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Non Zero-Sum Stopping Games of Symmetric Markov Processes

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Summary. A non zero-sum stopping game of a symmetric Markov process is investigated. A system of quasi-variational inequalites (QVI) is introduced in terms of Dirichlet forms and the existence of extremal solutions of the system of QVI is discussed. Nash equilibrium points of the stopping game are obtained from solutions of the system of QVI.

0. Introduction

Zero-sum stopping games (so called Dynkin games) of Markov processes have been studied by many authors, e.g., Bensoussan-Friedman [2], Bismut [4], Dynkin [5], Friedman [6], Krylov [8, 9], Lepeltier-Maingueneau [10], Morimoto [11], Stettner [15], etc. Above all in [2], [6] and [8, 9] it has been known that the value function of a Dynkin game can be identified as a solution of a certain variational inequality and the saddle point of the game is constructed from the solution. J. Zabczyk largely extended their results to symmetric Markov processes by using theory of Dirichlet forms (cf. [16, 17]).

On the other hand in the study of non zero-sum stopping game of a diffusion processes Bensoussan-Friedman introduced a system of quasi-variational inequalities (QVI) and showed that a Nash equilibrium point of the game can be constructed from a regular solution of the system of QVI (cf. [3]). Regularity problems however remained open so far.

In the present paper, treating with a non zero-sum stopping game of a symmetric Markov process, we introduce a system of QVI of obstacle types in terms of a Dirichlet space and show the existence of Nash equilibrium points of the game by establishing direct relationship between the system of QVI and the stopping game. We can dispense with regularity arguments of the solution of the system by using potential theory of Dirichlet forms and Markov processes which was useful for the studies of various stopping problems such as in [12–14] and [16, 17].

Our system of QVI is different from Bensoussan-Friedman's but equivalent to it (cf. §6). To introduce our system of QVI the notion of α-reduced functions is necessary. We devote §1 to introduction of the notion cited from [7]. Main results are in §2.

1. Preliminaries

In the present section we introduce a notion of α-reduced functions in terms of theory of Dirichlet forms and its probabilistic interpretation according to [7]. Then we will formulate variational inequalities related to optimal stopping problems, which we will utilize in the following section to introduce a system of quasi-variational inequalities concerning non zero-sum stopping games for symmetric Markov processes.

Let $m(dx)$ be a nonnegative everywhere dense Radon measure on a locally compact Hausdorff space X with a countable base. Suppose that $(\Omega, \mathscr{B}, \mathscr{B}_t, P_x, X_t)$ is an m-symmetric Hunt process on X whose Dirichlet space $(\mathscr{F}, \mathscr{E})$ relative to $L^2(m(dx))$ is regular. Let us take any α-excessive function $\psi \in \mathscr{F}$, and any set $B \subset X$. We set

$$L_{\psi, B} = \{v \in \mathscr{F}\,; \tilde{v} \geqq \tilde{\psi} \text{ q.e. on } B\},$$

where $\tilde{v}$ denotes a quasi-continuous modification of a function $v \in \mathscr{F}$ then it can be seen that $L_{\psi, B}$ admits a unique element $\psi_B \in L_{\psi, B}$ minimizing $\mathscr{E}_\alpha(w, w)$ on $L_{\psi, B}$. Here $\mathscr{E}_\alpha(w, w) = \mathscr{E}(w, w) + \alpha(w, w)_m$, $\alpha > 0$. This ψ_B satisfies the following inequality:

$$\mathscr{E}_\alpha(\psi_B, v) \geqq 0, \quad {}^\forall v \in \mathscr{F},\ \tilde{v} \geqq 0 \text{ q.e. on } B. \tag{1.1}$$

Moreover ψ_B has the following properties:

$$\psi_B \quad \text{is } \alpha\text{-excessive}, \tag{1.2}$$

$$\psi_B \leqq \psi \quad m\text{-a.e.}, \tag{1.3}$$

$$\tilde{\psi}_B = \tilde{\psi} \quad \text{q.e. on } B \tag{1.4}$$

(cf. [7]). ψ_B is called the α-reduced function of ψ on B. We note that α-reduced function $\psi_B(x)$ can be written as $\psi_B(x) = U_\alpha \mu_B(x)$ by a unique measure μ_B supported by $\bar{B}$, where $U_\alpha \mu$ is a function uniquely determined by equality $\mathscr{E}_\alpha(U_\alpha \mu, v) = \int v(x)\, \mu(dx)$ for $v \in \mathscr{F} \cap C_0$.

We can see α-reduced functions from another point of view. Let us set

$$\mathscr{F}_{X-B} = \{u \in \mathscr{F}\,; \tilde{u} = 0 \text{ q.e. on } B\}$$

for each Borel set B. $\mathscr{F}_{X-B}$ being a closed subspace of the Hilbert space $(\mathscr{F}, \mathscr{E}_\alpha)$ $\mathscr{F}$ admits a orthogonal decomposition

$$\mathscr{F} = \mathscr{F}_{X-B} + \mathscr{H}_\alpha^B$$

where we denote by $\mathscr{H}_\alpha^B$ the orthogonal complement of $\mathscr{F}_{X-B}$. We can see by (1.1) and (1.4) that

$$\psi=(\psi-\psi_B)+\psi_B \tag{1.5}$$

represents the orthogonal decomposition of the α-excessive function $\psi\in\mathscr{F}$. That is, α-reduced function ψ_B on the set B is the projection of ψ on the space $\mathscr{H}_\alpha^B$.

Taking the symmetric Markov process $(\Omega,\mathscr{B},\mathscr{B}_t,P_x,X_t)$ associated with the Dirichlet space $(\mathscr{F},\mathscr{E})$ we set $\sigma_B=\inf\{t; X_t\in B\}$ for a Borel set B. Then $E_x[e^{-\alpha\sigma_B}\tilde{\psi}(X_{\sigma_B})]$ is a quasi-continuous modification of the α-reduced function ψ_B (cf. [7]).

Let us consider the following variational inequality (1.6) for a given function $\phi\in\tilde{\mathscr{F}}$:

$$\mathscr{E}_\alpha(u,v-u)\geqq 0, \qquad \forall v\geqq\phi \text{ a.e. } u\geqq\phi \text{ a.e.} \tag{1.6}$$

Then this inequality (1.6) has an unique solution $u\in\mathscr{F}$. Therefore, when we denote by $U(\phi)$ the solution of (1.6) for given function $\phi\in\mathscr{F}$ $U(\cdot)$ define an operator from $\mathscr{F}$ to $\mathscr{F}$. We note that $U(\phi)$ is α-excessive for each $\phi\in\mathscr{F}$ (cf. [12, 17]).

Let us set

$$u^*(x)=\sup_\tau E_x[e^{-\alpha\tau}\phi(X_\tau)],$$

where τ ranges over all stopping times. Then, owing to [12], it is known that $u^*(x)$ is a quasi-continuous modification of the solution $U(\phi)$ of (1.6) and

$$u^*(x)=E_x[e^{-\alpha\tau^*}\phi(X_{\tau^*})],$$
$$\tau^*=\inf\{t; U(\phi)(X_t)=\phi(X_t)\}.$$

Remark. Any function $f(x)$ is extended to a function on $X_\Delta=X\cup\Delta$ by setting $f(\Delta)=0$, where X_Δ is a one point compactification of X. When X is already compact Δ is regarded as an isolated point. Since (P_x,X_t) is a Hunt process $X_t=\Delta$ for each $t\geqq\zeta(\omega)$, where $\zeta(\omega)$ is a terminal time. Therefore we can define $E_x[e^{-\alpha\tau}f(X_\tau)]$ for any stopping time τ and any quasi-continuous function $f(x)\in\mathscr{F}$.

2. System of QVI and Nash Equilibrium Points

Let us assume that quasi-continuous functions $\phi_i,\psi_i\in\mathscr{F}$ such that $\phi_i\leqq\psi_i$ m-a.e., $i=1,2$ are given. We set

$$J_x^1(\tau_1,\tau_2)=E_x[e^{-\alpha\tau_1}\phi_1(X_{\tau_1})I_{\{\tau_1\leqq\tau_2\}}+e^{-\alpha\tau_2}\psi_1(X_{\tau_2})I_{\{\tau_2<\tau_1\}}] \tag{2.1}$$

$$J_x^2(\tau_1,\tau_2)=E_x[e^{-\alpha\tau_2}\phi_2(X_{\tau_2})I_{\{\tau_2\leqq\tau_1\}}+e^{-\alpha\tau_1}\psi_2(X_{\tau_1})I_{\{\tau_1<\tau_2\}}] \tag{2.2}$$

for any stopping times τ_1 and τ_2. Our problem is to look for a pair of stopping times (τ_1^*,τ_2^*) such that

$$J_x^1(\tau_1^*,\tau_2^*)\geqq J_x^1(\tau_1,\tau_2^*), \qquad {}^\forall\tau_1,$$
$$J_x^2(\tau_1^*,\tau_2^*)\geqq J_x^2(\tau_1^*,\tau_2), \qquad {}^\forall\tau_2.$$

Such a pair (τ_1^*, τ_2^*) is called a Nash equilibrium point of non zero-sum stopping game with pay-off functions J_x^1 and J_x^2. Corresponding to the above stopping game we consider the following system of quasi-variational inequalities (QVI) (2.3):

$$\begin{aligned} &\mathscr{E}_\alpha(u_1, v-u_1) \geqq 0, \quad \forall v \geqq \phi_1 \vee U(\psi_1)_{B(u_2,\phi_2)}, \quad u_1 \geqq \phi_1 \vee U(\psi_1)_{B(u_2,\phi_2)} \\ &\mathscr{E}_\alpha(u_2, v-u_2) \geqq 0, \quad \forall v \geqq \phi_2 \vee U(\psi_2)_{B(u_1,\phi_1)}, \quad u_2 \geqq \phi_2 \vee U(\psi_2)_{B(u_1,\phi_1)}, \end{aligned} \tag{2.3}$$

where $B(u_i, \phi_i) = \{x; \tilde{u}_i(x) = \phi_i(x)\}$, $U(\psi_i)$ is the solution of the variational inequality (1.6) for a given function ψ_i and $U(\psi_i)_{B(u_j,\psi_j)}$ is the α-reduced function of the α-excessive function $U(\psi_i)$ on the set $B(u_j, \psi_j)$, $i,j=1,2$, $i \neq j$.

Then we have the following existence theorem of the system of QVI (2.3).

Theorem 2.1. *There exists a pair of solutions* $(\bar{u}_1, \underline{u}_2)$ *and* $(\underline{u}_1, \bar{u}_2)$ *of* (2.3) *such that any solution* (u_1, u_2) *of* (2.3) *satisfies*

$$\phi_i \leqq \underline{u}_i \leqq u_i \leqq \bar{u}_i \leqq U(\psi_i), \qquad i=1,2.$$

Concerning existence of Nash equilibrium points of the non zero-sum stopping game with pay-off function J_x^1 and J_x^2 respectively defined by (2.1) and (2.2) we have the following result under the assumption (A):

$$\{x; \widetilde{U(\phi_i)} = \phi_i\} \subset \{x; \widetilde{U(\psi_j)} = \psi_j\}, \qquad i,j=1,2, \ i \neq j, \tag{A}$$

except a set of capacity zero.

Theorem 2.2. *Let* (u_1, u_2) *be a solution of* (2.3) *and* $\tau_i^* = \inf\{t;\ \tilde{u}_i(X_t) = \phi_i(X_t)\}$, $i=1,2$, *then under the assumption* (A) (τ_1^*, τ_2^*) *is a Nash equilibrium point of the non zero-sum stopping game with pay-off function* J_x^1 *and* J_x^2.

Remark 2.1. If ψ_i is α-excessive for each i, or if $\phi_i < 0$ for each i, then the assumption (A) is always satisfied.

Remark 2.2. When $\phi_i = -\psi_j$, $i \neq j$, $i=1,2$ in (2.1) and (2.2) the game is called a zero-sum stopping game or Dynkin game, which has been studied by J. Zabczyk in the context of symmetric Markov processes and Dirichlet forms (cf. [16, 17]).

Remark 2.3. In general pay-off functions are defined as follows:

$$\begin{aligned} J_x^i(\tau_1, \tau_2) = E_x\Big[&\int_0^{\tau_1 \wedge \tau_2} e^{-\alpha s} f_i(X_s)\, ds \\ &+ e^{-\alpha\tau_i} \phi_i(X_{\tau_i}) I_{\{\tau_i \leqq \tau_j\}} + e^{-\alpha\tau_j} \psi_i(X_{\tau_j}) I_{\{\tau_j < \tau_i\}} \Big] \end{aligned} \tag{2.4}$$

$i \neq j$, $i,j=1,2$. The non zero-sum stopping game with pay-off functions (2.4) can be reduced to the above mentioned our case by taking $\phi_i - G_\alpha f_i$ and $\psi_i - G_\alpha f_i$ in place of ϕ_i and ψ_i, i$=1,2$.

3. Existence of Solutions of a System of QVI

Let us introduce for a given function $w \in \mathscr{F}$ a closed convex subset

$$K_i(w) = \{v \in \mathscr{F}; v \geqq \phi_i \vee U(\psi_i)_{B(w, \phi_j)}\},$$

$i, j = i, 2$, $i \neq j$ of a Hilbert space $(\mathscr{F}, \mathscr{E}_\alpha)$. Then we consider the following variational inequality (3.1)

$$\mathscr{E}_\alpha(u, v-u) \geqq 0, \qquad \forall v \in K_i(w),\ u \in K_i(w) \tag{3.1}$$

for each i. Since (3.1) has a unique solution we can define an operator S_i from $\mathscr{F}$ to $\mathscr{F}$ such that $S_i w$ is the solution of (3.1) with the convex set $K_i(w)$ for each i. Let us set $T = S_2 S_1$, then T has the following properties.

Lemma 3.1. *T satisfies the following properties:*

i) *$Tu \leqq Tw$ if $\phi_2 \leqq u \leqq w$*

ii) *if $w_n \leqq w_{n+1}$ for each n and $\{w_n\}$ converges to w in $(\mathscr{F}, \mathscr{E}_\alpha)$, then $\{Tw_n\}$ converges to Tw in $(\mathscr{F}, \mathscr{E}_\alpha)$.*

For the proof of Lemma 3.1 we need the following lemma.

Lemma 3.2. *Let $\psi \in \mathscr{F}$ be an α-excessive function and $\{B_n\}$ be a sequence of subsets of X, then we have the following:*

i) *if $\{B_n\}$ is a nondecreasing sequence and $B = \bigcup_{n=1}^{\infty} B_n$, then $\psi_{B_n} \to \psi_B$ in $(\mathscr{F}, \mathscr{E}_\alpha)$.*

ii) *if $\{B_n\}$ is a nonincreasing sequence of quasi-closed Borel subsets and $B = \bigcap_{n=1}^{\infty} B_n$, then $\psi_{B_n} \to \psi_B$ in $(\mathscr{F}, \mathscr{E}_\alpha)$.*

Remark. A subset B of X is called quasi-closed if there exist a quasi-continuous function $g(x)$ and a closed subset F of R^1 such that $B = g^{-1}(F)$.

Proof of Lemma 3.2. i) We first note that $(\psi_{B_2})_{B_1} = \psi_{B_1}$ for $B_1 \subset B_2$ because of the definition of α-reduced functions and (1.4). Therefore it follows from (1.3) that $\psi_{B_1} \leqq \psi_{B_2}$ if $B_1 \subset B_2$. Thus $\{\psi_{B_n}\}$ is a nondecreasing sequence of α-excessive functions such that $\psi_{B_n} \leqq \psi_B$, which means that $\mathscr{E}_\alpha(\psi_{B_n}, \psi_{B_n})$ is a nondecreasing sequence dominated by $\mathscr{E}_\alpha(\psi_B, \psi_B)$. Accordingly $\{\psi_{B_n}\}$ converges strongly in $(\mathscr{F}, \mathscr{E}_\alpha)$ to a certain function $\psi_* \in \mathscr{F}$. It is clear that ψ_* is α-excessive and $\mathscr{E}_\alpha(\psi_*, \psi_*) \leqq \mathscr{E}_\alpha(\psi_B, \psi_B)$. Moreover there exists a subsequence of $\{\psi_{B_n}\}$ which converges a.e. to ψ_*. Then we have $\tilde{\psi}_*(x) \geqq \tilde{\psi}_{B_n}(x) = \tilde{\psi}(x) = \tilde{\psi}_B(x)$ q.e. on B_n for each n. Since $B = \bigcup_{n=1}^{\infty} B_n$ we obtain $\tilde{\psi}_*(x) \geqq \tilde{\psi}_B(x)$ q.e. on B. Hence we conclude that $\psi_* = \psi_B$.

ii) Let $\{B_n\}$ be a nonincreasing sequence of quasi-closed Borel subsets of X, then it is easy to see that $\{\psi_{B_n}\}$ is an $\mathscr{E}_\alpha$-convergent sequence. We denote by ψ_0 its limit. According to Theorem 3.1.4 in [7] $\mathscr{E}_\alpha$-convergent sequence $\{\psi_{B_n}\}$ of quasi-continuous functions has a subsequence which converges q.e. to $\tilde{\psi}_0$. Therefore to see that $\tilde{\psi}_0 = \tilde{\psi}_B$ q.e. it suffices to show that

$$E_x[e^{-\alpha\sigma_B} \tilde{\psi}(X_{\sigma_B})] = \lim_{n \to \infty} E_x[e^{-\alpha\sigma_n} \tilde{\psi}(X_{\sigma_n})] \quad \text{q.e.} \tag{3.2}$$

where $\sigma_n=\inf\{t; X_t\in B_n\}$ for each n and $\sigma_B=\inf\{t; X_t\in B\}$ (cf. Sect. 1). σ_n being nondecreasing we set $\sigma=\lim_{n\to\infty}\sigma_n$ to see (3.2). Since each B_n is a quasi-closed Borel set we have $X_{\sigma_n}\in B_n$ P_x a.s. on $\{\sigma<\infty\}$ q.e. x. Therefore $\lim_{n\to\infty} X_{\sigma_n}\in\bigcap_{k=1}^{\infty}B_k$ $=B$, P_x a.s. on $\{\sigma<\infty\}$ q.e. x. By using quasi-left continuity of our process we see that $P_x(X_\sigma\in B, \sigma<\infty)=P_x(\sigma<\infty)$ q.e. x, which means $\sigma\geqq\sigma_B$, P_x a.s., q.e. x. Converse inequality is obvious by definition of σ. Accordingly we have $\sigma=\sigma_B$ P_x a.s. q.e. x. Hence we obtain $E_x[e^{-\alpha\sigma_n}\tilde{\psi}(X_{\sigma_n})]\to E_x[e^{-\alpha\sigma_B}\tilde{\psi}(X_{\sigma_B})]$ q.e. x as $n\to\infty$ because of quasi-left continuity of our process and dominated convergence theorem.

Proof of Lemma 3.1. i) Since $\phi_2\leqq u\leqq w$ we have $B(u,\phi_2)\supset B(w,\phi_2)$. Therefore we have $U(\psi_1)_{B(u,\phi_2)}\geqq U(\psi_1)_{B(w,\phi_2)}$ in the same way as the proof of Lemma 3.2 i). Accordingly we obtain $K_1(u)\subset K_1(w)$, from which it follows that $S_1 u\geqq S_1 w$. It is obvious that $S_1 w\geqq\phi_1$. By repeating similar arguments as above with respect to S_2 we obtain $Tu=S_2 S_1 u\leqq S_2 S_1 w=Tw$.

ii) We first note that $\{B(w_n,\phi_2)\}$ is a nonincreasing sequence of quasi-closed subsets of X and $B(w,\phi_2)=\bigcap_{n=1}^{\infty}B(w_n,\phi_2)$. Therefore we see by Lemma 3.2 that $U(\psi_1)_{B(w_n,\phi_2)}$ converges in $(\mathscr{F},\mathscr{E}_\alpha)$ to $U(\psi_1)_{B(w,\phi_2)}$. Then we have $\phi_1\vee U(\psi_1)_{B(w_n,\phi_2)}\to\phi_1\vee U(\psi_1)_{B(w,\phi_2)}$ in $(\mathscr{F},\mathscr{E}_\alpha)$ by Anconna's lemma (cf. [1, 17]). Therefore we obtain $S_1 w_n\to S_1 w$ in $(\mathscr{F},\mathscr{E}_\alpha)$ by Theorem 5.1 in [14]. Moreover we have $S_1 w_n\downarrow S_1 w\geqq\phi_1$ q.e. Now we have $B(S_1 w_n,\phi_1)\subset B(S_1 w_{n+1},\phi_1)$ and $\bigcup_{n=1}^{\infty}B(S_1 w_n,\phi_1)=B(S_1 w,\phi_1)$. Therefore by similar arguments as above using Lemma 3.2 i) in place of ii) we conclude $S_2 S_1 w_n\to S_2 S_1 w$ in $(\mathscr{F},\mathscr{E}_\alpha)$.

Proof of Theorem 2.1. We first note that $\phi_2\leqq T\phi_2$. Then we obtain $T^{n-1}\phi_2\leqq T^n\phi_2$ by Lemma 3.1. It is clear that $T^n\phi_2\leqq U(\psi_2)$ because $T^n\phi_2$ is a solution of (3.1) with $K_2(S_1 T^{n-1}\phi_2)$. Thus $\{T^n\phi_2\}_n$ is a nondecreasing sequence of α-excessive functions dominated by α-excessive function $U(\psi_2)$. Therefore it has a (strong) limit point $T^\infty\phi_2$. We set $\underline{u}_2=T^\infty\phi_2$, then we have $T\underline{u}_2=\underline{u}_2$ because of Lemma 3.1 ii). Hence, setting $\bar{u}_1=S_1\underline{u}_2$, we see that $(\bar{u}_1,\underline{u}_2)$ is a solution of (2.3).

Let us define an operator $\hat{T}$ from $\mathscr{F}$ to $\mathscr{F}$ by $\hat{T}=S_1 S_2$, then it follows that $\hat{T}u\leqq\hat{T}w$ for $\phi_1\leqq u\leqq w$ and that $\hat{T}w_n\to\hat{T}w$ in $(\mathscr{F},\mathscr{E}_\alpha)$ for nondecreasing sequence $\{w_n\}_n$ which converges to w in $(\mathscr{F},\mathscr{E}_\alpha)$. Therefore, setting $\underline{u}_1=\hat{T}^\infty\phi_1$ and $\bar{u}_2=S_2\underline{u}_1$, we can see by similar arguments as above that $(\underline{u}_1,\bar{u}_2)$ is a solution of (2.3). It can be easily seen that $\phi_i\leqq\underline{u}_i\leqq u_i\leqq\bar{u}_i\leqq U(\psi_i)$, $i=1,2$, using the above mentioned monotone property of T and $\hat{T}$.

4. A Variational Inequality and a Stopping Problem

Suppose that a Borel quasi-closed subset B and quasi-continuous functions ϕ and $\psi\in\mathscr{F}$ such that $\phi\leqq\psi$ m-a.e. and ψ is α-excessive are given. Then we consider the following variational inequality.

$$\mathscr{E}_\alpha(u, v-u) \geqq 0, \qquad \forall\, v \in K(B),\ u \in K(B), \tag{4.1}$$

where $K(B)=\{v\in\mathscr{F};\ v\geqq\phi\vee\psi_B\ m\text{-a.e.}\}$. We note that the unique solution u of (4.1) is the excessive majorant of $\phi\vee\psi_B$ (cf. [12, 14, 17]). Therefore by our assumptions we have the following lemma.

Lemma 4.1. *The solution u of* (4.1) *is α-excessive and satisfies*

$$\phi \leqq u \leqq \psi \qquad m\text{-a.e.} \tag{4.2}$$

$$\tilde{u} = \psi \qquad \text{q.e. on } B. \tag{4.3}$$

Let us set $\sigma_1=\inf\{t;\tilde{u}(X_t)=\phi(X_t)\}$, $\sigma_2=\inf\{t;\tilde{u}(X_t)=\tilde{\psi}_B(X_t)\}$, $\sigma_3=\inf\{t;X_t\in B\}$, $\sigma^*=\inf\{t;\tilde{u}(X_t)=\phi\vee\tilde{\psi}_B(X_t)\}=\sigma_1\wedge\sigma_2$ and $\tau^*=\sigma_1\wedge\sigma_3$. Then, owing to the theorem in [12] *we have*

$$\begin{aligned}\tilde{u}(x) &= \sup_\tau E_x[e^{-\alpha\tau}(\phi\vee\tilde{\psi}_B)(X_\tau)] \\ &= E_x[e^{-\alpha\sigma^*}(\phi\vee\tilde{\psi}_B)(X_{\sigma^*})] \\ &= E_x[e^{-\alpha\sigma^*}\tilde{u}(X_{\sigma^*})] \qquad \text{q.e.}\end{aligned} \tag{4.4}$$

Now, using the properties of an α-reduced function ψ_B, we have the following lemma useful for the proof of Theorem 2.2.

Lemma 4.2. *We have*

$$\tilde{u}(x)=E_x[e^{-\alpha\tau^*}\tilde{u}(X_{\tau^*})]=E_x[e^{-\alpha\tau^*}(\phi\vee\tilde{\psi}_B)(X_{\tau^*})] \qquad \text{q.e.}$$

Proof. By (3.4) we have

$$\begin{aligned}\tilde{u}(x) &= E_x[e^{-\alpha\sigma^*}(\phi\vee\tilde{\psi}_B)(X_{\sigma^*})] \\ &= E_x[e^{-\alpha\sigma_1}\phi(X_{\sigma_1})I_{\{\sigma_1\leqq\sigma_2\}}+e^{-\alpha\sigma_2}\tilde{\psi}_B(X_{\sigma_2})I_{\{\sigma_2<\sigma_1\}}] \qquad \text{q.e.}\end{aligned}$$

Since $\psi_B(x)$ is an α-reduced function of ψ on B we obtain

$$\begin{aligned}\psi_B(x) &= E_x[e^{-\alpha\sigma_3}\psi(X_{\sigma_3})] \\ &= E_x[e^{-\alpha\sigma_3\wedge\sigma_1}E_{X(\sigma_3\wedge\sigma_1)}[e^{-\alpha\sigma_3}\psi(X_{\sigma_3})]] \\ &= E_x[e^{-\alpha\sigma_3\wedge\sigma_1}\tilde{\psi}_B(X(\sigma_3\wedge\sigma_1))] \qquad \text{q.e.}\end{aligned}$$

We further note that $\sigma_2\leqq\sigma_3$, P_x a.s. q.e. x, $P_x(X_{\sigma_3}\in B;\ \sigma_3<\infty)=0$ q.e. x and $\tilde{u}(x)=\tilde{\psi}_B(x)=\psi(x)$ q.e. on B. Therefore we obtain

$$\begin{aligned}&E_x[e^{-\alpha\sigma_2}\tilde{\psi}_B(X_{\sigma_2})I_{\{\sigma_2<\sigma_1\}}] \\ &= E_x[e^{-\alpha\sigma_3\wedge\sigma_1}\tilde{\psi}_B(X_{\sigma_3\wedge\sigma_1})I_{\{\sigma_2<\sigma_1\}}] \\ &= E_x[e^{-\alpha\sigma_3}\tilde{\psi}_B(X_{\sigma_3})I_{\{\sigma_3<\sigma_1\}}+e^{-\alpha\sigma_1}\tilde{\psi}_B(X_{\sigma_1})I_{\{\sigma_2<\sigma_1\leqq\sigma_3\}}] \\ &\leqq E_x[e^{-\alpha\sigma_3}\tilde{u}(X_{\sigma_3})I_{\{\sigma_3<\sigma_1\}}+e^{-\alpha\sigma_1}\tilde{u}(X_{\sigma_1})I_{\{\sigma_2<\sigma_1\leqq\sigma_3\}}] \qquad \text{q.e.}\end{aligned}$$

Hence we have

$$\begin{aligned}\tilde{u}(x)&\leqq E_x[e^{-\alpha\sigma_1}\tilde{u}(X_{\sigma_1})I_{\{\sigma_1\leqq\sigma_3\}}+e^{-\alpha\sigma_3}\tilde{u}(X_{\sigma_3})I_{\{\sigma_3<\sigma_1\}}]\\&\leqq E_x[e^{-\alpha\sigma_1\wedge\sigma_3}\tilde{u}(X_{\sigma_1\wedge\sigma_3})]\\&=E_x[e^{-\alpha\tau^*}\tilde{u}(X_{\tau^*})]\quad\text{q.e.}\end{aligned}$$

Converse inequality $\tilde{u}(x)\geqq E_x[e^{-\alpha\tau^*}\tilde{u}(X_{\tau^*})]$ q.e. is obvious because u is α-excessive.

5. Proof of Theorem 2.2

Now we give the proof of Theorem 2.2. Let (u_1,u_2) be a solution of the system of QVI (2.3). We set

$$\tau_i^*=\inf\{t;X_t\in B(u_i,\phi_i)\},\qquad i=1,2.$$

We note that $B(u_i,\phi_i)$, $i=1,2$, are quasi-closed and therefore $P_x(X_{\tau_i^*}\in B(u_i,\phi_i),\ \tau_i^*<\infty)=0$, q.e., $i=1,2$. Then we have

$$\begin{aligned}J_x^1(\tau_1^*,\tau_2^*)&=E_x[e^{-\alpha\tau_1^*}\phi_1(X_{\tau_1^*})I_{\{\tau_1^*\leqq\tau_2^*\}}+e^{-\alpha\tau_2^*}\psi_1(X_{\tau_2^*})I_{\{\tau_2^*<\tau_1^*\}}]\\&=E_x[e^{-\alpha\tau_1^*}\tilde{u}_1(X_{\tau_1^*})I_{\{\tau_1^*\leqq\tau_2^*\}}+e^{-\alpha\tau_2^*}\tilde{u}_1(X_{\tau_2^*})I_{\{\tau_2^*<\tau_1^*\}}]\\&=E_x[e^{-\alpha\tau_1^*\wedge\tau_2^*}\tilde{u}_1(X_{\tau_1^*\wedge\tau_2^*})],\quad\text{q.e.}\end{aligned}$$

because $\tilde{u}_1(x)=\psi_1(x)$ q.e. on $B(u_2,\phi_2)$ by Lemma 4.1. We obtain by Lemma 4.1 and Lemma 4.2

$$\begin{aligned}E_x[e^{-\alpha\tau_1^*\wedge\tau_2^*}\tilde{u}_1(X_{\tau_1^*\wedge\tau_2^*})]\quad&=\tilde{u}_1(x)\\&\geqq E_x[e^{-\alpha\tau_1\wedge\tau_2^*}\tilde{u}_1(X_{\tau_1\wedge\tau_2^*})]\\&\geqq E_x[e^{-\alpha\tau_1}\phi_1(X_{\tau_1})I_{\{\tau_1\leqq\tau_2^*\}}+e^{-\alpha\tau_2^*}\psi_1(X_{\tau_2^*})I_{\{\tau_2^*<\tau_1\}}]\\&=J_x^1(\tau_1,\tau_2^*),\qquad\text{q.e.}\end{aligned}$$

for any stopping time τ_1. Thus we obtain $J_x^1(\tau_1^*,\tau_2^*)\geqq J_x^1(\tau_1,\tau_2^*)$ q.e. x, for each τ_1. In the same way we have $J_x^2(\tau_1^*,\tau_2^*)\geqq J_x^2(\tau_1^*,\tau_2)$ q.e. x, for each stopping time τ_2.

6. Equivalence of Two Systems of QVI

Bensoussan-Friedman introduced the following system of QVI in [3]:

$$\begin{aligned}&\mathscr{E}_\alpha(w_1,v-w_1)\geqq 0,\qquad\forall v\in\hat{K}_1(w_2),\ w_1\in\hat{K}_1(w_2)\\&\mathscr{E}_\alpha(w_2,v-w_2)\geqq 0,\qquad\forall v\in\hat{K}_2(w_1),\ w_2\in\hat{K}_2(w_1),\end{aligned}\tag{6.1}$$

where $\hat{K}_i(w_j)=\{v\in\mathscr{F};\ \tilde{v}=\psi_i$ q.e. on $\{\tilde{w}_j=\phi_j\}$ and $v\geqq\phi_i\}$, $i\neq j$, $i,j=1,2$ and ϕ_i and ψ_i are given quasi-continuous functions belonging to $\mathscr{F}$ such that $\phi_i\leqq\psi_i$ and ψ_i is α-excessive for each i. We are now going to show equivalence of (6.1) and our system of QVI (2.3) in the case that ψ_i is α-excessive for each i. It is sufficient to prove the equivalence of the variational inequality (4.1) and the following variational inequality (6.2).

$$\begin{aligned}&\mathscr{E}_\alpha(u, v-u)\geqq 0 \quad \forall\, v\in K_B,\ u\in K_B,\\ &K_B=\{v\in\mathscr{F};\ \tilde{v}=\psi \text{ q.e. on } B \text{ and } v\geqq\phi\}.\end{aligned} \tag{6.2}$$

Proposition 6.1. (4.1) *and* (6.2) *have common unique solution.*

Proof. Suppose that u (resp. w) is the solution of (4.1) (resp. (6.2)). Since $\tilde{u}=\psi$ q.e. on B and $u\geqq\phi$ m-a.e. u belong to K_B. Therefore we have

$$\mathscr{E}_\alpha(w, w)\leqq\mathscr{E}_\alpha(u, u). \tag{6.3}$$

On the other hand we can show that $w\in K_B$ as follows. Since $|w-\psi_B|$, $w-\psi_B\in\mathscr{F}_{X-B}$ we have

$$\begin{aligned}&\mathscr{E}_\alpha(\psi_B+|w-\psi_B|, \psi_B+|w-\psi_B|)\\ &\quad=\mathscr{E}_\alpha(\psi_B, \psi_B)+2\mathscr{E}_\alpha(\psi_B, |w-\psi_B|)+\mathscr{E}_\alpha(|w-\psi_B|, |w-\psi_B|)\\ &\quad\leqq\mathscr{E}_\alpha(\psi_B, \psi_B)+\mathscr{E}_\alpha(w-\psi_B, w-\psi_B)\\ &\quad=\mathscr{E}_\alpha(w, w)+2\mathscr{E}_\alpha(\psi_B, \psi_B-w)\\ &\quad=\mathscr{E}_\alpha(w, w).\end{aligned}$$

It can be easily seen that $\psi_B+|w-\psi_B|\in K_B$. Therefore we have $w=\psi_B+|w-\psi_B|$, which means $w\geqq\psi_B$ m-a.e. Accordingly we obtain $w\geqq\phi\vee\psi_B$. That is $w\in K(B)$. Hence we have

$$\mathscr{E}_\alpha(u, u)\leqq\mathscr{E}_\alpha(w, w). \tag{6.4}$$

From (6.3) and (6.4) it follows that $u=w$.

7. Example

We give a simple example of the solutions of system of QVI (2.3) in the case of absorbing Brownian motion on the interval $(0, 1)$, where the Dirichlet form is

$$\mathscr{E}(u, v)=\frac{1}{2}\int_0^1 \frac{du}{dx}\frac{dv}{dx}dx$$

with the domain $\mathscr{F}=\left\{v;\ v \text{ is absolutely continuous}, \int_0^1\left|\frac{dv}{dx}\right|^2 dx<\infty \text{ and } v(0)=v(1)=0\right\}$. We note that Theorem 2.1 and Theorem 2.2 are valid for $\alpha=0$ in the present case because $\mathscr{E}$ is already coercive.

Let ψ_i, ϕ_i, $i=1, 2$ be functions as follows:

$$\psi_1(x)=\begin{cases}\frac{8}{3}x & 0<x\leqq\frac{3}{8}\\ 1 & \frac{3}{8}<x\leqq\frac{5}{8}\\ -\frac{8}{3}x+\frac{8}{3} & \frac{5}{8}<x<1,\end{cases}$$

$$\phi_1(x)=\begin{cases}\frac{4}{3}x & 0<x\leqq\frac{3}{8}\\ \frac{1}{2} & \frac{3}{8}<x\leqq\frac{5}{8}\\ -\frac{4}{3}x+\frac{4}{3} & \frac{5}{8}<x<1,\end{cases}$$

$$\psi_2(x)=\begin{cases}4x & 0<x\leqq\frac{1}{4}\\ 1 & \frac{1}{4}<x\leqq\frac{3}{4}\\ -4x+4 & \frac{3}{4}<x<1,\end{cases}$$

$$\phi_2(x)=\begin{cases}4x & 0<x\leqq\frac{1}{8}\\ \frac{1}{2} & \frac{1}{8}<x\leqq\frac{7}{8}\\ -4x+4 & \frac{7}{8}<x<1.\end{cases}$$

Then extremal solutions $(\underline{u}_1, \bar{u}_2)$ and $(\bar{u}_1, \underline{u}_2)$ are the followings

$$\underline{u}_1(x)=\begin{cases}\frac{8}{3}x & 0<x<\frac{1}{8}\\ \frac{2}{3}x+\frac{1}{4} & \frac{1}{8}<x\leqq\frac{3}{8}\\ \frac{1}{2} & \frac{3}{8}<x\leqq\frac{5}{8}\\ -\frac{2}{3}x+\frac{11}{12} & \frac{5}{8}<x\leqq\frac{7}{8}\\ -\frac{8}{3}x+\frac{8}{3} & \frac{7}{8}<x<1,\end{cases}$$

$$\bar{u}_2(x)=\begin{cases}4x & 0<x<\frac{1}{8}\\ 2x+\frac{1}{4} & \frac{1}{8}<x<\frac{3}{8}\\ 1 & \frac{3}{8}<x<\frac{5}{8}\\ -2x+\frac{9}{4} & \frac{5}{8}<x<\frac{7}{8}\\ -4x+4 & \frac{7}{8}<x<1,\end{cases}$$

$$\bar{u}_1(x)=\psi_1(x) \quad \text{and} \quad \underline{u}_2(x)=\phi_2(x).$$

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