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# Reflected Brownian Motion with Skew Symmetric Data in a Polyhedral Domain*

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**Summary.** This paper is concerned with the characterization and invariant measures of certain reflected Brownian motions (RBM's) in polyhedral domains. The kind of RBM studied here behaves like $d$-dimensional Brownian motion with constant drift $\mu$ in the interior of a simple polyhedron and is instantaneously reflected at the boundary in directions that depend on the face that is hit. Under the assumption that the directions of reflection satisfy a certain skew symmetry condition first introduced in Harrison-Williams [9], it is shown that such an RBM can be characterized in terms of a family of submartingales and that it reaches non-smooth parts of the boundary with probability zero. In [9], a purely analytic problem associated with such an RBM was solved. Here the exponential form solution obtained in [9] is shown to be the density of an invariant measure for the RBM. Furthermore, if the density is integrable over the polyhedral state space, then it yields the unique stationary distribution for the RBM. In the proofs of these results, a key role is played by a dual process for the RBM and by results in [9] for reflected Brownian motions on smooth approximating domains.

## 1. Introduction

This paper is concerned with certain $d$-dimensional diffusion processes called reflected Brownian motions (or RBM's) that have applications in queueing and storage theory [6, 7, 13, 19, 22]. An RBM behaves like $d$-dimensional Brownian motion with constant drift in the interior of a simple polyhedron and is instantaneously reflected at the boundary of the polyhedron in directions that depend on the face that is hit. Under the assumption that the directions of reflection satisfy a certain skew symmetry condition first introduced in [9], it is shown that such an RBM can be characterized in terms of a family of

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submartingales and that it reaches non-smooth parts of the boundary (e.g., edges and corners for $d=3$) with probability zero. In [9], a purely analytic problem associated with such an RBM was investigated. It is shown here that the solution of that analytic problem is the density of an invariant measure for the RBM and furthermore, if the density is integrable over the polyhedral state space then it yields the unique stationary distribution for the RBM. This invariant density has an explicit exponential form, which is the constant function when the drift is zero. In the proofs of these results, a key role is played by a dual process for the RBM.

The notation used here is consistent with that in [9]. In particular, the data for an RBM are as follows (primes denote transposes, vectors without primes are column vectors, and diag(·) denotes the vector formed by the diagonal elements of a square matrix):

(a) integers $k \geqq d \geqq 1$,

(b) a $k \times d$ matrix $N$ such that $\mathrm{diag}(NN')=1$ and $N$ contains an invertible $d \times d$ submatrix $\bar{N}$,

(c) a $k \times d$ matrix $Q$ such that

$$\mathrm{diag}(QN')=0,$$

(d) a vector $b=(b_1, \ldots, b_k)' \in \mathbb{R}^k$, and

(e) a *drift vector* $\mu \in \mathbb{R}^d$.

Let $n_i'$ and $q_i'$ denote the $i^{\text{th}}$ rows of the matrices $N$ and $Q$ respectively $(i=1, \ldots, k)$; thus $n_i$ and $q_i$ are both $d$-dimensional column vectors. Let $\bar{G}$ denote the convex polyhedron defined by:

$$\bar{G} \equiv \{x \in \mathbb{R}^d \colon \; Nx \geqq b\}.$$

It is assumed that the interior $G$ is non-empty and that this representation of $\bar{G}$ is irreducible. That is, for any matrix $\tilde{N}$ and column vector $\tilde{b}$ formed by removing one of the rows of $N$ and the corresponding row element of $b$, the set $\{x \in \mathbb{R}^d \colon \tilde{N} x \geqq \tilde{b}\}$ is strictly larger than $\bar{G}$. This is equivalent to the assumption that each of the *faces*

$$F_i \equiv \{x \in \bar{G} \colon \; x \cdot n_i = b_i\}, \qquad i=1, \ldots, k,$$

has dimension $d-1$. The reader will observe that $n_i$ is a unit vector normal to $F_i$ that points into the interior $G$, whereas $q_i$ is a vector parallel to $F_i$. The vector $v_i \equiv n_i + q_i$ is called the *direction of reflection* associated with face $F_i$; $n_i$ and $q_i$ are called the *normal component* and *tangential component* respectively of $v_i$. A *vertex* of $\bar{G}$ is a point $x \in \partial G$ where $d$ or more of the faces $F_i$ intersect. A mild non-degeneracy assumption is made here, namely that the polyhedron $\bar{G}$ is *simple*, i.e., *precisely d* faces meet at each vertex.

The requirement that $N$ contain an invertible $d \times d$ submatrix means that no line can lie entirely within the polyhedron $\bar{G}$. That is, the boundary of the polyhedron must bound each dimension in at least one direction; this is of course automatic if $\bar{G}$ is bounded.

Loosely speaking, an RBM associated with these data is a strong Markov process with continuous sample paths in $\bar{G}$ that (a) behaves like $d$-dimensional Brownian motion with constant drift $\mu$ in $G$, (b) is reflected at the boundary of $G$ in the direction $v_i$ on $F_i$ and (c) spends zero time (in the sense of Lebesgue measure) on the boundary of $G$. Without further restrictions on the data, there need not exist a well defined process satisfying these conditions. Indeed, such processes do not fall within the realm of the general theory of multi-dimensional diffusions because the boundary of the state space is not smooth and the directions of reflection are discontinuous across non-smooth parts of the boundary. However, some instances of such processes with various restrictions on the data have been studied. When $\bar{G}$ is a two-dimensional wedge and $\mu=0$, the questions of existence, uniqueness and characterization of such a process were resolved in [18]. Even this simple case required non-trivial analysis and led to some surprising results such as the possibility of sufficient reflection toward the corner forcing any continuous strong Markov process satisfying (a) and (b) to be absorbed there. The case of a general polygon $\bar{G}$ in $\mathbb{R}^2$, can be reduced to that of a wedge by localization. Several authors have given sufficient conditions for a path-by-path construction of an RBM from a $d$-dimensional Brownian motion. These range from the simplest case of normal reflection treated by Tanaka [17], through cases requiring the polyhedron and vectors of reflection to be suitably approximable by smooth domains and vector fields as in Lions and Sznitman [10], to a construction on an orthant given by Harrison and Reiman [8] where $Q$ has non-positive entries and spectral radius strictly less than one. In [22], de Zelicourt gave abstract sufficient conditions for certain RBM's to exist as diffusion approximations to queueing and storage processes. One of these conditions requires that the RBM's do not reach any non-smooth parts of the boundary. A few concrete examples of two-dimensional RBM's for which this holds were given in [22]. In this paper it is shown that when the following skew symmetry condition (1.1) holds, the $d$-dimensional RBM associated with $N$, $Q$, $b$ and $\mu$ does not reach the non-smooth parts of the boundary of $G$:

$$NQ'+QN'=0. \tag{1.1}$$

A submartingale characterization is given for this RBM in Theorem 1.1 below. This is in the spirit of Stroock and Varadhan's [15] approach to diffusion processes on smooth domains with smooth boundary conditions. For the statement of Theorem 1.1, the following notation and terminology is needed.

Let $S$ denote the union of $G$ with the *smooth* part of the boundary of $G$, and let $\Omega$ denote the set of continuous functions $\omega$: $[0,\infty)\to\bar{G}$ satisfying $\omega(0)\in S$. Suppose $\Omega$ is endowed with the $\sigma$-algebra $\mathscr{F}=\sigma\{\omega(s)\colon 0\leqq s<\infty\}$ generated by the coordinate maps, and for each $t\in[0,\infty)$, let $\mathscr{F}_t=\sigma\{\omega(s)\colon 0\leqq s\leqq t\}$. A function $T$: $\Omega\to[0,\infty]$ is a stopping time (relative to $\{\mathscr{F}_t\}$) if $\{T\leqq t\}\in\mathscr{F}_t$ for all $t\geqq 0$. The $\sigma$-field $\mathscr{F}_T$ associated with such a $T$ is defined by:

$$\mathscr{F}_T\equiv\{A\in\Omega\colon A\cap\{T\leqq t\}\in\mathscr{F}_t \text{ for all } t\geqq 0\}.$$

For each $(t, \omega) \in [0, \infty) \times \Omega$, define

$$X(t, \omega) \equiv X_t(\omega) = \omega(t). \tag{1.2}$$

Let $C_c^2(\bar{G})$ denote the set of functions that are twice continuously differentiable with compact support in some domain containing $\bar{G}$. Let $\Delta$ and $\nabla$ denote the Laplace and gradient operators respectively in $\mathbb{R}^d$, and define the differential operators

$$L = \tfrac{1}{2}\Delta + \mu \cdot \nabla \quad \text{in } G, \tag{1.3}$$

$$D_i = v_i \cdot \nabla \quad \text{on } F_i. \tag{1.4}$$

For typographical convenience, the dependence of $L$ on $\mu$ has been suppressed. Finally, define a differential operator $D$ on $\partial G$ by (a) setting $D = D_i$ at all points on face $F_i$ that are not also on some other face, and (b) setting $D$ to zero at the intersections of faces. Define the stopping time $\tau$ by:

$$\tau = \inf\{t \geqq 0:\ \omega(t) \notin S\}.$$

**Theorem 1.1.** *Fix $N$, $Q$, $b$ and $\mu$, and suppose* (1.1) *holds. Then for each $x \in S$, there is a unique probability measure $P_x$ on $(\Omega, \mathscr{F})$ that has the following two properties.*

(i) $P_x(\omega(0) = x) = 1$.

(ii) *For each $f \in C_c^2(\bar{G})$ that satisfies*

$$Df \geqq 0 \quad \text{on } \partial G, \tag{1.5}$$

*we have*

$$\left\{ f(\omega(t)) - \int_0^t Lf(\omega(s))\, ds,\ t \geqq 0 \right\} \tag{1.6}$$

*is a submartingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$.*

*Moreover, for each $x \in S$,*

(iii) $P_x(\tau < \infty) = 0$.

It follows from the uniqueness and (iii) that $\{P_x,\ x \in S\}$ is Feller continuous and has the strong Markov property ([15], p. 196). The RBM (associated with $N$, $Q$, $b$ and $\mu$) is then defined to be the strong Markov process on $(\Omega, \mathscr{F})$ associated with (1.2) and the family of probability measures $\{P_x,\ x \in S\}$. When it is necessary to stress the dependence on $S$ and/or $\mu$, the qualifiers "in $S$" and "with drift $\mu$" will be used. In the above characterization, the points in $\bar{G} \setminus S$ have been excised from $\bar{G}$ to yield the reduced state space $S$. For the purpose of studying the invariant measures, there is no loss of generality in doing this since by property (iii) and the strong Markov property, an RBM for which (1.1) holds will never return to the non-smooth part of the boundary $\bar{G} \setminus S$ once it has escaped from there. For $d \geqq 3$, the question of how to construct and characterize an RBM starting from a point in the singular set $\bar{G} \setminus S$ is an interesting open problem. The case $d = 2$ is covered by [18].

For a two-dimensional convex polygon $\bar{G}$, (1.1) is equivalent to the requirement of a constant angle of reflection over the entire boundary [9]. Moreover, by applying the results in [18] locally and using the connectedness of the boundary, it follows that in this two-dimensional case if $\bar{G}$ is bounded then (1.1) is also necessary for Theorem 1.1(iii) to hold. The question of whether the same is true for $d \geqq 3$ is an open problem. Implicit in this is the problem of determining general conditions for existence of an RBM with given data.

In [9], a purely analytic problem was studied. It was shown there that (1.1) holds if and only if for each $\mu \in \mathbb{R}^d$ there is a solution of the exponential form $p(x) = \exp\{\gamma(\mu) \cdot x\}$, where $\gamma(\mu) \in \mathbb{R}^d$, to the following *basic adjoint relation:*

$$\int_G p L f \, dx + \tfrac{1}{2} \int_{\partial G} p D f \, d\sigma = 0 \qquad \text{for all } f \in C_c^2(\bar{G}). \tag{BAR}$$

Here $dx$ denotes integration with respect to Lebesgue measure on $\mathbb{R}^d$ and $d\sigma$ denotes integration with respect to surface measure on $\partial G$. The vector $\gamma(\mu)$ is unique and is given by the formula:

$$\gamma(\mu) = 2(I - \bar{N}^{-1} \bar{Q})^{-1} \mu, \tag{1.7}$$

where $\bar{N}$ denotes the invertible $d \times d$ submatrix of $N$ referred to in specifying the data of the RBM, and $\bar{Q}$ denotes the corresponding $d \times d$ submatrix of $Q$. Although it may at first appear that $\gamma(\mu)$ depends on the choice of $\bar{N}$, in fact it does not because (1.1) implies that $\bar{N}^{-1} \bar{Q}$ is independent of the particular choice of $\bar{N}$ [9]. The following complements to the formal results of [9] are proved in this paper.

**Theorem 1.2.** *Fix $N$, $Q$, $b$ and $\mu$. Assume* (1.1) *holds. Consider the measure $\rho$ on $S$ whose density function with respect to Lebesgue measure is $p(x) = \exp\{\gamma(\mu) \cdot x\}$. Then the RBM's associated with $(N, Q, b, \mu)$ and $(N, -Q, b, \gamma(\mu) - \mu)$ are in duality relative to $\rho$ and $\rho$ is an invariant measure for these two processes.*

By duality here we mean "strong duality", although "weak duality" (cf. Eq. (3.3)) would have sufficed for our purposes. For further details on duality, the reader is referred to [4].

**Corollary 1.1.** *Assume the hypotheses of Theorem* 1.2 *hold. Suppose $\mu \in \mathbb{R}^d$ is such that*

$$C(\mu) = \int_S \exp\{\gamma(\mu) \cdot x\} \, dx < \infty. \tag{1.8}$$

*Then the RBM in $S$ with drift $\mu$ has a unique stationary distribution. This stationary distribution is absolutely continuous with respect to Lebesgue measure and has density function $\{C(\mu)\}^{-1} \exp\{\gamma(\mu) \cdot x\}$.*

Note that if $\bar{G}$ is bounded, then (1.8) automatically holds for all $\mu \in \mathbb{R}^d$.

By Theorem 1.2, the dual process associated with an RBM satisfying (1.1) is also an RBM for which the skew symmetry condition holds. In particular, the dual process has *constant* drift $\gamma(\mu) - \mu$ and adjoint directions of reflection: $\hat{v}_i = n_i - q_i$. On the other hand, a reflected Brownian motion with constant drift $\mu$ in a *smooth* bounded $d$-dimensional domain with *smooth* reflection field on the boundary has a unique stationary measure of the form $p_\mu(x) \, dx$, where $p_\mu$ is a smooth strictly positive probability density. The associated dual process is a reflected Brownian motion with *constant* drift if and only if $\nabla p_\mu / p_\mu$ is a constant vector [11], i.e., if and only if $p_\mu(x) = C(\mu)\exp(\gamma(\mu)\cdot x)$ for some $\gamma(\mu) \in \mathbb{R}^d$ and $C(\mu) > 0$. In [9], it was shown that the latter holds for *all* $\mu \in \mathbb{R}^d$ if and only if a smooth analogue of the skew symmetry condition (1.1) holds. This and the results in [9] suggest that for a *simple polyhedron* an analogous result should hold, but with *some* $\mu \in \mathbb{R}^d$ in place of *all* $\mu \in \mathbb{R}^d$. Thus, it is conjectured here that assuming $\bar{G}$ is bounded and $\mu \in \mathbb{R}^d$, then relative to a stationary measure an RBM associated with $(N, Q, b, \mu)$ has a dual process that is an RBM with *constant* drift if and only if (1.1) holds. Of course, the *if* part follows from Theorem 1.1. An implicit problem for the *only if* part is that of determining suitable conditions (other than (1.1)) under which there is a well defined RBM.

The remainder of this paper is organized as follows. The existence and uniqueness result, Theorem 1.1, is proved in Sect. 2. By means of a Girsanov transformation, the proof is reduced to the case where the drift is zero. Then, Theorem 1.1 is proved when $\bar{G}$ is a cone using explicit knowledge of the cases $d=1$ (a half-line) and $d=2$ (a wedge [18]), and an induction argument involving some scaling properties and an associated dual process [16, 20]. The result for a cone is then applied locally and combined with a piecing together argument to deduce Theorem 1.1 for a general simple polyhedron. In Sect. 3, Theorem 1.2 and Corollary 1.1 are proved. The approach adopted there is to approximate $G$ by smooth bounded domains with smooth vector fields on their boundaries. These domains and vector fields are chosen such that the associated reflected Brownian motions have stationary densities proportional to $p$ and these approximating processes converge weakly to the RBM in $S$. The property that $p$ is the density of an invariant measure for these processes is preserved in the limit. Corollary 1.1 follows by a simple ergodic argument for finite invariant measures.

## 2. Existence and Uniqueness of the RBM

For the remainder of this paper it is assumed that the skew symmetry condition (1.1) holds. Also, the following additional notation and terminology is needed. For each $i$ and $j$, let $F_{ij} = F_i \cap F_j$, the intersection of two (possibly non-distinct) faces, and define

$$F_i^0 = F_i \setminus \bigcup_{j \neq i} F_{ij},$$

so that $\bigcup_i F_i^0$ is the smooth part of the boundary of $G$. Then, $S = G \cup (\bigcup_i F_i^0)$ and $\bar{G} \setminus S = \bigcup_i \bigcup_{j \neq i} F_{ij}$. If $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}, P)$ is a filtered probability space and $M$: $[0, \infty) \times \Omega \to \mathbb{R}^d$, then we say $M$ is a ($d$-dimensional) (sub)martingale on $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}, P)$ if each real-valued component $M_i$, $i = 1, \ldots, d$, of $M$ is a (sub)martingale on $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}, P)$. If $M(0) = m_0 \in \mathbb{R}^d$ $P$-a.s., then $M$ is a *local* (sub)martingale on $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}, P)$ if there is a sequence of stopping times $\{\sigma_m, m = 1, 2, \ldots\}$ relative to $\{\mathscr{G}_t\}$ such that as $m \to \infty$, $\sigma_m \uparrow \infty$ $P$-a.s. and for each $m$,

$M(\cdot \wedge \sigma_m)$ is a (sub)martingale on $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}, P)$. A process $\{Y(t), t \geqq 0\}$ defined on $(\Omega, \mathscr{G}, \{\mathscr{G}_t\})$ is said to be adapted if $Y(t) \in \mathscr{G}_t$ for each $t \geqq 0$. A process defined on a probability space is said to be continuous or increasing (i.e., non-decreasing) if it has that property almost surely. It is said to be unique if it is unique up to indistinguishability.

Several times in the sequel it will be necessary to approximate $G$ by smooth bounded domains. For this, let $\{G_m, m=1,2,\ldots\}$ be a sequence of non-empty bounded domains with $C^3$ boundaries such that for each $m$ and $i$:

$$G_m \subset G_{m+1} \subset G, \qquad \partial G_m \cap F_i^0 \neq \emptyset, \qquad \partial G_m \cap (\bar{G} \setminus S) = \emptyset;$$

and

$$G = \bigcup_m G_m, \qquad S \cap \partial G = \bigcup_m (\partial G_m \cap \partial G).$$

For each domain $G_m$, let $n_m$ denote the inward unit normal vector field on $\partial G_m$ and let $u_m$ be a $C^2$ vector field on $\partial G_m$ such that $u_m \cdot n_m = 1$ and $u_m = v_i$ on $\partial G_m \cap F_i^0$ for all $i$. The symbol $|\cdot|$ will be used to denote the Euclidean norm.

The following lemma enables us to reduce the proof of Theorem 1.1 to the case $\mu = 0$. Here $\mathscr{F}^\mu$ denotes the completion of $\mathscr{F}$ with respect to $P_x^\mu$ and $\mathscr{F}_t^\mu$ denotes the augmentation of $\mathscr{F}_t$ by the $P_x^\mu$-null sets in $\mathscr{F}^\mu$. This completion and augmentation are introduced for technical reasons. In particular, for each fixed $t$, the stochastic integral $\int_0^t 1_G(\omega(s))\, d\omega(s)$ in (2.1) is defined as an $\mathscr{F}_t$-measurable random variable, but in the proof of Lemma 2.1 we shall need a continuous version of the stochastic process defined by these integrals. Such a version is only known to be adapted to the family of augmented $\sigma$-fields $\{\mathscr{F}_t^{\mu_0}, t \geqq 0\}$ where $\mu_0 \equiv 0$. (The superscript $\mu_0$ is used here to denote 0 so as to avoid conflict with the standard practice of using the superscript 0 to denote a raw (unaugmented) $\sigma$-field.)

**Lemma 2.1.** *Let $x \in S$. If $P_x^0$ satisfies conditions* (i)–(iii) *of Theorem* 1.1 *for $\mu = 0$, then for any fixed $\mu \in \mathbb{R}^d$,*

$$\left\{\alpha(t) \equiv \exp\left(\mu \cdot \int_0^t 1_G(\omega(s))\, d\omega(s) - \tfrac{1}{2}|\mu|^2 t\right), \ t \geqq 0\right\} \tag{2.1}$$

*is a martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x^0)$ and there is a unique probability measure $P_x^\mu$ on $(\Omega, \mathscr{F})$ such that:*

$$\frac{dP_x^\mu}{dP_x^0} = \alpha(t) \qquad \text{on } \mathscr{F}_t \text{ for all } t \geqq 0, \tag{2.2}$$

*and $P_x^\mu$ satisfies* (i)–(iii) *of Theorem* 1.1 *for this $\mu$. Conversely, if $P_x^\mu$ satisfies conditions* (i)–(iii) *of Theorem* 1.1 *for some $\mu \in \mathbb{R}^d$, then $\{(\alpha(t))^{-1}, t \geqq 0\}$ is a martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x^\mu)$ and there is a unique probability measure $P_x^0$ on $(\Omega, \mathscr{F})$ such that*

$$\frac{dP_x^0}{dP_x^\mu} = (\alpha(t))^{-1} \qquad \text{on } \mathscr{F}_t \text{ for all } t \geqq 0, \tag{2.3}$$

*and $P_x^0$ satisfies conditions* (i)–(iii) *of Theorem* 1.1 *for $\mu = 0$.*

*Proof.* Suppose $P_x^0$ satisfies conditions (i)–(iii) of Theorem 1.1 for $\mu=0$. Let $\{G_m\}$ and $\{u_m\}$ be the sequences of approximating domains and vector fields described above. For each $m$,

$$T_m \equiv \inf\{t \geqq 0: \omega(t) \notin G_m \cup (\partial G_m \cap \partial G)\} \tag{2.4}$$

is a stopping time relative to $\{\mathscr{F}_t\}$. By Doob's stopping theorem, condition (ii) of Theorem 1.1 holds with $t \wedge T_m$ and $P_x^0$ in place of $t$ and $P_x$, respectively. For each $m$ such that $x \in \bar{G}_m$, let $P_x^m$ denote the probability measure on $(\Omega, \mathscr{F})$ associated with the driftless reflected Brownian motion in $\bar{G}_m$ that has reflection vector field $u_m$ on $\partial G_m$ and starting point $x$. Then, since $G_m \subset G$ and $u_m = v_i$ on $\partial G_m \cap F_i^0$ for each $i$, it follows from the submartingale characterization of $P_x^m$ on $\mathscr{F}_{T_m}$ [15, Theorem 5.6] that:

$$P_x^0 = P_x^m \quad \text{on } \mathscr{F}_{T_m}.$$

In words this says that the RBM in $S$ associated with $P_x^0$ behaves like that in $\bar{G}_m$ up to the time $T_m$. In particular, since the following depend only on the history up to the time $T_m$ and are true for $P_x^m$ [15], they are also true for $P_x^0$. On the probability space $(\Omega, \mathscr{F}^{\mu_0}, P_x^0)$ where $\mu_0 \equiv 0$, we have

(a) $\left\{ \int_0^{t \wedge T_m} 1_G(\omega(s))\, d\omega(s), \mathscr{F}_t,\ t \geqq 0 \right\}$ is a $d$-dimensional martingale that has a continuous martingale version $M^m \equiv \{M^m(t), \mathscr{F}_t^{\mu^0},\ t \geqq 0\}$ with mutual variation process:

$$\langle M_i^m, M_j^m \rangle_t = \delta_{ij}(t \wedge T_m) \quad \text{for } i, j \in \{1, \ldots, d\},$$

(b) there is a continuous, increasing, $\{\mathscr{F}_t^{\mu_0}\}$-adapted, real-valued process $\{V^m(t),\ t \geqq 0\}$ satisfying the following three properties $P_x^0$-a.s.

$V^m(0) = 0$,
$V^m$ can only increase at those times $t$ for which $\omega(t) \in (\partial G_m \cap \partial G)$,
$V^m(t) = V^m(T_m)$ for all $t \geqq T_m$,

(c) the following decomposition holds $P_x^0$-a.s.:

$$\omega(t \wedge T_m) = x + M^m(t) + \sum_{i=1}^{k} v_i \int_0^t 1_{F_i^0}(\omega(s))\, dV^m(s) \quad \text{for all } t \geqq 0.$$

Moreover, $V^m$ is uniquely determined by (b)–(c).

Now, as $m \uparrow \infty$, $G_m \cup (\partial G_m \cap \partial G) \uparrow S$ and by Theorem 1.1(iii), $T_m \uparrow \infty$ $P_x^0$-a.s. By letting $m \to \infty$ in the above, we obtain on $(\Omega, \mathscr{F}^{\mu_0}, P_x^0)$:

(a′) $\left\{ \int_0^t 1_G(\omega(s))\, d\omega(s), \mathscr{F}_t,\ t \geqq 0 \right\}$ is a local martingale that has a continuous local martingale version $B \equiv \{B(t), \mathscr{F}_t^{\mu_0},\ t \geqq 0\}$ with mutual variation process:

$$\langle B_i, B_j \rangle_t = \delta_{ij}\, t,$$

(b′) $V(t) \equiv \limsup_{m \to \infty} V^m(t \wedge T_m)$ for all $t \geqq 0$ defines a continuous, increasing, $\{\mathscr{F}_t^{\mu_0}\}$-adapted process such that $P_x^0$-a.s.

$V(0)=0$,

$V$ can only increase at those times $t$ for which $\omega(t)\in\partial G\cap S$,

(c′) the following decomposition holds $P_x^0$-a.s.

$$\omega(t)=x+B(t)+\sum_{i=1}^{k} v_i\int_0^t 1_{F_i^0}(\omega(s))\,dV(s) \qquad \text{for all } t\geqq 0. \tag{2.5}$$

In the definition of $V$, the lim sup there is $P_x^0$-a.s. equal to the limit as $m\to\infty$ since by uniqueness the $V^m$'s are consistent.

Now (see e.g., [3], Sects. 2.12, 2.13, 8.4), (a′) characterizes $B$ under $P_x^0$ as a $d$-dimensional Brownian motion starting from the origin, and for $\mu\in\mathbb{R}^d$, $\{\alpha(t), \mathscr{F}_t, t\geqq 0\}$ is a $P_x^0$-martingale that has a continuous martingale version on $(\Omega, \mathscr{F}^{\mu_0}, \{\mathscr{F}_t^{\mu_0}\}, P_x^0)$ which satisfies the following $P_x^0$-a.s.

$$\alpha(t)=1+\mu\cdot\left(\int_0^t \alpha(s)\,dB(s)\right) \qquad \text{for all } t\geqq 0. \tag{2.6}$$

Then (2.2) is a special case of Girsanov's formula. This uniquely determines a probability measure $P_x^\mu$ on $(\Omega, \mathscr{F})$, i.e., $P_x^\mu$ can be uniquely extended from a finitely additive set function defined by (2.2) on $\bigcup_{t\in\mathbb{R}_+}\mathscr{F}_t$ to a probability measure on $\mathscr{F}=\bigvee_{t\in\mathbb{R}_+}\mathscr{F}_t$ ([3], Sect. 8.4).

Suppose $f\in C_c^2(\bar G)$ such that $Df\geqq 0$ on $\partial G$. Then by the local semimartingale decomposition (2.5) of $\omega$ and Itô's formula we have $P_x^0$-a.s.

$$\begin{aligned} f(\omega(t))-f(\omega(0)) &= \int_0^t \nabla f(\omega(s))\cdot dB(s)+\sum_{i=1}^{k}\int_0^t v_i\cdot\nabla f(\omega(s))\,1_{F_i^0}(\omega(s))\,dV(s)\\ &\quad +\tfrac{1}{2}\int_0^t \Delta f(\omega(s))\,ds\\ &= \int_0^t \nabla f(\omega(s))\cdot dB(s)+\int_0^t Df(\omega(s))\,dV(s)+\tfrac{1}{2}\int_0^t \Delta f(\omega(s))\,ds. \end{aligned} \tag{2.7}$$

Note that this gives an explicit form for the submartingale in condition (ii) of Theorem 1.1. Let $\mu\in\mathbb{R}^d$ and let $L$ be given by (1.3) for this $\mu$. Define

$$\chi(t)=f(\omega(t))-\int_0^t Lf(\omega(s))\,ds. \tag{2.8}$$

Then by combining (2.6)–(2.8) with the product rule of stochastic calculus we obtain $P_x^0$-a.s.

$$\begin{aligned} \alpha(t)\,\chi(t) &= \alpha(0)\,\chi(0)+\int_0^t \alpha(s)\,d\chi(s)+\int_0^t \chi(s)\,d\alpha(s)+\langle\alpha,\chi\rangle_t\\ &= f(\omega(0))+\int_0^t \alpha(s)\,\nabla f(\omega(s))\cdot dB(s)+\int_0^t \alpha(s)\,Df(\omega(s))\,dV(s)\\ &\quad -\int_0^t \alpha(s)\,\mu\cdot\nabla f(\omega(s))\,ds+\int_0^t \chi(s)\,d\alpha(s)+\int_0^t \alpha(s)\,\mu\cdot\nabla f(\omega(s))\,ds\\ &= f(\omega(0))+\int_0^t \alpha(s)\,\nabla f(\omega(s))\cdot dB(s)+\int_0^t \chi(s)\,d\alpha(s)+\int_0^t \alpha(s)\,Df(\omega(s))\,dV(s). \end{aligned}$$

In the line above, the last term is non-negative since $\alpha \geqq 0$, $Df \geqq 0$ on the support $\partial G$ of $V$, and $V$ is increasing. The second and third terms are local martingales on $(\Omega, \mathscr{F}^{\mu_0}, \{\mathscr{F}_t^{\mu_0}\}, P_x^0)$. Hence, $\alpha(\cdot)\chi(\cdot)$ is a local submartingale on $(\Omega, \mathscr{F}^{\mu_0}, \{\mathscr{F}_t^{\mu_0}\}, P_x^0)$. However, since $\alpha(\cdot)$ is a continuous martingale and $\chi(\cdot)$ is bounded on each bounded time interval, it follows that $\alpha(\cdot)\chi(\cdot)$ is in fact a submartingale on $(\Omega, \mathscr{F}^{\mu_0}, \{\mathscr{F}_t^{\mu_0}\}, P_x^0)$ ([2], Proposition 1.8). By the definition of $P_x^\mu$ and since $\chi(\cdot)$ is adapted to $\{\mathscr{F}_t\}$, it follows that $\chi(\cdot)$ is a submartingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x^\mu)$. Moreover, since $P_x^0$ and $P_x^\mu$ are mutually absolutely continuous over any finite time horizon, we have

$$P_x^0(\tau \leqq t) = 0 \quad \text{for all } t \geqq 0 \Leftrightarrow P_x^\mu(\tau \leqq t) = 0 \quad \text{for all } t \geqq 0.$$

But this is equivalent to:

$$P_x^0(\tau < \infty) = 0 \Leftrightarrow P_x^\mu(\tau < \infty) = 0.$$

The left equality above holds by Theorem 1.1 (iii) for $P_x^0$, and hence so does the right equality. It follows that $P_x^\mu$ satisfies conditions (i)–(iii) of Theorem 1.1.

The converse is proved similarly. In particular, under $P_x^\mu$, there is a continuous version of $\xi(t) \equiv \int_0^t 1_G(\omega(s))\, d\omega(s)$ that defines a Brownian motion with drift $\mu$ and

$$\exp\{-\mu \cdot (\xi(t) - \mu t) - \tfrac{1}{2}|\mu|^2 t\} = (\alpha(t))^{-1}$$

defines a martingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x^\mu)$. □

*Proof of Theorem 1.1.* First observe that it suffices to prove existence of a solution $P_x$ of (i)–(iii) for each $x \in S$. To see this, suppose $P_x$ is such a solution and $P_x^*$ is any solution of (i)–(ii). Let $G_m$, $u_m$, $T_m$ be as in Lemma 2.1. Then by the uniqueness of the solution $P_x^m$ of the submartingale problem associated with $L$ and $u_m$ on $\bar{G}_m$, we have

$$P_x^* = P_x^m = P_x \quad \text{on } \mathscr{F}_{T_m},$$

and so for each $t \geqq 0$,

$$\begin{aligned} P_x^*(\tau \leqq t) &= \lim_{m \to \infty} P_x^*(T_m \leqq t) \\ &= \lim_{m \to \infty} P_x(T_m \leqq t) \\ &= P_x(\tau \leqq t) \\ &= 0. \end{aligned}$$

Hence $P_x^*$ also satisfies (iii). Similarly, for any $A \in \mathscr{F}_t$ and all $s > t$,

$$P_x^*(A \cap \{\tau > s\}) = P_x(A \cap \{\tau > s\}).$$

By letting $s \uparrow \infty$ and using the fact that $P_x^*$ and $P_x$ satisfy (iii), we conclude that $P_x^* = P_x$ on $\bigcup_{t \in \mathbb{R}_+} \mathscr{F}_t$ and hence on $\mathscr{F}$.

A second simplification is that we may assume $\mu=0$. For, by Lemma 2.1, there is a (unique) solution of (i)–(iii) for some $\mu\in\mathbb{R}^d$ if and only if there is a (unique) solution for $\mu=0$. Thus, for the remainder of this proof, we shall assume $\mu=0$ and focus on proving the existence of a solution of (i)–(iii) for each $x\in S$.

We first consider the case where $\bar{G}$ is a $d$-dimensional polyhedral cone with vertex at the origin and give a proof by induction on the dimension $d$. In the course of this proof and later, in the extension to the general case of a simple polyhedron $\bar{G}$, we shall need an estimate of how quickly an RBM escapes from a neighborhood of a vertex. Consequently, the following two propositions will be proved by induction on $d$.

**Proposition 2.1.** *Let $(N, Q, b=0,\ \mu=0)$ be the data for a driftless $d$-dimensional RBM satisfying the skew symmetry condition* (1.1) *with $k=d$. In particular, $N$ and $Q$ are $d\times d$ matrices, $N=\tilde{N}$ is invertible, and $\bar{G}=\{x\in\mathbb{R}^d\colon Nx\geqq 0\}$ is a polyhedral cone with vertex at the origin. Then for each $x\in S$ there is a probability measure $P_x$ on $(\Omega,\mathscr{F})$ satisfying* (i)–(iii) *of Theorem* 1.1.

*Remark.* By the preceding discussion, any such $P_x$ is uniquely determined by (i)–(ii).

Assuming Proposition 2.1 holds, let $\mathscr{F}^x$ denote the completion of $\mathscr{F}$ with respect to the probability measure $P_x$ and let $\mathscr{F}_t^x$ denote the augmentation of $\mathscr{F}_t$ by the $P_x$-null sets in $\mathscr{F}^x$. By similar reasoning to that in the proof of Lemma 2.1, for each $x\in S$ there is a unique pair of continuous adapted $d$-dimensional processes $B$ and $V$ on $(\Omega,\mathscr{F}^x,\{\mathscr{F}_t^x\},P_x)$ such that the following hold.

(a) $B$ is a driftless $d$-dimensional Brownian motion starting from the origin.

(b) For each $i\in\{1,\dots,d\}$, the $i^{\text{th}}$ component $V_i$ of $V$ is an increasing process such that $P_x$-a.s.

$V_i(0)=0$, and

$V_i$ can only increase at those times $t$ for which $\omega(t)\in F_i^0$.

Thus (cf. Lemma 2.1), $V_i(t)=\left(\int_0^t 1_{F_i^0}(\omega(s))\,dV(s)\right)_i$.

(c) The following decomposition of $\omega$ holds $P_x$-a.s.

$$\omega(t)=x+B(t)+(N'+Q')\,V(t)\qquad\text{for all } t\geqq 0.$$

**Proposition 2.2.** *For each $\beta>0$, there is $t>0$ and $\delta\in(0,\beta)$ such that for each $x\in S$ satisfying $|x|<\delta$,*

$$P_x\{\max_{0\leqq s\leqq t}|B(s)|\leqq\beta,\ |V(t)|\leqq\beta\}\geqq\delta. \tag{2.9}$$

*In general, $t$ and $\delta$ will depend on $\beta$, $N$, $Q$ and $d$.*

For the induction proof of these propositions, we consider $d=1$ first. In this case, $\bar{G}=S$ is either $[0,+\infty)$ or $(-\infty,0]$, and it is well known [7] that there is a solution of (i)–(iii) for each $x\in S$. For the proof of Proposition 2.2, by symmetry, we may suppose $\bar{G}=[0,\infty)$. Then $V$ can be represented explicitly in terms of $B$ [2, 7]:

$$V(t)=(-\min_{0\leqq s\leqq t}(x+B(s)))^{+}$$
$$\leqq \max_{0\leqq s\leqq t}|B(s)|.$$

Hence the left member of (2.9) is equal to

$$P_x\{\max_{0\leqq s\leqq t}|B(s)|\leqq\beta\}.$$

Under $P_x$, $B$ is a one-dimensional Brownian motion starting from the origin, and so the above probability is the same for all $x\in\bar{G}$ and is strictly positive for any $\beta>0$ and $t\geqq 0$. Hence, Proposition 2.2 holds for $d=1$.

For the induction step, suppose $d\geqq 2$ and Propositions 2.1 and 2.2 hold in all dimensions less than $d$. Let $(N, Q, b=0, \mu=0)$ be data as described in Proposition 2.1. Our candidate for a solution to (i)–(iii) is obtained as follows. Consider the sequence $\{G_m, m=1,2,\dots\}$ of bounded $C^3$ domains defined at the beginning of this section. We shall make a particular choice of vector field $u_m$ on $\partial G_m$. Since the matrix $N$ for the polyhedral cone $\bar{G}$ is $d\times d$ invertible and (1.1) holds, it follows from the proof of Lemma 3.2 in [9] that $u_m\equiv v_i$ on $\partial G_m\cap F_i^0$, $i=1,\dots,d$ can be uniquely extended to a $C^2$ vector field $u_m\equiv n_m+q_m$ on $\partial G_m$ such that the following skew symmetry condition holds:

$$n_m(\sigma^*)\cdot q_m(\sigma)+q_m(\sigma^*)\cdot n_m(\sigma)=0 \qquad \text{for all } \sigma,\sigma^*\in\partial G_m. \tag{2.10}$$

For each $m$, let $\{P_x^m, x\in\bar{G}_m\}$ denote the family of probability measures on $(\Omega,\mathscr{F})$ associated with the driftless reflected Brownian motion in $\bar{G}_m$ having reflection vector field $u_m$ on $\partial G_m$. More precisely, for each $x\in\bar{G}_m$, $P_x^m$ is the unique probability measure on $(\Omega,\mathscr{F})$ satisfying the following three properties [15].

(I) $P_x^m(\omega(0)=x)=1$.

(II) For each $f\in C_c^2(\bar{G})$ that satisfies

$$u_m\cdot\nabla f\geqq 0 \qquad \text{on } \partial G_m$$

we have

$$\left\{f(\omega(t))-\tfrac{1}{2}\int_0^t \Delta f(\omega(s))\,ds,\ t\geqq 0\right\}$$

is a submartingale on $(\Omega,\mathscr{F},\{\mathscr{F}_t\},P_x^m)$.

(III) $P_x^m(\omega(t)\in\bar{G}_m$ for all $t\geqq 0)=1$.

Let $T_m$ be given by (2.4). Each $x\in S$ is in $\bar{G}_m$ for all $m$ sufficiently large. For such $m$'s, the probability measures $P_x^m$ on $\mathscr{F}_{T_m}$ are consistent, i.e., $P_x^l=P_x^m$ on $\mathscr{F}_{T_m}$ for all $l\geqq m$, and so induce a *finitely* additive set function $P_x$ on the ring $\bigcup_m\mathscr{F}_{T_m}$. However, $P_x$ need not be *countably* additive on $\bigcup_m\mathscr{F}_{T_m}$ and so is not necessarily extendable to a probability measure on $\mathscr{F}_\tau$. Intuitively, this corresponds to the fact that we can construct paths of an RBM prior to the time $\tau$ by taking pathwise limits of RBM's in $\bar{G}_m$ stopped on exit from

$G_m \cup (\partial G_m \cap \partial G)$, but as these paths approach $\bar{G} \backslash S$ they need not have a single limit point in $\bar{G} \backslash S$, and so on $\{\tau < \infty\}$ they need not be continuously extendable to $[0, \tau]$. Eventually we will show that under the skew symmetry condition, $P_x$ can be extended to a probability measure on $\mathscr{F}$ with $\tau = \infty$ $P_x$-a.s. However, since we do not know this a priori, we must initially define an extension of $P_x$ on a sufficiently rich probability space, namely, one that allows killing at the time $\tau$.

For this, let $S^\partial = S \cup \{\partial\}$ where $\partial$ is a (cemetery) point isolated from $\bar{G}$. Let $\Omega^\partial$ denote the set of all right continuous functions $\omega$: $[0, \infty) \to S^\partial$ such that $\omega$ is continuous on $[0, \tau)$ where $\tau = \inf\{s \geqq 0: \omega(s) \notin S\}$ and $\omega(s) = \partial$ for all $s \geqq \tau$. The $\sigma$-fields $\mathscr{F}^\partial$, $\mathscr{F}_t^\partial$ and $\mathscr{F}_T^\partial$ are defined on $\Omega^\partial$ in the same way as those without the $\partial$'s are defined on $\Omega$. With $\omega \in \Omega^\partial$, (2.4) defines an extension of $T_m$ to a stopping time on $(\Omega^\partial, \mathscr{F}^\partial, \{\mathscr{F}_t^\partial\})$. The probability measures $P_x^m$ are concentrated on $\{\omega \in \Omega: \omega(s) \in \bar{G}_m \subset S$ for all $s \geqq 0\}$ and so can be uniquely extended to probability measures on $(\Omega^\partial, \mathscr{F}^\partial)$. These extended measures will again be denoted by $P_x^m$. For each $x \in S$, the consistent sequence $\{P_x^m|_{\mathscr{F}_{T_m}^\partial}, m = 1, 2, \ldots\}$ induces a unique probability measure $P_x^\tau$ on $(\Omega^\partial, \mathscr{F}^\partial)$ such that $P_x^\tau = P_x^m$ on $\mathscr{F}_{T_m}^\partial$ for all $m$ sufficiently large that $x \in G_m$, and $P_x^\tau(\omega(t) = \partial$ for all $t \geqq \tau) = 1$. In particular, $P_x^\tau$ is uniquely characterized by the following four properties. Here we adopt the usual convention that functions defined on $\bar{G}$ are automatically extended to be zero at the cemetery $\partial$.

(i′) $P_x^\tau(\omega(0) = x) = 1$.

(ii′) For each $m$ and each $f \in C_c^2(\bar{G})$ that satisfies

$$Df \geqq 0 \quad \text{on } \partial G,$$

we have

$$\left\{ f(\omega(t \wedge T_m)) - \tfrac{1}{2} \int_0^{t \wedge T_m} \Delta f(\omega(s))\, ds,\ t \geqq 0 \right\}$$

is a submartingale on $(\Omega^\partial, \mathscr{F}^\partial, \{\mathscr{F}_t^\partial\}, P_x^\tau)$.

(iii′) $P_x^\tau(\omega(t) \in S$ for $0 \leqq t \leqq T_m) = 1$.

(iv′) $P_x^\tau(\omega(t) = \partial$ for all $t \geqq \tau) = 1$.

We define $P_\partial^\tau$ to be the unit mass at $\omega_\partial \equiv \partial$. Then the family $\{P_x^\tau, x \in S^\partial\}$ has the strong Markov property. To prove the existence of a solution of (i)–(iii) of Theorem 1.1, it suffices to show that for each $x \in S$,

$$P_x^\tau(\tau < \infty) = 0. \tag{2.11}$$

For then $P_x^\tau$ induces a suitable probability measure $P_x$ on $(\Omega, \mathscr{F})$ by

$$P_x(A) = P_x^\tau(A \cap \{\tau = \infty\}) \quad \text{for all } A \in \mathscr{F}.$$

For the proof of (2.11), we first prove that for each $z \in \bar{G} \backslash \{0\}$ there are nonempty open balls $U_1(z) \subset U(z)$ centered at $z$ in $\mathbb{R}^d$, and $t(z) > 0$ and $\delta(z) > 0$, such that $U(z)$ has radius less than $\frac{1}{2}$ and for all $x \in U_1(z) \cap S$,

$$P_x^\tau\{\omega(t \wedge T_{U(z)}) \in S \text{ for all } t \in [0, \infty)\} = 1 \tag{2.12}$$

and

$$P_x^\tau\{T_{U(z)} \geqq t(z)\} \geqq \delta(z), \tag{2.13}$$

where $T_{U(z)} \equiv \inf\{t \geqq 0: \omega(t) \notin U(z)\}$. For this, fix $z \in \bar{G} \setminus \{0\}$. If $z \in G$, then for $m$ sufficiently large, $z \in G_m$ and there is a non-empty open ball $U(z)$ centered at $z$ of radius less than $\frac{1}{2}$ such that $U(z) \subset G_m$. Then, for $x \in U(z)$, since $P_x^\tau = P_x^m$ on $\mathscr{F}_{T_m}$ and the RBM associated with $P_x^m$ behaves like driftless $d$-dimensional Brownian motion up to the time $T_{U(z)} < T_m$ $P_x^m$-a.s., it follows that there is a non-empty open ball $U_1(z) \subset U(z)$, and $t(z) > 0$ and $\delta(z) > 0$, such that (2.12)–(2.13) hold for all $x \in U_1(z)$.

For $z \in \partial G \setminus \{0\}$, let $k(z)$ denote the maximum number of faces of the polyhedral cone $\bar{G} = \{x \in \mathbb{R}^d: Nx \geqq 0\}$ that contain $z$. Note, since $z \neq 0$, we have $1 \leqq k(z) < d$. By relabelling the faces if necessary, we may denote the faces containing $z$ by $F_1, \ldots, F_{k(z)}$. Let $N(z)$ (resp. $Q(z)$) denote the $k(z) \times d$ matrix whose rows are given by the normals (resp. $q$-vectors) associated with these faces. Let $U(z)$ be an non-empty open ball in $\mathbb{R}^d$ centered at $z$ and of radius less than $\frac{1}{2}$ such that $U(z)$ is disjoint from all the faces of $\bar{G}$ except $F_1, \ldots, F_{k(z)}$. Since $\bar{G}$ is simple, the normals to the faces $F_1, \ldots, F_{k(z)}$ are linearly independent ([1], Theorem 12.14) and so generate a vector space $H(z)$ of dimension $k(z)$. Any vector $x \in \mathbb{R}^d$ can be uniquely decomposed: $x = \tilde{x} + \hat{x}$ where $\tilde{x}$ is the orthogonal projection of $x$ on $H(z)$ and $\hat{x}$ is the orthogonal projection of $x$ on the orthogonal complement of $H(z)$. Indeed, by performing a change of basis if necessary, we may view $\tilde{x}$ as a vector in $\mathbb{R}^{k(z)}$ and $\hat{x}$ as a vector in $\mathbb{R}^{d-k(z)}$ so that $x = (\tilde{x}, \hat{x})$ and $H(z)$ is identified with $\mathbb{R}^{k(z)}$. Let $\tilde{N}$ and $\tilde{Q}$ denote the $k(z) \times k(z)$ matrices whose rows are respectively given by $\tilde{n}'_1, \ldots, \tilde{n}'_{k(z)}$ and $\tilde{q}'_1, \ldots, \tilde{q}'_{k(z)}$; and let $\hat{Q}$ denote the $k(z) \times (d - k(z))$ matrix whose rows are given by $\hat{q}'_1, \ldots, \hat{q}'_{k(z)}$. Then $(\tilde{N}, \tilde{Q}, \tilde{b} = 0, \tilde{\mu} = 0)$ are data for a $k(z)$-dimensional RBM in the polyhedron $\bar{G}_z = \{\tilde{x} \in \mathbb{R}^{k(z)}: \tilde{N}\tilde{x} \geqq 0\}$. Since (1.1) is assumed to hold and

$$\tilde{n}_i \cdot \tilde{q}_j = n_i \cdot q_j \qquad \text{for all } i, j \in \{1, \ldots, k(z)\},$$

it follows that this data satisfies the skew symmetry condition.

Let $S_z$ denote the smooth part of $\bar{G}_z$ and let $\tilde{\Omega}$ denote the set of continuous functions $\tilde{\omega}: [0, \infty) \to \mathbb{R}^{k(z)}$ satisfying $\tilde{\omega}(0) \in S_z$. Let $\tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t$ be defined on $\tilde{\Omega}$ in the same way that $\mathscr{F}$ and $\mathscr{F}_t$ are defined on $\Omega$. Since the data $(\tilde{N}, \tilde{Q}, \tilde{b} = 0, \tilde{\mu} = 0)$ satisfy the skew symmetry condition, by the induction assumption, for this data and each $\tilde{x} \in S_z$ there is a probability measure $\tilde{P}_{\tilde{x}}$ on $(\tilde{\Omega}, \tilde{\mathscr{F}})$ corresponding to the RBM starting from $\tilde{x}$. Moreover, by the discussion preceding Proposition 2.2, the following decomposition holds $\tilde{P}_{\tilde{x}}$-a.s.:

$$\tilde{\omega}(t) = \tilde{x} + \tilde{B}(t) + (\tilde{N}' + \tilde{Q}')\, \tilde{V}(t) \qquad \text{for all } t \geqq 0,$$

where $\tilde{B}$ and $\tilde{V}$ are continuous adapted $k(z)$-dimensional processes on $(\tilde{\Omega}, \tilde{\mathscr{F}}^{\tilde{x}}, \{\tilde{\mathscr{F}}_t^{\tilde{x}}\}, \tilde{P}_{\tilde{x}})$ with the following properties.

(a) $\tilde{B}$ is a driftless $k(z)$-dimensional Brownian motion starting from the origin, and

(b) for each $i$, $\tilde{V}_i$ is an increasing process such that $\tilde{P}_{\tilde{x}}$-a.s.

$\tilde{V}_i(0) = 0$ and

$\tilde{V}_i$ can only increase at those times $t$ for which $\tilde{\omega}(t) \in \tilde{F}_i^0$, where $\tilde{F}_i^0$ is the part of $\tilde{F}_i = \{\tilde{y} \in \bar{G}_z: \tilde{n}_i \cdot \tilde{y} = 0\}$ that does not meet any other face of $\bar{G}_z$.

Here the completion $\tilde{\mathscr{F}}^{\tilde{x}}$ and augmentation $\tilde{\mathscr{F}}_t^{\tilde{x}}$ are defined in the obvious way.

Let $\hat{B}$ be a $(d-k(z))$-dimensional Brownian motion starting from the origin defined on some probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}_0)$ such that $\hat{B}$ is independent of $\tilde{B}$ and $\tilde{V}$. For $x=(\tilde{x}, \hat{x}) \in U(z) \cap S$, define $\bar{X}=(\tilde{X}, \hat{X})$ such that for each $(t, \tilde{\omega}, \hat{\omega}) \in [0, \infty) \times \tilde{\Omega} \times \hat{\Omega}$,

$$\tilde{X}(t, \tilde{\omega}, \hat{\omega}) = \tilde{x} + \tilde{B}(t, \tilde{\omega}) + (\tilde{N}' + \tilde{Q}')\, \tilde{V}(t, \tilde{\omega}), \tag{2.14}$$

$$\hat{X}(t, \tilde{\omega}, \hat{\omega}) = \hat{x} + \hat{B}(t, \hat{\omega}) + \hat{Q}'\, \tilde{V}(t, \tilde{\omega}). \tag{2.15}$$

Define $\bar{\Omega} = \tilde{\Omega} \times \hat{\Omega}$, $\bar{\mathscr{F}} = \tilde{\mathscr{F}} \times \hat{\mathscr{F}}$, $\bar{P}_x = \tilde{P}_x \times \hat{P}_0$,

$$\bar{T}_m = \inf\{t \geqq 0\colon\ \bar{X}(t) \notin G_m \cup (\partial G_m \cap \partial G)\},$$

$$\bar{T}_{U(z)} = \inf\{t \geqq 0\colon\ \bar{X}(t) \notin U(z)\}.$$

Note that for $m$ sufficiently large, $x \in \bar{G}_m$ and $\bar{P}_x$-a.s.: $\bar{X}(t \wedge \bar{T}_m) \in \bar{G}_m$ for all $t \geqq 0$. It follows from (2.14)–(2.15) and Itô's formula that, for $x \in \bar{G}_m$, the probability measure induced on the canonical space $(\Omega, \mathscr{F})$ by $\bar{X}(\cdot \wedge \bar{T}_m)$ under $\bar{P}_x$ satisfies the submartingale characterization of $P_x^m$ on $\mathscr{F}_{T_m}$, and hence agrees with $P_x^\tau$ on $\mathscr{F}_{T_m}$.

Now, for $m$ sufficiently large that $x \in \bar{G}_m$, on $\{\bar{T}_m < \infty\}$ we have $\bar{X}(\bar{T}_m) \in \partial G_m \backslash (F_1^0 \cup \ldots \cup F_{k(z)}^0)$ $\bar{P}_x$-a.s. Then, since $\limsup_{m \to \infty} (U(z) \cap (\partial G_m \backslash (F_1^0 \cup \ldots \cup F_{k(z)}^0))) = \emptyset$ and $\bar{X}$ has continuous paths in $S_z \times \mathbb{R}^{d-k(z)}$, we have:

$$\bar{P}_x(\bar{T}_m \geqq \bar{T}_{U(z)} \text{ for some } m,\ \bar{T}_{U(z)} < \infty) + \bar{P}_x(\lim_m \bar{T}_m = \bar{T}_{U(z)} = \infty) = 1.$$

By the definition of $P_x^\tau$, the above also holds with $P_x^\tau$, $T_m$, and $T_{U(z)}$, in place of $\bar{P}_x$, $\bar{T}_m$, and $\bar{T}_{U(z)}$, respectively, and hence

$$P_x^\tau\{\omega(t \wedge T_{U(z)}) \in S \text{ for all } t \geqq 0\} = 1, \tag{2.16}$$

and the probability measure induced on $(\Omega, \mathscr{F})$ by $\bar{X}(\cdot \wedge \bar{T}_{U(z)})$ under $\bar{P}_x$ agrees with $P_x^\tau$ on $\mathscr{F}_{T_{U(z)}}$.

Let $\bar{B} = (\tilde{B}, \hat{B})$. Then, from the representation (2.14)–(2.15), it follows that there is $\beta(z) \in (0, 1/2)$ such that the open ball $U_2(z)$ in $\mathbb{R}^d$ centered at $z$ and of radius $\beta(z)$ is contained in $U(z)$, and for each $x \in U_2(z) \cap S$ we have $\bar{P}_x$-a.s.

$$\{\max_{0 \leqq s \leqq t} |\bar{B}(s)| \leqq \beta(z),\ |\tilde{V}(t)| \leqq \beta(z)\} \subset \{\bar{X}(s) \in U(z) \text{ for all } 0 \leqq s \leqq t\}.$$

It is proved below that there is $t(z) > 0$ and $\delta(z) \in (0, \beta(z))$ such that

$$\bar{P}_x\{\max_{0 \leqq s \leqq t(z)} |\bar{B}(s)| \leqq \beta(z),\ |\tilde{V}(t(z))| \leqq \beta(z)\} \geqq \delta(z) \tag{2.17}$$

for all $x \in U_1(z) \cap S$ where $U_1(z) = \{x \in \mathbb{R}^d\colon |x - z| < \delta(z)\}$. It then follows that

$$\bar{P}_x\{\bar{T}_{U(z)} \geqq t(z)\} \geqq \delta(z) \qquad \text{for all } x \in U_1(z) \cap S. \tag{2.18}$$

Then by (2.16) and the remarks following it, we see that (2.12)–(2.13) hold. For the proof of (2.17), note that for each $x \in U_2(z) \cap S$,

$$\bar{P}_x\{\max_{0 \leqq s \leqq t} |\bar{B}(s)| \leqq \beta(z),\ |\tilde{V}(t)| \leqq \beta(z)\}$$
$$\geqq \hat{P}_0\left\{\max_{0 \leqq s \leqq t} |\hat{B}(s)| \leqq \frac{\beta(z)}{2}\right\} \tilde{P}_{\tilde{x}}\left\{\max_{0 \leqq s \leqq t} |\tilde{B}(s)| \leqq \frac{\beta(z)}{2},\ |\tilde{V}(t)| \leqq \frac{\beta(z)}{2}\right\}.$$

Then by the properties of the $(d-k(z))$-dimensional Brownian motion $\hat{B}$ and the induction hypothesis that Proposition 2.2 holds in dimension $k(z) < d$, it follows that there is $t(z) > 0$ and $\delta(z) \in (0, \frac{1}{2}\beta(z))$ such that (2.17) holds.

Now, let $0 < \varepsilon < K < \infty$ and define $\bar{G}_{\varepsilon K} = \{x \in \bar{G}: \varepsilon \leqq |x| \leqq K\}$. Then $\{U_1(z): z \in \bar{G}_{\varepsilon K}\}$ is an open cover of the compact set $\bar{G}_{\varepsilon K}$ and so has a finite subcover $U_1(z_1), \ldots, U_1(z_l)$, say. Let $t_{\varepsilon K} = \min_{i=1}^{l} t(z_i)$ and $\delta_{\varepsilon K} = \min_{i=1}^{l} \delta(z_i)$. Define $\tau_{\varepsilon K} = \inf\{t \geqq 0: \omega(t) \notin \bar{G}_{\varepsilon K}\}$. Fix $x \in \bar{G}_{\varepsilon K} \cap S$. We inductively define sequences $\{i(j), j = 1, 2, \ldots\}$ and $\{\sigma_j, j = 1, 2, \ldots\}$ on $\Omega^\partial$, and an increasing family of $P_x^\tau$-null sets $\{N_j, j = 1, 2, \ldots\}$ in $\mathscr{F}^\partial$ as follows. Let $i(1) \in \{1, \ldots, l\}$ such that $x \in U_1(z_{i(1)})$, and define $\sigma_1 = \inf\{t \geqq 0: \omega(t) \notin U(z_{i(1)})\} \wedge \tau_{\varepsilon K}$. By (2.12), $P_x^\tau\{\omega(t \wedge \sigma_1) \in S$ for all $t \geqq 0\} = 1$. Hence, there is a $P_x^\tau$-null set $N_1 \in \mathscr{F}^\partial$ such that

$$\omega(\sigma_1) \in \bar{G}_{\varepsilon K} \cap S \quad \text{ for all } \omega \in \{\sigma_1 < \infty\} \cap N_1^c.$$

Here the superscript $c$ is used to denote the complement of a set in $\Omega^\partial$. Suppose $j \geqq 2$ and $i(j-1)$, $\sigma_{j-1}$ and $N_{j-1}$ have been defined such that $P_x^\tau(\omega(t \wedge \sigma_{j-1}) \in S$ for all $t \geqq 0) = 1$, and $N_{j-1}$ is a $P_x^\tau$-null set such that $\omega(\sigma_{j-1}) \in \bar{G}_{\varepsilon K} \cap S$ for each $\omega \in \{\sigma_{j-1} < \infty\} \cap N_{j-1}^c$. Then on $\{\sigma_{j-1} = \infty\} \cup N_{j-1}$, define $i(j) = i(j-1)$ and let $\sigma_j = \sigma_{j-1}$, and on $\{\sigma_{j-1} < \infty\} \cap N_{j-1}^c$, let $i(j)$ be such that $\omega(\sigma_{j-1}) \in U_1(z_{i(j)})$ and define $\sigma_j = \inf\{t \geqq \sigma_{j-1}: \omega(t) \notin U(z_{i(j)})\} \wedge \tau_{\varepsilon K}$. Then it follows from (2.12) that $P_x^\tau(\omega(t \wedge \sigma_j) \in S$ for all $t \geqq 0) = 1$ and hence there is a $P_x^\tau$-null set $N_j \supset N_{j-1}$ such that $\omega(\sigma_j) \in \bar{G}_{\varepsilon K} \cap S$ for all $\omega \in \{\sigma_j < \infty\} \cap N_j^c$.

We shall now prove that

$$P_x^\tau\{\sigma_j = \tau_{\varepsilon K} \text{ some } j,\ \tau_{\varepsilon K} < \infty\} + P_x^\tau\{\lim_{j \to \infty} \sigma_j = \tau_{\varepsilon K} = \infty\} = 1. \tag{2.19}$$

From this it follows that

$$P_x^\tau(\omega(t \wedge \tau_{\varepsilon K}) \in S \text{ for all } t \geqq 0) = 1. \tag{2.20}$$

Note that since $\sigma_j \leqq \tau_{\varepsilon K}$, to prove (2.19) it suffices to show that

$$\{\sigma_j < \tau_{\varepsilon K} \text{ for all } j\} \subset \{\lim_{j \to \infty} \sigma_j = \infty\} \quad P_x^\tau\text{-a.s.}. \tag{2.21}$$

For this, note that by the strong Markov property we have $P_x^\tau$-a.s.

$$\sum_{j=1}^{\infty} P_x^\tau(\sigma_j < \infty,\ \sigma_{j+1} - \sigma_j \geqq t_{\varepsilon K} | \mathscr{F}_{\sigma_j}) \geqq \sum_{j=1}^{\infty} 1_{\{\sigma_j < \tau_{\varepsilon K}\}} P_{\omega(\sigma_j)}^\tau(T_{U(z_{i(j+1)})} \geqq t_{\varepsilon K})$$
$$\geqq \sum_{j=1}^{\infty} 1_{\{\sigma_j < \tau_{\varepsilon K}\}} \delta_{\varepsilon K},$$

where the last inequality follows by (2.13). Thus, $P_x^\tau$-a.s.

$$\{\sigma_j<\tau_{\varepsilon K} \text{ for all } j\}=\left\{\sum_{j=1}^{\infty} 1_{\{\sigma_j<\tau_{\varepsilon K}\}}\delta_{\varepsilon K}=\infty\right\}$$

$$\subset\left\{\sum_{j=1}^{\infty} P_x^\tau(\sigma_j<\infty,\ \sigma_{j+1}-\sigma_j\geqq t_{\varepsilon K}\,|\,\mathscr{F}_{\sigma_j})=\infty\right\}.$$

By an extension of the Borel-Cantelli lemma ([5], Corollary 2.3), the last set above is $P_x^\tau$-a.s. equal to

$$\{\sigma_j<\infty,\ \sigma_{j+1}-\sigma_j\geqq t_{\varepsilon K} \text{ for infinitely many } j\}.$$

Hence (2.21) and so (2.19)–(2.20) hold.

It follows from (2.20) and the characterization (i′)–(iv′) that for each $x\in S$, $P_x^\tau$ is the unique probability measure on $(\Omega^\partial, \mathscr{F}^\partial)$ satisfying the following four properties.

(i″) $P_x^\tau(\omega(0)=x)=1$.

(ii″) For each $f\in C_c^2(\bar{G})$ that satisfies

$$Df\geqq 0 \quad \text{on } \partial G$$

and each $0<\varepsilon<|x|<K<\infty$, we have

$$\left\{f(\omega(t\wedge\tau_{\varepsilon K}))-\tfrac{1}{2}\int_0^{t\wedge\tau_{\varepsilon K}} \Delta f(\omega(s))\,ds,\ t\geqq 0\right\}$$

is a submartingale on $(\Omega^\partial, \mathscr{F}^\partial, \{\mathscr{F}_t^\partial\}, P_x^\tau)$.

(iii″) For each $0<\varepsilon<|x|<K<\infty$,

$$P_x^\tau(\omega(t\wedge\tau_{\varepsilon K})\in S \text{ for all } t\geqq 0)=1.$$

(iv″) $\tau=\lim_{\varepsilon\downarrow 0}\tau_{\varepsilon\varepsilon^{-1}}$, $P_x^\tau$-a.s. and

$$P_x^\tau(\omega(t)=\partial \text{ for all } t\geqq\tau)=1.$$

For the proofs of (2.11) and Proposition 2.2, the following two lemmas are needed. The first of these describes a "scaling" property of the family of probability measures $\{P_x^\tau,\ x\in S\}$. It is analogous to Proposition 2.9 in [16] and Lemma 4.3 in [20]. The main inequality (2.32) used in the proof of the second lemma is analogous to (3.28) in [16].

For notational convenience, we define $\lambda^{-1}\{\partial\}=\{\partial\}$ for any $\lambda>0$.

**Lemma 2.2.** *Let $x\in S$ and $\lambda>0$. Then for each $A\in\mathscr{F}^\partial$,*

$$P_x^\tau(A)=P_{\lambda x}^\tau(\lambda^{-1}\omega(\lambda^2\,\cdot)\in A). \tag{2.22}$$

*Proof.* For each $A\in\mathscr{F}^\partial$, let $Q_x(A)$ denote the right member of (2.22). By the characterization of $P_x^\tau$, to prove (2.22) it suffices to verify that $Q_x$ satisfies (i″)–(iv″) with $Q_x$ in place of $P_x^\tau$.

Properties (i″), (iii″) and (iv″) for $Q_x$ follow easily from those for $P^\tau_{\lambda x}$ and the facts that $\lambda S = S$,

$$\tau_{\lambda\varepsilon, \lambda K}(\omega) = \lambda^2 \tau_{\varepsilon K}(\lambda^{-1}\omega(\lambda^2 \cdot)) \quad \text{for } 0<\varepsilon<K<\infty, \tag{2.23}$$

and

$$\tau(\omega) = \lambda^2 \tau(\lambda^{-1}\omega(\lambda^2 \cdot)). \tag{2.24}$$

For (ii″), if $f \in C^2_c(\bar{G})$ satisfies $Df \geqq 0$ on $\partial G$ then so does $f(\lambda^{-1}\cdot)$, since $\bar{G}$ is a cone with vertex at the origin and the directions of reflection are constant on each face. Then by applying property (ii″) of $P^\tau_{\lambda x}$ to $f(\lambda^{-1}\cdot)$ and performing a change of variable in the time integration (from $s$ to $\lambda^{-2}s$), we conclude that for $0<\varepsilon<|x|<K<\infty$,

$$\left\{ f(\lambda^{-1}\omega(\lambda^2 t \wedge \tau_{\lambda\varepsilon, \lambda K}(\omega))) - \tfrac{1}{2} \int_0^{\tau(t,\omega)} (\Delta f)(\lambda^{-1}\omega(\lambda^2 s))\, ds,\ \mathscr{F}^\partial_{\lambda^2 t},\ t \geqq 0 \right\}$$

is a $P^\tau_{\lambda x}$-submartingale, where $\tau(t,\omega) = t \wedge \lambda^{-2}\tau_{\lambda\varepsilon, \lambda K}(\omega)$. Then by the definition of $Q_x$ and (2.23), it follows that

$$\left\{ f(\omega(t \wedge \tau_{\varepsilon K})) - \tfrac{1}{2} \int_0^{t \wedge \tau_{\varepsilon K}} \Delta f(\omega(s))\, ds,\ \mathscr{F}^\partial_t,\ t \geqq 0 \right\}$$

is a $Q_x$-submartingale. □

For each $t \in [0,\infty)$, define the extended real-valued random variable $A(t) \equiv A(t,\cdot)$ on $\Omega^\partial$ by

$$A(t,\omega) = \begin{cases} \displaystyle\int_0^t \frac{1}{|\omega(s)|^2}\, ds & \text{for } t<\tau, \\ \infty & \text{for } t \geqq \tau. \end{cases} \tag{2.25}$$

Note that $A(t)<\infty$ on $\{\omega: t<\tau(\omega)\}$. For each $t \in [0,\infty)$, let

$$\sigma(t) = \inf\{s \geqq 0:\ A(s) > t\}.$$

From the next lemma it follows that for each $x \in S$, $P^\tau_x\{\sigma(t)<\tau \text{ for all } t \in [0,\infty)\} = 1$.

For ease of notation, in the sequel $E^\tau_x$ will be used to denote expectation with respect to $P^\tau_x$.

**Lemma 2.3.** *For each $x \in S$ we have*

$$P^\tau_x(A(\tau-) \equiv \lim_{t \uparrow \tau} A(t) = \infty) = 1. \tag{2.26}$$

*Proof.* Define

$$\phi(x) = E^\tau_x[\exp(-\int_{[0,\tau)} |\omega(s)|^{-2}\, ds)] \quad \text{for each } x \in S. \tag{2.27}$$

To prove (2.26), it suffices to show that $\phi \equiv 0$ on $S$. Now, by Lemma 2.2 and (2.24), for each $\lambda>0$ and $x \in S$,

$$\phi(\lambda x) = E^\tau_x[\exp(-\int_{[0,\lambda^2\tau(\omega))} |\lambda\omega(\lambda^{-2}s)|^{-2}\, ds)] = \phi(x), \tag{2.28}$$

where the last equality follows by a change of variable in the integration. By the strong Markov property and (2.20), for $x \in S$ satisfying $|x|=1$ and any $t>0$ we have

$$\phi(x)=E_x^\tau\left[\exp\left(-\int_0^{\eta\wedge t}|\omega(s)|^{-2}\,ds\right)\phi(\omega(\eta\wedge t))\right], \tag{2.29}$$

where

$$\eta \equiv \inf\{s \geqq 0\colon\ \omega(s)\notin \bar{G}_{1/2,2}\} \tag{2.30}$$

and

$$\bar{G}_{1/2,2}\equiv\{z\in\bar{G}\colon\ \tfrac{1}{2}\leqq|z|\leqq 2\}.$$

Let

$$K\equiv\sup\{\phi(x)\colon\ x\in S\}.$$

By (2.28), $K=\sup\{\phi(x)\colon x\in S,\ |x|=1\}$, and so from (2.29) we obtain

$$K\leqq \sup_{|x|=1} E_x^\tau\left[\exp\left(-\int_0^{\eta\wedge t}|\omega(s)|^{-2}\,ds\right)\right]K, \tag{2.31}$$

where the supremum is over $x\in S$ satisfying $|x|=1$. The following crucial estimate will be used to show $K=0$. It is proved below that there is $t_d>0$ and $\delta_d>0$ such that

$$\inf_{|x|=1} P_x^\tau\{\eta\geqq t_d\}\geqq\delta_d. \tag{2.32}$$

Loosely speaking this means that with positive $P_x^\tau$-probability (uniformly bounded away from zero for all $x\in S$ satisfying $|x|=1$), $\omega$ does not hit the surfaces $\{y\in S\colon |y|=\frac{1}{2}\}$ or $\{y\in S\colon |y|=2\}$ "too quickly". In particular, the reflection near the non-smooth part of the boundary does not give "too large a push" towards these surfaces. To prove this, consider $z\in\bar{G}$ such that $|z|=1$. Recall that the ball $U(z)$ of (2.13) has radius less than $\frac{1}{2}$, so that for each $x\in U_1(z)\cap S$ by (2.13) we have

$$P_x^\tau\{\eta\geqq t(z)\}\geqq P_x^\tau\{T_{U(z)}\geqq t(z)\}\geqq\delta(z)>0. \tag{2.33}$$

Now $\{U_1(z)\colon z\in\bar{G},\ |z|=1\}$ is an open cover of the compact set $\{z\in\bar{G}\colon |z|=1\}$ and so it has a finite subcover $U_1(z_1),\dots,U_1(z_l)$, say. Set $\delta_d=\min_{i=1}^{l}\delta(z_i)$ and $t_d=\min_{i=1}^{l}t(z_i)$. Since each $x\in S$ satisfying $|x|=1$ is contained in $U_1(z_{i(x)})$ for some $i(x)\in\{1,\dots,l\}$, it follows from (2.33) that (2.32) holds.

Now, by applying (2.32), we obtain

$$\begin{aligned}\sup_{|x|=1}E_x^\tau\left[\exp\left(-\int_0^{\eta\wedge t_d}|\omega(s)|^{-2}\,ds\right)\right]&\leqq\sup_{|x|=1}\left(P_x^\tau\{\eta<t_d\}\cdot 1+P_x^\tau\{\eta\geqq t_d\}\,e^{-t_d/4}\right)\\&=\sup_{|x|=1}\left(1-P_x^\tau\{\eta\geqq t_d\}\,(1-e^{-t_d/4})\right)\\&\leqq 1-\delta_d(1-e^{-t_d/4})<1.\end{aligned}$$

It follows from this and (2.31) that $K=0$, as desired. □

Having established Lemmas 2.2 and 2.3, we now prove that (2.11) holds. First consider the case $d=2$, for which $\bar{G}$ is a two-dimensional wedge. To interpret the skew symmetry condition, suppose that the boundary of $\bar{G}$ has been oriented with unit tangent vector $e_i$ pointing in the positive direction on the side $F_i$, $i=1,2$. For each side $F_i$, define an *angle of reflection* $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by the relationship $q_i = e_i \tan\theta_i$. Then as shown in [9], the condition (1.1) is equivalent to the condition: $\theta_1 = \theta_2$. It was shown in [18] that when this condition holds, for each $x \in S$ there is a unique probability measure $P_x$ on $(\Omega, \mathscr{F})$ satisfying properties (i)–(iii) of Theorem 1.1. By the construction of $P_x^\tau$, it follows that $P_x = P_x^\tau$ on $\mathscr{F}$ and hence (2.11) holds for $d=2$.

Now suppose $d \geqq 3$. To prove (2.11) in this case, we extend the definition of $X$ to $\Omega^\partial$:

$$X(t, \omega) = \omega(t) \qquad \text{for all } t \geqq 0 \text{ and } \omega \in \Omega^\partial,$$

and define $Y$ on $\Omega^\partial$ by the following time-change of $X$

$$Y(t) = \begin{cases} X_{\sigma(t)} & \text{for } 0 \leqq t < A(\tau-) \\ \partial & \text{for } t \geqq A(\tau-). \end{cases} \tag{2.34}$$

By Lemma 2.3, for each $x \in S$ we have $P_x^\tau$-a.s.

$$Y(t) = X_{\sigma(t)} \in S \qquad \text{for all } t \geqq 0. \tag{2.35}$$

To define processes using the functions $X$ and $Y$, we need to specify associated probability measures on $(\Omega^\partial, \mathscr{F}^\partial)$. In particular, different processes can be defined using the same function $X$ (or $Y$) but different probability measures on $(\Omega^\partial, \mathscr{F}^\partial)$. For reference purposes, let $Y^\tau$ denote the strong Markov process in $S$ associated with $Y$ and the family of probability measures $\{P_x^\tau, x \in S\}$.

In the sequel, the process $\hat{Y}^\tau$ defined by adjoint (or dual) data to that for $Y^\tau$ will be needed. Note that if the direction of reflection $v_i = n_i + q_i$ on each face $F_i$ is replaced by the adjoint direction of reflection $\hat{v}_i \equiv n_i - q_i$, then (1.1) still holds. For each $x \in S$, let $\hat{P}_x^\tau$ denote the probability measure on $(\Omega^\partial, \mathscr{F}^\partial)$ characterized in the same way (i″)–(iv″) as $P_x^\tau$, but with $\hat{v}_i$ in place of $v_i$ for each $i$. Then (2.26) holds with $\hat{P}_x^\tau$ in place of $P_x^\tau$ and so (2.35) holds $\hat{P}_x^\tau$-a.s. Let $\hat{Y}^\tau$ denote the strong Markov process in $S$ associated with $Y$ and the family of probability measures $\{\hat{P}_x^\tau, x \in S\}$. It is shown below that $Y^\tau$ and $\hat{Y}^\tau$ are in duality relative to the measure $\nu$ defined on $S$ by:

$$\nu(dx) = |x|^{-2}\, dx. \tag{2.36}$$

Note that $\nu$ is a *Radon* measure on $S$ since we are considering the case $d \geqq 3$.

Recall that for each $x \in \bar{G}_m$, $P_x^m$ is the probability measure on $(\Omega^\partial, \mathscr{F}^\partial)$ such that the strong Markov process $X^m$ associated with $X$ and $\{P_x^m, x \in \bar{G}_m\}$ is a realization of the driftless reflected Brownian motion in $\bar{G}_m$ having reflection vector field $u_m$ on $\partial G_m$, where $u_m \equiv n_m + q_m$ agrees with $v_i$ on $\partial G_m \cap F_i^0$ for each $i \in \{1, \dots, d\}$ and satisfies (2.10). Similarly, let $\{\hat{P}_x^m, x \in \bar{G}_m\}$ denote the family of probability measures on $(\Omega^\partial, \mathscr{F}^\partial)$ associated with the driftless reflected Brownian motion in $\bar{G}_m$ having the adjoint reflection vector field $\hat{u}_m \equiv n_m - q_m$ on $\partial G_m$, and let $\hat{X}^m$ denote the realization associated with $X$ and $\{\hat{P}_x^m, x \in \bar{G}_m\}$. Since $\bar{G}_m \cap \{0\} = \emptyset$, it follows from property (III) of $P_x^m$ and the definition of $A$ that for each $x \in \bar{G}_m$, $P_x^m$-a.s.: $A(\tau) = \infty$ and (2.35) holds. Similarly, this is true with $\hat{P}_x^m$ in place of $P_x^m$. Let $Y^m$ (respectively $\hat{Y}^m$) denote the strong Markov process in $\bar{G}_m$ associated with $Y$ and the family of probability measures $\{P_x^m, x \in \bar{G}_m\}$ (respectively $\{\hat{P}_x^m, x \in \bar{G}_m\}$).

By [11], Sect. 5 and [12], $X^m$ and $\hat{X}^m$ are in duality relative to their common stationary distribution. It was shown in [9] that under the condition (2.10), this stationary distribution is the uniform distribution on $\bar{G}_m$. Then by the nature of the time-change defining $Y$ from $X$, it follows that $Y^m$ and $\hat{Y}^m$ are in duality relative to the measure $\nu$. In particular, they are in weak duality relative to $\nu$, i.e., for all continuous functions $f$ and $g$ having compact support in $\mathbb{R}^d$,

$$\int_{\bar{G}_m} E_x^m[f(Y(t))]\, g(x)\, |x|^{-2}\, dx = \int_{\bar{G}_m} f(x)\, \hat{E}_x^m[g(Y(t))]\, |x|^{-2}\, dx \quad \text{for all } t \geqq 0. \tag{2.37}$$

Here and below, $E_x^m$ and $\hat{E}_x^m$ denote the expectations relative to $P_x^m$ and $\hat{P}_x^m$, respectively. Next it is proved that (2.37) continues to hold with $S$, $E_x^\tau$ and $\hat{E}_x^\tau$ in place of $\bar{G}_m$, $E_x^m$ and $\hat{E}_x^m$, respectively. For $T_m$ defined by (2.4) and each $x \in \bar{G}_m$,

$$E_x^m[f(Y(t));\, t \leqq A(T_m)] = E_x^\tau[f(Y(t));\, t \leqq A(T_m)] \tag{2.38}$$

and

$$P_x^m(t > A(T_m)) = P_x^\tau(t > A(T_m)), \tag{2.39}$$

because $Y^m$ and $Y^\tau$ have the same behavior up to the time $A(T_m)$. Hence,

$$\begin{aligned} &\Big| \int_{\bar{G}_m} E_x^m[f(Y(t))]\, g(x)\, |x|^{-2}\, dx - \int_S E_x^\tau[f(Y(t))]\, g(x)\, |x|^{-2}\, dx \Big| \\ &\quad = \Big| \int_{\bar{G}_m} \{E_x^m[f(Y(t));\, t > A(T_m)] - E_x^\tau[f(Y(t));\, t > A(T_m)]\}\, g(x)\, |x|^{-2}\, dx \\ &\qquad - \int_{S \setminus \bar{G}_m} E_x^\tau[f(Y(t))]\, g(x)\, |x|^{-2}\, dx \Big| \\ &\quad \leqq |f|_\infty \Big(2 \int_S P_x^\tau(t > A(T_m))\, |g(x)|\, |x|^{-2}\, dx + \int_{S \setminus \bar{G}_m} |g(x)|\, |x|^{-2}\, dx\Big), \end{aligned} \tag{2.40}$$

where in the last line above, $|f|_\infty \equiv \max_{x \in \mathbb{R}^d} |f(x)|$ and (2.39) has been used. Now for each $x \in S$, $T_m \uparrow \tau$ $P_x^\tau$-a.s. so that by (2.26): $P_x^\tau(t > A(T_m)) \to 0$ as $m \to \infty$. By combining this with $\lim_{m \to \infty} (S \setminus \bar{G}_m) = \emptyset$ and the fact that $g$ has compact support, we conclude that the expression in (2.40) tends to zero as $m \to \infty$. The same is true with $\hat{E}_x^m$ and $\hat{E}_x^\tau$ in place of $E_x^m$ and $E_x^\tau$, respectively. Thus, by letting $m \to \infty$ in (2.37) we obtain

$$\int_S E_x^\tau[f(Y(t))]\, g(x)\, |x|^{-2}\, dx = \int_S f(x)\, \hat{E}_x^\tau[g(Y(t))]\, |x|^{-2}\, dx \quad \text{for all } t \geqq 0. \tag{2.41}$$

It follows that $\nu$ is an invariant measure for $Y^\tau$ and $\hat{Y}^\tau$ and that these processes are in weak duality relative to $\nu$.

The proof of (2.11) can now be completed using an argument similar to that in [16]. For this, consider the canonical Markov chain defined on the path space $\Gamma \equiv S^{\mathbb{N}}$ with one-step transition probabilities: $\pi(x, dy) = P_x^\tau[Y(1) \in dy]$ for $x \in S$. By (2.41), $\nu$ is an invariant measure for this Markov chain. Let $P_\nu$ denote the law on $\Gamma$ of the chain with initial ($\sigma$-finite) measure $\nu$. By applying the Hopf decomposition theorem to the shift operator acting on the set of $P_\nu$-

integrable functions on $\Gamma$, we obtain a decomposition of $\Gamma$ into a conservative part $\mathscr{C}$ and a dissipative part $\mathscr{D}$ (see Revuz [14], Theorem 2.3, p. 124). To see the implications of this, let $w=(w(0), w(1), w(2), \dots)$ denote a generic element of $\Gamma$. Since $d \geqq 3$, the open ball $U(0,r)$, centered at the origin 0 and of radius $r \in (0, \infty)$, has finite $\nu$-measure. Thus the function $f$ defined on $\Gamma$ by $f(w) = 1_{U(0,1)}(w(0))$ is $P_\nu$-integrable, and so are the functions $g_j$, $j=2,3,\dots$ defined on $\Gamma$ by $g_j(w)=1_{U_c(j)}(w(0))$ where $U_c(j)=U(0,1) \setminus U(0,1/j)$. Then it follows from the Hopf theorem that [14] (p. 124)

$$P_\nu\text{-a.e. on } \mathscr{D}: \sum_{i=0}^{\infty} 1_{U(0,1)}(w(i)) < \infty, \tag{2.42}$$

and for each $j \geqq 2$

$$P_\nu\text{-a.e. on } \mathscr{C}: \sum_{i=0}^{\infty} 1_{U_c(j)}(w(i)) = 0 \text{ or } \infty. \tag{2.43}$$

Thus $\mathscr{C}$ is included $P_\nu$-a.e. in the set

$$\left\{\forall j \geqq 2, \sum_{i=0}^{\infty} 1_{U_c(j)}(w(i)) = 0\right\} \cup \left\{\exists j \geqq 2: \sum_{i=0}^{\infty} 1_{U_c(j)}(w(i)) = \infty\right\}. \tag{2.44}$$

By combining (2.42)–(2.44) we obtain

$$P_\nu(\limsup_{i \to \infty} |w(i)| = 0) = 0. \tag{2.45}$$

Thus, for $\nu$-a.e. $x \in S$

$$P_x^\tau(\limsup_{i \to \infty} |Y(i)| = 0) = 0. \tag{2.46}$$

Fix an $x$ for which (2.46) holds and let $j \geqq 2$ be a fixed positive integer. Define $\sigma_1 = \inf\{t \geqq 0: X_t \notin U(0,1/j) \cap S\}$, $\tau_1 = \inf\{t \geqq \sigma_1: X_t \notin S \setminus U(0, 1/(2j))\}$, and define $\sigma_i$ and $\tau_i$ for $i \geqq 2$ inductively such that $\sigma_i = \inf\{t \geqq \tau_{i-1}: X_t \notin U(0,1/j) \cap S\}$ and $\tau_i = \inf\{t \geqq \sigma_i: X_t \notin S \setminus U(0,1/(2j))\}$. Then,

$$\{\liminf_{t \uparrow \tau} |X_t| = 0\} \cap \{\limsup_{t \uparrow \tau} |X_t| \geqq 1/j\} \subset \left\{\sum_{i=1}^{\infty} 1_{\{\sigma_i < \infty\}} = \infty\right\}. \tag{2.47}$$

For any $t > 0$, by the strong Markov property we have $P_x^\tau$-a.s.

$$\sum_{i=1}^{\infty} P_x^\tau(\sigma_i < \infty, \tau_i - \sigma_i \geqq t \mid \mathscr{F}_{\sigma_i}) \geqq \sum_{i=1}^{\infty} 1_{\{\sigma_i < \infty\}} P_{\omega(\sigma_i)}^\tau(\eta_j \geqq t) \tag{2.48}$$

where $\eta_j = \inf\{s \geqq 0: w(s) \notin \bar{G}_j\}$ and $\bar{G}_j = \left\{z \in \bar{G}: \frac{1}{2j} \leqq |z| \leqq \frac{2}{j}\right\}$. By the scaling lemma (Lemma 2.2) and inequality (2.32), we have

$$\inf_z P_z^\tau(\eta_j \geqq j^{-2} t_d) \geqq \delta_d \tag{2.49}$$

where the infimum is over all $z \in S$ satisfying $|z| = 1/j$. It then follows from (2.47)-(2.49) that $P_x^\tau$-a.s.

$$\{\liminf_{t\uparrow\tau} |X_t| = 0\} \cap \{\limsup_{t\uparrow\tau} |X_t| \geqq 1/j\} \subset \left\{ \sum_{i=1}^{\infty} P_x^\tau(\sigma_i < \infty, \tau_i - \sigma_i \geqq j^{-2} t_d | \mathscr{F}_{\sigma_i}) = \infty \right\}.$$

By an extension of the Borel-Cantelli lemma ([5], Corollary 2.3), the last set above is $P_x^\tau$-a.s. equal to

$$\{\sigma_i < \infty, \tau_i - \sigma_i \geqq j^{-2} t_d \text{ for infinitely many } i\} \subset \{\tau = \infty\}.$$

Combining this with (2.46), since $j$ was arbitrary we see that $P_x^\tau$-a.s.

$$\{\liminf_{t\uparrow\tau} |X_t| = 0\} = \bigcup_{j=1}^{\infty} \{\liminf_{t\uparrow\tau} |X_t| = 0\} \cap \{\limsup_{t\uparrow\tau} |X_t| \geqq 1/j\} \subset \{\tau = \infty\}. \tag{2.50}$$

But, by Lemma 2.3 and the definition of $A$,

$$\{\tau < \infty\} \subset \{\liminf_{t\uparrow\tau} |X_t| = 0\} \; P_x^\tau\text{-a.s.}$$

Combining this with (2.50) yields (2.11) for $v$-a.e. $x \in S$. In fact (2.11) holds for *each* $x \in S$. To see this, fix $x \in S$. Then there is $r > 0$ (depending on $x$) such that the open ball $U(x,r)$ in $\mathbb{R}^d$, centered at $x$ and of radius $r$, meets at most one face $F_i$ of $\bar{G}$ and is a positive distance from the origin. Since $v$ is uniformly equivalent to Lebesgue measure on $U(x,r)$, by using spherical polar coordinates centered at $x$ and Fubini's theorem, it follows from the fact that (2.11) holds $v$-a.e. on $S$ that there is $s \in (0,r)$ such that

$$\int_{\partial U(x,s) \cap S} P_y^\tau(\tau < \infty)\, d\sigma(y) = 0 \tag{2.51}$$

and hence

$$P_y^\tau(\tau < \infty) = 0 \qquad \text{for } \sigma\text{-a.e. } y \in \partial U(x,s) \cap S, \tag{2.52}$$

where $\sigma$ denotes surface measure on the boundary $\partial U(x,s)$ of $U(x,s)$. Now, by the strong Markov property

$$P_x^\tau(\tau < \infty) = \int_{\partial U(x,s) \cap S} \kappa(dy)\, P_y^\tau(\tau < \infty), \tag{2.53}$$

where $\kappa$ denotes the hitting distribution of $X$ on $\partial U(x,s) \cap S$ under $P_x^\tau$. Up to the time of hitting $\partial U(x,s) \cap S$, $X$ under $P_x^\tau$ behaves like a driftless Brownian motion that is reflected in the direction $v_i$ on the hyperplane containing $F_i$. It follows that $\kappa$ is absolutely continuous with respect to $\sigma$ and hence by (2.52)-(2.53), (2.11) holds. Thus, Proposition 2.1 holds with $P_x = P_x^\tau$ on $(\Omega, \mathscr{F})$. To complete the induction step, we need to prove Proposition 2.2 for $d \geqq 2$.

It follows from the skew symmetry condition that $I + N^{-1}Q$ is invertible [9] and so $N' + Q'$ is invertible, since $N$ is invertible. Thus, from the semimartingale representation (c) following Proposition 2.1, for each $x \in S$ we have $P_x$-a.s.

$$V(t) = (N' + Q')^{-1}(\omega(t) - x - B(t)) \qquad \text{for all } t \geqq 0. \tag{2.54}$$

Let $M$ denote the matrix norm of $(N'+Q')^{-1}$ so that for all $x\in\mathbb{R}^d$, $|(N'+Q')^{-1}x|\leqq M|x|$. Recall from the proof of Lemma 2.3 that for

$$\eta\equiv\inf\{t\geqq 0\colon |\omega(t)|<\tfrac{1}{2} \text{ or } |\omega(t)|>2\},$$

there is $t_d>0$ and $\delta_d>0$ such that

$$\inf_x P_x(\eta\geqq t_d)\geqq\delta_d \tag{2.55}$$

where the infimum is over all $x\in S$ satisfying $|x|=1$. Moreover, by the scaling lemma (Lemma 2.2), we have for each $\lambda>0$, $t\geqq 0$ and $x\in S$,

$$P_{\lambda x}(\eta_\lambda\geqq\lambda^2 t)=P_x(\eta\geqq t), \tag{2.56}$$

where $\eta_\lambda\equiv\inf\{s\geqq 0\colon |\omega(s)|<\lambda/2 \text{ or } |\omega(s)|>2\lambda\}$. Now, fix $\beta>0$ and let $\delta=\min(\beta,\beta/(4M),\delta_d/2)$. If $|\omega(t)|\leqq 2\delta$, $|x|\leqq\delta$ and $|B(t)|\leqq\delta$, then (2.54) implies that

$$|V(t)|\leqq\beta. \tag{2.57}$$

For each $r>0$, let $\tau_r=\inf\{s\geqq 0\colon |\omega(s)|\geqq r\}$. Then for each $x\in S$ satisfying $|x|<\delta$, by the strong Markov property and (2.55)–(2.56) we have for all $t\leqq t_d$,

$$\begin{aligned} P_x(\tau_{2\delta}<\delta^2 t) &\leqq P_x(\tau_\delta<\delta^2 t;\, P_{\omega(\tau_\delta)}\{\eta_\delta<\delta^2 t\}) \\ &\leqq P_x(\tau_\delta<\delta^2 t; P_{\delta^{-1}\omega(\tau_\delta)}\{\eta<t\}) \\ &\leqq 1-\delta_d<1. \end{aligned}$$

Then for each $x\in S$ satisfying $|x|<\delta$, by (2.57) we have for all $t\leqq t_d$,

$$\begin{aligned} P_x\{\max_{0\leqq s\leqq\delta^2 t}|B(s)|\leqq\beta, |V(\delta^2 t)|\leqq\beta\} &\geqq P_x\{\max_{0\leqq s\leqq\delta^2 t}|B(s)|\leqq\delta, \tau_{2\delta}\geqq\delta^2 t\} \\ &\geqq P_x\{\max_{0\leqq s\leqq\delta^2 t}|B(s)|\leqq\delta\}-P_x\{\tau_{2\delta}<\delta^2 t\} \\ &\geqq P_x\{\max_{0\leqq s\leqq\delta^2 t}|B(s)|\leqq\delta\}-(1-\delta_d). \end{aligned}$$

Under $P_x$, $B$ is a $d$-dimensional Brownian motion starting from the origin. Thus, for all sufficiently small $t\in(0,t_d)$, the probability in the last line above exceeds $(1-\frac{1}{2}\delta_d)$, uniformly in $x$. Thus, for some $t^*\in(0,t_d)$, for all $x\in S$ satisfying $|x|<\delta$, we have

$$P_x\{\max_{0\leqq s\leqq\delta^2 t^*}|B(s)|\leqq\beta, |V(\delta^2 t^*)|\leqq\beta\}\geqq\tfrac{1}{2}\delta_d\geqq\delta>0,$$

and so (2.9) holds with $t=\delta^2 t^*$. This completes the proof of the induction step and hence of Propositions 2.1 and 2.2.

To complete the proof of Theorem 1.1, we need to consider the general case where $\bar{G}$ is a polyhedron. We must prove existence of a solution of (i)–(iii) for each $x\in S$ and data $(N,Q,b,\mu=0)$ satisfying (1.1). Since each point in $\bar{G}$ may be viewed locally as being in a polyhedral cone, it follows from Propositions 2.1 and 2.2 and their proofs (see especially (2.12)–(2.13)), that for each $z\in\bar{G}$ there are non-empty open balls $U_1(z)\subset U(z)$ centered at $z$ in $\mathbb{R}^d$, and $t(z)>0$ and $\delta(z)>0$, such that the following properties hold. Here $T_{U(z)}=\inf\{t\geqq 0\colon \omega(t)$

$\notin U(z)\}$. The ball $U(z)$ does not meet any face of $\bar{G}$ other than those containing $z$, and for each $x \in U_1(z) \cap S$ there is a probability measure $P_x$ on $(\Omega, \mathscr{F}_{T_{U(z)}})$ characterized by three conditions, namely, (i)–(ii) of Theorem 1.1 with $t \wedge T_{U(z)}$ in place of $t$ there, and the condition: $P_x(\omega(t \wedge T_{U(z)}) \in S$ for all $t \geqq 0) = 1$. Moreover, for each $x \in U_1(z) \cap S$,

$$P_x\{T_{U(z)} \geqq t(z)\} \geqq \delta(z). \tag{2.58}$$

If $\bar{G}$ is compact, there is a finite cover of $\bar{G}$ by sets $U_1(z_1), \ldots, U_1(z_l)$ for some $z_1, \ldots, z_l \in \bar{G}$. By successive conditioning [15], for any $x \in S$ we can build a solution of (i)–(ii) from the locally defined $P_x$'s. The fact that this procedure yields a probability measure on $\mathscr{F}$, or in other words (iii) holds, follows from the estimate (2.58) by a Borel-Cantelli argument similar to that used to prove (2.20). Now if $\bar{G}$ is unbounded, we can cover it by countably many balls $\{U_1(z_i), i = 1, 2, \ldots\}$ such that for each $i$, (2.58) holds with $z = z_i$, $t(z_i) = t$ and $\delta(z_i) = \delta$ where $t > 0$ and $\delta > 0$ are independent of $i$. This uniform estimate follows because $\bar{G}$ is convex and has only finitely many faces. More precisely, outside of some compact set we can apply a version of the Brownian-like scaling property of Lemma 2.2 on each face extending to infinity together with the spatial homogeneity of Brownian motion in $G$ to obtain the desired uniformity. Then for $x \in S$, the existence of $P_x$ satisfying (i)–(ii) follows in the same way as when $\bar{G}$ is compact. □

It follows from (2.41) and the nature of the time-change defining $Y$ from $X$ that when $\bar{G}$ is a polyhedral cone (i.e., $N$ is a $d \times d$ invertible matrix) and $\mu = 0$, the strong Markov processes associated with $X$ and the families of probability measures $\{P_x, x \in S\}$ and $\{\hat{P}_x, x \in S\}$ are in weak duality with respect to Lebesgue measure on $S$. In the next section, a generalization of this (Theorem 1.2) is proved.

## 3. Invariant Measure and Dual Process

*Proof of Theorem 1.2.* Assume that the hypotheses of Theorem 1.2 hold. Consider the sequence $\{G_m, m = 1, 2, \ldots\}$ of bounded $C^3$ domains defined at the beginning of Sect. 2. Recall that $n_m$ denotes the inward unit normal vector field on $\partial G_m$. Then since (1.1) holds and $N$ contains an invertible $d \times d$ submatrix, it follows from Lemma 3.2 in [9] that $u_m \equiv v_i$ on $\partial G_m \cap F_i^0$ for $i = 1, \ldots, k$, can be uniquely extended to a $C^2$ vector field $u_m = n_m + q_m$ on $\partial G_m$ such that the following holds:

$$n_m(\sigma^*) \cdot q_m(\sigma) + q_m(\sigma^*) \cdot n_m(\sigma) = 0 \qquad \text{for all } \sigma, \sigma^* \in \partial G_m. \tag{3.1}$$

For each $m$, let $\{P_x^m, x \in \bar{G}_m\}$ denote the family of probability measures on $(\Omega, \mathscr{F})$ associated with the reflected Brownian motion in $\bar{G}_m$ having drift $\mu$ and reflection vector field $u_m$ on $\partial G_m$, i.e., for each $x \in \bar{G}_m$, $P_x^m$ is characterized by (I)–(III) (following (2.10)) of Sect. 2, with $L$ in place of $\Delta/2$ there. Let $X^m$ denote the realization of this process associated with $X$ and $\{P_x^m, x \in \bar{G}_m\}$. It was shown in [9] that under condition (3.1), $X^m$ has a unique stationary distribution with density proportional to $p(x)=\exp\{\gamma(\mu)\cdot x\}$, where $\gamma(\mu)$ is given by (1.7). Let $\{\hat{P}_x^m,\ x\in\bar{G}_m\}$ denote the family of probability measures on $(\Omega,\mathscr{F})$ associated with the reflected Brownian motion in $\bar{G}_m$ having drift $\gamma(\mu)-\mu$ and the adjoint reflection vector field $\hat{u}_m\equiv n_m-q_m$ on $\partial G_m$, and let $\hat{X}^m$ denote the realization associated with $X$ and $\{\hat{P}_x^m,\ x\in\bar{G}_m\}$. Then by [11, 12], $X^m$ and $\hat{X}^m$ are in duality relative to the common invariant measure $\rho(dx)=p(x)\,dx$. In particular, for all continuous functions $f$ and $g$ having compact support in $\mathbb{R}^d$,

$$\int_{\bar{G}_m} g(x)\,E_x^m[f(X(t))]\,p(x)\,dx=\int_{\bar{G}_m} f(x)\,\hat{E}_x^m[g(X(t))]\,p(x)\,dx \quad \text{for all } t\geqq 0. \tag{3.2}$$

Here $E_x^m$ and $\hat{E}_x^m$ denote expectations with respect to $P_x^m$ and $\hat{P}_x^m$, respectively. Now $\{P_x,\ x\in S\}$ denotes the family of probability measures on $(\Omega,\mathscr{F})$ for the RBM associated with the data $(N,Q,b,\mu)$. Let $\{\hat{P}_x,\ x\in S\}$ denote the family for the RBM associated with $(N,-Q,b,\gamma-\mu)$. Note that (1.1) also holds for the latter. Let $E_x$ and $\hat{E}_x$ denote the expectations with respect to $P_x$ and $\hat{P}_x$ respectively. Then since $\tau=\infty$ $P_x$-a.s. and $\hat{P}_x$-a.s. for each $x\in S$, it follows by the same kind of argument as used to deduce (2.41) from (2.37) that the following holds for all continuous functions $f$ and $g$ having compact support in $\mathbb{R}^d$,

$$\int_S g(x)\,E_x[f(X(t))]\,p(x)\,dx=\int_S f(x)\,\hat{E}_x[g(X(t))]\,p(x)\,dx \quad \text{for all } t\geqq 0. \tag{3.3}$$

Hence the RBM's associated with $\{P_x,\ x\in S\}$ and $\{\hat{P}_x,\ x\in S\}$ are in weak duality relative to the measure $\rho(dx)\equiv p(x)\,dx$ and this measure is invariant for both processes. Indeed, the processes are in strong duality relative to $\rho$. This is a consequence of the fact that for each $x\in S$, each Borel set $A\subset S$, and each $t>0$,

$$P_x(X(t)\in A)>0 \quad \text{if and only if } m(A)>0, \tag{3.4}$$

where $m$ denotes Lebesgue measure on $\mathbb{R}^d$. The above follows from the properties of the reflected Brownian motions (with drift $\mu$) in $\{\bar{G}_m\}$ and the fact that for each $x\in S$, $X$ behaves the same under $P_x$ and $P_x^m$ until the time $T_m$, and $T_m\to\infty$ $P_x$-a.s. as $m\to\infty$. □

*Proof of Corollary 1.1.* Suppose the hypotheses of the Corollary hold. Then by Theorem 1.2 and the normalization of $p$, $\{C(\mu)\}^{-1}\exp\{\gamma(\mu)\cdot x\}$ is a stationary density for the RBM associated with $(N,Q,b,\mu)$. Uniqueness follows from (3.4) since this implies the RBM is ergodic in $S$ and has at most one stationary distribution (finite invariant measure) ([21], pp. 388–390). □

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