mathfrak{F}_k^s$ then $F^2 \in \mathfrak{F}_k^+$ and, by Corollary 1.1,

$$\varrho(F^n, F^{n+2}) = \varrho((F^2)^l, (F^2)^{l+1}) \leqq cl^{-1} = cn^{-1}.$$

If now $n = 2l+1$, $l \in \mathbb{N}$, then using (1.6) it is not difficult to show that

$$\varrho(F^n, F^{n+2}) \leqq \varrho((F^2)^l, (F^2)^{l+1}) \leqq cn^{-1}.$$

The inequality (1.8) is proved. The inequality (1.9) can be easily derived from (1.8) with the help of the triangle inequality. To obtain (1.10) it is necessary to attract in addition the inequality (1.7).

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random vectors with a common distribution $F \in \mathfrak{F}_k$ and let $(\mu, \nu) \in \mathbb{Z}^2$ be a random vector, independent of $\{\xi_i\}_{i=1}^{\infty}$ and having non-negative integer coordinates. Denote

$$U = \mathscr{L}(\mu),\ V = \mathscr{L}(\nu),\ G = \mathscr{L}(\xi_1 + \ldots + \xi_\mu),\ H = \mathscr{L}(\xi_1 + \ldots + \xi_\nu).$$

It is well known that

$$G = \sum_{s=0}^{\infty} \mathbf{P}\{\mu = s\} F^s, \quad H = \sum_{s=0}^{\infty} \mathbf{P}\{\nu = s\} F^s. \tag{1.11}$$

We shall show that the following upper bound for the uniform distance between the distributions G and H holds true, if $F \in \mathfrak{F}_k^s$.

Theorem 1.3. *If $F \in \mathfrak{F}_k^s$ then*

$$\varrho(G, H) \leqq \inf \mathbf{E} \min\left\{\frac{ck}{\sqrt{\nu+1}} + c(k)\frac{|\mu - \nu|}{\nu+1},\ 1\right\}, \tag{1.12}$$

and if $F \in \mathfrak{F}_k^+$ then

$$\varrho(G, H) \leqq \inf \mathbf{E} \min\left\{c(k)\frac{|\mu - \nu|}{\nu+1},\ 1\right\}. \tag{1.13}$$

Here the lower bound is taken over all possible two-dimensional joint distributions $\mathscr{L}((\mu, \nu)) \in \mathfrak{F}_2$ such that $\mathscr{L}(\mu) = U$, $\mathscr{L}(\nu) = V$.

Proof. Let $F \in \mathfrak{F}_k^s$. From (1.9), (1.10) it follows that for all $n, m \in \mathbb{Z}$ such that $n \geqq 0$, $m \geqq 0$ the inequality

$$\varrho(F^n, F^m) \leqq \min\left\{\frac{ck}{\sqrt{n+1}} + c(k)\,\frac{|m-n|}{n+1},\ 1\right\} \tag{1.14}$$

is valid. Let $\mathbf{x} \in \mathbb{R}^k$. Then (1.14) implies that

$$\begin{aligned}|G(\mathbf{x}) - H(\mathbf{x})| &= \left|\sum_{m,n}\left(\mathbf{P}\left\{\sum_{i=1}^{m} \xi_i < \mathbf{x}\right\} - \mathbf{P}\left\{\sum_{i=1}^{n} \xi_i < \mathbf{x}\right\}\right)\mathbf{P}\{\mu=m, \nu=n\}\right| \\ &\leqq \sum_{m,n} \min\left\{\frac{ck}{\sqrt{n+1}} + c(k)\,\frac{|m-n|}{n+1},\ 1\right\}\mathbf{P}\{\mu=m, \nu=n\} \\ &= \mathbf{E}\min\left\{\frac{ck}{\sqrt{\nu+1}} + c(k)\,\frac{|\mu-\nu|}{\nu+1},\ 1\right\}.\end{aligned}$$

Hence the inequality (1.12) is proved. The inequality (1.13) can be obtained in a similar way. If $V = E_n$ then Theorem 1.3 turns into the following result.

Corollary 1.3. *If* $F \in \mathfrak{F}_k^s$, $n \in \mathbb{N}$ *then*

$$\varrho(G, F^n) \leqq \mathbf{E}\min\left\{\frac{ck}{\sqrt{n}} + c(k)\left|\frac{\mu}{n} - 1\right|,\ 1\right\} \tag{1.15}$$

and if $F \in \mathfrak{F}_k^+$ *then*

$$\varrho(G, F^n) \leqq \mathbf{E}\min\left\{c(k)\left|\frac{\mu}{n} - 1\right|,\ 1\right\}. \tag{1.16}$$

The inequalities (1.15), (1.16) show that if n is large and $\mathbf{E}\left|\frac{\mu}{n} - 1\right|$ is small then the distribution G of a sum of random number of summands does not considerably differ from the distribution of the sum of n summands and we can find the upper bounds for the closeness of above mentioned distributions which are uniform with respect to classes $\mathfrak{F}_k^s$ and $\mathfrak{F}_k^+$. With the help of the inequalities (1.1)–(1.4) we can also obtain analogous inequalities for other classes of distributions.

The distributions of sums of a random number of i.i.d. summands were considered in many articles (see, for example, [3, 6, 9, 12, 13]). In particular, there can be found results about the closeness of studying distributions to the distributions of sums of a non-random number of random variables (the so-called transfer theorems). In several works the number of summands is not supposed independent of these summands (see, e.g., [9, p. 418], [13]). But it should be noted that the statements contained in the most papers are either of qualitative character or proved under additional moment restrictions. We do not mention here numerous works about the central limit theorem for random sums.

As an example of the application of the inequality (1.15) we give an estimate for the closeness of F^n, where $F \in \mathfrak{F}_k^s$, to the accompanying distribution $e(nF) \in \mathfrak{D}_k$. It should be noted that this estimate is independent of $F \in \mathfrak{F}_k^s$.

Theorem 1.4. *For any* $n \in \mathbb{N}$ *the inequality*

$$\sup_{F \in \mathfrak{F}_k^s} \varrho(F^n, e(nF)) \leqq c(k)\, n^{-1/2} \tag{1.17}$$

is valid.

Proof. It is sufficient to note that the distribution $e(nF)$ satisfies all conditions imposed on the distribution G from (1.11) if $\mathscr{L}(\mu)=e(nE_1)$ is the Poisson distribution with expectation $\mathbf{E}\mu=\mathbf{D}\mu=n$. In view of (1.15),

$$\begin{aligned}\varrho(F^n, e(nF)) &\leqq ckn^{-1/2}+c(k)n^{-1}\mathbf{E}|\mu-n| \\ &= ckn^{-1/2}+c(k)n^{-1}\mathbf{E}|\mu-\mathbf{E}\mu| \\ &\leqq ckn^{-1/2}+c(k)n^{-1}\sqrt{\mathbf{D}\mu} \\ &\leqq c(k)n^{-1/2}.\end{aligned}$$

The closeness of n-fold convolutions of arbitrary distributions from $\mathfrak{F}_k$ to corresponding accompanying laws was studied by Présman [10]. He proved that

$$\inf_{\mathbf{a}\in\mathbb{R}^k} \varrho(F^n, E_{n\mathbf{a}}e(nFE_{-\mathbf{a}}))\leqq c(k)n^{-1/3} \tag{1.18}$$

for any $F\in\mathfrak{F}_k$. It is known that the inequality (1.18) is unimprovable with respect to order and cannot be essentially strengthened in general case. The inequality (1.17) shows that such sharpening is possible for symmetric distributions F.

One-dimensional variants of above mentioned results can be found in [2], §§4–6, Chap. V, see also [11, 16, 17]. There can be also found the proofs of unimprovability of some of them with respect to order (of course, for this it is sufficient to consider the one-dimensional case). For example, in order to verify the fact that the inequalities (1.1), (1.3), (1.10) cannot be sharpened it is sufficient to consider the distribution $F=E_{-1}/2+E_1/2$, but for the inequalities (1.8), (1.9) and for Corollary 1.1 one should analyze the case when $F=\Phi$ is the standard normal distribution $\mathscr{N}(0,1)$.

Consider now the question about the unimprovability of the inequality (1.4). We shall show that the dependence of the right-hand side of this inequality on α is close to unimprovable one. As an example we consider the distribution $F=(1-\alpha)(E_{-1}/2+E_1/2)+\alpha\Phi\in\mathfrak{F}_1^\alpha$. It can be easily proved that

$$\varrho(F^n, F^{n+1})\geqq c(1-\alpha)^n n^{-1/2}.$$

Assume that the question is how little must be $\alpha=\alpha(n)$ in order to ensure the validity of the inequality $\varrho(F^n, F^{n+1})\leqq c(k)n^{-1}$ for all $F\in\mathfrak{F}_k^{\alpha(n)}$. In view of (1.4), we can choose $\alpha(n)=ckn^{-1}\ln^3 n$. On the other hand, the above example says that $\alpha(n)$ cannot be chosen less than $cn^{-1}\ln n$.

By (1.4), in the Theorem 1.1 conditions for any $X\subset\mathbb{R}^k$, representable in the form

$$X=\{\mathbf{x}\in\mathbb{R}^k: x_j<y_j,\ y_j\in\mathbb{R}^1, j=1,\dots,k\},$$

the inequality

$$|F^n\{X\}-F^{n+1}\{X\}|\leqq c(k)(n^{-1}+\exp(-n\alpha+ck\ln^3 n)) \tag{1.19}$$

holds. The question about the possibility of the extension of the inequality (1.19) on arbitrary convex sets $X\subset\mathbb{R}^k$ remains open. However, this inequality can be easily extended on convex polytopes with a number of vertices, bounded by $c(k)$. In fact, if the distribution $F=\mathscr{L}(\xi)$ satisfies the Theorem 1.1 conditions then the same may be said about the distribution $F^*=\mathscr{L}(A\xi)$ where $A:\mathbb{R}^k\to\mathbb{R}^k$ is an arbitrary linear operator. Using this observation, we can easily obtain the inequality (1.19) for the sets X, representable in the form

$$X=\{\mathbf{x}\in\mathbb{R}^k:(\mathbf{x},\mathbf{t}_j)<y_j,\mathbf{t}_j\in\mathbb{R}^k,\ y_j\in\mathbb{R}^1,\ j=1,\dots,k\}. \qquad (1.20)$$

Consider now a non-degenerate k-dimensional simplex Y given as the intersection of $k+1$ half-spaces

$$Y=\bigcap_{j=1}^{k+1} A_j$$

where $A_j=\{\mathbf{x}\in\mathbb{R}^k:(\mathbf{x},\mathbf{t}_j)\leqq y_j\}$, $\mathbf{t}_j\in\mathbb{R}^k$, $y_j\in\mathbb{R}^1$ and let $\xi\in\mathbb{R}^k$ be a random vector with $\mathscr{L}(\xi)=H\in\mathfrak{F}_k$. Then, denoting $\bar{A}_j=\mathbb{R}^k\backslash A_j$ and using the formula (1.5), Chap. IV [5], we obtain

$$\begin{aligned}H\{Y\}&=1-H\left\{\bigcup_{j=1}^{k+1}\bar{A}_j\right\}=1-\sum_{j=1}^{k+1}H\{\bar{A}_j\}\\&+\sum_{j_1<j_2}H\{\bar{A}_{j_1}\bar{A}_{j_2}\}+\dots+(-1)^l\sum_{j_1<\dots<j_l}H\{\bar{A}_{j_1}\dots\bar{A}_{j_l}\}\\&+\dots+(-1)^{k+1}H\{\bar{A}_1\bar{A}_2\dots\bar{A}_{k+1}\}. \qquad (1.21)\end{aligned}$$

Note that from the assumption about the non-degeneracy of the simplex Y it follows that $\bar{A}_1\bar{A}_2\dots\bar{A}_{k+1}=\Phi$ and so the last summand in the right-hand side of (1.21) is equal to zero. All of the rest sets $X=\bar{A}_{j_1}\dots\bar{A}_{j_l}$, $l<k+1$, may be written as in (1.20). Applying (1.21) for $H=F^n$, $H=F^{n+1}$ and using the inequality (1.19) we obtain

$$|F^n\{Y\}-F^{n+1}\{Y\}|\leqq c(k)(n^{-1}+\exp(-n\alpha+ck\ln^3 n)).$$

If now Y is a convex polytope representable as an union of m simplexes with empty interiorities of their intersections then

$$|F^n\{Y\}-F^{n+1}\{Y\}|\leqq c(k)m(n^{-1}+\exp(-n\alpha+ck\ln^3 n)). \qquad (1.22)$$

The right-hand side of (1.22) increases together with a growth of m. Therefore we have no possibility to obtain the inequality (1.19) for arbitrary convex sets by a passage to a limit.

Remark 1.1. The inequality (1.22) may be considered as a generalization of the inequality (1.19) for a larger class of sets. Of course, analogous generalizations can be written out for other inequalities mentioned above.

The rest of the paper is devoted to the proof of Theorem 1.1. We shall use the methods which are multidimentional analogues of the so-called triangular functions method developed and firstly applied by Arak [1], see also [2]. In the sequel we shall essentially use the results and the methods from [1]. Note, however, that in the multidimensional case it will be necessary to revise the arguments used in [2] to prove the one-dimensional version of Theorem 1.1. In particular, somewhere we shall refuse to apply the Parseval equality and shall use Lindeberg's method based on the Taylor formula (see [8, p. 82]). In order to obtain the possibility to apply this formula we shall change the triangular functions, used in [1,2], by their infinitely differentiable analogues. The technical details connected with above mentioned changes of methods are discussed in [18].

2. Auxiliary Results

Let $\mathfrak{M}_k$ be the set of all finite charges defined on $\mathfrak{B}_k$. For an arbitrary $\mathfrak{B}_k$-measurable bounded function f and arbitrary $\mu \in \mathfrak{M}_k$; $F, G \in \mathfrak{F}_k$ denote

$$\Gamma_f(\mu) = \sup_{\mathbf{z} \in \mathbb{R}^k} \left| \int_{\mathbb{R}^k} f(\mathbf{x}-\mathbf{z}) \mu\{d\mathbf{x}\} \right|,$$

$$\varrho_f(F, G) = \Gamma_f(F-G). \tag{2.1}$$

In the sequel we shall use the following evident inequalities that are valid for any $\mu \in \mathfrak{M}_k$; $F, G, H \in \mathfrak{F}_k$:

$$\Gamma_f(\mu * H) = \Gamma_f(\mu H) \leqq \Gamma_f(\mu), \tag{2.2}$$

$$\varrho_f(FH, GH) \leqq \varrho_f(F, G). \tag{2.3}$$

Define now the function $\varphi(x)$, $x \in \mathbb{R}^1$, setting

$$\varphi(x) = \left(\int_0^1 e^{-\frac{1}{y} - \frac{1}{1-y}} dy \right)^{-1} \int_0^x e^{-\frac{1}{y} - \frac{1}{1-y}} dy \quad \text{for} \quad 0 < x < 1,$$

$$\varphi(x) = 0 \quad \text{for} \quad x \leqq 0 \quad \text{and} \quad \varphi(x) = 1 \quad \text{for} \quad x \geqq 1.$$

The function φ is infinitely differentiable on the real line. For $z, \tau, h, x \in \mathbb{R}^1$, $0 < \tau \leqq h$, set

$$\begin{aligned}
&\varphi_{z,\tau}(x) = \varphi((x-z)/\tau), \\
&f_{z,h,\tau}(x) = f^{(0)}_{z,h,\tau}(x) = \varphi_{z,\tau}(x) - \varphi_{z,h}(x), \\
&f^{(1)}_{z,h,\tau}(x) = \varphi_{z,h}(x) - \varphi_{z+h-\tau,\tau}(x), \\
&g_{z,h,\tau}(x) = \varphi_{z,\tau}(x) - \varphi_{z+h-\tau,\tau}(x) = f^{(0)}_{z,h,\tau}(x) + f^{(1)}_{z,h,\tau}(x).
\end{aligned} \tag{2.4}$$

Denote by Ξ the collection of vectors $\mathbf{J} \in \mathbb{R}^k$ having coordinates which can be equal either to zero or to unit. For $F, G \in \mathfrak{F}_k$; $\mu \in \mathfrak{M}_k$; $\mathbf{J} \in \Xi$; $\mathbf{z}, \mathbf{h}, \boldsymbol{\tau}, \mathbf{x} \in \mathbb{R}^k$, $\mathbf{0} < \boldsymbol{\tau} \leqq \mathbf{h}$, we set

$$\begin{aligned}
&f^{(\mathbf{J})}_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}) = \prod_{p=1}^{k} f^{(J_p)}_{z_p,h_p,\tau_p}(x_p), \\
&g_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}) = \prod_{p=1}^{k} g_{z_p,h_p,\tau_p}(x_p) = \sum_{\mathbf{J} \in \Xi} f^{(\mathbf{J})}_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}), \\
&\varrho^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(F, G) = \varrho_f(F, G), \ \Gamma^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(\mu) = \Gamma_f(\mu) \quad \text{where} \quad f = f^{(\mathbf{J})}_{\mathbf{0},\mathbf{h},\boldsymbol{\tau}}, \\
&d_{\mathbf{h},\boldsymbol{\tau}}(F, G) = \varrho_{g_{\mathbf{0},\mathbf{h},\boldsymbol{\tau}}}(F, G).
\end{aligned} \tag{2.5}$$

Let us argree to omit a superscript (J) if $\mathbf{J} = \mathbf{0}$ (as a rule, we shall estimate $\varrho_{\mathbf{h},\boldsymbol{\tau}}(\cdot,\cdot) = \varrho^{(\mathbf{0})}_{\mathbf{h},\boldsymbol{\tau}}(\cdot,\cdot)$ only, keeping in mind the possibility to use the symmetry arguments). Note that the characteristics just introduced can be non-zero if $\mathbf{0} < \boldsymbol{\tau} < \mathbf{h}$ only, since if $\tau_j = h_j$ for some j then $f^{(\mathbf{J})}_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}) \equiv 0$. It can be shown that if $\mathbf{0} < \boldsymbol{\tau} < \mathbf{h}$ then $\varrho^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(\cdot,\cdot)$ and $d_{\mathbf{h},\boldsymbol{\tau}}(\cdot,\cdot)$ are metrics in $\mathfrak{F}_k$ (see, e.g., [14], Theorem 2.2). It is easy to check that for all $\boldsymbol{\tau}, \mathbf{h} \in \mathbb{R}^k$, $\mathbf{0} < \boldsymbol{\tau} < \mathbf{h}$; $\mathbf{z} \in \mathbb{R}^k$; $j, p = 1, \ldots, k$; $F, G \in \mathfrak{F}_k$ the following inequalities are valid:

$$d_{\mathbf{h},\boldsymbol{\tau}}(F,G) \leqq \sum_{\mathbf{J}\in\Xi} \varrho^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(F,G). \tag{2.6}$$

$$\sup_{\mathbf{x}} \left|\frac{\partial f_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}}{\partial x_j}(\mathbf{x})\right| \leqq \frac{c}{\tau_j}, \ \sup_{\mathbf{x}} \left|\frac{\partial^2 f_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}}{\partial x_j \partial x_p}(\mathbf{x})\right| \leqq \frac{c}{\tau_j \tau_p}, \tag{2.7}$$

$$\Gamma_{\mathbf{h},\boldsymbol{\tau}}(F) \leqq Q(F,\mathbf{h}) \tag{2.8}$$

(see (2.4), (2.5)). We shall need the following auxiliary results.

Lemma 2.1 [18]. *Let* $F,G,U\in\mathfrak{F}_k$. *For* $\mathbf{h}\in\mathbb{R}^k$, $\mathbf{h}>\mathbf{0}$, *denote* $\gamma^{(j)}_{\mathbf{h}}=Q(U^{(j)},h_j)$, $\gamma_{\mathbf{h}}=\min\limits_{1\leqq j\leqq k}\gamma^{(j)}_{\mathbf{h}}$. *Suppose that for all* $\mathbf{J}\in\Xi$ *and for all* $\boldsymbol{\tau},\mathbf{h}\in\mathbb{R}^k$ *such that* $\mathbf{0}<\boldsymbol{\tau}<\mathbf{h}$; $\gamma^{(j)}_{\mathbf{h}}\leqq 4\gamma^{(j)}_{\boldsymbol{\tau}}$, $j=1,\dots,k$, *the following inequality is valid*

$$\varrho^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(F,G) \leqq c(k)\varepsilon_1\gamma^{\alpha}_{\mathbf{h}}(|\ln\gamma_{\mathbf{h}}|+1)^{\beta}+\varepsilon_2(\boldsymbol{\tau})$$

where $\alpha>0$, $\beta\geqq 0$, $\varepsilon_1\geqq 0$ *and* $\varepsilon_2(\boldsymbol{\tau})$ *is a non-increasing non-negative function of a parameter* $\boldsymbol{\tau}\in\mathbb{R}^k$. *Then for all* $\mathbf{J}\in\Xi$ *and for all* $\boldsymbol{\tau},\mathbf{h}\in\mathbb{R}^k$ *such that* $\mathbf{0}<\boldsymbol{\tau}<\mathbf{h}$ *the inequality*

$$\varrho^{(\mathbf{J})}_{\mathbf{h},\boldsymbol{\tau}}(F,G) \leqq c(k,\alpha,\beta)\varepsilon_1 \prod_{j=1}^{k}\left((\gamma^{(j)}_{\mathbf{h}})^{\frac{\alpha}{k}}(|\ln\gamma^{(j)}_{\mathbf{h}}|+1)^{\frac{\beta}{k}}\right)$$

$$+\varepsilon_2(\boldsymbol{\tau})\prod_{j=1}^{k}\left(\frac{1}{\ln 2}\ln\frac{\gamma^{(j)}_{\mathbf{h}}}{\gamma^{(j)}_{\boldsymbol{\tau}}}+1\right)$$

holds.

We shall denote by

$$[X]_\tau=\{\mathbf{y}\in\mathbb{R}^k:\inf_{\mathbf{x}\in X}\|\mathbf{x}-\mathbf{y}\|\leqq\tau\},\ \tau>0,$$

the closed τ-neighbourhood of a set $X\subset\mathbb{R}^k$. Define the set

$$K_m(\mathbf{u})=\left\{\sum_{i=1}^{l} n_i u_i : n_i\in\mathbb{Z},\ |n_i|\leqq m,\ i=1,\dots,l\right\}\subset\mathbb{R}^1 \tag{2.9}$$

for $\mathbf{u}\in\mathbb{R}^l$, $m\in\mathbb{N}$. The following lemma is a special case of Corollary 3.1 [18].

Lemma 2.2. *Let* $r\in\mathbb{N}$ *and let a distribution* $L_0\in\mathfrak{F}_1$ *be represented in the form* $L_0=s_0U_0+v_0R_0$ *where* $0\leqq v_0\leqq 1$, $s_0=1-v_0$; U_0, $R_0\in\mathfrak{F}_1$ *and there exist such* $\tau_0\geqq 0$, $l\in\mathbb{N}$, $\mathbf{u}\in\mathbb{R}^l$ *that* $U_0\{[-\tau_0,\tau_0]\}=R_0\{[K_1(\mathbf{u})]_{\tau_0}\}=1$ *and*

$$\int_{-\infty}^{\infty} xU_0\{dx\}=0,\quad rs_0\int_{-\infty}^{\infty} x^2U_0\{dx\}=\sigma^2.$$

Then for any $y\geqq 0$ *and for all integers* $d\geqq rv_0$ *the inequality*

$$(L_0)^r\{\mathbb{R}^1\backslash[K_d(\mathbf{u})]_{d\tau_0+y}\}$$
$$\leqq 2\max\left\{\exp\left(-\frac{y^2}{4\sigma^2}\right),\quad \exp\left(-\frac{y}{8\tau_0}\right)\right\}$$
$$+\max\left\{\exp\left(-\frac{(d-rv_0)^2}{4rv_0}\right),\quad \exp\left(-\frac{d-rv_0}{4}\right)\right\}$$

holds.

Lemma 2.3 below is an analogue of Lemma 4.1, Chap. III [2], obtained by Arak [1].

Lemma 2.3 [18]. *Let* $0<\tau<h$; $l, m\in\mathbb{N}$; $\mathbf{u}\in\mathbb{R}^l$; $z\in\mathbb{R}^1$. *There exist the functions* $w(x)$ *and* $g(t)$ *having the following properties:*

a) $0\leqq w(x)\leqq f_{z,h,\tau}(x)$ *for all* $x\in\mathbb{R}^1$;

b) $w(x)=f_{z,h,\tau}(x)$ *for* $x\in[K_m(\mathbf{u})]_{m\tau}$;

c) *the function* $w(x)$ *is infinitely differentiable on the real line and*

$$\sup_x\left|\frac{d^p}{dx^p}w(x)\right|\leqq\frac{c(p)}{\tau^p},\quad p=1,2,\ldots;$$

d) *for all* $t\in\mathbb{R}^1$ *the inequality* $|\hat{w}(t)|\leqq g(t)$ *is valid where*

$$\hat{w}(t)=\int_{-\infty}^{\infty}e^{itx}w(x)\,dx$$

is the Fourier transform of the function w;

e) *the function* $g(t)$ *is even, does not increase for* $t\geqq 0$ *and takes a constant value* $g(0)$ *for* $|t|\leqq 2h^{-1}$;

f) $\int_{-\infty}^{\infty} g(t)\,dt\leqq cl^2\ln(lm+1)$.

For $\mathbf{h}\in\mathbb{R}^k$, $\mathbf{h}>\mathbf{0}$, and for an arbitrary finite measure Δ defined on $\mathfrak{B}_k$ we introduce the following characteristics:

$$v_j(\Delta,h_j)=\int_{\mathbb{R}^k}\min\{1,x_j^2h_j^{-2}\}\Delta\{dx\},\ j=1,\ldots,k; \tag{2.10}$$

$$\omega(\Delta,\mathbf{h})=\max_{1\leqq j\leqq k}v_j(\Delta,h_j); \tag{2.11}$$

$$v(\Delta,\mathbf{h})=\int_{[-\mathbf{h},\mathbf{h}]}\sum_{j=1}^{k}x_j^2h_j^{-2}\Delta\{d\mathbf{x}\}+\Delta\{\mathbb{R}^k\backslash[-\mathbf{h},\mathbf{h}]\}. \tag{2.12}$$

It is not difficult to check that $v(y\Delta,\mathbf{h})=yv(\Delta,\mathbf{h})$ for any $y>0$ and

$$\omega(\Delta,\mathbf{h})\leqq v(\Delta,\mathbf{h})\leqq k\omega(\Delta,\mathbf{h}). \tag{2.13}$$

Lemma 2.4. *Let* $H\in\mathfrak{F}_1$, $y>0$, $\tau>0$, $D=e(yH)\in\mathfrak{D}_1$, $\gamma=Q(D,\tau)$. *Then*

a) $\gamma\leqq c(v(yH,\tau))^{-1/2}$; (2.14)

b) *there exist* $l\in\mathbb{N}$, $\mathbf{u}\in\mathbb{R}^l$ *such that*

$$l\leqq c(|\ln\gamma|+1), \tag{2.15}$$

$$yH\{\mathbb{R}^1 \backslash [K_1(\mathbf{u})]_\tau\} \leqq c(|\ln \gamma| + 1)^3. \tag{2.16}$$

The statement a) was proved by Le Cam [7] and the assertions b) were obtained by Arak [1], see also [2], Chap. III, Theorem 3.3. The following lemma may be easily deduced from [4], Lemma 6.1.

Lemma 2.5. *Let* $F \in \mathfrak{F}_k$, $\mathbf{h} \in \mathbb{R}^k$, $\mathbf{h} > \mathbf{0}$. *Then*

$$Q(F, \mathbf{h}) \leqq c(k) \int_{[-\mathbf{h}^{-1}, \mathbf{h}^{-1}]} |\hat{F}(\mathbf{t})| d\mathbf{t} \prod_{j=1}^{k} h_j \tag{2.17}$$

where $\mathbf{h}^{-1} \in \mathbb{R}^k$ *is a vector with coordinates* h_j^{-1}, $j = 1, \ldots, k$.

For $\mathbf{h} \in \mathbb{R}^k$, $\mathbf{h} > \mathbf{0}$, introduce the class $\Lambda(\mathbf{h})$ of functions $g(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^k$, representable in the form

$$g(\mathbf{t}) = \prod_{j=1}^{k} g_j(t_j)$$

where the functions $g_j \in L^1(\mathbb{R}^1)$, $j = 1, \ldots, k$, are even, non-negative, non-increasing for $t_j \geqq 0$, and equal to $g_j(0)$ for $|t_j| \leqq h_j^{-1}$.

Lemma 2.6 [18]. *Let* $H \in \mathfrak{F}_k^+$, $\mathbf{h} \in \mathbb{R}^k$, $\mathbf{h} > \mathbf{0}$. *Then for any* $g \in \Lambda(\mathbf{h})$ *the inequality*

$$\int_{\mathbb{R}^k} g(\mathbf{t}) \hat{H}(\mathbf{t}) d\mathbf{t} \leqq c(k) Q(H, \mathbf{h}) \int_{\mathbb{R}^k} g(\mathbf{t}) d\mathbf{t} \tag{2.18}$$

holds. In particular,

$$\int_{[-\mathbf{h}^{-1}, \mathbf{h}^{-1}]} \hat{H}(\mathbf{t}) d\mathbf{t} < \leqq c(k) Q(H, \mathbf{h}) \prod_{j=1}^{k} h_j^{-1}. \tag{2.19}$$

3. Beginning of the Proof of Theorem 1.1

Let $F \in \mathfrak{F}_k^s$, $n \in \mathbb{N}$ and let for all $\mathbf{t} \in \mathbb{R}^k$ the inequality $\hat{F}(\mathbf{t}) \geqq -1 + \alpha$, $0 < \alpha < 2$, be valid, i.e. $F \in \mathfrak{F}_k^\alpha$. Denote

$$\Delta = nF, \quad D = e(nF) \in \mathfrak{D}_k \cap \mathfrak{F}_k^+. \tag{3.1}$$

It is not difficult to see that for all $\mathbf{t} \in \mathbb{R}^k$ the following relations hold true:

$$\hat{D}(\mathbf{t}) = \exp(n(\hat{F}(\mathbf{t}) - 1)),$$

$$|\hat{F}(\mathbf{t})| \leqq \max\{\exp(\hat{F}(\mathbf{t}) - 1), \quad e^{-\alpha}\}, \tag{3.2}$$

$$|\hat{F}^{n+1}(\mathbf{t})| \leqq |\hat{F}^n(\mathbf{t})| \leqq \max\{\hat{D}(\mathbf{t}), e^{-n\alpha}\}, \tag{3.3}$$

$$|\hat{F}^n(\mathbf{t}) - \hat{F}^{n+1}(\mathbf{t})| \leqq \max\{cn^{-1}, 2e^{-n\alpha}\}. \tag{3.4}$$

In order to prove the inequalities (3.2)–(3.4) it is sufficient to consider separately the cases $\hat{F}(\mathbf{t}) \geqq 0$ and $\hat{F}(\mathbf{t}) < 0$. The inequality (3.4) shows that an analogue of the inequality (1.4) is valid for the uniform distance between characteristic functions of distributions to be compared.

For $\mathbf{v}\in\mathbb{R}^k$, $\mathbf{v}>\mathbf{0}$, set

$$\gamma_{\mathbf{v}}^{(j)}=Q(D^{(j)},v_j),\quad \gamma_{\mathbf{v}}=\min_{1\leqq j\leqq k}\gamma_{\mathbf{v}}^{(j)}. \tag{3.5}$$

The inequalities (2.17), (2.19), (3.3) imply that

$$\begin{aligned}Q(F^n,\mathbf{h})&\leqq c(k)\int_{[-\mathbf{h}^{-1},\mathbf{h}^{-1}]}|\hat{F}^n(\mathbf{t})|d\mathbf{t}\prod_{j=1}^{k}h_j\\ &\leqq c(k)\int_{[-\mathbf{h}^{-1},\mathbf{h}^{-1}]}(\hat{D}(\mathbf{t})+e^{-n\alpha})d\mathbf{t}\prod_{j=1}^{k}h_j\\ &\leqq c(k)\,(Q(D,\mathbf{h})+e^{-n\alpha})\\ &\leqq c(k)\,(\gamma_{\mathbf{h}}+e^{-n\alpha}).\end{aligned} \tag{3.6}$$

for any $\mathbf{h}\in\mathbb{R}^k$, $\mathbf{h}>\mathbf{0}$.

Fix $\tau\in\mathbb{R}^k$, $\tau>\mathbf{0}$, and apply the statement of item b) of Lemma 2.4 to the distributions $D^{(j)}=e(nF^{(j)})\in\mathfrak{D}_1$. By (2.15), (2.16), (3.5), for $j=1,\dots,k$ there exist $l_j\in\mathbb{N}$, $\mathbf{u}^{(j)}\in\mathbb{R}^{l_j}$ such that

$$l_j\leqq c(|\ln\gamma_{\tau}^{(j)}|+1), \tag{3.7}$$

$$F^{(j)}\{\mathbb{R}^1\backslash[K_1(\mathbf{u}^{(j)})]_{\tau_j}\}\leqq cn^{-1}(|\ln\gamma_{\tau}^{(j)}|+1)^3. \tag{3.8}$$

For $m\in\mathbb{N}$ we define the sets $K_m\subset\mathbb{R}^k$ as direct products of the sets $[K_m(\mathbf{u}^{(j)})]_{m\tau_j}$:

$$K_m=\bigotimes_{j=1}^{k}[K_m(\mathbf{u}^{(j)})]_{m\tau_j}. \tag{3.9}$$

Remark 3.1. The sets K_m are symmetric with respect to zero, contain the point $\mathbf{0}$ and grow when m increases. In addition, if $\mathbf{x}\in K_{m_1}$, $\mathbf{y}\in K_{m_2}$ then $\mathbf{x}+\mathbf{y}\in K_{m_1+m_2}$.

Setting

$$s=F\{[-\tau,\tau]\},\ v=F\{K_1\backslash[-\tau,\tau]\},\ w=F\{\mathbb{R}^k\backslash K_1\}, \tag{3.10}$$

we represent F as a mixture of distributions $U, R, P\in\mathfrak{F}_k^s$ concentrated on the disjoint sets $[-\tau,\tau]$, $K_1\backslash[-\tau,\tau]$, $\mathbb{R}^k\backslash K_1$, respectively:

$$\begin{gathered}F=sU+vR+wP,\\ U\{[-\tau,\tau]\}=R\{K_1\backslash[-\tau,\tau]\}=P\{\mathbb{R}^k\backslash K_1\}=1.\end{gathered} \tag{3.11}$$

Define also the probability measures $L, W\in\mathfrak{F}_k^s$ and the numbers b, q by the relations

$$bL=sU+vR,\quad qW=vR+wP. \tag{3.12}$$

It is evident that

$$F=bL+wP=sU+qW, \tag{3.13}$$

$$s+v+w=1,\quad b=s+v=1-w,\quad q=v+w. \tag{3.14}$$

The numbers and the distributions just introduced depend on the choice of a parameter τ. From (3.5), (3.8)–(3.10) it follows that

$$w \leqq c n^{-1} \sum_{j=1}^{k} (|\ln \gamma_{\tau}^{(j)}|+1)^3 \leqq ckn^{-1}(|\ln \gamma_{\tau}|+1)^3. \tag{3.15}$$

Set

$$\Delta_{\tau}=nbL=\Delta-nwP. \tag{3.16}$$

The relations (2.10), (2.12), (3.15), (3.16) imply that for $\boldsymbol{\delta}\in\mathbb{R}^k$, $\boldsymbol{\delta}>\mathbf{0}$, the following inequalities are valid

$$v_j(\Delta,\delta_j) \leqq v_j(\Delta_{\tau},\delta_j)+ck(|\ln \gamma_{\tau}|+1)^3, \quad j=1,\dots,k; \tag{3.17}$$

$$v(\Delta,\boldsymbol{\delta}) \leqq v(\Delta_{\tau},\boldsymbol{\delta})+ck^2(|\ln \gamma_{\tau}|+1)^3. \tag{3.18}$$

In addition, by (3.1), (2.12), (3.11)–(3.14) we have the equality

$$v(\Delta,F)=ns \int_{\mathbb{R}^k} \sum_{j=1}^{k} x_j^2 \tau_j^{-2} U\{d\mathbf{x}\}+nq. \tag{3.19}$$

For $\mathbf{h}\in\mathbb{R}^k$, $\mathbf{h}>\mathbf{0}$; $G,H\in\mathfrak{F}_k$ define

$$\varrho_{\mathbf{h}}(H,G)=\varrho_{\mathbb{1}_{[0,\mathbf{h}]}}(H,G) \tag{3.20}$$

(here and further $\mathbb{1}_A$ is the indicator function of a set A). Instead of Theorem 1.1 we shall prove the following more general result.

Theorem 3.1. *For any* $\mathbf{h}\in\mathbb{R}^k$ *such that* $\mathbf{h}>\mathbf{0}$ *the inequality*

$$\varrho_{\mathbf{h}}(F^n,F^{n+1}) \leqq c(k)n^{-1} \prod_{j=1}^{k} \left((\gamma_{\mathbf{h}}^{(j)})^{\frac{1}{3k}}(|\ln \gamma_{\mathbf{h}}^{(j)}|+1)^2\right) + c(k)\exp(-n\alpha+ck(\ln n+1)^3) \tag{3.21}$$

is valid.

Theorem 1.1 may be easily derived from Theorem 3.1 since

$$\varrho(F^n,F^{n+1}) \leqq \sup_{\mathbf{h}} \varrho_{\mathbf{h}}(F^n,F^{n+1}).$$

Note that the inverse assertion is false because the factor after $c(k)n^{-1}$ in the right-hand side of (3.21) can be small, but (1.4) implies only that

$$\varrho_{\mathbf{h}}(F^n,F^{n+1}) \leqq c(k)(n^{-1}+\exp(-n\alpha+ck\ln^3 n)).$$

We can interprete the inequality (3.21) as a non-uniform bound for $\varrho_{\mathbf{h}}(F^n,F^{n+1})$, taking into account the possible smallness of $\gamma_{\mathbf{h}}^{(j)}$ for sufficiently small h_j, $j=1,\dots,k$.

To prove Theorem 3.1 we shall need the following Lemmas 3.1 and 3.2.

Lemma 3.1. *For all* $\mathbf{J}\in\Xi$ *and for all* $\boldsymbol{\tau},\mathbf{h}\in\mathbb{R}^k$ *such that* $\mathbf{0}<\boldsymbol{\tau}<\mathbf{h}$ *the following inequalities holds:*

$$\begin{aligned}\varrho_{\mathbf{h},\boldsymbol{\tau}}^{(\mathbf{J})}(F^n,F^{n+1}) &\leqq c(k)(n^{-1}\gamma_{\mathbf{h}} v(\Delta,\boldsymbol{\tau})+e^{-n\alpha}) \\ &\leqq c(k)(n^{-1}\gamma_{\mathbf{h}}(v(\Delta_{\tau},\boldsymbol{\tau})+(|\ln \gamma_{\tau}|+1)^3)+e^{-n\alpha}),\end{aligned} \tag{3.22}$$

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}^{(\mathbf{J})}(F^n,F^{n+1}) \leqq c(k)\left\{n^{-1}\left(\gamma_{\mathbf{h}}(|\ln \gamma_{\tau}|+1)^3+\frac{(|\ln \gamma_{\tau}|+1)^{3k}}{(v(\Delta_{\tau},\mathbf{h}))^{1/2}}\right) + \exp(-n\alpha+ck(|\ln \gamma_{\tau}|+1)^3)\right\}. \tag{3.23}$$

Lemma 3.2. *For all* $\mathbf{J}\in\Xi$ *and for all* $\boldsymbol{\tau},\mathbf{h}\in\mathbb{R}^k$ *such that* $\mathbf{0}<\boldsymbol{\tau}<\mathbf{h}$; $\gamma_{\mathbf{h}}^{(j)}\leqq 4\gamma_{\boldsymbol{\tau}}^{(j)}$, $j=1,\dots,k$, *the inequality*

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}^{(\mathbf{J})}(F^n,F^{n+1})\leqq c(k)n^{-1}\gamma_{\mathbf{h}}^{1/3}(|\ln\gamma_{\mathbf{h}}|+1)^{2k}$$
$$+c(k)\exp(-n\alpha+ck(|\ln\gamma_{\boldsymbol{\tau}}|+1)^3) \tag{3.24}$$

is valid.

The proof of Lemma 3.1 will be carried out in Sect. 4. Lemma 3.2 will be proved in Sect. 5 with the help of Lemma 3.1. Finally, in Sect. 6 we shall derive Theorem 3.1 from Lemma 3.2.

4. Proof of Lemma 3.1

We shall prove Lemma 3.1 for $\mathbf{J}=\mathbf{0}$ only. The fact that this does not imply the loss of generality follows from the symmetry argument. We assume below that $\mathbf{h},\boldsymbol{\tau}\in\mathbb{R}^k$ are fixed (although arbitrary) vectors satisfying the Lemma 3.1 conditions.

First we shall prove the inequality (3.22). Using (3.13) and the definition of the metric $\varrho_{\mathbf{h},\boldsymbol{\tau}}(\cdot,\cdot)$ we find that

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,F^{n+1})\leqq s\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,UF^n)+q\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,WF^n). \tag{4.1}$$

It follows from (2.1), (2.2), (2.5), (2.8), (3.6) that for any $G\in\mathfrak{F}_k$ the following inequalities hold true:

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,GF^n)\leqq\max\{\Gamma_{\mathbf{h},\boldsymbol{\tau}}(F^n),\quad\Gamma_{\mathbf{h},\boldsymbol{\tau}}(GF^n)\}$$
$$\leqq Q(F^n,\mathbf{h})\leqq c(k)(\gamma_{\mathbf{h}}+e^{-n\alpha}). \tag{4.2}$$

In particular,

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,WF^n)\leqq c(k)(\gamma_{\mathbf{h}}+e^{-n\alpha}). \tag{4.3}$$

Let us estimate $\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n,UF^n)$. For this we consider the integral

$$I=\int_{\mathbb{R}^k}f(\mathbf{x})(F^n-UF^n)\{d\mathbf{x}\} \tag{4.4}$$

where $f=f_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}$, $\mathbf{z}\in\mathbb{R}^k$, and introduce independent random vectors $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^k$ with distributions $\mathscr{L}(\boldsymbol{\xi})=U$, $\mathscr{L}(\boldsymbol{\eta})=F^n$. Then the integral I can be rewritten as

$$I=\mathbf{E}(f(\boldsymbol{\eta})-f(\boldsymbol{\eta}+\boldsymbol{\xi}))=\mathbf{E}(f(\boldsymbol{\eta})-f(\boldsymbol{\eta}+\boldsymbol{\xi}))\Psi_{\mathbf{z}}(\boldsymbol{\eta}) \tag{4.5}$$

where $\Psi_{\mathbf{z}}$ is the indicator function of the parallelepiped $[\mathbf{z}-\boldsymbol{\tau},\mathbf{z}+\mathbf{h}+\boldsymbol{\tau}]$. We have used that $U\{[-\boldsymbol{\tau},\boldsymbol{\tau}]\}=1$ and $f(\mathbf{x})=0$ for $\mathbf{x}\notin[\mathbf{z},\mathbf{z}+\mathbf{h}]$.

Let us apply under the expectation sign in the right-hand side of (4.5) the Taylor formula to the function $f^*(\alpha)=f(\boldsymbol{\eta}+\alpha\boldsymbol{\xi})$, $\alpha\in\mathbb{R}^1$. Note that $\mathbf{E}\boldsymbol{\xi}=\mathbf{0}$ since $U\in\mathfrak{F}_k^s$. Using (2.7) and taking into account the mutual independence of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ we obtain that

$$|I| = |\mathbf{E}\{\{f(\boldsymbol{\eta}) - f(\boldsymbol{\eta}) - \sum_{j=1}^{k} \xi_j \frac{\partial f}{\partial x_j}(\mathbf{x})\Big|_{\mathbf{x}=\boldsymbol{\eta}}$$

$$-\frac{1}{2}\sum_{j=1}^{k}\sum_{p=1}^{k} \xi_j \xi_p \frac{\partial^2 f}{\partial x_j \partial x_p}(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\eta}+\theta\boldsymbol{\xi}}\Big\} \Psi_{\mathbf{z}}(\boldsymbol{\eta})\}\Big|$$

$$\leqq c\mathbf{E}\Psi_{\mathbf{z}}(\boldsymbol{\eta})\mathbf{E}\left(\sum_{j=1}^{k}\sum_{p=1}^{k} |\xi_j|\tau_j^{-1}|\xi_p|\tau_p^{-1}\right)$$

$$\leqq c(k)\mathbf{E}\Psi_{\mathbf{z}}(\boldsymbol{\eta})\mathbf{E}\sum_{j=1}^{k} \xi_j^2\tau_j^{-2}. \tag{4.6}$$

Since $\mathbf{0}<\boldsymbol{\tau}<\mathbf{h}$, with the help of (3.6) it can be easily deduced that

$$\begin{aligned}\mathbf{E}\Psi_{\mathbf{z}}(\boldsymbol{\eta}) &\leqq Q(\mathscr{L}(\boldsymbol{\eta}), \mathbf{h}+2\boldsymbol{\tau})\\ &\leqq c(k)Q(F^n, \mathbf{h})\\ &\leqq c(k)(\gamma_{\mathbf{h}}+e^{-n\alpha}).\end{aligned} \tag{4.7}$$

Using (4.4), (4.6), (4.7), we obtain

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, UF^n) \leqq c(k)(\gamma_{\mathbf{h}}+e^{-n\alpha})\sum_{j=1}^{k} \mathbf{E}\xi_j^2\tau_j^{-2}. \tag{4.8}$$

Since $\mathscr{L}(\boldsymbol{\xi})=U$, the inequality (3.22) follows immediately from (4.1), (4.3), (4.8), (3.18), (3.19).

Let us pass on to the proof of the inequality (3.23). Suppose, at first, that $w>1/2$. Then, applying (3.15) and the inequality (4.2) for $G=F$, we get

$$\begin{aligned}\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, F^{n+1}) &\leqq c(k)(\gamma_{\mathbf{h}}+e^{-n\alpha}) \leqq c(k)w(\gamma_{\mathbf{h}}+e^{-n\alpha})\\ &\leqq c(k)n^{-1}(|\ln\gamma_{\boldsymbol{\tau}}|+1)^3(\gamma_{\mathbf{h}}+e^{-n\alpha}).\end{aligned} \tag{4.9}$$

Obviously, (4.9) implies (3.23) if $w>1/2$. Therefore we shall further assume that

$$w \leqq \tfrac{1}{2}. \tag{4.10}$$

By (3.13), we have

$$\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, F^{n+1}) \leqq b\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, LF^n) + w\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, PF^n) \tag{4.11}$$

and from (3.15), (4.2) it follows that

$$w\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, PF^n) \leqq c(k)n^{-1}(|\ln\gamma_{\boldsymbol{\tau}}|+1)^3(\gamma_{\mathbf{h}}+e^{-n\alpha}). \tag{4.12}$$

Let us estimate $b\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, LF^n)$. It can be easily derived from (3.13) that

$$F^n = \sum_{r=0}^{n} C_n^r b^r w^{n-r} L^r P^{n-r} \tag{4.13}$$

where $C_n^r = n!/((n-r)!r!)$ are binomial coefficients. Using (2.1), (2.3), (2.5), (4.13) we find

$$\varrho_{\mathbf{h},\tau}(F^n, LF^n) = \Gamma_{\mathbf{h},\tau}(F^n - LF^n)$$

$$\leqq \sum_{r=0}^{n} C_n^r b^r w^{n-r} \varrho_{\mathbf{h},\tau}(L^r P^{n-r}, L^{r+1} P^{n-r})$$

$$\leqq \sum_{r=0}^{n} C_n^r b^r w^{n-r} \varrho_{\mathbf{h},\tau}(L^r, L^{r+1}) \tag{4.14}$$

By (3.12), the distribution L may be written as

$$L = s_0 U + v_0 R \quad \text{where} \quad s_0 = sb^{-1}, v_0 = vb^{-1}. \tag{4.15}$$

Note that (3.10), (3.14), (4.10) imply $1 \geqq b = 1 - w \geqq 1/2$ and, hence

$$s \leqq s_0 \leqq 2s, \quad v \leqq v_0 \leqq 2v. \tag{4.16}$$

Fix $r \in \mathbb{N}$, $j \in \mathbb{N}$, where $1 \leqq r \leqq n$, $1 \leqq j \leqq k$, and apply Lemma 2.2 for $L_0 = L^{(j)}$, $U_0 = U^{(j)}$, $R_0 = R^{(j)}$, $\tau_0 = \tau_j$, $l = l_j$, $\mathbf{u} = \mathbf{u}^{(j)}$,

$$\sigma^2 = \sigma_j^2 = rs_0 \int_{-\infty}^{\infty} x_j^2 U^{(j)}\{dx_j\} = rs_0 \int_{\mathbb{R}^k} x_j^2 U\{d\mathbf{x}\}. \tag{4.17}$$

By this lemma, the inequality

$$(L^{(j)})^r \{\mathbb{R}^1 \setminus [K_d(\mathbf{u}^{(j)})]_{d\tau_j + y_j}\}$$
$$\leqq 2\exp(-y_j/8\tau_j) + \exp(-d/8) \tag{4.18}$$

is valid, if

$$y_j \geqq \sigma_j^2 / 2\tau_j, \quad d \geqq 2rv_0. \tag{4.19}$$

In addition, from (2.13), (2.14), (3.1), (3.5) it follows that

$$\gamma_\tau \leqq c(\omega(\Delta, \tau))^{-1/2} \leqq c\sqrt{k}(v(\Delta, \tau))^{-1/2} \tag{4.20}$$

or

$$v(\Delta, \tau) \leqq ck\gamma_\tau^{-2} \tag{4.21}$$

Let

$$m \in \mathbb{N}, \quad m \geqq c_0 k \gamma_\tau^{-2} \tag{4.22}$$

where c_0 is a sufficiently large absolute constant. We shall apply the inequality (4.18) for $y_j = m\tau_j/2$, $d = m/2$. The validity of the inequalities (4.19) is ensured by a suitable choice of c_0 with regard to the relations $r \leqq n$, $v \leqq q$, (3.19), (4.16), (4.17), (4.21). Setting now $y_j = m\tau_j/2$, $d = m/2$ we obtain from (4.18) that

$$(L^{(j)})^r \{\mathbb{R}^1 \setminus [K_m(\mathbf{u}^{(j)})]_{m\tau_j}\} \leqq 3e^{-m/16} \tag{4.23}$$

From (3.9), (4.23) it follows that

$$L^r \{\mathbb{R}^k \setminus K_m\} \leqq 3ke^{-m/16} \tag{4.24}$$

for each $r \in \mathbb{N}$ such that $1 \leqq r \leqq n$. The inequality (4.24) is also valid for $r = 0$ since $L^0 = E$ and $\mathbf{0} \in K_m$.

Let us estimate $\varrho_{\mathbf{h},\tau}(L^r, L^{r+1})$ for $r = 0, 1, \ldots, n$. For this we take an arbitrary $\mathbf{z} \in \mathbb{R}^k$ and consider the integral

$$I_0 = \int_{\mathbb{R}^k} f_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x})\,(L^r - L^{r+1})\{d\mathbf{x}\}\,. \tag{4.25}$$

By Lemma 2.3, for $j=1,\dots,k$ there exist the functions $w_j(x_j)$, $g_j(t_j)$ such that

a) $0 = w_j(x_j) \leqq f_{z_j,h_j,\tau_j}(x_j) \leqq 1$ for all $x_j \in \mathbb{R}^1$;

b) $w_j(x_j) = f_{z_j,h_j,\tau_j}(x_j)$ for $x_j \in [K_{m+1}(\mathbf{u}^{(j)})]_{(m+1)\tau_j}$;

c) the functions $w_j(x_j)$ are infinitely differentiable on the real line and

$$\sup_{x_j}\left|\frac{d^p}{dx_j^p}\,w_j(x_j)\right| \leqq \frac{c(p)}{\tau_j^p}, \quad p=1,2,\dots; \tag{4.26}$$

d) $|\hat{w}_j(t_j)| \leqq g_j(t_j)$ for all $t_j \in \mathbb{R}^1$;

e) the functions $g_j(t_j)$ are even, do not increase for $t_j \geqq 0$ and take constant values $g_j(0)$ for $|t_j| \leqq 2h_j^{-1}$;

f)
$$\int_{-\infty}^{\infty} g_j(t_j)\,dt_j \leqq c l_j^2 \ln(l_j(m+1)+1)\,. \tag{4.27}$$

For $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{t} \in \mathbb{R}^k$ set

$$w(\mathbf{x}) = \prod_{j=1}^{k} w_j(x_j), \quad h(\mathbf{x}) = f_{\mathbf{z},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}) - w(\mathbf{x}), \quad g(\mathbf{t}) = \prod_{j=1}^{k} g_j(t_j)\,. \tag{4.28}$$

By property a) of the functions w_j, for all $\mathbf{x} \in \mathbb{R}^k$ the inequalities

$$0 \leqq h(\mathbf{x}) \leqq 1 \tag{4.29}$$

are valid, and from (2.7), (4.28) and from properties a), c) it follows that

$$\sup_{\mathbf{x}}\left|\frac{\partial h}{\partial x_j}(\mathbf{x})\right| \leqq \frac{c}{\tau_j}, \quad \sup_{\mathbf{x}}\left|\frac{\partial^2 h}{\partial x_j \partial x_p}(\mathbf{x})\right| \leqq \frac{c}{\tau_j \tau_p} \tag{4.30}$$

for all $j, p \in \mathbb{N}$ such that $1 \leqq j \leqq k$, $1 \leqq p \leqq k$. In addition, property b) and (3.9) imply that

$$h(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \in K_{m+1}\,. \tag{4.31}$$

Represent I_0 as a sum

$$I_0 = s_0 I_1 + v_0 I_2 + I_3 \tag{4.32}$$

where

$$I_1 = \int_{\mathbb{R}^k} h(\mathbf{x})\,(L^r - UL^r)\{d\mathbf{x}\}, \tag{4.33}$$

$$I_2 = \int_{\mathbb{R}^k} h(\mathbf{x})\,(L^r - RL^r)\{d\mathbf{x}\}, \tag{4.34}$$

$$I_3 = \int_{\mathbb{R}^k} w(\mathbf{x})\,(L^r - L^{r+1})\{d\mathbf{x}\} \tag{4.35}$$

(see (4.25), (4.28)).

Let us estimate I_1 and I_2. For this we introduce independent random vectors $\xi, \boldsymbol{\eta}, \zeta \in \mathbb{R}^k$ with distributions $\mathscr{L}(\xi) = U$, $\mathscr{L}(\boldsymbol{\eta}) = L^r$, $\mathscr{L}(\zeta) = R$ and define a random event Y_1 by

$$Y_1 = \{\boldsymbol{\eta} \notin K_m\}\,. \tag{4.36}$$

In view of (4.24),

$$\mathbf{E}\mathbb{1}_{Y_1} = \mathbf{P}\{Y_1\} \leqq 3ke^{-m/16}. \tag{4.37}$$

It can be easily seen that

$$\mathbf{E}\boldsymbol{\xi} = \mathbf{0}, \tag{4.38}$$

since $U = \mathscr{L}(\boldsymbol{\xi}) \in \mathfrak{F}_k^s$. It follows from (2.9), (3.9), (3.11), (3.12) that $L\{K_1\} = U\{K_1\} = R\{K_1\} = 1$. Hence, if $\boldsymbol{\eta} \in K_m$ then, using (4.31) and Remark 3.1, we obtain that $h(\boldsymbol{\eta}) = h(\boldsymbol{\eta} + \boldsymbol{\xi}) = h(\boldsymbol{\eta} + \boldsymbol{\zeta}) = 0$. Taking (4.36) into account, we get

$$h(\boldsymbol{\eta}) = h(\boldsymbol{\eta})\mathbb{1}_{Y_1}, \quad h(\boldsymbol{\eta} + \boldsymbol{\xi}) = h(\boldsymbol{\eta} + \boldsymbol{\xi})\mathbb{1}_{Y_1}, \quad h(\boldsymbol{\eta} + \boldsymbol{\zeta}) = h(\boldsymbol{\eta} + \boldsymbol{\zeta})\mathbb{1}_{Y_1}. \tag{4.39}$$

Applying (4.30), (4.33), (4.36)–(4.39), the Taylor formula and the mutual independence of $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, we find that

$$\begin{aligned} |I_1| &= |\mathbf{E}(h(\boldsymbol{\eta}) - h(\boldsymbol{\eta} + \boldsymbol{\xi}))| \\ &= |\mathbf{E}(h(\boldsymbol{\eta}) - h(\boldsymbol{\eta} + \boldsymbol{\xi}))\mathbb{1}_{Y_1}| \\ &= \frac{1}{2}\left|\mathbf{E}\left(\mathbb{1}_{Y_1} \sum_{j=1}^{k} \sum_{p=1}^{k} \xi_j \xi_p \frac{\partial^2 h}{\partial x_j \partial x_p}(\mathbf{x})\bigg|_{\mathbf{x} = \boldsymbol{\eta} + \theta\boldsymbol{\xi}}\right)\right| \\ &\leqq c\left|\mathbf{E}\left(\mathbb{1}_{Y_1} \sum_{j=1}^{k} \sum_{p=1}^{k} |\xi_j \xi_p| \tau_j^{-1} \tau_p^{-1}\right)\right| \\ &\leqq c(k) \sum_{j=1}^{k} \mathbf{E}\xi_j^2 \tau_j^{-2} \mathbf{E}\mathbb{1}_{Y_1} \\ &\leqq c(k) e^{-m/16} \sum_{j=1}^{k} \mathbf{E}\xi_j^2 \tau_j^{-2}. \end{aligned} \tag{4.40}$$

With the help of (4.29), (4.34), (4.37), (4.39) we can similarly derive the inequality

$$\begin{aligned} |I_2| &= |\mathbf{E}(h(\boldsymbol{\eta}) - h(\boldsymbol{\eta} + \boldsymbol{\zeta}))| \\ &= |\mathbf{E}\{(h(\boldsymbol{\eta}) - h(\boldsymbol{\eta} + \boldsymbol{\zeta}))\mathbb{1}_{Y_1}\}| \\ &\leqq \mathbf{P}\{Y_1\} \leqq 3ke^{-m/16}. \end{aligned} \tag{4.41}$$

Now from (3.14), (3.19), (4.16), (4.40), (4.41) and from the fact that $\mathscr{L}(\boldsymbol{\xi}) = U$ it follows that

$$|s_0 I_1 + v_0 I_2| \leqq c(k) n^{-1} v(\Delta, \boldsymbol{\tau}) e^{-m/16}. \tag{4.42}$$

Let us pass on to the estimation of $|I_3|$. Put

$$\beta = \frac{\alpha - 2w}{1 - w} = \frac{\alpha - 2w}{b}. \tag{4.43}$$

If $\beta < 0$ then the inequality (3.23) follows immediately from the inequalities $\varrho_{\mathbf{h},\boldsymbol{\tau}}(F^n, F^{n+1}) \leqq 1$ and (3.15). Therefore we further assume that

$$\beta \geqq 0. \tag{4.44}$$

If for some $\mathbf{t}\in\mathbb{R}^k$ we have $\hat{L}(\mathbf{t})\leqq 0$ then, using the fact that $F\in\mathfrak{F}_k^\alpha$ and the relations (3.11), (4.15), (4.43), it is not difficult to show that

$$0\geqq\hat{L}(\mathbf{t})=b^{-1}(s\hat{U}(\mathbf{t})+v\hat{R}(\mathbf{t}))\geqq b^{-1}(\hat{F}(\mathbf{t})-w)$$
$$\geqq\frac{-1+\alpha-w}{1-w}=-1+\beta.$$

Hence $\beta\leqq 1$ and $|\hat{L}(\mathbf{t})|\leqq(1-\beta)\mathbb{1}_{\{0\leqq\beta\leqq 1\}}$. If now $\hat{L}(\mathbf{t})>0$ then $|\hat{L}(\mathbf{t})|=\hat{L}(\mathbf{t})\leqq\exp(\hat{L}(\mathbf{t})-1)$. Thus, for all $\mathbf{t}\in\mathbb{R}^k$ we have

$$|\hat{L}(\mathbf{t})|\leqq\max\left\{\exp(\hat{L}(\mathbf{t})-1),(1-\beta)\mathbb{1}_{\{0\leqq\beta\leqq 1\}}\right\}. \tag{4.45}$$

Define the distribution $D^*\in\mathfrak{D}_k\cap\mathfrak{F}_k^+$ with characteristic function

$$\hat{D}^*(\mathbf{t})=\exp(r(\hat{L}(\mathbf{t})-1)/2). \tag{4.46}$$

From (4.45), (4.46) it follows that

$$|\hat{L}^r(\mathbf{t})-\hat{L}^{r+1}(\mathbf{t})|\leqq\max\left\{\frac{c}{r+1}\hat{D}^*(\mathbf{t}),\ 2(1-\beta)^r\mathbb{1}_{\{0\leqq\beta\leqq 1\}}\right\} \tag{4.47}$$

for all $\mathbf{t}\in\mathbb{R}^k$.

By the Parseval equality, (4.28), (4.35), (4.47) and in view of property d) of the functions w_j and g_j, the following inequality holds:

$$|I_3|=|(2\pi)^{-k}\int_{\mathbb{R}^k}\hat{w}(-\mathbf{t})(\hat{L}^r(\mathbf{t})-\hat{L}^{r+1}(\mathbf{t}))d\mathbf{t}|\leqq I_4+I_5, \tag{4.48}$$

where

$$I_4=\frac{c(k)}{r+1}\int_{\mathbb{R}^k}g(\mathbf{t})\hat{D}^*(\mathbf{t})d\mathbf{t}, \tag{4.49}$$

$$I_5=c(k)(1-\beta)^r\mathbb{1}_{\{0\leqq\beta\leqq 1\}}\int_{\mathbb{R}^k}g(\mathbf{t})d\mathbf{t}. \tag{4.50}$$

Property e) of the functions g_j ensures the function g to belong to the class $A(\mathbf{h})$ and to satisfy the Lemma 2.6 conditions. By this lemma,

$$\int_{\mathbb{R}^k}g(\mathbf{t})\hat{D}^*(\mathbf{t})d\mathbf{t}\leqq c(k)Q(D^*,\mathbf{h})\int_{\mathbb{R}^k}g(\mathbf{t})d\mathbf{t} \tag{4.51}$$

(see (2.18)). From (4.27), (4.28) it follows that

$$\int_{\mathbb{R}^k}g(\mathbf{t})d\mathbf{t}\leqq c(k)\prod_{j=1}^{k}l_j^2\ln(l_j(m+1)+1). \tag{4.52}$$

In view of (4.46), $D^*=e(rL/2)$. Using the assertion a) of Lemma 2.4 we obtain

$$Q(D^*,\mathbf{h})\leqq c(k)((r+1)v(L,\mathbf{h}))^{-1/2} \tag{4.53}$$

(if $r=0$ then (4.53) follows from the evident inequalities $Q(D^*,\mathbf{h})\leqq 1$, $v(L,\mathbf{h})\leqq k$).

Define $m\in\mathbb{N}$ more exactly, setting $m=[c_0k\gamma_\tau^{-2}]+1$ where $[\cdot]$ means the integer part of a number. It is evident that (4.22) is then satisfied. With the help of (4.21) we get

$$v(\Delta,\tau)e^{-m/16}\leqq ck\gamma_\tau^{-2}\exp(-c_0k\gamma_\tau^{-2}/16)$$
$$\leqq c\exp(-ck\gamma_\tau^{-2}) \tag{4.54}$$

and (3.5), (3.7), (4.52) imply

$$\int_{\mathbb{R}^k} g(\mathbf{t})\,d\mathbf{t}\leqq c(k)(|\ln\gamma_\tau|+1)^{3k}. \tag{4.55}$$

In view of (4.49)–(4.51), (4.53), (4.55), the inequalities

$$I_4\leqq c(k)(|\ln\gamma_\tau|+1)^{3k}(r+1)^{-3/2}(v(L,\mathbf{h}))^{-1/2}, \tag{4.56}$$
$$I_5\leqq c(k)(|\ln\gamma_\tau|+1)^{3k}(1-\beta)^r\mathbb{1}_{\{0\leqq\beta\leqq1\}} \tag{4.57}$$

hold true. From (4.25), (4.32), (4.42), (4.48), (4.54), (4.56), (4.57) it follows that

$$\varrho_{\mathbf{h},\tau}(L^r,L^{r+1})\leqq c(k)\{n^{-1}\exp(-ck\gamma_\tau^{-2})$$
$$+(|\ln\gamma_\tau|+1)^{3k}((r+1)^{-3/2}(v(L,\mathbf{h}))^{-1/2}+(1-\beta)^r\mathbb{1}_{\{0\leqq\beta\leqq1\}})\}. \tag{4.58}$$

Let us return to the formula (4.14). By using (3.14), (4.10), the binomial formula and the Hölder inequality it is not difficult to show that

$$\sum_{r=0}^{n} C_n^r b^r w^{n-r}(r+1)^{-3/2}\leqq cn^{-3/2}. \tag{4.59}$$

In addition, with the help of (3.14), (3.16), (4.43) we obtain

$$\sum_{r=0}^{n} C_n^r b^r w^{n-r}(1-\beta)^r=(b(1-\beta)+w)^n=(1-\alpha+2w)^n, \tag{4.60}$$
$$(1-\alpha+2w)^n\mathbb{1}_{\{0\leqq\beta\leqq1\}}\leqq\exp(-n(\alpha-2w)), \tag{4.61}$$
$$v(L,\mathbf{h})\geqq n^{-1}v(\Delta_\tau,\mathbf{h}). \tag{4.62}$$

Now the inequality (3.23) follows immediately from (4.14), (4.58)–(4.62), (3.15). Hence Lemma 3.1 is completely proved.

5. Proof of Lemma 3.2

Let $\tau,\mathbf{h}\in\mathbb{R}^k$ satisfy the Lemma 3.2 conditions, i.e. $\mathbf{0}<\tau<\mathbf{h}$ and $\gamma_{\mathbf{h}}^{(j)}\leqq4\gamma_\tau^{(j)}$ for $j=1,\ldots,k$. Taking into account the obvious symmetry of the situation considered in Lemma 3.2 with respect to a parameter $\mathbf{J}$ we shall consider again the case $\mathbf{J}=\mathbf{0}$ only and shall prove the inequality (3.24) for $\varrho_{\mathbf{h},\tau}(F^n,F^{n+1})=\varrho_{\mathbf{h},\tau}^{(\mathbf{0})}(F^n,F^{n+1})$. Put

$$\kappa_{\mathbf{h}}=\gamma_{\mathbf{h}}^{-2/3}(|\ln\gamma_{\mathbf{h}}|+1)^{2k}. \tag{5.1}$$

We shall construct the numbers $r\in\mathbb{Z}$, $0\leqq r\leqq k$; $\varepsilon_1,\ldots,\varepsilon_r$, using the following rule. We shall denote by $\tau^{(j)}\in\mathbb{R}^k$, $j=1,\ldots,r+1$, the vectors having the coordinates $\tau_l^{(j)}=\varepsilon_l$ for $l<j$ and $\tau_l^{(j)}=\tau_l$ for $l\geqq j$. Suppose that the numbers $\varepsilon_1,\ldots,\varepsilon_{j-1}$ are already constructed. So, we know the vectors $\tau^{(1)},\ldots,\tau^{(j)}$ (of course, for $j=1$ we have constructed the vector $\tau^{(1)}=\tau$).

Consider separately three possible situations:

a) $$v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, \tau_j) \leqq \kappa_{\mathbf{h}}\,; \tag{5.2}$$

b) $$v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, h_j) < \kappa_{\mathbf{h}} < v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, \tau_j)\,; \tag{5.3}$$

c) $$v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, h_j) \geqq \kappa_{\mathbf{h}}\,. \tag{5.4}$$

In the case a) we set $\varepsilon_j = \tau_j$. In the case b) we take as ε_j the solution of the equation

$$v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, \varepsilon_j) = \kappa_{\mathbf{h}}\,. \tag{5.5}$$

This solution exists and $\tau_j < \varepsilon_j < h_j$ because the function $v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, u)$ is non-increasing and continuous with respect to u, $u > 0$ (see (2.10)). Finally, in the case c) we denote $r = j-1$ and complete the construction. If on constructing the numbers ε_j, $j = 1, \dots, k$, the case c) does not appear, we take $r = k$.

Denote by $\mathbf{h}^{(j)} \in \mathbb{R}^k$, $j = 1, \dots, r$, the vectors with the coordinates $h_l^{(j)} = \varepsilon_l$ for $l = j$ and $h_l^{(j)} = h_l$ for $l \neq j$. Then

$$f_{\mathbf{0}, \mathbf{h}, \boldsymbol{\tau}^{(j)}} = f_{\mathbf{0}, \mathbf{h}, \boldsymbol{\tau}^{(j+1)}} + f_{\mathbf{0}, \mathbf{h}^{(j)}, \boldsymbol{\tau}^{(j)}}\,.$$

Hence

$$\varrho_{\mathbf{h}, \boldsymbol{\tau}^{(j)}}(F^n, F^{n+1}) \leqq \varrho_{\mathbf{h}, \boldsymbol{\tau}^{(j+1)}}(F^n, F^{n+1}) + \varrho_{\mathbf{h}^{(j)}, \boldsymbol{\tau}^{(j)}}(F^n, F^{n+1})\,. \tag{5.6}$$

With the help of an induction one can easily derive from (5.6) the inequality

$$\begin{aligned} \varrho_{\mathbf{h}, \boldsymbol{\tau}}(F^n, F^{n+1}) &= \varrho_{\mathbf{h}, \boldsymbol{\tau}^{(1)}}(F^n, F^{n+1}) \\ &\leqq \sum_{j=1}^{r} \varrho_{\mathbf{h}^{(j)}, \boldsymbol{\tau}^{(j)}}(F^n, F^{n+1}) + \varrho_{\mathbf{h}, \boldsymbol{\tau}^{(r+1)}}(F^n, F^{n+1})\,. \end{aligned} \tag{5.7}$$

From the Lemma 3.2 conditions and from the definition of the vector $\boldsymbol{\tau}^{(j)}$ it follows that

$$|\ln \gamma_{\boldsymbol{\tau}^{(j)}}| \leqq |\ln \gamma_{\mathbf{h}}| + \ln 4\,, \quad j = 1, \dots, r+1\,, \tag{5.8}$$

(we use (3.5) and the fact that $\tau_j \leqq \varepsilon_j < h_j$).

Consider more explicitly the process of construction of numbers ε_j. If the case a) occurs for a number j then, of course,

$$\varrho_{\mathbf{h}^{(j)}, \boldsymbol{\tau}^{(j)}}(F^n, F^{n+1}) = 0 \tag{5.9}$$

since $h_j^{(j)} = \tau_j^{(j)} = \varepsilon_j = \tau_j$. In addition, in view of (3.17), (5.2), (5.8),

$$\begin{aligned} v_j(\Delta, \varepsilon_j) &= v_j(\Delta, \tau_j) \\ &\leqq v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, \tau_j) + ck(|\ln \gamma_{\boldsymbol{\tau}^{(j)}}| + 1)^3 \\ &\leqq \kappa_{\mathbf{h}} + ck(|\ln \gamma_{\mathbf{h}}| + 1)^3\,. \end{aligned} \tag{5.10}$$

If now the case b) occurs then, by (2.10), (2.11), (2.13), (5.5), we have:

$$v(\Delta_{\boldsymbol{\tau}^{(j)}}, \mathbf{h}^{(j)}) \geqq v_j(\Delta_{\boldsymbol{\tau}^{(j)}}, h_j^{(j)}) = \kappa_{\mathbf{h}}\,. \tag{5.11}$$

Using the inequality (3.23) and taking into account (5.1), (5.8), (5.11), we find that

$$\varrho_{\mathbf{h}^{(j)},\boldsymbol{\tau}^{(j)}}(F^n,F^{n+1})\leqq c(k)n^{-1}\left\{\gamma_{\mathbf{h}^{(j)}}(|\ln\gamma_{\boldsymbol{\tau}^{(j)}}|+1)^3+\frac{(|\ln\gamma_{\boldsymbol{\tau}^{(j)}}|+1)^{3k}}{(v(\Delta_{\boldsymbol{\tau}^{(j)}},\mathbf{h}^{(j)}))^{1/2}}\right\}$$
$$+c(k)\exp(-n\alpha+ck(|\ln\gamma_{\boldsymbol{\tau}^{(j)}}|+1)^3)$$
$$\leqq c(k)\,(n^{-1}\gamma_{\mathbf{h}}^{1/3}(|\ln\gamma_{\mathbf{h}}|+1)^{2k}$$
$$+\exp(-n\alpha+ck(|\ln\gamma_{\mathbf{h}}|+1)^3)) \tag{5.12}$$

(it is clear that $\mathbf{h}^{(j)}\leqq\mathbf{h}$ and $\gamma_{\mathbf{h}^{(j)}}\leqq\gamma_{\mathbf{h}}$). The relations (3.17), (5.5), (5.8) imply now that

$$v_j(\Delta,\varepsilon_j)\leqq v_j(\Delta_{\boldsymbol{\tau}^{(j)}},\varepsilon_j)+ck(|\ln\gamma_{\boldsymbol{\tau}^{(j)}}|+1)^3$$
$$\leqq\kappa_{\mathbf{h}}+ck(|\ln\gamma_{\mathbf{h}}|+1)^3\,. \tag{5.13}$$

If the construction of the numbers $\varepsilon_1,\dots,\varepsilon_r$ is completed by the realization of the case c) for $j=r+1$ then, in view of (2.11), (2.13), (5.4),

$$v(\Delta_{\boldsymbol{\tau}^{(r+1)}},\mathbf{h})\geqq v_{r+1}(\Delta_{\boldsymbol{\tau}^{(r+1)}},h_{r+1})\geqq\kappa_{\mathbf{h}}\,. \tag{5.14}$$

Using again the inequality (3.23) and taking (5.1), (5.8), (5.14) into account we obtain

$$\varrho_{\mathbf{h},\boldsymbol{\tau}^{(r+1)}}(F^n,F^{n+1})\leqq c(k)n^{-1}\gamma_{\mathbf{h}}^{1/3}(|\ln\gamma_{\mathbf{h}}|+1)^{2k}$$
$$+c(k)\exp(-n\alpha+ck(|\ln\gamma_{\mathbf{h}}|+1)^3)\,. \tag{5.15}$$

In this case the inequality (3.24) follows from (5.7), (5.9), (5.12), (5.15) (of course, $|\ln\gamma_{\mathbf{h}}|\leqq|\ln\gamma_{\boldsymbol{\tau}}|$).

If, on the contrary, for all $j=1,\dots,k$ we deal with the cases a) or b) then $r=k$ and $\tau^{(r+1)}=(\varepsilon_1,\dots,\varepsilon_r)$. According to (2.11), (2.13), (5.10), (5.13), we have

$$v(\Delta_{\boldsymbol{\tau}^{(r+1)}})\leqq k\max_{1\leqq j\leqq k}v_j(\Delta,\varepsilon_j)$$
$$\leqq k\kappa_{\mathbf{h}}+ck^2(|\ln\gamma_{\mathbf{h}}|+1)^3\,. \tag{5.16}$$

Applying the inequality (3.22) of Lemma 3.1 and taking (5.1), (5.16) into account we find that

$$\varrho_{\mathbf{h},\boldsymbol{\tau}^{(r+1)}}(F^n,F^{n+1})\leqq c(k)(n^{-1}\gamma_{\mathbf{h}}v(\Delta,\tau^{(r+1)})+e^{-n\alpha})$$
$$\leqq c(k)(n^{-1}\gamma_{\mathbf{h}}^{1/3}(|\ln\gamma_{\mathbf{h}}|+1)^{2k}+e^{-n\alpha})\,. \tag{5.17}$$

The inequality (3.24) can be derived from (5.7), (5.9), (5.12), (5.17). Thus, Lemma 3.2 is completely proved.

6. Proof of Theorem 3.1

Let $\mathbf{h}\in\mathbb{R}^k$, $\mathbf{h}>\mathbf{0}$. Assume that $\gamma_{\mathbf{h}}<2n^{-2}$. Then from (2.1), (3.5), (3.6), (3.20) it follows that

$$\begin{aligned}\varrho_{\mathbf{h}}(F^n, F^{n+1}) &\leqq \max\{\Gamma_{\mathbb{1}_{[0,\mathbf{h}]}}(F^n), \Gamma_{\mathbb{1}_{[0,\mathbf{h}]}}(F^{n+1})\}\\ &\leqq \Gamma_{\mathbb{1}_{[0,\mathbf{h}]}}(F^n) = Q(F^n, \mathbf{h}) \leqq c(k)(\gamma_{\mathbf{h}} + e^{-n\alpha})\\ &\leqq c(k)(n^{-1}\gamma_{\mathbf{h}}^{1/2} + e^{-n\alpha}) \leqq c(k)\left(n^{-1}\prod_{j=1}^{k}(\gamma_{\mathbf{h}}^{(j)})^{\frac{1}{2k}} + e^{-n\alpha}\right). \qquad (6.1)\end{aligned}$$

The inequality (6.1) implies (3.21) for $\gamma_{\mathbf{h}} < 2n^{-2}$.

Let now $\gamma_{\mathbf{h}} \geqq 2n^{-2}$. Define the set

$$T = \{j \in \mathbb{N} : 1 \leqq j \leqq k,\ Q(D^{(j)}, 0) < 2n^{-2}\}. \qquad (6.2)$$

For $j \in T$ we can find the points τ_j^* such that

$$0 < \tau_j^* \leqq h_j, \quad n^{-2} \leqq Q(D^{(j)}, \tau_j^*) \leqq 2n^{-2}. \qquad (6.3)$$

The existence of τ_j^* follows from elementary properties of one-dimensional concentration functions. Let $\delta > 0$ be an arbitrary number which is less than $\min\{\min\{\tau_j^*, j \in T\}, \min\{h_j, j = 1, \ldots, k\}\}$. Define the vectors $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\delta)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\delta) \in \mathbb{R}^k$, setting

$$\begin{aligned}&\varepsilon_j = \delta, \quad j = 1, \ldots, k;\\ &\tau_j = \tau_j^*, \quad j \in T; \qquad (6.4)\\ &\tau_j = \delta, \quad j = 1, \ldots, k, \quad j \notin T.\end{aligned}$$

By (2.4), (2.5), (6.2), (6.4) we get that

$$f_{\mathbf{0},\mathbf{h},\boldsymbol{\tau}}(\mathbf{x}) = f_{\mathbf{0},\mathbf{h},\boldsymbol{\varepsilon}}(\mathbf{x}) - \sum_{j \in T} f_{0,\tau_j,\varepsilon_j}(x_j) \prod_{l=1}^{j-1} f_{0,h_l,\tau_l}(x_l) \prod_{l=j+1}^{k} f_{0,h_l,\varepsilon_l}(x_l). \qquad (6.5)$$

From the definition of concentration functions and from (2.4), (2.5), (6.5) it is not difficult to deduce that for $\mathbf{J} = \mathbf{0}$

$$\varrho_{\mathbf{h},\boldsymbol{\varepsilon}(\delta)}^{(\mathbf{J})}(F^n, F^{n+1}) \leqq \varrho_{\mathbf{h},\boldsymbol{\tau}}^{(\mathbf{J})}(F^n, F^{n+1}) + \sum_{j \in T} Q((F^{(j)})^n, \tau_j^*). \qquad (6.6)$$

For arbitrary $\mathbf{J} \in \Xi$ the validity of the inequality (6.6) is established in a similar way.

It is clear that if $F \in \mathfrak{F}_k^\alpha$ then $F^{(j)} \in \mathfrak{F}_1^\alpha$ for all $j = 1, \ldots, k$. Applying the inequality (3.6) to the distributions $F^{(j)}$ and using (3.5), (6.3) we obtain that if $j \in T$ and $\gamma_{\mathbf{h}} \geqq 2n^{-2}$ then

$$\begin{aligned}Q((F^{(j)})^n, \tau_j^*) &\leqq c(Q(D^{(j)}, \tau_j^*) + e^{-n\alpha})\\ &\leqq c(n^{-2} + e^{-n\alpha}) \leqq c(n^{-1}\gamma_{\mathbf{h}}^{1/2} + e^{-n\alpha})\\ &\leqq c(k)\left(n^{-1}\prod_{j=1}^{k}(\gamma_{\mathbf{h}}^{(j)})^{\frac{1}{2k}} + e^{-n\alpha}\right). \qquad (6.7)\end{aligned}$$

Lemmas 2.1, 3.2 imply that for all $\mathbf{J} \in \Xi$

$$\begin{aligned}\varrho_{\mathbf{h},\boldsymbol{\tau}}^{(\mathbf{J})}(F^n, F^{n+1}) &\leqq c(k)n^{-1}\prod_{j=1}^{k}((\gamma_{\mathbf{h}}^{(j)})^{\frac{1}{3k}}(|\ln\gamma_{\mathbf{h}}^{(j)}| + 1)^2)\\ &+ c(k)\exp(-n\alpha + ck(|\ln\gamma_{\boldsymbol{\tau}}| + 1)^3)\prod_{j=1}^{k}\left(\frac{1}{\ln 2}\ln\frac{\gamma_{\mathbf{h}}^{(j)}}{\gamma_{\boldsymbol{\tau}}^{(j)}} + 1\right). \qquad (6.8)\end{aligned}$$

In view of (3.5), (6.2)–(6.4) for all $j=1,\dots,k$ we have $\gamma_\tau^{(j)} \geqq n^{-2}$ and, hence

$$\ln \frac{\gamma_{\mathbf{h}}^{(j)}}{\gamma_\tau^{(j)}} \leqq |\ln \gamma_\tau^{(j)}| \leqq |\ln \gamma_\tau| \leqq 2 \ln n. \tag{6.9}$$

It is clear that

$$\varrho_{\mathbf{h}}(F^n, F^{n+1}) \leqq \limsup_{\delta \to 0} d_{\mathbf{h}, \varepsilon(\delta)}(F^n, F^{n+1}), \tag{6.10}$$

$$d_{\mathbf{h}, \varepsilon(\delta)}(F^n, F^{n+1}) \leqq \sum_{\mathbf{J} \in \Xi} \varrho_{\mathbf{h}, \varepsilon(\delta)}^{(\mathbf{J})}(F^n, F^{n+1}) \tag{6.11}$$

(see (2.4)–(2.6), [18], p. 85). From (6.6)–(6.11) we can easily derive the inequality (3.21) for $\gamma_{\mathbf{h}} \geqq 2n^{-2}$. Theorem 3.1 is completely proved. We have already noted that Theorem 1.1 follows from it.

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Received May 6, 1987; in revised form February 25, 1988