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Remarks on Limit Theorems for Nonlinear Functionals of Gaussian Sequences

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Summary. Limit theorems for sums of nonlinear functionals of Gaussian sequences typically obtain as limit distribution that of a single term in an expansion given by Dobrushin [1] for a process subordinate to a Gaussian process. Here we show how one can obtain limit theorems of this type where the limit distribution is that of a full expansion of Dobrushin's type.

Introduction

Let $\{\xi_n\}$, $n = \dots, -1, 0, 1, \dots$ be a strictly stationary sequence. In dealing with limit laws one often defines the new sequences

$$S_n^N = A_N^{-1} \sum_{j \in B_n^N} \xi_j, \qquad N = 1, 2, \dots,$$

where

$$B_n^N = \{j \mid nN \leqq j < (n+1)N\}$$

and A_N is an appropriate norming constant. In investigating possible limit laws one is led to sequences $\{\xi_n\}$ with the following property. The joint distribution of

$$\xi_{n_1}, \dots, \xi_{n_k}$$

is the same as that of

$$S_{n_1}^N, \dots, S_{n_k}^N$$

for all $N = 1, 2, \dots$ and integers $n_1, \dots, n_k$ with $A_N = N^\alpha$ for some parameter $\alpha > 0$. The parameter α is called the self-similarity parameter for the sequence $\{\xi_n\}$ and the sequence itself is referred to as a self-similar process.

Dobrushin has characterized the self-similar processes subordinate to a Gaussian random spectral measure Z_G with spectral distribution G. For Borel

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sets A with $G(A)<\infty$, the random variables $Z_G(A)$ have the following properties:

(i) The random variables $Z_G(A)$ are jointly Gaussian complex-valued.

(ii) $EZ_G(A)=0$,
$EZ_G(A)\overline{Z_G(B)}=G(A\cap B)$.

(iii) $\sum_{j=1}^{n} Z_G(A_j)=Z_G\left(\bigcup_{j=1}^{n} A_j\right)$ if the sets $A_1,\ldots,A_n$ are disjoint.

(iv) $Z_G(A)=\overline{Z_G(-A)}$.

Dobrushin [1] has shown that if $\{\xi_n\}$ has the form

$$\xi_j=\sum_{n=1}^{\infty}\frac{1}{n!}\int \exp\left\{ij\sum_{s=1}^{n}x_s\right\}\frac{\exp\left\{i\sum_{s=1}^{n}x_s\right\}-1}{i\left(\sum_{s=1}^{n}x_s\right)} f_n(x_1,\ldots,x_n)Z_G(dx_1)\ldots Z_G(dx_n) \tag{1}$$

with

$$f_n(\lambda x_1,\ldots,\lambda x_n)=\lambda^{1-\frac{n\alpha}{2}-\beta}f_n(x_1,\ldots,x_n)$$

and

$$G(\lambda A)=\lambda^{\alpha}G(A),$$

then $\{\xi_n\}$ is a self-similar process with self-similarity parameter β. The integrals in the representation (1) are multiple Wiener-Ito integrals. One has to check whether the formula (1) is meaningful for a sequence f_n in the sense that the variance of (1) is finite. If we assume that the functions f_n are symmetric functions of the variables $x_1,\ldots,x_n$ the variance of (1) is given by

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int |f_n(x_1,\ldots,x_n)|^2 K\left(\sum_{s=1}^{n}x_s\right)G(dx_1)\ldots G(dx_n)$$

where K is the Fejer kernel

$$\mathbf{K}(\mu)=\frac{\sin^2\frac{\mu}{2}}{\left(\frac{\mu}{2}\right)^2}.$$

The usual non Gaussian limit theorems for partial sums of nonlinear functionals of Gaussian sequences (see Dobrushin and Major [2]; Giraitis and Surgailis [5]; Major [3, 4], Taqqu [8, 9], and Rosenblatt [6, 7]) correspond to a single term in the expansion (1). In this paper, we will show that there are limit theorems in which the limiting distribution corresponds to a complete expansion of the type (1).

Let X_n, $n=\ldots,-1,0,1,\ldots$, $EX_n=0$, $EX_n^2=1$, be a Gaussian stationary process with correlation function $r(n)=EX_0X_n$ satisfying

$$r(n)\cong|n|^{-\alpha},\qquad 0<\alpha<1, \tag{2}$$

as $|n|\to\infty$. Let H be the spectral distribution function corresponding to (2) so that

$$r(n)=\int_{-\pi}^{\pi} e^{inx} H(dx).$$

The random spectral measure of the process $\{X_n\}$ is Z_H. Let us consider the process $\{Y_n\}$ subordinate to $\{X_n\}$ given by

$$Y_n=\sum_{k>k_0}\frac{c_k}{k!}\int \exp[in(x_1+\dots+x_k)]\,|x_1|^{\gamma_k}\dots|x_k|^{\gamma_k}\, Z_H(dx_1)\dots Z_H(dx_k) \tag{3}$$

with

$$\gamma_k=\frac{1}{k}-\frac{\alpha}{2}-\frac{\beta}{k},\qquad k=1,2,\dots.$$

and k_0 the greatest integer less than or equal to $2(1-\beta)/\alpha$.

For the representation (3) to make sense we require that

$$\sum_{k>k_0}\frac{|c_k|^2}{k!}\int |x_1|^{2\gamma_k}\dots|x_k|^{2\gamma_k} H(dx_1)\dots H(dx_k)<\infty \tag{4}$$

since it is the variance of (3). This requires at the very least that for any index k for which $c_k\neq 0$ that

$$a_k=\int |x|^{2\gamma_k} H(dx)<\infty.$$

This will be finite if one has $1>\beta$. Let us assume that $2\gamma_k+\alpha=2(1-\beta)/k<1/k$ so that

$$U_n^{(k)}=\int e^{in(x_1+\dots+x_k)}\frac{e^{i(x_1+\dots+x_k)}-1}{i(x_1+\dots+x_k)}|x_1|^{\gamma_k}\cdots|x_k|^{\gamma_k}\, |x_1|^{\frac{\alpha-1}{2}}\dots|x_k|^{\frac{\alpha-1}{2}}\, W(dx_1)\dots W(dx_k)\, D^{-k/2}$$

is well-defined where W is the random spectral measure of the white-noise process and $D=2\Gamma(\alpha)\cos\left(\frac{\pi}{2}\alpha\right)$. This will happen if

$$\beta>\frac{1}{2}.$$

Notice that

$$V_n=\sum_{k>k_0}\frac{c_k}{k!}U_n^{(k)} \tag{5}$$

is well-defined if $1>\beta>\frac{1}{2}$ and

$$\sum_{k>k_0}\frac{|c_k|^2}{k!}\int\frac{\sin^2\frac{1}{2}\left(\sum_1^k x_j\right)}{\left(\frac{1}{2}\sum_1^k x_j\right)^2}|x_1|^{2\gamma_k+\alpha-1}\dots|x_k|^{2\gamma_k+\alpha-1}dx_1\dots dx_k\, D^{-k}<\infty.$$

Notice that if (2) is satisfied, there is a constant

$$B > 2\Gamma(\alpha) \tag{6}$$

such that

$$\left|\int e^{ipx} H(dx)\right| < B(1+|p|)^{-\alpha}$$

for all integral p. Let

$$A = 4B\left\{\Gamma(\alpha)\cos\left(\frac{\pi}{2}\alpha\right)\right\}^{-1}.$$

We wish to show that under appropriate conditions, the partial sums of the process $\{Y_n\}$ given by (3) will converge in distribution to the process $\{V_n\}$ given by (5). More specifically, our object is to prove the following theorem.

Theorem. *Let $\{X_n\}$ be a stationary Gaussian process with mean zero and covariance function $r(n)$ satisfying* (2). *If $\{Y_n\}$ is the process* (3) *subordinate to $\{X_n\}$ and satisfies*

$$\sum_{k>k_0} \frac{|c_k|^2}{k!} A^k \Gamma\left(\frac{2}{k}(1-\beta)\right)^k < \infty \tag{7}$$

(see (8)) *then the partial sums*

$$N^{-\beta} \sum_{j\in B_n^N} Y_j, \qquad n=1, \ldots, p,$$

converge in distribution to that of the random variables

$$V_n, \; n=1, \ldots, p,$$

if $1>\beta>\frac{1}{2}$. If $2(1-\beta)/\alpha<1$ or if

$$H(dx) = \left(\frac{c}{|x|^{1-\alpha}} + f(x)\right) dx, \qquad 0<\alpha<1,$$

with the constant $c=D^{-1}$ and f a function of bounded variation, the sum in (3), (5), *and* (7) *can be taken from* 1 *to* ∞.

The following simple lemma is helpful in the derivation.

Lemma. *Under condition* (2) *and $\gamma_k<0$,*

$$\left|\int e^{ipx}|x|^{2\gamma_k} H(dx)\right|$$

$$< B\left\{\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{2}{k}(1-\beta)\right)} + \frac{\Gamma\left(\frac{2}{k}(1-\beta)\right)}{\Gamma(\alpha)}\left(1+\frac{\sin\pi\left(\alpha-\frac{2}{k}(1-\beta)\right)}{\sin\pi\alpha}\right)\right\} \tag{8}$$

$$\left\{\cos\left(\frac{\pi}{2}\eta\right)\right\}^{-1}(1+|p|)^{-2\gamma_k-\alpha}.$$

In estimating (8) let us note that for k sufficiently large $2\gamma_k = -\eta$ is negative. It is clearly of some interest to look at the Fourier coefficients of $|x|^{-\eta}$. Now

$$\begin{aligned}\int_{-\pi}^{\pi} e^{inx}|x|^{-\eta}dx &= \int_{-\pi}^{\pi} \cos nx|x|^{-\eta}dx \\ &= \int_{-\pi/2n}^{\pi/2n} \cos nx\left\{\sum_{k=-n+1}^{n-1}(-1)^k\left|x+\frac{k\pi}{n}\right|^{-\eta}\right\}dx \\ &\quad +(-1)^n\left[\int_{\pi-\frac{\pi}{2n}}^{\pi} + \int_{-\pi}^{-\pi+\frac{\pi}{2n}}\right]\cos nx|x|^{-\eta}dx. \qquad (9)\end{aligned}$$

For $n=0$ we have

$$2\int_0^{\pi} x^{-\eta}dx = \frac{2}{1-\eta}\pi^{1-\eta}.$$

For $n \neq 0$, the first part of the expression on the right of (9) is

$$\int_{-\pi/2}^{\pi/2} \cos x\left\{\sum_{k=-n+1}^{n-1}(-1)^k|x+k\pi|^{-\eta}\right\}dx\, n^{\eta-1}$$

and for large n this can be approximated by

$$\begin{aligned}&\int_{-\pi/2}^{\pi/2} \cos x \sum_{k=-\infty}^{\infty}(-1)^k|x+k\pi|^{-\eta}dx\, n^{\eta-1} \\ &\quad = \int_{-\infty}^{\infty} \cos x|x|^{-\eta}dx\, n^{\eta-1} = 2\Gamma(1-\eta)\cos\left(\frac{\pi}{2}(1-\eta)\right)n^{\eta-1}. \qquad (10)\end{aligned}$$

If k is small and $2\gamma_k$ is positive a_k is automatically finite. If $2\gamma_k$ is negative we can still show that

$$\int |x|^{-\eta}H(dx) < \infty.$$

Let

$$g_m(x) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \frac{\sin^2\frac{m}{2}(x-y)}{\sin^2\frac{1}{2}(x-y)}|y|^{-\eta}dy,$$

the Cesaro one sum of the truncated Fourier series of $|x|^{-\eta}$. Let the c_k's be the Fourier coefficients of $|x|^{-\eta}$. Now

$$\begin{aligned}\int_{-\pi}^{\pi} g_m(x)dH(x) &= \frac{1}{2\pi}\sum_{|k|\leqq m} c_k\left(1-\frac{|k|}{m}\right)r_{-k} \\ &\leqq \frac{1}{2\pi}\sum_k |c_k|\,|r_{-k}|.\end{aligned}$$

Further $g_m(x) \geqq 0$ for all x and

$$\lim_{m\to\infty} g_m(x) = |x|^{-\eta}$$

for all $x, 0<|x|\leqq\pi$. Therefore by Fatou's lemma

$$\int_{-\pi}^{\pi}|x|^{-\eta}dH(x)\leqq\liminf_{m\to\infty}\int_{-\pi}^{\pi}g_m(x)dH(x)$$
$$\leqq\frac{1}{2\pi}\sum|c_k||r_{-k}|$$

and this last expression is finite by virtue of (2) and (10) since

$$\eta-1-\alpha=-\frac{2}{k}(1-\beta)-1.$$

We shall now show that

$$\int_{-\pi}^{\pi}e^{ipx}|x|^{-\eta}H(dx)=\frac{1}{2\pi}\sum_j c_j r_{p-j}. \tag{11}$$

Let us consider

$$\int_{-\pi}^{\pi}e^{ipx}g_m(x)dH(x)=\frac{1}{2\pi}\sum_{|k|\leqq m}c_k\left(1-\frac{|k|}{m}\right)r_{p-k}. \tag{12}$$

The equality (11) will follow from (12) on applying the Lebesgue convergence theorem with $m\to\infty$, if we can show that

$$g_m(x)\leqq K_1+K_2\left(\frac{1}{m^2}+x^2\right)^{-\eta/2} \tag{13}$$

for some constants $K_1, K_2>0$, First it is clear that for appropriate constants $L_1, L_2>0$ one has

$$\frac{1}{2\pi m}\frac{\sin^2\frac{m}{2}u}{\sin^2\frac{1}{2}u}\leqq L_1+L_2\frac{m}{\pi}\frac{1}{1+m^2u^2}$$

for $|u|\leqq\frac{\pi}{2}$.

Now

$$\int_{-\infty}^{\infty}\frac{m}{\pi}\frac{1}{1+m^2(x-y)^2}|y|^{-\eta}dy=L_3\operatorname{Re}\int_{-\infty}^{\infty}|\lambda|^{\eta-1}e^{-\left|\frac{\lambda}{m}\right|}e^{iy\lambda}d\lambda$$

for an appropriate constant $L_3>0$. But

$$\operatorname{Re}\int_0^{\infty}\lambda^{\eta-1}e^{-\lambda\left(\frac{1}{m}-iy\right)}d\lambda=\operatorname{Re}\left(\frac{1}{m}-iy\right)^{-\eta}\Gamma(\eta)$$

and

$$\operatorname{Re}\left(\frac{1}{m}-iy\right)^{-\eta}=\cos\eta\theta\cdot\left(\frac{1}{m^2}+y^2\right)^{-\eta/2}$$

with

$$\theta = \arc\cos\left(\frac{1}{m}\Big/\{m^{-2}+y^2\}^{\frac{1}{2}}\right).$$

This gives us the inequality (13) and consequently the relation (11). From (6) and (10) it follows that

$$\frac{1}{2\pi}\sum_j |c_j r_{p-j}| \leqq \frac{B}{\pi}\Gamma(1-\eta)\cos\left(\frac{\pi}{2}(1-\eta)\right)\sum_j (1+|j|)^{\eta-1}(1+|p-j|)^{-\alpha}. \tag{14}$$

The sum on the right side of inequality (14) can be partitioned into three parts

$$\left(\sum_{j=0}^{p}+\sum_{j<0}+\sum_{j>p}\right)(1+|j|)^{\eta-1}(1+|p-j|)^{-\alpha}=S_1+S_2+S_3.$$

An estimate for S_1 is given by

$$\begin{aligned}\sum_{j=0}^{p} |j/p|^{\eta-1}\left|\frac{p-j}{p}\right|^{-\alpha} & |p|^{\eta-1-\alpha}\\ &\cong \int_0^1 u^{\eta-1}(1-u)^{-\alpha}\,du\,|p|^{\eta-\alpha}\\ &\equiv \frac{\Gamma(\eta)\Gamma(1-\alpha)}{\Gamma(\eta+1-\alpha)}|p|^{\eta-\alpha}.\end{aligned}$$

if $p \neq 0$. A corresponding estimate for S_2 is

$$\begin{aligned}&\int_{-\infty}^{0} |u|^{\eta-1}|1-u|^{-\alpha}\,du\,|p|^{\eta-\alpha}\\ &\quad=\int_0^1 |1-v|^{\eta-1}|v|^{1-\eta}|v|^{\alpha}|v|^{-2}\,dv\,|p|^{\eta-\alpha}\\ &\quad=\int_0^1 |1-v|^{\eta-1}|v|^{\alpha-\eta-1}\,dv\,|p|^{\eta-\alpha}\\ &\quad=\frac{\Gamma(\eta)\Gamma(\alpha-\eta)}{\Gamma(\alpha)}|p|^{\eta-\alpha}\end{aligned}$$

if $p \neq 0$. In a similar manner one obtains the estimate

$$\frac{\Gamma(\alpha-\eta)\Gamma(1-\alpha)}{\Gamma(1-\eta)}|p|^{\eta-\alpha}$$

for S_3 if $p \neq 0$. These estimates imply that the right hand side of (14) is bounded by

$$\begin{aligned}\frac{B}{\pi}\Gamma(1-\eta)\cos\left(\frac{\pi}{2}(1-\eta)\right)&\left\{\frac{\Gamma(\eta)\Gamma(1-\alpha)}{\Gamma(\eta+1-\alpha)}\right.\\ &\left.+\frac{\Gamma(\eta)\Gamma(\alpha-\eta)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\eta)\Gamma(1-\alpha)}{\Gamma(1-\eta)}\right\}|p|^{\eta-\alpha}\end{aligned} \tag{15}$$

if $p \neq 0$. The relation $\Gamma(\eta)\Gamma(1-\eta)=\pi/\sin(\pi\eta)$ $\left(\text{recalling that } \eta=\alpha-\frac{2}{k}(1-\beta)\right)$ together with the bound (15) imply (8).

Let us now consider the proof of the theorem. The terms corresponding to different k in (3) are orthogonal to each other. The estimate of the lemma and (7) imply that Y_n is well-defined. Notice that

$$\begin{aligned}
S_n^N(Y) &= N^{-\beta} \sum_{j \in B_n^N} Y_j \\
&= \sum_{k>k_0} \frac{c_k}{k!} \int e^{\mathrm{i}nN(x_1+\ldots+x_k)} \frac{e^{iN(x_1+\ldots+x_k)}-1}{e^{i(x_1+\ldots+x_k)}-1} \\
&\quad \prod_{s=1}^{k} (|x_s|^{\gamma_k} Z_G(dx_s)) \\
&= \sum_{k>k_0} \frac{c_k}{k!} \int e^{\mathrm{i}n(\mu_1+\ldots+\mu_k)} \frac{e^{i(\mu_1+\ldots+\mu_k)}-1}{\{e^{i\frac{1}{N}(\mu_1+\ldots+\mu_k)}-1\}N} \\
&\quad Z_{{}_kH_N}(d\mu_1)\ldots Z_{{}_kH_N}(d\mu_k)
\end{aligned}$$

with

$$\begin{aligned}
&{}_kH(dx)=|x|^{2\gamma_k}H(dx), \\
&{}_kH_N(A)=N^{\tilde{\alpha}_k}\,{}_kH(N^{-1}A), \qquad \tilde{\alpha}_k=2\gamma_k+\alpha=\frac{2}{k}(1-\beta).
\end{aligned}$$

The measures ${}_kH_N(\cdot)$ tend locally weakly to the measures ${}_kG(\cdot)$ where ${}_kG(\cdot)$ has a spectral density

$$D^{-1}|x|^{\tilde{\alpha}_k-1}$$

with

$$D_k=2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right).$$

Let

$$K_N(x_1,\ldots,x_k)=\frac{e^{i(x_1+\ldots+x_k)}-1}{\{e^{i\frac{1}{N}(x_1+\ldots+x_k)}-1\}N}.$$

Just as in the case of the paper of Dobrushin and Major [2] one can show that the sequence of measures

$$\begin{aligned}
{}_k\mu_N(A)=\int_A \left|\sum_{p=1}^{l} \beta_p e^{\mathrm{i}np(x_1+\ldots+x_k)}\right|^2 |K_N(x_1,\ldots,x_k)|^2 \\
{}_kH_N(dx_1)\ldots{}_kH_N(dx_k)
\end{aligned} \tag{16}$$

tends weakly to the finite measure

$$\begin{aligned}
{}_k\mu(A)=\int_A \left|\sum_{p=1}^{l} \beta_p e^{\mathrm{i}np(x_1+\ldots+x_k)}\right| p^2 \left|K\left(\sum_{j=1}^{k} x_j\right)\right|^2 \\
G_k(dx_1)\ldots G_k(dx_k)
\end{aligned} \tag{17}$$

as $N\to\infty$, $k>k_0$.

We shall now show that the distribution of

$$\sum_{k=k_0+1}^{m} \frac{c_k}{k!} \int e^{in(\mu_1+\ldots+\mu_k)} K_N(\mu_1, \ldots, \mu_k) \\ Z_{kH_N}(d\mu_1) \ldots Z_{kH_N}(d\mu_k) \tag{18}$$

tends to that of

$$\sum_{k=k_0+1}^{m} \frac{c_k}{k!} \int e^{in(\mu_1+\ldots+\mu_k)} K\left(\sum_{j=1}^{k} \mu_j\right) \\ \prod_{j=1}^{k} (|\mu_j|^{\gamma_k+\frac{\alpha-1}{2}} W(d\mu_j)) \tag{19}$$

for each positive integer m. Let us now use the notation of Sect. 4 of the paper of Dobrushin [1]. Let $h_k \in H^k_{G_k}$, $k_0 < k \leqq m$. Then the distribution of

$$\sum_{k=k_0+1}^{m} \int h_k(x_1, \ldots, x_k) Z_{kH_N}(dx_1) \ldots Z_{kH_N}(dx_k) \tag{20}$$

tends to that of

$$\sum_{k=k_0+1}^{m} \int h_k(x_1, \ldots, x_k) Z_{G_k}(dx_1) \ldots Z_{G_k}(dx_k) \tag{21}$$

as $N \to \infty$. This follows since the integrals of (20) are polynomials in the random variables $Z_{kH_N}(B)$ (the Bs are the one-dimensional projections of the level sets of the functions h_k). Since the joint distributions of the random variables $Z_{kH_N}(B)$ tend to the joint distribution of the random variables $Z_{G_k}(B)$, it follows that the distribution of (20) tends to that of (21). Given any $\varepsilon > 0$ there is an $h_k \in H^k_{G_k}$ such that

$$\int_{R^k} |K_N(x_1, \ldots, x_k) - h_k(x_1, \ldots, x_k)|^2 \quad {}_kH_N(dx_1) \ldots {}_kH_N(dx_k) < \varepsilon,$$

$k_0 < k \leqq m$, and

$$\int_{R^k} \left| K\left(\sum_{j=1}^{k} x_j\right) - h_k(x_1, \ldots, x_k) \right|^2 G_k(dx_1) \ldots G_k(dx_k) < \varepsilon.$$

The sets of inequalities follow from the convergence of the measures (16) to (17) and the convergence of $K_N(x_1, \ldots, x_k)$ to $K\left(\sum_{j=1}^{k} x_j\right)$ on every finite interval. The last comments imply that the distribution of (18) tends to the distribution of (19) as $N \to \infty$ for each positive m.

The proof of the theorem will be complete if we can show that the variance of the remainder

$$\sum_{k=m}^{\infty} \frac{c_k}{k!} \int e^{in(\mu_1+\ldots+\mu_k)} K_N(\mu_1, \ldots, \mu_k) \\ Z_{kH_N}(d\mu_1) \ldots Z_{kH_N}(d\mu_k) \tag{22}$$

can be made uniformly small as $N \to \infty$ if m is fixed but sufficiently large. The variance of

$$\int e^{in(\mu_1 + \ldots + \mu_k)} K_N(\mu_1, \ldots, \mu_k) Z_{{}_kH_N}(d\mu_1) \ldots Z_{{}_kH_N}(d\mu_k)$$

is

$$\int |K_N(\mu_1, \ldots, \mu_k)|^2 {}_kH_N(d\mu_1) \ldots {}_kH_N(d\mu_k). \tag{23}$$

If

$$\rho(j) = \int e^{ijx} |x|^{2\gamma_k} H(dx)$$

then (23) by the argument on p. 34 of [2] can be rewritten

$$\frac{1}{N^{2-k\tilde{\alpha}}} \sum_{|j|<N} (N - |j|) \rho(j)^k.$$

Using the lemma, one can see that this is bounded in absolute value by

$$A^k \Gamma \left(\frac{2}{k}(1-\beta)\right)^k \sum_{|j|<N} \left(1 - \frac{|j|}{N}\right) \left|\frac{j}{N}\right|^{-2(1-\beta)} N^{-1}. \tag{24}$$

The assumption (7) together with the estimate (24) implies that the variance of (22) is uniformly small as $N \to \infty$ if m is sufficiently large.

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