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Probability Theory and Related Fields

Sufficient Moment and Truncated Moment Conditions for the Law of the Iterated Logarithm

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Summary. Various sufficient conditions for the law of the iterated logarithm are given extending the main result of the author's previous paper [16] and Kolmogoroff's law of iterated logarithm. As a byproduct we give a unified approach to various old and new stability results

1. Introduction

Throughout the whole paper (X_n) will be a sequence of independent random variables satisfying

$$E(X_i)=0, \qquad E(X_i^2)<\infty \quad (i\in\mathbb{N}).$$

We will use the following notation

$$S_n:=\sum_{i=1}^n X_i, \qquad B_n:=\sum_{i=1}^n E(X_i^2)$$

$$\log_+ x:=\begin{cases}\log x & \text{if } x\geqq e,\\ 0 & \text{if } x\leqq e,\end{cases} \qquad \log_2 x:=\log_+(\log_+ x)$$

$$\Gamma_n(\varepsilon):=E(I_{\{|X_n|>\varepsilon\sqrt{B_n/\log_2 B_n}\}}X_n^2) \qquad (\varepsilon>0)$$

$$L_n(\varepsilon):=B_n^{-1}\sum_{i=1}^n E(I_{\{|X_i|>\varepsilon\sqrt{B_n/\log_2 B_n}\}}X_i^2)\leqq 1.$$

We always assume that

$$\lim_{n\to\infty} B_n=\infty.$$

In [16] we have proved the following

Theorem 1.1. *Let* $2<p\leqq 3$ *and assume that*

(i) $$\sum_{n=1}^\infty (B_n\log_2 B_n)^{-p/2}E(|X_n|^p)<\infty$$

(ii) $$\limsup_{n\to\infty}\frac{B_{n+1}}{B_n}<\infty.$$

Then the law of the iterated logarithm holds, i.e.

$$\limsup_{n\to\infty}(2B_n \log_2 B_n)^{-1/2} S_n = 1 \quad a.s.$$

Besides its proof the novel point of this theorem consists in the absence of the condition

$$(*) \qquad \lim_{n\to\infty} L_n(\varepsilon)=0.$$

This condition appeared explicitly or implictly in *all* previous results about the law of the iterated logarithm. It is a strengthend Lindeberg type condition and implies therefore the central limit theorem, whereas in [16] examples were given which satisfy the assumptions of the above theorem but not the central limit theorem. It was also shown in [16] that Theorem 1.1 is in a certain sense sharpe. Nevertheless two important questions remained unanswered:

1. Does Theorem 1.1 also hold when $p>3$? (For $3<p\leqq 4$ this was already shown in [16] under the additional condition $E(X_i^3)=0$.)

2. Can the essential condition 1.1 (i) be weakened if we assume also

$$\sum_{n=1}^{\infty} P\{|X_n|>\varepsilon\sqrt{B_n \log_2 B_n}\} \qquad (\varepsilon>0)?$$

This condition excludes the counterexamples in [16] and is almost necessary.

Independently and with quite different methods Alt [1] obtained a partial answer to question 1 in the setting of Banach space valued random variables and Einmahl [4] answered question 1 completely in the affirmative by giving very general a.s. invariance principles. The method of [4] however doesn't seem to be adequate to solve question 2. In this paper we answer both questions in the affirmative. Though Kolmogoroff's law of the iterated logarithm is too weak to prove our results by the usual truncation technique, his ideas are fundamental for this paper. The method of [16] is based on convergence estimates in the central limit theorem and therefore entirely different.

The plan of our paper is as follows. In Sect. 2 we introduce the technique for the proof of the lower class part in the law of the iterated logarithm and give very general lower class results for symmetrically distributed random variables. In Sect. 3 we use an exponential inequality similar to Teicher [13], Lemma 1 to prove various old and new stability results. In particular we reprove a famous theorem of Egorov [5] in a completely new way. In Sect. 4 we combine results of the two preceding sections (which are completely independent of each other) to prove our main results. We can even replace the moment condition by weaker truncated moment conditions.

2. A Lower Class Result for Symmetric Random Variables

All our lower class results are based on a very refined application of the subsequent exponential inequality which is an immediate consequence of Kolmogoroff [7], Hilfssatz IV (unfortunately the corresponding statements in textbooks are not strong enough).

Lemma 2.1. *For any* $\varepsilon>0$ *there exists* $K_\varepsilon>0$ *and* $\delta>0$ *such that for any finite sequence* $(\tilde{X}_i)_{m<i\leqq n}$ *of independent random variables satisfying*

(i) $$K_\varepsilon^{-1}\leqq \tilde{B}_{m,n}:=\sum_{i=m+1}^{n} E(\tilde{X}_i^2)<\infty, \qquad E(\tilde{X}_i)=0$$

(ii) $$|\tilde{X}_i|\leqq K_\varepsilon\sqrt{\tilde{B}_{m,n}/\log_2 \tilde{B}_{m,n}}$$

we have

$$P\left\{\sum_{i=m+1}^{n}\tilde{X}_i>(1-\varepsilon)\sqrt{2\tilde{B}_{m,n}\log_2 \tilde{B}_{m,n}}\right\}\geqq(\log_+\tilde{B}_{m,n})^{\delta-1}.$$

The next lemma is fundamental for this and the last section.

Lemma 2.2. *Let* $0<\varepsilon$, $\delta_0<1$, $\rho:=\frac{(1-\varepsilon)^2}{2}K_\varepsilon$ *and* $d>0$ *be such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta_0-1}\Gamma_n(\rho)\,L_n^d(\varepsilon^3\rho)<\infty$$

(ii) $$\sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta-1}E(X_n^2)=\infty \qquad (\delta>0).$$

Denoting

$$A_n:=\{|X_n|\leqq\rho\sqrt{B_n/\log_2 B_n}\},$$
$$\tilde{X}_n:=I_{A_n}X_n-E(I_{A_n}X_n),$$
$$\tilde{S}_{m,n}:=\sum_{i=m+1}^{n}\tilde{X}_i$$

we have

$$\limsup_{k\to\infty}(2B_{n_k}\log_2 B_{n_k})^{-1/2}\tilde{S}_{n_{k-1},n_k}\geqq(1-\varepsilon)^3 \quad a.s.$$

for any sequence (n_k) *satisfying*

(iii) $$\varepsilon^{-1}B_{n_k}\leqq B_{n_{k+1}}\leqq\varepsilon^{-3}B_{n_k+1} \qquad (k\in\mathbb{N}).$$

For the proof we need the following

Lemma 2.3. *Let* (a_n) *be an increasing sequence in* $]0,\infty[$, (b_n) *a decreasing sequence in* $[0,\infty[$ *such that*

$$C a_{n_k}\leqq a_{n_{k+1}}\leqq C' a_{n_k+1} \qquad (k\in\mathbb{N})$$

for some $C'>C>1$. *Then we have*

$$(C-1)C^{-1}\sum_{k=2}^{\infty}b_{n_k}\leqq\sum_{i=n_1+1}^{\infty}a_i^{-1}(a_i-a_{i-1})b_i\leqq(C'-1)\sum_{k=1}^{\infty}b_{n_k}.$$

Proof. The assertion follows from

$$\begin{aligned}\sum_{k=2}^{\infty} b_{n_k} &\leqq (C-1)\,C^{-1}\sum_{k=1}^{\infty} a_{n_{k+1}}^{-1}(a_{n_{k+1}}-a_{n_k})\,b_{n_{k+1}}\\ &=(C-1)\,C^{-1}\sum_{k=1}^{\infty} b_{n_{k+1}}\,a_{n_{k+1}}^{-1}\sum_{i=n_k+1}^{n_{k+1}}(a_i-a_{i-1})\\ &\leqq (C-1)\,C^{-1}\sum_{i=n_1+1}^{\infty} a_i^{-1}(a_i-a_{i-1})\,b_i\\ &\leqq (C-1)\,C^{-1}\sum_{k=1}^{\infty} b_{n_k}\,a_{n_k+1}^{-1}\sum_{i=n_k+1}^{n_{k+1}} a_i-a_{i-1}\\ &\leqq (C-1)\,C^{-1}(C'-1)\sum_{k=1}^{\infty} b_{n_k}.\end{aligned}$$

Proof of Lemma 2.2. We denote

$$B_{m,n}:=\sum_{i=m+1}^{n} E(X_i^2),\qquad \tilde{B}_{m,n}:=\sum_{i=m+1}^{n} E(\tilde{X}_i^2),$$
$$I:=\{k\geqq 2:\tilde{B}_{n_{k-1},n_k}<(1-\varepsilon)\,B_{n_{k-1},n_k}\}$$
$$J:=\{k\geqq 2:k\notin I,\tilde{B}_{n_{k-1},n_k}\geqq(1-\varepsilon)^{-2}\,K_\varepsilon^{-1}\}.$$

By (iii) we have

$$(1-\varepsilon)\,B_{n_k}\leqq B_{n_k}-B_{n_{k-1}}=B_{n_{k-1},n_k}.$$

Clearly $k\in I$ if and only if

$$2\sum_{i=n_{k-1}+1}^{n_k}\Gamma_i(\rho)\geqq\sum_{i=n_{k-1}+1}^{n_k}\Gamma_i(\rho)+(E(I_{A_i}X_i))^2>\varepsilon\,B_{n_{k-1},n_k}\geqq\varepsilon(1-\varepsilon)\,B_{n_k}.$$

By the definition of $L_m(\varepsilon^3\rho)$ and $\Gamma_i(\varepsilon)$ we have

$$L_m(\varepsilon^3\rho)\geqq B_{n_k}^{-1}\sum_{i=n_{k-1}+1}^{m}\Gamma_i(\rho)\qquad(k\geqq 2,n_{k-1}<m\leqq n_k).\tag{1}$$

For any $k\in I$ we define

$$m_k:=\inf\left\{m\leqq n_k:\sum_{i=m+1}^{n_k}\Gamma_i(\rho)\leqq\tfrac{1}{4}\varepsilon(1-\varepsilon)\,B_{n_k}\right\}$$

and we get

$$\sum_{i=m_k}^{n_k}\Gamma_i(\rho)>\tfrac{1}{4}\varepsilon(1-\varepsilon)\,B_{n_k}\tag{2}$$

$$\sum_{i=n_{k-1}+1}^{m_k}\Gamma_i(\rho)>\tfrac{1}{4}\varepsilon(1-\varepsilon)\,B_{n_k}.\tag{3}$$

Together with (1) we get from (3) that

$$L_m(\varepsilon^3\rho)\geqq\tfrac{1}{4}\varepsilon(1-\varepsilon)\qquad(k\in I,m_k\leqq m\leqq n_k).\tag{4}$$

Using (2), (4) and (i) we can conclude

$$(\tfrac{1}{4}\varepsilon(1-\varepsilon))^{d+1}\sum_{k\in I}(\log_+ B_{n_k})^{\delta_0-1} \leqq \sum_{k\in I} B_{n_k}^{-1}(\log_+ B_{n_k})^{\delta_0-1}\sum_{i=m_k}^{n_k}\Gamma_i(\rho)\,L_i^d(\varepsilon^3\rho)$$
$$\leqq \sum_{i=1}^{\infty} B_i^{-1}(\log_+ B_i)^{\delta_0-1}\,\Gamma_i(\rho)\,L_i^d(\varepsilon^3\rho)<\infty.$$

Thus we have

$$\sum_{k\in I}(\log_+ B_{n_k})^{\delta-1}<\infty$$

for any $0<\delta<\delta_0$. On the other hand (ii) and 2.3 yield

$$\sum_{k=1}^{\infty}(\log_+ B_{n_k})^{\delta-1}=\infty.$$

Putting both things together we obtain

$$\sum_{k\in J}(\log_+ B_{n_k})^{\delta-1}=\infty \qquad (\delta>0). \tag{5}$$

Since for any $k\in J$ and $i\leqq n_k$

$$\begin{aligned}|\tilde{X}_i| &\leqq 2\rho\sqrt{B_i/\log_2 B_i}\\ &\leqq (1-\varepsilon)^2 K_\varepsilon\sqrt{B_{n_k}/\log_2 B_{n_k}}\\ &\leqq (1-\varepsilon)K_\varepsilon\sqrt{B_{n_{k-1},n_k}/\log_2 B_{n_{k-1},n_k}}\\ &\leqq K_\varepsilon\sqrt{\tilde{B}_{n_{k-1},n_k}/\log_2 \tilde{B}_{n_{k-1},n_k}}\end{aligned}$$

we may apply Lemma 2.1 to find $\delta>0$ such that for any $k\in J$

$$P\{\tilde{S}_{n_{k-1},n_k}>(1-\varepsilon)\sqrt{2\tilde{B}_{n_{k-1},n_k}\log_2\tilde{B}_{n_{k-1},n_k}}\}\geqq(\log_+\tilde{B}_{n_{k-1},n_k})^{\delta-1}.$$

Since also

$$\begin{aligned}\tilde{B}_{n_{k-1},n_k} &\geqq (1-\varepsilon)B_{n_{k-1},n_k}\\ &\geqq (1-\varepsilon)^2 B_{n_k}\end{aligned}$$

for any $k\in J$, this implies

$$P\{\tilde{S}_{n_{k-1},n_k}>(1-\varepsilon)^3\sqrt{2B_{n_k}\log_2 B_{n_k}}\}\geqq(\log_+ B_{n_k})^{\delta-1}$$

whence together with (5) we obtain

$$\sum_{k\in J}P\{\tilde{S}_{n_{k-1},n_k}>(1-\varepsilon)^3\sqrt{2B_{n_k}\log_2 B_{n_k}}\}=\infty.$$

By the converse to the Borel-Cantelli Lemma the assertion follows.

The main result of this section now reads as follows

Theorem 2.4. *Assume that the* X_n *are symmetrically distributed and that for any* $\rho>0$ *there exist* $\delta_\rho>0$ *and* $d>0$ *such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta_\rho-1}\,\Gamma_n(\rho)\,L_n^d(\rho)<\infty$$

(ii) $$\sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta-1}\,E(X_n^2)=\infty \qquad (\delta>0).$$

Then we have

$$\limsup_{n\to\infty}(2B_n\log_2 B_n)^{-1/2}S_n\geqq 1 \quad a.s.$$

Proof. Since the S_n are also symmetrically distributed we need only show that

$$\limsup_{n\to\infty}(2B_n\log_2 B_n)^{-1/2}|S_n|\geqq 1 \quad \text{a.s.} \tag{1}$$

To this end let $0<\varepsilon<1$ be given. By [16], Lemma 3.3 there exists a sequence (n_k) satisfying

$$\varepsilon^{-2}B_{n_k}\leqq B_{n_{k+1}}\leqq\varepsilon^{-6}B_{n_k+1} \qquad (k\in\mathbb{N}).$$

Using the notation introduced in Lemma 2.2 and denoting $\tilde S_n:=\tilde S_{0,n}$, $a_n:=\sqrt{2B_n\log_2 B_n}$ we have

$$\{\tilde S_{n_{k-1},n_k}>(1-\varepsilon)^3 a_{n_k}\text{ i.o.}^1\}\subset\{\tilde S_{n_k}>(1-\varepsilon)^4 a_{n_k}\text{ i.o.}\}\cup\{\tilde S_{n_{k-1}}<-\varepsilon(1-\varepsilon)^3 a_{n_k}\text{ i.o.}\},$$
$$\{\tilde S_{n_{k-1}}<-\varepsilon(1-\varepsilon)^3 a_{n_k}\text{ i.o.}\}\subset\{\tilde S_{n_{k-1}}<-(1-\varepsilon)^3 a_{n_{k-1}}\text{ i.o.}\}$$

and therefore

$$\{\tilde S_{n_{k-1},n_k}>(1-\varepsilon)^3 a_{n_k}\text{ i.o.}\}\subset\{|\tilde S_n|>(1-\varepsilon)^4 a_n\text{ i.o.}\} \tag{2}$$

Lemma 2.2 and (2) now yield (1)

$$\limsup_{n\to\infty}a_n^{-1}|\tilde S_n|\geqq(1-\varepsilon)^4 \quad \text{a.s.} \tag{3}$$

Using Lemma 1 of Teicher [12] we get from (3) that

$$\limsup_{n\to\infty}a_n^{-1}|S_n|\geqq(1-\varepsilon)^5 \quad \text{a.s.}$$

Making $\varepsilon>0$ arbitrary small the assertion (1) follows.

Since $\Gamma_n(\varepsilon)\leqq\varepsilon^{1-\frac{p}{2}}B_n^{1-\frac{p}{2}}(\log_2 B_n)^{\frac{p}{2}-1}E(|X_n|^p)$ for any $p>2$ we get

Corollary 2.5. *Let the* X_n *be symmetrically distributed. If there exist* $\delta_0>0$ *and* $p>2$ *such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-p/2}(\log_+ B_n)^{\delta_0-1}\,E(|X_n|^p)<\infty$$

(ii) $$\sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta-1}\,E(X_n^2)=\infty \qquad (\delta>0)$$

[1] i.o. = infinitely often

then we have

$$\limsup_{n\to\infty}(2B_n\log_2 B_n)^{-1/2}S_n \geqq 1 \quad a.s.$$

Corollary 2.6. *Let (b_n) be a sequence $\mathbb{R}$ with $\sum_{n=1}^{\infty} b_n^2=\infty$, (Y_n) be an independent sequence of random variables with $P\{Y_n=1\}=P\{Y_n=-1\}=-1/2$ and $X_n:=b_n Y_n$. If there exist $\delta_0>0$ and $p>2$ such that*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} B_n^{-p/2}(\log_+ B_n)^{\delta_0-1}|b_n|^p<\infty$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{\delta-1} b_n^2=\infty \qquad (\delta>0)$$

then we have

$$\limsup_{n\to\infty}(2B_n\log_2 B_n)^{-1/2}S_n=1 \quad a.s.$$

Proof. Even for an arbitrary sequence (b_n) with $\sum_{n=1}^{\infty} b_n^2=\infty$ it follows from the work of Marcinkiewicz and Zygmund [8] that

$$\limsup_{n\to\infty}(2B_n\log_2 B_n)^{-1/2}S_n \leqq 1 \quad \text{a.s.}$$

The "$\geqq$" half is of course an immediate consequence of Corollary 2.5.

The condition (ii) in the above theorem and corollaries looks somewhat complicated. But it is already satisfied if

$$\limsup_{n\to\infty}\frac{B_{n+1}}{B_n}<\infty.$$

This is a consequence of

Lemma 2.7. *Let (a_n) be a sequence in $]0,\infty[$ increasing to ∞ such that*

$$M:=\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}<\infty.$$

Then we have

$$\sum_{n=2}^{\infty} a_n^{-1}(\log_+ a_n)^{-1}(\log_2 a_n)^{-1}(a_n-a_{n-1})=\infty.$$

In particular, we have

$$\sum_{n=2}^{\infty} a_n^{-1}(\log_+ a_n)^{\delta-1}(a_n-a_{n-1})=\infty$$

for any $\delta\geqq 0$.

Proof. Let $\tilde{M}:=2M>1$ and define inductively

$$n_1:=\inf\left\{n\in\mathbb{N}: \frac{a_{m+1}}{a_m}\leqq \tilde{M}^2\ \forall\, m\geqq n\right\}$$

$$n_{k+1}:=\inf\{n\in\mathbb{N}: a_n\geqq \tilde{M}\, a_{n_k}\}.$$

Then we have for any $k \geqq 2$

$$\tilde{M}\, a_{n_k} \leqq a_{n_{k+1}} \leqq \tilde{M}^3\, a_{n_k}$$

and therefore

$$a_{n_k} \leqq \tilde{M}^{3k}\, a_{n_1}. \tag{1}$$

Applying now Lemma 2.3 to (a_n) and $b_n := (\log_+ a_n)^{-1}(\log_2 a_n)^{-1}$ we see that the assertion is equivalent to

$$\sum_{k=1}^{\infty} (\log_+ a_{n_k})^{-1}(\log_2 a_{n_k})^{-1} = \infty.$$

But using (1) this follows from

$$\sum_{k=1}^{\infty} (\log_+(a_{n_1}\tilde{M}^{3k}))^{-1}(\log_2(a_{n_1}\tilde{M}^{3k}))^{-1} \geqq C \sum_{k=1}^{\infty} (3k \log 3k)^{-1} = \infty.$$

Remark. If we would strengthen condition (ii) a little bit then we could weaken condition (i) considerably in the above theorem and corollaries. For instance,

$$\limsup_{n\to\infty} (2B_n \log_2 B_n)^{-1/2} S_n \geqq 1 \quad \text{a.s.}$$

holds for symmetrically distributed random variables if for any $\rho > 0$ there exists $d > 0$ with

(i′) $\quad \sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{-1}(\log_2 B_n)^{-1} \Gamma_n(\rho) L_n^d(\rho) < \infty$

(ii′) $\quad \sum_{n=1}^{\infty} B_n^{-1}(\log_+ B_n)^{-1}(\log_2 B_n)^{-1} E(X_n^2) = \infty \qquad (\delta > 0)$

(note that (ii′) is again quite weak by the last lemma). This was essentially proved in a first draft of the article. But since in our main result Theorem 4.1 the above condition (ii) is in a certain sense necessary and since (i′) is too weak for the "$\leqq$" half of the LIL the above arrangement seems to be better if one doesn't want to write down the same arguments twice.

3. Stability Results

Throughout the whole section (a_n) will be a sequence in $]0, \infty[$ increasing to infinity. In what follows the subsequent exponential inequality will be fundamental. It is closely related to Teicher [13], Lemma 1.

Lemma 3.1. *Let $m < n$. If*

$$|X_i| \leqq C_n a_n \qquad (i = m+1, \ldots, n)$$

then we have

$$P\{S_{m,n} \geqq a_n\} \leqq \exp(-\tfrac{1}{2}\kappa_{m,n})$$

where $S_{m,n} := S_n - S_m$ *and* $\kappa_{m,n}$ *is the unique solution of the equation*

$$\kappa_{m,n} = B_{m,n}^{-1} a_n^2 \exp(-\kappa_{m,n} C_n) \qquad (B_{m,n} := B_n - B_m).$$

Proof. Since the function $f(t) := t$ is strictly increasing and

$$g(t) := B_{m,n}^{-1} a_n^2 \exp(-t\, C_n)$$

is strictly decreasing on $[0, \infty[$ and

$$f(0)=0, \qquad g(0)=B_{m,n}^{-1} a_n^2 > 0,$$
$$\lim_{t\to\infty} f(t)=\infty, \qquad \lim_{t\to\infty} g(t) \quad =0$$

there exists exactly one $\kappa_{m,n}$ such that $f(\kappa_{m,n})=g(\kappa_{m,n})$, i.e. the above equation holds.

Using $1+x \leqq \exp(x)$ we have for any $t>0$

$$E(\exp(t X_i)) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X_i^k) \leqq 1 + \sum_{k=0}^{\infty} \frac{t^{k+2}}{(k+2)!} E(X_i^2)(C_n a_n)^k$$
$$\leqq 1 + \frac{t^2}{2} E(X_i^2) \exp(t\, C_n a_n) \leqq \exp(\tfrac{1}{2} t^2 E(X_i^2) \exp(t\, C_n a_n)),$$

$$E(\exp(t S_{m,n})) = \prod_{i=m+1}^{n} E(\exp(t X_i)) \leqq \exp(\tfrac{1}{2} t^2 B_{m,n} \exp(t\, C_n a_n)),$$

$$P\{S_{m,n} \geqq a_n\} \leqq \exp(-t\, a_n) E(\exp(t S_{m,n})) \leqq \exp(\tfrac{1}{2} t^2 B_{m,n} \exp(t\, C_n a_n) - t\, a_n),$$

Inserting $a_n^{-1} \kappa_{m,n}$ for t into the last inequality we obtain

$$P\{S_{m,n} \geqq a_n\} \leqq \exp(\tfrac{1}{2} a_n^{-2} B_{m,n} \kappa_{m,n}^2 \exp(\kappa_{m,n} C_n) - \kappa_{m,n}) = \exp(-\tfrac{1}{2} \kappa_{m,n}).$$

Lemma 3.2 (Ottaviani-Skorohod). *For any* $\alpha, \varepsilon > 0$ *and* $m, n \in \mathbb{N}$ *with* $m<n$ *we have*

$$P\{\max_{m<k\leqq n} |S_{m,k}| > \alpha+\varepsilon\} \cdot \min_{m \leqq k < n} P\{|S_{k,n}| \leqq \varepsilon\} \leqq P\{|S_{m,n}| > \alpha\}.$$

A proof of this well known lemma can be found in Chung [3], p. 120.

Theorem 3.3. *Let* $M>1$ *and* (n_k) *be a sequence satisfying*

(i) $$M a_{n_k} \leqq a_{n_{k+1}} \leqq M^6 a_{n_k+1} \qquad (k \in \mathbb{N}).$$

Let further (C_n) *be a sequence in* $]0, \infty[$ *such that* $(C_n a_n)$ *is increasing and*

(ii) $$\sum_{n=1}^{\infty} P\{|X_n| > \tfrac{1}{2} C_n a_n\} < \infty.$$

Let $\kappa_{m,n} > 0$ *be the unique solution of*

$$\kappa_{m,n} = B_{m,n}^{-1} a_n^2 \exp(-\kappa_{m,n} C_n).$$

Finally assume that

(iii) $$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} \kappa_{n_{k-1}, n_k}) < \infty$$

holds. Then we have

$$\limsup_{k\to\infty} a_{n_k}^{-1} \max_{n_{k-1}<m\leqq n_k} |S_m| \leqq \left(1-\frac{1}{M}\right) \quad a.s.$$

and in particular

$$\limsup_{n\to\infty} |a_n^{-1} S_n| \leqq \left(1-\frac{1}{M}\right)^{-1} M^6 \quad a.s.$$

Proof. For any $n\in\mathbb{N}$ we set

$$\bar{X}_n(\omega) := \begin{cases} X_n(\omega) & \text{if } |X_n(\omega)| \leqq \frac{1}{2} C_n a_n \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{X}_n := \bar{X}_n - E(\bar{X}_n)$$

and define $\tilde{B}_{m,n}, \tilde{S}_{m,n}, \tilde{S}_n$ as usual. Obviously we have

$$|\tilde{X}_i| \leqq C_n a_n \qquad (i=1,\ldots,n).$$

Since $\tilde{B}_{m,n} \leqq B_{m,n}$ we have $\kappa_{m,n} \leqq \tilde{\kappa}_{m,n}$ where $\tilde{\kappa}_{m,n} > 0$ satisfies

$$\tilde{\kappa}_{m,n} = \tilde{B}_{m,n}^{-1} a_n^2 \exp(-\tilde{\kappa}_{m,n} C_n).$$

Applying Lemma 3.1 to $(\tilde{X}_n)$ we obtain

$$P\{\tilde{S}_{n_{k-1},n_k} \geqq a_{n_k}\} \leqq \exp(-\tfrac{1}{2}\tilde{\kappa}_{n_{k-1},n_k}) \leqq \exp(-\tfrac{1}{2}\kappa_{n_{k-1},n_k}) \tag{1}$$

and analogously

$$P\{\tilde{S}_{n_{k-1},n_k} \leqq -a_{n_k}\} \leqq \exp(-\tfrac{1}{2}\kappa_{n_{k-1},n_k}). \tag{2}$$

From (iii) we get $\lim_{k\to\infty} \kappa_{n_{k-1},n_k} = \infty$ and therefore for any $\varepsilon>0$ there exists $k_\varepsilon\in\mathbb{N}$ such that

$$a_{n_k}^{-2} B_{n_{k-1},n_k} \leqq \varepsilon \qquad (k \geqq k_\varepsilon).$$

Hence for any $k \geqq k_\varepsilon$ we have using (i)

$$B_{n_k} \leqq B_{n_{k_\varepsilon}-1} + \sum_{j=k_\varepsilon}^{k} B_{n_{j-1},n_j}$$

$$\leqq B_{n_{k_\varepsilon}-1} + \varepsilon a_{n_k}^2 \sum_{j=0}^{\infty} M^{-2}$$

whence using again (i)

$$\limsup_{n\to\infty} a_n^{-2} B_n \leqq M^{12} \limsup_{k\to\infty} a_{n_k}^{-2} B_{n_k} \leqq M^{12}(1-M^{-2})^{-1}\varepsilon.$$

Making $\varepsilon>0$ arbitrarily small we obtain

$$\lim_{n\to\infty} a_n^{-2} B_n = 0.$$

By Tschebyscheff's inequality this implies

$$\lim_{k\to\infty} \max_{n_{k-1}<m\leqq n_k} P\{|\tilde{S}_{m,n_k}| > \varepsilon a_{n_k}\} = 0$$

for any $\varepsilon>0$. Hence for any $\varepsilon>0$ there exists $k_\varepsilon\in\mathbb{N}$ such that

$$\min_{n_{k-1}<m\leqq n_k} P\{|\tilde{S}_{m,n_k}|\leqq\varepsilon\, a_{n_k}\}\geqq\tfrac{1}{2}$$

for any $k\geqq k_\varepsilon$. Together with (2) and Lemma 3.2 this entails for any $\varepsilon>0$ and $k\geqq k_\varepsilon$

$$\begin{aligned} P\{\max_{n_{k-1}<m\leqq n_k}|\tilde{S}_{n_{k-1},m}|>(1+\varepsilon)\,a_{n_k}\} &\leqq 2P\{|\tilde{S}_{n_{k-1},n_k}|>a_{n_k}\}\\ &\leqq 4\exp(-\tfrac{1}{2}\kappa_{n_{k-1},n_k}) \end{aligned}$$

whence together with (iii) we obtain

$$\sum_{k=2}^{\infty} P\{\max_{n_{k-1}<m\leqq n_k}|\tilde{S}_{n_{k-1},m}|>(1+\varepsilon)\,a_{n_k}\}<\infty.$$

Thus, by the Borel-Cantelli Lemma for almost all $\omega\in\Omega$ there exists $k_{\varepsilon,\omega}\in\mathbb{N}$ such that for any $k\geqq k_{\varepsilon,\omega}$

$$\max_{n_{k-1}<m\leqq n_k}|\tilde{S}_{n_{k-1},m}(\omega)|\leqq(1+\varepsilon)\,a_{n_k}.$$

Hence for almost all $\omega\in\Omega$, $k\geqq k_{\varepsilon,\omega}$ and $n_{k-1}<m\leqq n_k$ we have

$$\begin{aligned} a_m^{-1}|\tilde{S}_m(\omega)| &\leqq a_{n_k}^{-1}|\tilde{S}_{n_{k_{\varepsilon,\omega}}}\omega)|+a_{n_k}^{-1}\sum_{j=1}^{k}(1+\varepsilon)\,a_{n_j}\\ &\leqq a_{n_k}^{-1}|\tilde{S}_{n_{k_{\varepsilon,\omega}}}(\omega)|+(1+\varepsilon)\,a_{n_k}^{-1}\frac{a_{n_k}}{1-M^{-1}}. \end{aligned}$$

Making $\varepsilon>0$ arbitrarily small we obtain

$$\limsup_{k\to\infty} a_{n_k}^{-1}\max_{n_{k-1}<m\leqq n_k}|\tilde{S}_m|\leqq(1-M^{-1})^{-1}\quad\text{a.s.}\tag{3}$$

By (ii) and the Borel-Cantelli Lemma this implies

$$\limsup_{k\to\infty} a_{n_k}^{-1}\max_{n_{k-1}<m\leqq n_k}|S_n-b_n|\leqq\left(1-\frac{1}{M}\right)^{-1}\quad\text{a.s.}$$

where $b_n:=\sum_{i=1}^{n}E(\bar{X}_i)$. The assertion follows now from

$$a_n^{-1}b_n\leqq\left(a_n^{-2}\sum_{i=1}^{n}E(\bar{X}_i^2)\right)^{1/2}\leqq(a_n^{-2}B_n)^{1/2}\xrightarrow{n\to\infty}0.$$

Corollary 3.4. *Let* (n_k) *and* $M>1$ *be such that*

(i) $$M\,a_{n_k}\leqq a_{n_{k+1}}\leqq M^6\,a_{n_k+1}\qquad(k\in\mathbb{N}).$$

If there exists $d>0$ *such that*

(ii) $$\sum_{n=1}^{\infty}P\{|X_n|>\varepsilon\,a_n\}<\infty\qquad(\varepsilon>0)$$

(iii) $$\sum_{k=2}^{\infty}(a_{n_k}^{-2}B_{n_{k-1},n_k})^d<\infty$$

then we have

$$\lim_{n\to\infty} a_n^{-1} S_n = 0 \quad a.s.$$

Proof. Because of (ii) we can find (C_n) such that $(C_n a_n)$ is increasing

$$\lim_{n\to\infty} C_n = 0 \tag{1}$$

$$\sum_{n=0}^{\infty} P\{|X_n| > \tfrac{1}{2} C_n a_n\} < \infty. \tag{2}$$

Let $\varepsilon > 0$ be given and define $\kappa_{m,n}$ by

$$\kappa_{m,n} = B_{m,n}^{-1}(\varepsilon a_n)^2 \exp(-\kappa_{m,n} \varepsilon^{-1} C_n).$$

By Theorem 3.3 we have to show

$$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} \kappa_{n_{k-1}, n_k}) < \infty. \tag{3}$$

To this end we choose k_0 so large such that for any $k \geqq k_0$

$$2d\varepsilon^{-1} C_{n_k} \leqq \tfrac{1}{2}, \qquad \varepsilon B_{n_{k-1}, n_k}^{-1/2} a_{n_k} \geqq \log_+(\varepsilon B_{n_{k-1}, n_k}^{-1/2} a_{n_k}). \tag{4}$$

If there would exist $k \geqq k_0$ with

$$\kappa_{n_{k-1}, n_k} < 2d \log_+(B_{n_{k-1}, n_k}^{-1} \varepsilon^2 a_{n_k}^2)$$

then we would get from (4) the contradiction

$$\begin{aligned} \kappa_{n_{k-1}, n_k} &\geqq B_{n_{k-1}, n_k}^{-1} \varepsilon^2 a_{n_k}^2 \exp(-2d \log_+(B_{n_{k-1}, n_k}^{-1} \varepsilon^2 a_{n_k}^2) \varepsilon^{-1} C_{n_k}) \\ &\geqq B_{n_{k-1}, n_k}^{-1/2} \varepsilon a_{n_k} \geqq \log_+(\varepsilon B_{n_{k-1}, n_k}^{-1/2} a_{n_k}). \end{aligned}$$

Hence we have for any $k \geqq k_0$

$$\kappa_{n_{k-1}, n_k} > 2d \log_+(B_{n_{k-1}, n_k}^{-1} \varepsilon^2 a_{n_k}^2).$$

Using (iii) the assertion (3) follows from

$$\sum_{k=k_0}^{\infty} \exp(-\tfrac{1}{2} \kappa_{n_{k-1}, n_k}) \leqq \sum_{k=k_0}^{\infty} (a_{n_k}^{-2} B_{n_{k-1}, n_k})^d < \infty.$$

The above theorem and its first corollary look very technical, but Corollary 3.4 will be one important ingredient for the upper class part in the proof of our main result 4.1 and it implies a famous result of Egorov [5] which has its origin in Teicher [11]. For this purpose we need the following

Lemma 3.5. *Let (b_n) be a sequence in $[0, \infty[$ and $d \in \mathbb{N}$.*

(a) For any $m, n \in \mathbb{N}$ with $m \leqq n$ we have

$$\left(\sum_{i=m}^{n} b_i\right)^d \leqq d! \sum_{i=m}^{n} b_i \left(\sum_{j=m}^{n} b_j\right)^{d-1}.$$

(b) Let further (n_k) be a sequence and $M>1$ be such that $M a_{n_k} \leqq a_{n_{k+1}} \leqq M^6 a_{n_k+1}$ for any $k \in \mathbb{N}$. Then the following conditions are equivalent

$$\text{(i)} \qquad \sum_{k=1}^{\infty} a_{n_k}^{-1} \left(\sum_{i=1}^{n_k} b_i\right)^d < \infty$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} a_n^{-1} b_n \left(\sum_{i=1}^{n} b_i\right)^{d-1} < \infty.$$

Proof. (a) follows from

$$\begin{aligned}\left(\sum_{i=m}^{n} b_i\right)^d &\leqq d! \sum_{m \leqq j_1 \leqq \ldots \leqq j_d \leqq n} b_{j_1} b_{j_2} \ldots b_{j_d} \\ &= d! \sum_{i=m}^{n} b_i \sum_{m \leqq j_1 \leqq \ldots \leqq j_{d-1} \leqq i} b_{j_1} b_{j_2} \ldots b_{j_{d-1}} \\ &\leqq d! \sum_{i=m}^{n} b_i \left(\sum_{j=m}^{i} b_j\right)^{d-1}.\end{aligned}$$

For the proof of (b) we assume first that (ii) holds. Setting $n_1 := 0$ and using (a), property (i) follows from

$$\begin{aligned}\sum_{k=1}^{\infty} a_{n_k}^{-1} \left(\sum_{i=1}^{n_k} b_i\right)^d &\leqq d! \sum_{j=1}^{\infty} b_j \left(\sum_{i=1}^{j} b_i\right)^{d-1} \sum_{n_k \geqq j} a_{n_k}^{-1} \\ &\leqq d! \sum_{j=1}^{\infty} b_j \left(\sum_{i=1}^{j} b_i\right)^{d-1} a_j^{-1} (1-M^{-1})^{-1}.\end{aligned}$$

Assume now that (i) holds. Then (ii) follows from

$$\begin{aligned}\sum_{n=n_1+1}^{\infty} a_n^{-1} b_n \left(\sum_{i=1}^{n} b_i\right)^{d-1} &= \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} a_n^{-1} b_n \left(\sum_{i=1}^{n} b_i\right)^{d-1} \\ &\leqq \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_k} M^6 a_{n_{k+1}}^{-1} b_n \left(\sum_{i=1}^{n_{k+1}} b_i\right)^{d-1} \\ &\leqq M_6 \sum_{k=1}^{\infty} a_{n_{k+1}}^{-1} \left(\sum_{i=1}^{n_{k+1}} b_i\right)^d.\end{aligned}$$

Corollary 3.6. *Assume that there exists $d \geqq 1$ such that*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} P\{|X_n| > \varepsilon a_n\} < \infty \qquad (\varepsilon > 0)$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} a_n^{-2d} B_n^{d-1} E(X_n^2) < \infty.$$

Then we have

$$\lim_{n \to \infty} a_n^{-1} S_n = 0 \quad a.s.$$

Proof. By [16], Lemma 3.3 there exists a sequence (n_k) satisfying 3.4(i). Assume first that $d \in \mathbb{N}$. Applying Lemma 3.5 with $b_n := E(X_n^2)$ we see that (ii) implies condition (iii) of Corollary 3.4. Hence in this case the assertion follows from Corollary 3.4. If $p-1<d\leqq p$ for a $p\in\mathbb{N}$ then setting $I:=\{n\in\mathbb{N}\colon B_n>a_n^2\}$ the full assertion follows from the already shown special case by observing

$$\sum_{n\in\mathbb{N}\setminus I} a_n^{-2p} B_n^{p-1} E(X_n^2) \leqq \sum_{n=1}^{\infty} a_n^{-2d} B_n^{d-1} E(X_n^2)$$

$$\sum_{n\in I} a_n^{-2p+2} B_n^{p-2} E(X_n^2) \leqq \sum_{n=1}^{\infty} a_n^{-2d} B_n^{d-1} E(X_n^2).$$

Corollary 3.6 is slightly weaker then Egorov's original statement. Though his result can be deduced in full strength Corollary 3.4, we prefer the above somewhat simpler formulation which is just as good in all resonable applications. As a new application of Egorov's theorem we give now a strengthened and widely generalized version of a classical result of Brunk, Chung and Prohoroff (cf. [2], p. 333 or [10], p. 154). By more elementary methods a weaker result was obtained in Wittmann [17]. Again we need a real variable lemma:

Lemma 3.7. *Assume that $a_n<a_{n+1}$ for any $n\in\mathbb{N}$ and let (b_n) be a sequence in $[0,\infty[$. Then for any $p\in\mathbb{N}$*

$$\sum_{n=2}^{\infty} a_n^{-1}(a_n-a_{n-1})^{1-p} b_n^p<\infty$$

implies

$$\sum_{n=1}^{\infty} a_n^{-p} b_n \left(\sum_{i=1}^{n} b_i\right)^{p-1}<\infty.$$

Proof. Setting $q:=(p-1)^{-1}p$ and $a_0:=0$ Hölder's inequality yields

$$\sum_{i=1}^{n} b_i=\sum_{i=1}^{n}(a_i-a_{i-1})^{1/q} b_i (a_i-a_{i-1})^{-1/q}$$

$$\leqq\left(\sum_{i=1}^{n} a_i-a_{i-1}\right)^{1/q}\left(\sum_{i=1}^{n} b_i^p (a_i-a_{i-1})^{-p/q}\right)^{1/p} \leqq a_n^{1/q}\left(\sum_{i=1}^{n}(a_i-a_{i-1})^{1-p} b_i^p\right)^{1/p}. \quad (1)$$

Let now $M>1$ be given. Then by [16], Lemma 3.3 there exists a sequence (n_k) with

$$M\, a_{n_k} \leqq a_{n_{k+1}} \leqq M^3\, a_{n_k+1} \qquad (k\in\mathbb{N}).$$

Applying Lemma 3.5 with $d=1$ our assumption yields

$$\sum_{k=1}^{\infty} a_{n_k}^{-1} \sum_{i=1}^{n_k}(a_i-a_{i-1})^{1-p} b_i^p<\infty$$

whence taking (1) into account we get

$$\sum_{k=1}^{\infty} a_{n_k}^{-p}\left(\sum_{i=1}^{n_k} b_i\right)^p \leqq \sum_{k=1}^{\infty} a_{n_k}^{-p} a_{n_k}^{p/q} \sum_{i=1}^{n_k}(a_i-a_{i-1})^{1-p} b_i^p<\infty.$$

Applying Lemma 3.5 once more with $d=p$ the assertion follows.

Corollary 3.8. *Let $p \geqq 1$ and assume that*

(i) $$a_n < a_{n+1} \quad (n \in \mathbb{N})$$

(ii) $$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon a_n\} < \infty \quad (\varepsilon > 0)$$

(iii) $$\sum_{n=2}^{\infty} a_n^{-2}(a_n^2 - a_{n-1}^2)^{1-p}(E(X_n^2))^p < \infty.$$

Then we have

$$\lim_{n\to\infty} a_n^{-1} S_n = 0 \quad a.s.$$

Proof. By Lemma 3.7 condition 3.6(ii) follows from 3.8(iii) if $p \in \mathbb{N}$. For the proof of the general case let $d-1 < p \leqq d$ with $d \in \mathbb{N}$ and set $I := \{n \geqq 2\colon a_n^2 - a_{n-1}^2 < E(X_n^2)\}$. Then the assertion follows from the special case by observing

$$\sum_{n\in\mathbb{N}\setminus I} a_n^2(a_n^2 - a_{n-1}^2)^{1-d} E(X_n^2)^d \leqq \sum_{n=2}^{\infty} a_n^2(a_n^2 - a_{n-1}^2)^{1-p} E(X_n^2)^p$$

$$\sum_{n\in I} a_n^2(a_n^2 - a_{n-1}^2)^{1-(d-1)} E(X_n^2)^{d-1} \leqq \sum_{n=2}^{\infty} a_n^2(a_n^2 - a_{n-1}^2)^{1-p} E(X_n^2)^p.$$

The next corollary is a wide generalization of the sufficiency part in Prohoroff's [9] law of large numbers.

Corollary 3.9. *Let (n_k) and $M > 1$, L, $C > 0$ be such that*

(i) $$M a_{n_k} \leqq a_{n_{k+1}} \leqq M^6 a_{n_k+1} \quad (k \in \mathbb{N})$$

(ii) $$|X_n| \leqq C(\log_2 a_n)^{-1} a_n \quad (n \in \mathbb{N})$$

(iii) $$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4LC} B_{n_{k-1},n_k}^{-1} a_{n_k}^2) < \infty$$

(iv) $$\sum_{k=1}^{\infty} (\log_+ a_{n_k})^{-L} < \infty.$$

Note that the last condition is always satisfied for $L > 1$ by (i).

Then we have

$$\limsup_{k\to\infty} a_{n_k}^{-1} \max_{n_{k-1} < m \leqq n_k} |S_m| \leqq \left(1 - \frac{1}{M}\right), \qquad \limsup_{n\to\infty} a_n^{-1} |S_n| \leqq \left(1 - \frac{1}{M}\right)^{-1} M^6 \quad a.s.$$

Proof. Setting $C_n := 2C(\log_2 a_n)^{-1}$ and defining $\kappa_{m,n}$ as in Theorem 3.3 we have to verify the assumptions of Theorem 3.3. Conditions 3.3(i), (ii) are obvious. Now if $\kappa_{n_{k-1},n_k} \leqq 2L \log_2 a_{n_k}$ then

$$\kappa_{n_{k-1},n_k} \geqq (B_{n_{k-1},n_k})^{-1} a_{n_k}^2 \exp(-C_n 2L \log_2 a_{n_k}) = e^{-4CL}(B_{n_{k-1},n_k})^{-1} a_{n_k}^2.$$

Hence

$$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2}\kappa_{n_{k-1},n_k}) \leqq \sum_{k=2}^{\infty} \exp(-L \log_2 a_{n_k}) + \sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4CL}(B_{n_{k-1},n_k})^{-1} a_{n_k}^2).$$

Since the last two sums are finite by (iii) and (iv) also condition 3.3(iii) follows and the proof is complete.

Corollary 3.10. *Let* (n_k) *and* $M>1$, L, $C>0$ *be such that condition* (i), (ii), (iv) *of Corollary* 3.9 *are satisfied. Instead of* (iii) *we assume the stronger condition*

$$\text{(iii')} \qquad \sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4LC} B_{n_k}^{-1} a_{n_k}^2) < \infty.$$

Then we have

$$\limsup_{n\to\infty} a_n^{-1} |S_n| \leqq M^6 \quad a.s.$$

Proof. We use the technique of Tomkins [15]. To this end we fix $p \in \mathbb{N}$ and define for any $0 \leqq i \leqq p$ a subsequence $n(i)_k := n_{kp+i}$ satisfying

$$M^p a_{n(i)_k} \leqq a_{n(i)_{k+1}} \leqq M^{6p} a_{n(i)_k+1} \qquad (k \in \mathbb{N})$$

$$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4LC} B_{n(i)_{k-1}, n(i)_k}^{-1} a_{n(i)_k}^2) \leqq \sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4LC} B_{n(i)_k}^{-1} a_{n(i)_k}^2) < \infty.$$

Thus the assumptions of Corollary 3.9 are fulfilled for the subsequences $(n(i)_k)$. Hence

$$\limsup_{k\to\infty} a_{n(i)_k}^{-1} \max_{n(i)_{k-1} < m \leqq n(i)_k} |S_m| \leqq (1 - M^p) \quad \text{a.s.}$$

Since for any $m \geqq n_{p+1}$ there exists $0 \leqq i \leqq p$, $k \geqq 1$ with $n(i)_k{}^{-1} < m \leqq n(i)_k$ we obtain from (i) that

$$\limsup_{m\to\infty} a_m^{-1} |S_m| \leqq \frac{M^6}{1 - M^{-6p}} \quad \text{a.s.}$$

Letting p tend to infinity the assertion follows.

In most of the above results there appear certain sequences (n_k). With the aid of the next lemma, which is an immediate consequence of Wittmann [18], Lemma 3.2 one can nearly always remove these subsequences in the assumptions.

Lemma 3.11. *If* $M := \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty$ *then for any* $\varepsilon > 0$ *and any sequence* (b_n) *in* $\mathbb{R}_+$ *satisfying*

$$\sum_{n=2}^{\infty} a_n^{-1} (a_n - a_{n-1}) b_n < \infty$$

there exists a subsequence (n_k) *such that*

$$\sum_{k=1}^{\infty} b_{n_k} < \infty$$

$$(1+\varepsilon) a_{n_k} \leqq a_{n_{k+1}} \leqq (1+\varepsilon)^6 M^3 a_{n_k} \qquad (k \in \mathbb{N}).$$

The essential difference between 3.11 and 2.3 is that the sequence (b_n) need no more be increasing.

As an example of applicability we give the following (slightly weaker) variant of Prohoroff's law of large numbers.

Corollary 3.12. *Assume that there exists $C>0$ such that*

(i) $$|X_n| \leqq C(\log_2 n)^{-1} n \qquad (k \in \mathbb{N})$$

(ii) $$\sum_{n=1}^{\infty} n^{-1} \exp(-\varepsilon n^2 (B_{2n} - B_n)^{-1}) < \infty \qquad (\varepsilon > 0)$$

Then we have

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s.$$

Proof. Let $\varepsilon>0$ be given. Then by Lemma 3.11 and (ii) there exists a subsequence (m_k) such that

$$\sum_{k=1}^{\infty} \exp(-\varepsilon^2 m_k^2 (B_{2m_k} - B_{m_k})^{-1}) < \infty \tag{1}$$

$$\sqrt[6]{\tfrac{3}{2}}\, m_k \leqq m_{k+1} \leqq \tfrac{3}{2} m_k \qquad (k \in \mathbb{N}). \tag{2}$$

Setting $n_k = 2m_k$ we get from (2) that $n_k - n_{k-1} \leqq m_{k-1} < m_k$ and therefore $B_{n_k} - B_{n_{k-1}} \leqq B_{2m_k} - B_{m_k}$. Hence (1) implies

$$\sum_{k=1}^{\infty} \exp(-\varepsilon^2 n_k^2 (B_{n_k} - B_{n_{k-1}})^{-1}) < \infty.$$

Setting $L := 2$ and $a_n := \varepsilon n 2 e^{4LC}$ we see that 3.9 (iii) is satisfied. Obviously (ii) holds and (i) follows from (2). Since for $L>1$ condition 3.9 (iv) is always satisfied we obtain

$$\limsup_{n\to\infty} \frac{S_n}{n} \leqq \varepsilon 2 e^{4LC} \left(1 - \frac{1}{M}\right)^{-1} M^6 \quad \text{a.s.}$$

Making $\varepsilon>0$ arbitrarily small the assertion follows.

4. The Law of the Iterated Logarithm

The two main streams are now flowing together in our main

Theorem 4.1. *Assume that for any $\varepsilon>0$ there exists $d>0$ such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-1} (\log_2 B_n)^{-d} \Gamma_n(\varepsilon) L_n^d(\varepsilon) < \infty$$

(ii) $$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon \sqrt{B_n \log_2 B_n}\} < \infty$$

(iii) $$\sum_{n=1}^{\infty} B_n^{-1} (\log_+ B_n)^{\varepsilon - 1} E(X_n^2) = \infty.$$

Then the law of the iterated logarithm holds, i.e.

$$\limsup_{n\to\infty} (2B_n \log_2 B_n)^{-1/2} S_n = 1 \quad a.s.$$

Addendum to Theorem 4.1. *If* 4.1 (i), (ii) *and the law of the iterated logarithm hold, then* (iii) *must also hold.*

Proof. Let $0<\varepsilon<1$ be given and use (i) to choose $d>0$ such that

$$\sum_{i=1}^{\infty} B_i^{-1}(\log_2 B_i)^{-d-1}\Gamma_i((1+\varepsilon)^{-3}\varepsilon)\,L_i^d((1+\varepsilon)^{-3}\varepsilon)<\infty. \tag{1}$$

We denote for any $n\in\mathbb{N}$

$$\begin{aligned} a_n &:= \sqrt{2B_n\log_2 B_n} \\ A_n &:= \{|X_n|>\varepsilon\sqrt{B_n/\log_2 B_n}\} \\ \check{X}_n &:= I_{A_n}X_n - E(I_{A_n}X_n) \\ \hat{X}_n &:= X_n-\check{X}_n \\ M &:= 1+\varepsilon \end{aligned}$$

and define $\check{S}_n$ and $\hat{S}_n$ as usual. By [16], Lemma 3.3 there exists a sequence (n_k) such that

$$M\,a_{n_k}\leqq a_{n_{k+1}}\leqq M^3\,a_{n_k+1} \qquad (k\in\mathbb{N}). \tag{2}$$

Since $\frac{a_n}{a_m}\leqq\frac{B_n}{B_m}\leqq\frac{a_n^2}{a_m^2}$ for any $m\leqq n$, we get from (2) that

$$M B_{n_k}\leqq B_{n_{k+1}}\leqq M^6\,B_{n_k+1} \qquad (k\in\mathbb{N}). \tag{3}$$

Denoting

$$\check{B}_{m,n} := \sum_{i=m+1}^{n} E(\check{X}_i^2)\leqq\sum_{i=m+1}^{n}\Gamma_i(\varepsilon)$$

observing

$$B_m^{-1}\sum_{i=n_{k-1}+1}^{m}\Gamma_i(\varepsilon)\leqq(1+\varepsilon)^6\,L_m((1+\varepsilon)^{-3}\varepsilon) \qquad (n_{k-1}<m\leqq n_k)$$

and using Lemma 3.5 (a) we obtain from (1)

$$\begin{aligned} \sum_{k=2}^{\infty} a_{n_k}^{-2(d+1)}\check{B}_{n_{k-1},n_k}^{d+1} &\leqq \sum_{k=2}^{\infty} a_{n_k}^{-2(d+1)}(d+1)!\sum_{i=n_{k-1}+1}^{n_k} E(\check{X}_i^2)\,\check{B}_{n_{k-1},i}^d \\ &\leqq (1+\varepsilon)^{6(d+1)}(d+1)!\sum_{i=1}^{\infty} a_i^{-2(d+1)}\Gamma_i(\varepsilon)((1+\varepsilon)^6 B_i L_i((1+\varepsilon)^{-3}\varepsilon))^d \\ &\leqq K\sum_{i=1}^{\infty} B_i^{-1}(\log_2 B_i)^{-d-1}\Gamma_i((1+\varepsilon)^{-3}\varepsilon)\,L_i^d((1+\varepsilon)^{-3}\varepsilon)<\infty. \end{aligned}$$

Hence all the assumptions of Corollary 3.4 are fulfilled and we obtain

$$\lim_{n\to\infty} a_n^{-1}\check{S}_n=0 \quad \text{a.s.} \tag{4}$$

By the definition of $\hat{X}_n$ we have for any $n\in\mathbb{N}$

$$|\hat{X}_n|\leqq 2\varepsilon\sqrt{B_n/\log_2 B_n}\leqq 4\varepsilon(\log_2 a_n)^{-1}a_n. \tag{5}$$

Defining $\hat{B}_{m,n}$ as usual, setting $C := 4\varepsilon$, $L := 2$ and $\tilde{a}_n := (1+\varepsilon) e^{2LC} a_n$ we have

$$\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2} e^{-4LC} \hat{B}_{n_k}^{-1} \tilde{a}_{n_k}^2) \leqq \sum_{k=2}^{\infty} \exp(-(1+\varepsilon)^2 \log_2 B_{n_k}) < \infty.$$

Hence condition (iii') of Corollary 3.10 is satisfied for $(\hat{X}_n)$, L, C, M. Condition 3.9 (ii) follows from (5) and 3.9 (i) is obvious. Finally as already remarked 3.9 (iv) is always satisfied when $L > 1$. Now Corollary 3.10 entails

$$\limsup_{n\to\infty} \tilde{a}_n^{-1} |\hat{S}_n| \leqq M^6 \quad \text{a.s.}$$

and therefore

$$\limsup_{n\to\infty} a_n^{-1} |\hat{S}_n| \leqq (1+\varepsilon) e^{16\varepsilon} (1+\varepsilon)^6 \quad \text{a.s.} \tag{6}$$

Together with (4) and $\limsup_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} |E(I_{A_i} X_i)| \leqq \lim_{n\to\infty} a_n^{-1} \sqrt{B_n} = 0$ we obtain

$$\limsup_{n\to\infty} a_n |S_n| \leqq 1 \quad \text{a.s.} \tag{7}$$

by making $\varepsilon > 0$ arbitrarily small. We now assume that $\varepsilon > 0$ was chosen small enough such that

$$e^{16\varepsilon} (1+\varepsilon)^7 \leqq 2. \tag{8}$$

Using again [16], Lemma 3.3 we can find a sequence (m_k) such that

$$\varepsilon^{-1} B_{m_k} \leqq B_{m_{k+1}} \leqq \varepsilon^{-3} B_{m_k+1} \qquad (k \in \mathbb{N}).$$

Because of $a_{m_{k-1}} \leqq \varepsilon a_{m_k}$ and (6), (8) we have

$$\limsup_{k\to\infty} a_{m_k}^{-1} |\hat{S}_{m_{k-1}}| \leqq 2\varepsilon \quad \text{a.s.} \tag{9}$$

From Lemma 2.2 we get

$$\limsup_{k\to\infty} a_{m_k}^{-1} \hat{S}_{m_{k-1}, m_k} \geqq (1-\varepsilon)^3 \quad \text{a.s.}$$

Together with (9) this implies

$$\limsup_{k\to\infty} a_{m_k}^{-1} \hat{S}_{m_k} \geqq (1-\varepsilon)^3 - 2\varepsilon \quad \text{a.s.}$$

Letting $\varepsilon > 0$ tend to 0 and recalling (4) we obtain

$$\limsup_{n\to\infty} a_n^{-1} S_n \geqq 1.$$

Together with (7) the assertion now follows.

Proof of the Addendum. Assume that there exists $0 < \delta < 1$ such that

$$\sum_{n=1}^{\infty} B_n^{-1} (\log_+ B_n)^{\delta-1} E(X_n^2) < \infty. \tag{10}$$

We choose N, $M>1$ and (n_k) such that (2), (3) and

$$M^6<N<(1-\delta)^{-1/2} \tag{11}$$

hold. By Lemma 2.3, (10) is equivalent to

$$\sum_{k=1}^{\infty} (\log_+ B_{n_k})^{\delta-1}<\infty.$$

Setting $L:=1-\delta$ and $\bar{a}_n:=N^{-1}\bar{a}_n$ we have then also

$$\sum_{k=1}^{\infty} (\log_+ \bar{a}_{n_k})^{-L}<\infty. \tag{12}$$

We now choose $\varepsilon>0$ so small such that

$$e^{-16LN\varepsilon}N^{-2}\geqq 1-\delta.$$

Defining $\hat{X}_n$ as above and setting $C:=4N\varepsilon$ we get

$$\begin{aligned}\sum_{k=2}^{\infty} \exp(-\tfrac{1}{2}e^{-4LC}\hat{B}_{n_k}^{-1}\bar{a}_{n_k}^2) &\leqq \sum_{k=2}^{\infty} \exp(-e^{-16LN\varepsilon}N^{-2}\log_2 B_{n_k})\\ &\leqq \sum_{k=2}^{\infty} (\log_+ B_{n_k})^{\delta-1}<\infty.\end{aligned}$$

Hence condition (iii′) of Corollary 3.10 is satisfied for $(\hat{X}_n)$, $(\bar{a}_n)$, C, L, M. Condition 3.9(iv) was shown in (12) and 3.9(i) is trivial. Finally condition 3.9(ii) follows as in the proof of the theorem. From Corollary 3.10 we get

$$\limsup_{n\to\infty} \bar{a}_n^{-1}\hat{S}_n\leqq M^6<N \quad \text{a.s.}$$

Together with (4) we obtain

$$\limsup_{n\to\infty} a_n^{-1}S_n\leqq M^6 N^{-1}<1 \quad \text{a.s.}$$

Hence the law of the iterated logarithm cannot hold and the proof of the Addendum is complete.

Remark. If we would replace condition 4.1(i) by the stronger and less natural condition

$$\text{(i}'\text{)} \qquad \sum_{n=1}^{\infty} (B_n\log_2 B_n)^{-d-1}\Gamma_n(\varepsilon)\left(\sum_{i=1}^{n}\Gamma_i(\varepsilon)\right)^d<\infty$$

then the proof would be much simpler. A slightly weaker condition appears in the work of Egorov [5]. But in this paper also the restrictive condition (*) discussed in the introduction was imposed to show the law of the iterated logarithm. In [2], p. 345 Egorov's condition is replaced by a twice truncated moment condition. In this case condition (*) cannot be removed completely but the following much weaker condition would be sufficient

$$(**) \qquad \lim_{n\to\infty} B_n^{-1}\sum_{i=1}^{n} E(I_{\{|X_i|>\varepsilon\sqrt{B_n\log_2 B_n}\}}X_i^2)=0 \qquad (\varepsilon>0).$$

This can be shown by using the arguments of Sect. 2 for the "$\geqq$" part and the arguments of [2] for the "$\leqq$" part.

Since $\Gamma_n(\varepsilon) \leqq \varepsilon^{1-\frac{p}{2}} B_n^{1-\frac{p}{2}} (\log_2 B_n)^{\frac{p}{2}-1} E(|X_n|^p)$ for any $p>2$ and since $L_n(\varepsilon) \leqq 1$ we obtain the following

Corollary 4.2. *Assume that there exists* $p>2$ *and* $d>0$ *such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-p/2} (\log_2 B_n)^{-d} E(|X_n|^p) < \infty$$

(ii) $$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon \sqrt{B_n \log_2 B_n}\} < \infty \qquad (\varepsilon > 0)$$

(iii) $$\sum_{n=1}^{\infty} B_n^{-1} (\log_+ B_n)^{\varepsilon-1} E(X_n^2) = \infty \qquad (\varepsilon > 0).$$

Then the law of the iterated logarithm holds.

Remark. The above Corollary seems to be sharp in the following sense: For any sequence (a_n) such that

$$\lim_{n\to\infty} a_n^{-2} B_n^2 (\log_2 B_n)^d = 0$$

for any $d>0$. Then there exists a sequence (X_n) of independent random variables such that

$$E(X_n)=0, \qquad E(X_n^2)=1 \qquad (n \in \mathbb{N}) \tag{1}$$

$$\sum_{n=1}^{\infty} a_n^{-p} E(|X_n|^p) < \infty \tag{2}$$

$$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon \sqrt{B_n \log_2 B_n}\} < \infty \qquad (\varepsilon > 0) \tag{3}$$

$$\limsup_{n\to\infty} (B_n \log_2 B_n)^{-1/2} S_n = \infty \quad \text{a.s.} \tag{4}$$

A proof should go along the lines of [6], Sect. 5.

Since

$$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon \sqrt{B_n \log_2 B_n}\} \leqq \sum_{n=1}^{\infty} \varepsilon^{-p} (B_n \log_2 B_n)^{-p/2} E(|X_n|^p)$$

we obtain the following generalization of [16], Theorem 1.2

Corollary 4.3. *Assume that there exist* $p>2$ *such that*

(i) $$\sum_{n=1}^{\infty} B_n^{-p/2} (\log_2 B_n)^{-p/2} E(|X_n|^p) < \infty$$

(ii) $$\sum_{n=1}^{\infty} B_n^{-1} (\log_+ B_n)^{\varepsilon-1} E(X_n^2) = \infty \qquad (\varepsilon > 0).$$

Then the law of the iterated logarithm holds.

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