Probab. Th. Rel. Fields 75, 499–507 (1987)

Probability Theory and Related Fields

Decoupling of Banach-Valued Multilinear Forms in Independent Symmetric Banach-Valued Random Variables

Terry R. McConnell[1]* and Murad S. Taqqu[2]**

[1] Syracuse University, 200 Carnegie, Syracuse, NY 13244-1150, USA
[2] Boston University, 111 Cummington Street, Boston, MA 02215, USA

Summary. Let E be a Banach space and $\Phi: E \to \mathbb{R}_+$ be symmetric, continuous and convex. Let $\{U_i\}$ and $\{r_i\}$ be independent sequences of random variables having, respectively, $U(0,1)$ and symmetric Bernoulli distributions, and let $\{U_i^{(j)}\}$ and $\{r_i^{(j)}\}$ for $j=1,2,\ldots,d$ be independent copies of these sequences. We prove two-sided inequalities between the quantities

$$E\Phi\Big(\sum_{\underline{i}\in\mathbb{Z}_+^d} r_{i_1}\ldots r_{i_d} F_{\underline{i}}(U_{i_1},\ldots,U_{i_d})\Big)$$

and their "decoupled" versions

$$E\Phi\Big(\sum_{\underline{i}\in\mathbb{Z}_+^d} r_{i_1}^{(1)}\ldots r_{i_d}^{(d)} F_{\underline{i}}(U_{i_1}^{(1)},\ldots,U_{i_d}^{(d)})\Big),$$

for Bochner integrable $F_{\underline{i}}: [0,1]^d \to E$. This generalizes results of Kwapień and of Zinn.

1. Introduction

Decoupling inequalities for convex moments were introduced by the authors in [8] (see also [10]) as a tool in the study of multiple stochastic integration, and were extended in [9]. A number of further extensions and related results are now known ([2, 4, 5, 11]). In the present paper we prove a decoupling inequality which has direct application to the study of stochastic integration of vector-valued integrands. The proof is based on a convexity result (Lemma 2.2) which may be of independent interest.

A nonnegative convex function Φ defined on $\mathbb{R}$ or, more generally, on a Banach space E, is said to satisfy Δ_2 if there is a constant β such that $\Phi(2x) \leqq \beta\Phi(x)$, for each x in E. Let X_i be independent symmetric random

* Supported in part by NSF grant DMS 85-03775 and by the Sloan Foundation
** Supported in part by NSF grant ECS 84-08524

variables and let $\{X_i^{(j)}\}_{i=1}^{\infty}$ for $j=1,2,\ldots,d$ be independent copies of this sequence. Let $a_{\underline{i}}$, $\underline{i}\in\mathbb{Z}_+^d$ be an array of real numbers such that $a_{\underline{i}}=a_{\underline{j}}$ whenever $\underline{i}$ is a permutation of $\underline{j}$. Also assume that all but finitely many $a_{\underline{i}}$ vanish and that $a_{\underline{i}}$ vanishes when any pair of indices in $\underline{i}$ agree. Let $|\ |_\Phi$ denote any Orlicz norm on $L_\Phi(\Omega)$. In the case $E=\mathbb{R}$, it is known that if Φ satisfies Δ_2 then one has

$$|\sum a_{\underline{i}} X_{i_1}\ldots X_{i_d}|_\Phi \approx |\sum a_{\underline{i}} X_{i_1}^{(1)}\ldots X_{i_d}^{(d)}|_\Phi. \tag{1.1}$$

(Unless otherwise specified, all sums are over $\mathbb{Z}_+^d$.) Here and elsewhere in this paper two positive quantities are related by $\approx$ and said to be comparable if their ratio is bounded below and above by positive finite constants. The right-hand inequality in (1.1) was proved in [9]. For the left-hand inequality see [4].

The restriction that Φ satisfy Δ_2 rules out many interesting choices of Φ, for example $\Phi(x)=e^{|x|}$. The following result of Kwapień removes this restriction and, further, allows the $a_{\underline{i}}$ to take values in a Banach space E. It is convenient to introduce the notations $X_{\underline{i}}$ for $X_{i_1}X_{i_2}\ldots X_{i_d}$ and $\hat{X}_{\underline{i}}$ for $X_{i_1}^{(1)}X_{i_2}^{(2)}\ldots X_{i_d}^{(d)}$.

Theorem 1.1 ([4]). *Suppose Φ: $E\to\mathbb{R}_+$ is continuous, convex and satisfies $\Phi(x)=\Phi(-x)$. Let X_i, $X_i^{(j)}$ be real-valued symmetric random variables as above, and $a_{\underline{i}}$ as above but E-valued. Then*

$$E\Phi\left(d^{-d}\sum a_{\underline{i}}\hat{X}_{\underline{i}}\right)\leqq E\Phi\left(\sum a_{\underline{i}}X_{\underline{i}}\right)\leqq E\Phi\left(\frac{d^{3d}}{d!}\sum a_{\underline{i}}\hat{X}_{\underline{i}}\right). \tag{1.2}$$

We shall use only the special case of this theorem in which the X_i are the Rademacher functions r_i (independent symmetric Bernoulli random variables).

The following theorem is the main result of the present paper.

Theorem 1.2. *Let $F_{\underline{i}}$: $[0,1]^d\to E$, $\underline{i}\in\mathbb{Z}_+^d$, be Bochner integrable, symmetric under interchange of their arguments and vanish on the diagonals of $[0,1]^d$. Also assume $\underline{i}\to F_{\underline{i}}$ is symmetric. To avoid convergence questions, assume that all but finitely many $F_{\underline{i}}$ are identically zero. Let U_i be i.i.d. $U(0,1)$ random variables and $\{U_i^{(j)}\}$ be independent copies of this sequence. Then, with Φ as above,*

$$\begin{aligned} E\Phi\left(d^{-2d}\sum \hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1}^{(1)},\ldots,U_{i_d}^{(d)})\right) &\leqq E\Phi\left(\sum r_{\underline{i}}F_{\underline{i}}(U_{i_1},\ldots,U_{i_d})\right)\\ &\leqq E\Phi\left(\frac{d^{4d}}{d!}\sum \hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1}^{(1)},\ldots,U_{i_d}^{(d)})\right). \end{aligned} \tag{1.3}$$

This result with different constants was obtained by J. Zinn [11] when $E=\mathbb{R}$ and $\Phi(x)=|x|^p$. His proof works also in the non-convex cases $0<p<1$, but it does not generalize to the Banach-valued case.

Various choices of $F_{\underline{i}}$ in (1.3) lead to various generalizations of Theorem 1.1. For example, to extend Theorem 1.1 to the case in which the $a_{\underline{i}}$ and $X_{\underline{i}}$ take values in a commutative Banach *algebra E*, choose g_i: $[0,1]\to E$ such that $g_i(U_i)$ has the same distribution as X_i and use

$$F_{\underline{i}}(x_1,\ldots,x_d)=a_{\underline{i}}\,g_{i_1}(x_1)\,g_{i_2}(x_2)\ldots g_{i_d}(x_d).$$

(The same result holds in a noncommutative algebra if each $F_{\underline{i}}$ is symmetrized.)

All of our results translate directly into statements about multiple stochastic integrals of vector-valued integrands with respect to symmetric, infinitely divisible processes. This is because work of Le Page [6], and more recently Pisier and Marcus [7], implies that such integrals, whenever they exist, may be represented as limits of expressions of the sort occurring in (1.3).

We should emphasize here that the symmetrizing presence of the Rademacher functions is essential to our methods. In some special situations this represents no loss in generality, for the Rademacher functions may be inserted by means of martingale transform inequalities. For example, see Proposition 4.2 below and the discussion preceeding it. On the other hand, there are interesting decoupling phenomena which are not covered by our methods, most notably, the decoupling of random Fourier quadratic forms. For further insight on the effects of asymmetry see the example on p. 965 of [2].

The proof of Theorem 1.2 is given in Sect. 3 and is based on a convexity result which is proved in the next section.

In Sect. 4 we consider results analogous to (1.3) in which the summations extend only over the tetrahedron $T_+^d = \{\underset{\sim}{i} \in \mathbb{Z}_+^d : i_1 < i_2 < \ldots < i_d\}$. An example, due to J. Bourgain, is given which shows that such results do not hold for general E. Some additional, geometric, conditions on E are necessary. We recall two such conditions which are sufficient to give the desired result.

2. The Convexity Lemma

Throughout this section Ω denotes a fixed *finite* sample space on which, unless otherwise specified, all random variables under consideration are defined. Also E denotes a fixed finite dimensional topological vector space over $\mathbb{R}$, and Φ: $E \to \mathbb{R}_+$ a convex function. We need the following simple result about convex functions.

Lemma 2.1. *Fix $a, b \in E$ and let $g(\lambda) = E\Phi(a + \lambda r_1 b)$ where r_1 denotes a Rademacher function. Then g is nondecreasing on $\mathbb{R}_+$.*

Proof. Let $0 \leqq \lambda_1 < \lambda_2$ and put $p = \lambda_1/\lambda_2$, $p + q = 1$. Then $g(\lambda_1) \leqq p g(\lambda_2) + q g(0)$. But by Jensen's inequality $g(0) \leqq g(\lambda_2)$, hence $g(\lambda_1) \leqq g(\lambda_2)$.

Lemma 2.2. *Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_d)$ be a vector of mean 0 E-valued random variables and $\underset{\sim}{X}' = (X_1', \ldots, X_d')$ a vector with the same marginal distributions as $\underset{\sim}{X}$, but whose components are independent. Let $r_1, \ldots, r_d$ be a Rademacher sequence which is independent of $\underset{\sim}{X}$ and $\underset{\sim}{X}'$. Let Ψ: $E^{(d)} \to \mathbb{R}$ be convex. Then*

$$E\Psi\left(\frac{r_1 X_1'}{d}, \ldots, \frac{r_d X_d'}{d}\right) \leqq E\Psi(r_1 X_1, \ldots, r_d X_d) \tag{2.1}$$

and

$$E\Psi(X_1, \ldots, X_d) \leqq E\Psi(dX_1', \ldots, dX_d'). \tag{2.2}$$

Proof. Let π_1 denote the cyclic permutation of $\{1, 2, \ldots, d\}$, and $\pi_2, \ldots, \pi_d$ the successive powers of π_1 (π_d is the identity). Let $\underset{\sim}{X}^1, \underset{\sim}{X}^2, \ldots, \underset{\sim}{X}^d$ be independent copies of $\underset{\sim}{X}$. Set

$$\underline{W}_i=(X_1^{\pi_i(1)},\dots,X_k^{\pi_i(k)},\dots,X_d^{\pi_i(d)})$$

and note that each $\underline{W}_i$ has the same distribution as $\underline{X}'$. Thus, since $\underline{X}$ has mean zero, we have by Jensen's inequality,

$$E\Psi(\underline{X})\leqq E\Psi\left(\sum_{k=1}^{d}\underline{X}^k\right)=E\Psi\left(\sum_{i=1}^{d}\underline{W}_i\right)\leqq\frac{1}{d}\sum_{i=1}^{d}E\Psi(d\underline{W}_i)=E\Psi(d\underline{X}'),$$

which is (2.2).

To obtain (2.1) let $\underline{r}^1,\underline{r}^2,\dots,\underline{r}^d$ be Rademacher sequences which are independent of all other variables under consideration, and, in particular, independent of $\underline{r}$. Then

$$\begin{aligned}E\Psi\left(\frac{r_1X_1'}{d},\dots,\frac{r_dX_d'}{d}\right)&=E\Psi\left(\frac{1}{d}r_1^{\pi_1(1)}X_1^{\pi_1(1)},\dots,\frac{1}{d}r_d^{\pi_1(d)}X_d^{\pi_1(d)}\right)\\&\leqq E\Psi\left(\frac{1}{d}\sum_{i=1}^{d}r_i\,r_1^{\pi_i(1)}X_1^{\pi_i(1)},\dots,\frac{1}{d}\sum_{i=1}^{d}r_i\,r_d^{\pi_i(d)}X_d^{\pi_i(d)}\right)\\&=E\Psi\left(\frac{1}{d}\sum_{l=1}^{d}r_{i(l,1)}r_1^lX_1^l,\dots,\frac{1}{d}\sum_{l=1}^{d}r_{i(l,d)}r_d^lX_d^l\right),\end{aligned}$$

where $i(l,j)$ is defined by $i(l,j)=i$ if $\pi_i(j)=l$. For each fixed l the sequence $r_{i(l,j)}r_j^l$, $j=1,2,\dots,d$, has the same distribution as $\underline{r}$ and hence, for each l the sequence $\{r_{i(l,j)}r_j^lX_j^l\}$ has the same distribution as $\{r_jX_j\}$. Thus the expression above is dominated by

$$\frac{1}{d}\sum_{l=1}^{d}E\Psi(r_{i(l,1)}r_1^lX_1^l,\dots,r_{i(l,d)}r_d^lX_d^l)=E\Psi(r_1X_1,\dots,r_dX_d),$$

and the proof is complete.

3. Proof of Theorem 1.2

Suppose first that each $F_{\underline{i}}$ assumes only finitely many values. (We shall remove this restriction at the end of the section.) To obtain the left-hand inequality in (1.3) it is sufficient, by (1.2) and the independence of the r_i and U_i, to prove that

$$E\Phi(d^{-d}\sum\hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1}^{(1)},\dots,U_{i_d}^{(d)}))\leqq E\Phi(\sum\hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1},\dots,U_{i_d})). \tag{3.1}$$

(Note the replacement of $r_{\underline{i}}$ by $\hat{r}_{\underline{i}}$ on the right-hand side.)

For each $l=1,2,\dots$ put $I_j(l)=\{\underline{i}:\ i_j=l\}$ and $I_0(l)=\{\underline{i}:\ i_k\neq l,\ k=1,\dots,d\}$. With l fixed, set

$$X_j=\sum_{\underline{i}\in I_j(l)}\hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1},\dots,U_{i_d}),\qquad j=0,\dots,d.$$

Thus X_0 comprises all those terms which do not involve U_l, and X_j comprises those terms which contain U_l in the "j^{th} slot" of their $F_{\underline{i}}$. Also, let

$$X_j'=\sum_{\underline{i}\in I_j(l)}\hat{r}_{\underline{i}}F_{\underline{i}}(U_{i_1},\dots,U_l^{(j)},\dots,U_{i_d}),\qquad j=1,\dots,d,$$

and let $\mathscr{J}$ denote the σ-field generated by all variables except $r_l^1, \ldots, r_l^d, U_l$, and $U_l^1, \ldots, U_l^d$. Note that, conditional on $\mathscr{J}$, X_0 is a constant vector in E and $(X_1, \ldots, X_d)$ and $(X_1', \ldots, X_d')$ are symmetric sequences of E-valued random variables which satisfy the hypotheses of Lemma 2.1. Applying (2.1) with

$$\Psi(y_1, \ldots, y_d) = \Phi(X_0 + y_1 + \ldots + y_d)$$

we obtain

$$E(\Phi(\sum \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_{i_d})) | \mathscr{J}) \geqq E\left(\Phi\left(X_0 + \frac{1}{d} \sum_{j=1}^{d} \sum_{I_j(l)} \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_l^{(j)}, \ldots, U_{i_d})\right) \middle| \mathscr{J}\right).$$

We now apply this argument to each l in turn. At each stage, those terms which do not appear in the corresponding X_0 are multiplied by $1/d$. But this will happen for a given term exactly d times, once for each argument of its $F_{\underline{i}}$. This proves (3.1). The opposite inequality is proved the same way, using (2.2) instead of (2.1).

There remains only to remove the restriction on the $F_{\underline{i}}$. Note that all expectations in (1.3) are well-defined (possibly infinite) since the integrands are nonnegative random variables. Let $\mathscr{S}_n$ denote the n^{th} dyadic σ-field of $[0,1]^d$, and put $F_{\underline{i}}^n = E(F_{\underline{i}} | \mathscr{S}_n)$ which is well-defined as an E-valued function on $[0,1]^d$ since each $F_{\underline{i}}$ is Bochner integrable. Put

$$Z_n = \sum \hat{r}_{\underline{i}} F_{\underline{i}}^n(U_{i_1}, \ldots, U_{i_d})$$

and

$$Z_n' = \sum \hat{r}_{\underline{i}} F_{\underline{i}}^n(U_{i_1}^{(1)}, \ldots, U_{i_d}^{(d)}).$$

Recalling that each of these sums is actually finite, we obtain from the Martingale Convergence Theorem that

$$\sum \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_{i_d}) = \lim_{n \to \infty} Z_n, \text{ a.s.,}$$

and a similar statement for $\lim_{n \to \infty} Z_n'$. Moreover, we have

$$E\Phi(\sum \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_{i_d})) = \lim_{n \to \infty} E\Phi(Z_n)$$

and

$$E\Phi(\sum \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}^{(1)}, \ldots, U_{i_d}^{(d)})) = \lim_{n \to \infty} E\Phi(Z_n').$$

(One inequality results from Fatou's lemma and the other from Jensen's inequality applied to the conditional expectations given $\mathscr{S}_n$.) The desired results now follow from the special cases proved above.

4. Inequalities for Tetrahedronal Sums

It is often useful to have inequalities analogous to (1.3) in which the summation extends only over the tetrahedron

$$T_+^d = \{\underline{i} \in \mathbb{Z}_+^d : i_1 < i_2 < \ldots < i_d\}.$$

Unfortunately, such inequalities do not hold for general E even if the F_i are constant. This was noticed by J. Bourgain, and we thank him for communicating to us the following example.

Example 4.1. On the space $l_n^2 \otimes l_n^2$ of tensors of rank 2 over $(l_n^2)^*$ we consider two norms, the "projective" norm defined by

$$\|x\|_\wedge = \inf \sum_{i,j} \|\alpha_i\| \, \|\beta_j\|,$$

where the infimum extends over all representations $x = \sum_{i,j} \alpha_i \otimes \beta_j$ as a finite sum, and the "injective" norm defined by

$$\|x\|_\vee = \sup |\alpha^T x \beta|,$$

where the supremum extends over $\alpha, \beta \in (l_n^2)^*$ having norm one. Here each tensor x is identified with an $n \times n$ matrix via its expansion, $x = \sum_{i,j \leqq n} x_{ij} e_i \otimes e_j$, in terms of the standard basis $e_1, e_2, \ldots, e_n$ of l_n^2. We shall need the inequality

$$|\operatorname{tr}(x^T y)| \leqq \|x\|_\vee \, \|y\|_\wedge \tag{4.1}$$

which follows easily from the definitions.

Let $\{r_i'\}$ be an independent copy of the Rademacher sequence. Put

$$x_{ij} = \begin{cases} e_i \otimes e_j, & i < j, \ i \text{ even}, \ j \text{ odd}, \\ e_j \otimes e_i, & i < j, \ i \text{ odd}, \ j \text{ even, and} \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$E \Big\| \sum_{i<j \leqq n} r_i r_j x_{ij} \Big\|_\wedge^2 = E \Big\| \Big(\sum_{\substack{i \text{ even}, \\ \leqq n}} r_i e_i \Big) \otimes \Big(\sum_{\substack{j \text{ odd}, \\ \leqq n}} r_j e_j \Big) \Big\|_\wedge^2$$

$$= \Big(E \Big\| \sum_{\substack{i \text{ even}, \\ \leqq n}} r_i e_i \Big\|_{l_n^2}^2 \Big) \Big(E \Big\| \sum_{\substack{j \text{ odd}, \\ \leqq n}} r_j e_j \Big\|_{l_n^2}^2 \Big) \leqq n^2.$$

On the other hand, conditioning on r_i, i even, we obtain

$$E \Big\| \sum_{i<j \leqq n} r_i r_j' x_{ij} \Big\|_\wedge^2 \geqq E \Big\| \sum_{\substack{i<j \leqq n, \\ i \text{ even}, \, j \text{ odd}}} r_i r_j' (e_i \otimes e_j) \Big\|_\wedge^2 .$$

Now it is straightforward to check that there is an absolute constant α such that

$$E \left\| \sum_{i \neq j \leqq n} r_i r_j' \frac{e_i \otimes e_j}{i - j} \right\|_\vee^2 \leqq \alpha^2.$$

Thus, by (4.1) and Schwartz's inequality,

$$\begin{aligned}
&\Big(E\Big\| \sum_{\substack{i<j\leqq n,\\ i \text{ even}, j \text{ odd}}} r_i r_j'(e_i \otimes e_j)\Big\|_{\wedge}^2\Big)^{1/2} \\
&\geqq \frac{1}{\alpha} E\left|\operatorname{tr}\left(\left[\sum_{i \neq j} r_i r_j' \frac{(e_i \otimes e_j)}{i-j}\right]^T \Big[\sum_{\substack{i<j\leqq n,\\ i \text{ even}, j \text{ odd}}} r_i r_j'(e_i \otimes e_j)\Big]\right)\right| \\
&= \frac{1}{\alpha}\Big|\sum_{\substack{i<j\leqq n,\\ i \text{ even}, j \text{ odd}}} 1/(i-j)\Big| \\
&\geqq c\, n \log n, \qquad \text{for some constant } c.
\end{aligned}$$

It follows from this that Theorem 1.1 (hence Theorem 1.2) does not hold with summation over T_+^2 when $E = l^2 \hat{\otimes} l^2$.

In the remainder of this section we give sufficient conditions on E for tetrahedronal analogues of Theorem 1.2 to hold true. Let $\Delta_2(E)$ denote the set of all convex $\Phi\colon E \to \mathbb{R}_+$ satisfying a Δ_2 condition, $\Phi(-x) = \Phi(x)$, and $\Phi(0) = 0$. For any map $a\colon \mathbb{Z}_+^d \to E$ we set

$$Q_n(a) = \sum_{\substack{\underline{i} \in T_+^d,\\ i_d \leqq n}} a(\underline{i})\, r_{\underline{i}},$$

and

$$\hat{Q}_n(a) = \sum_{\substack{\underline{i} \in T_+^d,\\ i_d \leqq n}} a(\underline{i})\, \hat{r}_{\underline{i}}.$$

Note that both Q_n and $\hat{Q}_n$ are martingales. Also let $\{\varepsilon_{\underline{i}}\}_{\underline{i} \in \mathbb{Z}_+^d}$ be a family of symmetric i.i.d. Bernoulli random variables, and set

$$\bar{Q}_n(a) = \sum_{\substack{\underline{i} \in T_+^d,\\ i_d \leqq n}} a(\underline{i})\, \varepsilon_{\underline{i}}.$$

Szulga and Krakowiak [3] say that E satisfies the multilinear contraction principle (MCP) if there exists $1 \leqq p < \infty$ and $d \geqq 2$ such that

$$\|Q_n(a)\|_p \approx \|\bar{Q}_n(a)\|_p \approx \|\hat{Q}_n(a)\|_p, \tag{4.2}$$

for all n and $a\colon \mathbb{Z}_+^d \to E$. Moreover, they show that (4.2) holding for one such p and d implies (4.2) for all such p and for all such d.

Appealing to results of Pisier, the authors of [3] show further that a sufficient condition for E to satisfy MCP is that E should have local unconditional structure (L.U.S.T.) and not contain copies of l_n^∞ uniformly (see [3] for definitions).

The symmetry assumptions on the $F_{\underline{i}}$ in Theorem 1.2 were necessary only in order to apply Kwapień's result, Theorem 1.1. The rest of the argument, i.e., the proof of the inequality in (3.1) and of the reverse inequality, made no use of these assumptions. Here, inequalities (4.2) replace the result of Theorem 1.1. Therefore we may deduce the following result.

Theorem 4.1. *Let E have L.U.S.T. and not contain copies of l_n^∞ uniformly. Then under the hypotheses of Theorem 1.2 we have for each* $1 \leqq p < \infty$

$$\| \sum_{\underline{i} \in T_+^d} r_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_{i_d}) \|_p \approx \| \sum_{\underline{i} \in T_+^d} \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}^{(1)}, \ldots, U_{i_d}^{(d)}) \|_p. \tag{4.3}$$

We conjecture that the same result holds with the L^p norms replaced by the norms $| \ |_\Phi$, $\Phi \in \Delta_2(E)$.

Another family of Banach spaces for which results like (4.3) hold is the class UMD introduced by Burkholder [1]. A Banach space E is UMD (for *u*nconditional *m*artingale *d*ifferences) if for some $1 < p < \infty$ there is a constant c such that for every E-valued martingale difference sequence d_n and $\varepsilon_i \in \{\pm 1\}$ we have

$$\left\| \sum_{i=1}^n \varepsilon_i d_i \right\|_p \leqq c \| \sum_i d_i \|_p.$$

This is equivalent to the statement that for every $\Phi \in \Delta_2(\mathbb{R})$ one has

$$E \sup_n \Phi \left(\left\| \sum_{i=1}^n \varepsilon_i d_i \right\|_E \right) \approx E \sup_n \Phi \left(\left\| \sum_{i=1}^n d_i \right\|_E \right) \tag{4.4}$$

with the constants in $\approx$ depending only on Φ (see [1]).

Proposition 4.2. *Suppose E is UMD. Then for $1 < p < \infty$ one has*

$$\|Q_n(a)\|_p \approx \|\hat{Q}_n(a)\|_p, \tag{4.5}$$

for all n and a: $\mathbb{Z}_+^d \to E$.

Proof. For convenience we give details only in the case $d = 2$. We prove a stronger version of the right inequality. Put $d_j = (\sum_{i<j} a(i,j) r_i) r_j$ and let $\Phi \in \Delta_2(\mathbb{R})$. Then by (4.4) and Lévy's inequality

$$\begin{aligned} E\Phi(\|Q_n(a)\|) &\leqq E \sup_{m \leqq n} \Phi(\| \sum_{j \leqq m} d_j \|) \leqq c E \sup_{m \leqq n} \Phi(\| \sum_{j \leqq m} r_j' d_j \|) \leqq c' E\Phi(\| \sum_{j \leqq n} r_j' d_j \|) \\ &= E\Phi(\|\hat{Q}_n(a)\|). \end{aligned}$$

The left-hand inequality for $\Phi(x) = |x|_E^p$, $1 < p < \infty$, is proved similarly, using Doob's inequality in the last step.

For the same reasons as above we obtain

Theorem 4.2. *Suppose E is UMD. Then under the same hypotheses as in Theorem 1.2 we have for* $1 < p < \infty$,

$$\| \sum_{\underline{i} \in T_+^d} r_{\underline{i}} F_{\underline{i}}(U_{i_1}, \ldots, U_{i_d}) \|_p \approx \| \sum_{\underline{i} \in T_+^d} \hat{r}_{\underline{i}} F_{\underline{i}}(U_{i_1}^1, \ldots, U_{i_d}^d) \|_p.$$

We conclude by remarking that neither of Theorems 4.2 and 4.1 includes the other. For example, the space l^1 satisfies the hypotheses of Theorem 4.1 but is not UMD, and the spaces C_p for $1 < p < \infty$ are UMD but do not have L.U.S.T. It is therefore of interest to know the precise class of spaces E for which the tetrahedronal analogues of Theorem 1.1 carry over.

Acknowledgment. The authors wish to thank S. Kwapień and J. Bourgain for sharing their ideas, and the referees for their careful reading and constructive criticism of the manuscript.

References

1. Burkholder, D.L.: A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. Ann. Probab. **9**, 997–1011 (1981)
2. Krakowiak, W., Szulga, J.: Random multilinear forms. Ann. Probab. **14**, 955–973 (1986)
3. Krakowiak, W., Szulga, J.: Contraction principle for Banach space valued random multilinear forms. In: Case Western Reserve University. Preprint #85-34 (1985)
4. Kwapień, S.: Decoupling inequalities for polynomial chaos. In: Case Western Reserve University. Preprint #85-34 (1985)
5. Kwapień, S., Woyczynski, W.A.: Decoupling of martingale transforms and stochastic integrals for processes with independent increments. In: Case Western Reserve University. Preprint #85-34 (1985)
6. Le Page, R.: Multidimensional infinitely divisible measures and processes, part II. Lect. Notes Math. **860**, 279–284 (1981)
7. Marcus, M.B., Pisier, G.: Infinitely divisible measures on the space of continuous functions induced by random Fourier series and transforms. In: Harmonic analysis and probability. New York, Basel: Marcel Dekker (to appear)
8. McConnell, T., Taqqu, M.: Double integration with respect to symmetric stable processes. Preprint (1984)
9. McConnell, T., Taqqu, M.: Decoupling inequalities for multilinear forms in independent symmetric random variables. Ann. Probab. **14**, 943–954 (1986)
10. McConnell, T., Taqqu, M.: Dyadic approximation of double integrals with respect to symmetric stable processes. Stoch. Proc. Appl. **22**, 323–331 (1986)
11. Zinn, J.: Comparison of martingale difference sequences. Lect. Notes Math. **1153**, 453–457 (1985)

Received January 21, 1986