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Probability Theory and Related Fields

Pointwise Ergodic Theorems for the Symmetric Exclusion Process

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Summary. Almost sure convergence theorems are proved for Cesaro averages of continuous functions in the case of the symmetric exclusion processes in dimension $d \geqq 3$. For the occupation time of a single site the same result is proved in all dimensions.

1. Introduction

In 1970, Spitzer [11] introduced a wide class of Markov processes, now generally referred to as interacting particle systems. In general these processes have more than one invariant measure and therefore the term ergodic has been reserved for the exceptional situation in which there is only one such measure. Most of the attention for these processes has been focused on the analysis of the set of invariant measures and their domains of attraction; much less is known about the behaviour of the flow induced by these Markov semigroups under some initial measure ν. Of course, it follows from standard ergodic theory that ν is an extremal invariant measure if and only if for every bounded continuous f and for ν-almost all η

$$\mathbb{P}_\eta \left\{ \lim_{T \to \infty} 1/T \int_0^T f(\eta_s)\, ds = \int f\, d\nu \right\} = 1 \tag{A}$$

where $\mathbb{P}_\eta$ is the probability measure on the space of trajectories representing the process when it starts from the initial state η and η_s is the random configuration obtained at time s.

However more information is needed to decide when the above limit exists with probability one for some specified η. An early impetus to the study of this type of problem was given in [6].

In non ergodic situations, (i.e. when there is more than one invariant measure) it seems that the only results of this available concern the contact

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process ([4–6]). In the present paper we study the symmetric exclusion process on $\mathbb{Z}^d$. This process has been extensively studied and for background on it the reader is referred to [9].

Because of the existence of a dual process it has been possible to give a condition on η that ensures that the distribution of the process at time t starting from the configuration η converges to a given extremal invariant measure ([7, 8, 12]). It follows from the results of [3] that this last condition is satisfied for a large class of configurations. What we prove here is that if η is a configuration such that the distribution of η_t converges to an extremal ν then (A) holds if f depends only on whether one site is occupied or not. The same result is also proved for all continuous f when $d \geqq 3$. Proofs of these results are in Sect. 2. Finally, in Sect. 3 we remark that the proof of the second result applies to the voter model too, thus proving in dimensions greater than or equal to 3, one of the conjectures stated in the final remark of [2].

2. Symmetric Exclusion Process on $\mathbb{Z}^d$

To describe the symmetric exclusion process we let S be a finite or countable set and $p(x, y)$ a symmetric irreducible probability matrix on S. The state space of the process will be $X = \{0, 1\}^S$ and if $\eta \in X$ we interpret $\eta(x)$ as the number of particles present at site $x \in S$. Each particle will wait an exponentially distributed time of parameter one and then attempt a jump according to the matrix $p(x, y)$. The interaction between particles is given by the following rule: the attempted jump of a particle from x to y occurs if and only if y is vacant. In the sequel we will identify $\eta \in X$ with the following subset of S: $\{x \in S\colon \eta(x) = 1\}$ and μ_ρ will denote the product measure on X such that $\mu_\rho\{\eta\colon \eta(x) = 1\} = \rho$ for all $x \in S$, $0 \leqq \rho \leqq 1$. When S is the integer lattice $\mathbb{Z}^d$ and $p(x, y)$ is translation invariant the set of extremal invariant measures is known to be $\{\mu_\rho\}$ $0 \leqq \rho \leqq 1$ (see [7] and [12]). For our proof it is convenient to think of the symmetric exclusion process in terms of a stirring process [1]. We recall here that this last term means that to each pair of sites $\{x, y\}$ we attach an independent Poisson process with intensity $p(x, y)$ and that the configuration is changed at the times corresponding to this pair of sites by exchanging the occupation variables at x and y.

The formulation has the advantage of making clearest the association property of the symmetric exclusion by concentrating all the randomness on the random permutation generated by the Poisson processes of transpositions. Reading backwards in time from time t the Poisson processes of transpositions we define a random set $A_r^t(\omega)$ for $0 \leqq r \leqq t$ that satisfies

$$\{\omega\colon A_t^t(\omega) \subset \eta\} = \{\omega\colon \eta_t(\omega) \supset A\} \tag{1}$$

and of course $(A_r^t)_{0 \leqq r \leqq t}$ evolves according to a simple exclusion process with initial configuration A. If A is a singleton we will freely use x_s^t instead of $\{x\}_s^t$ when no confusion is possible.

Our first result deals with a particular function f namely: $f(\eta) = \eta(x)$ the occupation variable of a single site.

Theorem 2.1. *If η_t converges in distribution to μ_ρ then*

$$\lim_T \frac{1}{T}\int_0^T \eta_s(x)\,ds=\rho \text{ a.s.} \quad \textit{for all } x\in S.$$

Proof. Fix $x\in S$ and let k be a positive integer. Define $S_{k,n}$ by the formula

$$S_{k,n}=\sum_{i=0}^{n-1}\eta_{i/k}(x).$$

Then

$$\begin{aligned}
&E\left(\frac{S_{k,n}-E(S_{k,n})}{n}\right)^2\\
&=\frac{1}{n^2}\sum_{i=0}^{n-1}(\eta_{i/k}(x)-E(\eta_{i/k}(x)))^2+\frac{2}{n^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\operatorname{cov}(\eta_{i/k}(x),\eta_{j/k}(x))\\
&\leqq\frac{1}{n}+\frac{2}{n^2}\sum_{i=0}^{n-2}\sum_{j=1}^{n-i-1}\operatorname{cov}(\eta_{i/k}(x),\eta_{(i+j)/k}(x)).
\end{aligned}$$

However (1) implies that

$$\begin{aligned}
\operatorname{cov}(\eta_{i/k}(x),\eta_{(i+j)/k}(x))&=\operatorname{cov}(\eta(x_{i/k}^{i/k}),\eta(x_{(i+j)/k}^{(i+j)/k}))\\
&=\sum_y P\left(\frac{j}{k},x,y\right)\operatorname{cov}[\eta(x_{i/k}^{i/k}),\eta(y_{i/k}^{i/k})]
\end{aligned}$$

where $p(t,x,y)$ is, as usual, $\sum_{n=0}^{\infty}e^{-t}\frac{t^n}{n!}p^{(n)}(x,y)$.

By Lemma 4.12 in Chap. VIII of [9] the covariance of the random variables $\eta(x_{i/k}^{i/k})$ and $\eta(y_{i/k}^{i/k})$ is non positive if $x\neq y$, therefore

$$\operatorname{cov}(\eta_{i/k}(x),\eta_{(i+j)/k}(x))\leqq p\left(\frac{j}{k},x,x\right).$$

Hence

$$\begin{aligned}
E\left(\frac{S_{k,n}-E(S_{k,n})}{n}\right)^2&\leqq\frac{1}{n}+\frac{2}{n^2}\sum_{i=0}^{n-2}\sum_{j=1}^{n-i-1}p\left(\frac{j}{k},x,x\right)\\
&\leqq\frac{1}{n}+\frac{2}{n}\sum_{j=1}^{n}p\left(\frac{j}{k},x,x\right).
\end{aligned}$$

From Proposition P.6 in Chap. II of [10] we know that there exists a constant C such that $p\left(\frac{j}{k},x,x\right)\leqq C\left(\frac{j}{k}\right)^{-\frac{1}{2}}$ for all $j\geqq 1$. Using this upper bound we see that there is a constant L such that

$$E\left(\frac{S_{k,n}-E(S_{k,n})}{n}\right)^{-\frac{1}{2}}\leqq Ln^{-\frac{1}{2}} \quad \text{for all } n\in N.$$

Applying Theorem 3.7.3 in [13] we obtain

$$\lim\frac{S_{k,n}}{n}=\lim\frac{E(S_{k,n})}{n}=\rho \text{ a.s.} \tag{2}$$

It follows from the construction of the process that if $\eta_s(x) \neq \eta_{i/k}(x)$ for some $s \in \left[\frac{i}{k}, \frac{i+1}{k}\right)$ then there exists a y for which an event of the Poisson process associated to $\{x, y\}$ occurs in the time interval $\left[\frac{i}{k}, \frac{i+1}{k}\right)$. However these last events are independent for different values of i and they all have probability $\frac{1}{k}$. It then follows that

$$\limsup_T \frac{1}{T} \int \eta_s(x)\, ds \leq \lim_n \frac{S_{k,n}}{n} + \frac{1}{k} \text{ a.s.}$$

and

$$\liminf_T \frac{1}{T} \int \eta_s(x)\, ds \geq \lim_n \frac{S_{k,n}}{n} - \frac{1}{k} \text{ a.s.}$$

Since these two inequalities as well as (2) hold for all k the theorem is proved.

Our second theorem deals with arbitrary continuous functions but its proof requires the dimension to be at least 3:

Theorem 2.2. *Let $d \geq 3$ and suppose that η_t converges in distribution to μ_ρ as t tends to infinity. Then for all $f \in C(X)$,*

$$\lim \frac{1}{T} \int_0^T f(\eta_s)\, ds = \int f(\eta)\, d\mu_\rho(\eta) \text{ a.s.}$$

Proof. First note that the subspace generated by functions of the form $\prod_{x \in A} \eta(x)$, where A ranges over all finite subsets of $\mathbb{Z}^d$, is dense in $C(X)$. Hence it suffices to prove the theorem for these functions. To do so we introduce the following notation:

$$\eta(A) = \prod_{x \in A} \eta(x) \quad \text{and} \quad S_{k,n} = \sum_{i=0}^{n-1} \eta_{i/k}(A).$$

Then we proceed as in the proof of our first theorem and obtain

$$E\left(\frac{S_{k,n} - E(S_{k,n})}{n}\right)^2 \leq \frac{1}{n} + \frac{2}{n^2} \sum_{0 \leq i \leq n-2} \sum_{1 \leq l \leq n-i-1} \operatorname{cov}(\eta_{i/k}(A), \eta_{(i+l)/k}(A)).$$

To complete the proof as before it suffices to show the following lemma.

Lemma 2.3. *Suppose $d \geq 3$ and let A be a finite subset of $\mathbb{Z}^d$. Then there exists a constant C such that for any t and s and any $\eta \in X$*

$$|E_\eta(\eta_t(A)\, \eta_{t+s}(A)) - E_\eta(\eta_t(A))\, E_\eta(\eta_{t+s}(A))| \leq C |A|^2 s^{-\frac{1}{2}} \tag{3}$$

where $|A|$ denotes the cardinality of A.

Proof. By (1) the left hand side of (3) is equal to $\operatorname{cov}(\eta(A_t^t), \eta(A_{t+s}^{t+s}))$. To prove the inequality we will modify the random variable $\eta(A_{t+s}^{t+s})$ on a set of small probability and obtain another random variable independent of $\eta(A_t^t)$. For this purpose take another independent copy of the Poisson processes associated to pairs $\{x, y\}$. This defines another stirring process independent of the first one. The original and the new stirring processes will be called the ω-stirring process and the ω'-stirring process, respectively.

Now construct for the time index r such that $0 \leqq r \leqq t+s$ a new process $\bar{A}_r^{t+s}$ which evolves according to the ω-stirring process except when the following two conditions are met simultaneously:

1) $r \geqq s$,
2) $\{x, y\} \cap A_{r-s}^t \neq \emptyset$ and $\{x, y\} \cap \bar{A}_r^{t+s} \neq \emptyset$.

In this case the $\bar{A}_r^{t+s}$ process will exchange the occupation of x and y according to the ω'-stirring process. Since the A_r^t process is built only with the use of the ω-stirring process the processes $A^t_\cdot$ and $A_\cdot^{t+s}$ depend on independent Poisson processes. Hence they are independent, and

$$\operatorname{cov}(\eta(A_t^t), \eta(A_{t+s}^{t+s})) \leqq P(A_{t+s}^{t+s} \neq \bar{A}_{t+s}^{t+s}).$$

Noting that

$$\begin{aligned} P(A_{t+s}^{t+s} \neq \bar{A}_{t+s}^{t+s}) &\leqq P(\bar{A}_r^{t+s} \cap A_{r-s}^t \neq \emptyset \text{ for some } s \leqq r \leqq s+t) \\ &\leqq \sum_{x \in A} \sum_{y \in A} P(\{\bar{x}\}_r^{t+s} = \{y\}_{r-s}^t \text{ for some } r \geqq s) \\ &\leqq \sum_{x, y \in A} P(\text{a random walk starting at } x \text{ hits } y \text{ after time } s) \end{aligned}$$

we obtain the lemma as a consequence of a continuous time version of Proposition P.6 in Chap. II of [10].

3. Remarks

3.1. The method of proof that was used to prove Theorem 2.2 can be applied also to the voter model in dimension $d \geqq 3$ since the same method of duality gives the inequality (3).

3.2. The estimate (3) is crude and in particular the method to prove it is not good enough to show that Theorem 2.2 holds in dimension one or two. However after completion of this paper the following inequality was proved by one of us (E.A.):

For a symmetric exclusion process if A and B are finite disjoint subsets of Z^d then

$$E_\eta(\eta_t(A \cup B)) \leqq E_\eta(\eta_t(A))\, E_\eta(\eta_t(B))$$

for any $\eta \in X$ and any $t \geqq 0$. This will appear in a future paper and allows one to extend Theorem 2.2 to all dimensions by the method used in the proof of Theorem 2.1.

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