Probab. Th. Rel. Fields 81, 291–318 (1989)

Probability Theory and Related Fields

On the Hydrodynamic Limit of a Ginzburg-Landau Lattice Model

The Law of Large Numbers in Arbitrary Dimensions

J. Fritz*

Mathematical Institute, HAS, H-1364 Budapest, Pf. 127, Hungary

Summary. A lattice system of interacting diffusion processes is investigated. The evolution is attractive and time reversible, the spin satisfies a conservation law. It is shown that the rescaled spin field converges in probability to the corresponding solution to a nonlinear diffusion equation.

1. Introduction

We are going to investigate the hydrodnamic behaviour of interacting diffusion processes satisfying a conservation law. The configuration space will be defined as a set, Ω of real sequences labelled by the points of the d-dimensional cubic lattice, $\mathbb{Z}^d$. If $\omega \in \Omega$ and $k \in \mathbb{Z}^d$ then $\omega_k \in \mathbb{R}$ denotes the spin at site k. The energy of the spin at site k is given by

$$H_k(\omega) = V(\omega_k) + \frac{\alpha}{2} \sum_{|j-k|=1} (\omega_j - \omega_k)^2, \tag{1.1}$$

where α is a nonnegative constant, while $V\colon \mathbb{R} \to \mathbb{R}$ is a convex function having three continuous derivatives such that $|1 - V''(x)| < c$ for all $x \in \mathbb{R}$ with some $c < 1$, and V''' is bounded, too. The evolution is governed by the gradient of H,

$$\partial_k H = \partial_k H(\omega) = V'(\omega_k) + \alpha \sum_{|j-k|=1} (\omega_k - \omega_j), \tag{1.2}$$

in the following way. Along each oriented bond $k \to j$ there is a random current, J_{kj} of the spin, and the temporal evolution is determined by the associated conservation law. More exactly, for each positively oriented bond we are given a standard Wiener process, w_{kj}, they are independent, and $w_{kj} + w_{jk} = 0$ by convention. The currents admit a stochastic differential,

$$d J_{kj} = \tfrac{1}{2} [\partial_k H(\omega) - \partial_j H(\omega)]\, dt + d w_{kj}, \tag{1.3}$$

* Supported in part by the Hungarian National Foundation for Scientific Research, grant No. 819/1, and by the Mathematical Department of Rutgers University, N.S.F. grant DMR 8612369

and the evolution is defined by a system of stochastic differential equations,

$$d\omega_k + \sum_{|j-k|=1} dJ_{kj} = 0, \quad k \in \mathbb{Z}^d, \qquad \omega_k(0) = \sigma_k. \tag{1.4}$$

We are interested in the asymptotic behaviour of the rescaled spin field

$$S^\varepsilon(t, \varphi, \sigma^\varepsilon) = \varepsilon^d \sum_{k \in \mathbb{Z}^d} \varphi(\varepsilon k)\, \omega_k(t/\varepsilon^2) \tag{1.5}$$

as $\varepsilon \to 0$ and the initial configuration, σ^ε approaches some smooth profile, ρ_0: $\mathbb{R}^d \to \mathbb{R}$. We show that S^ε converges in probability to a nontrivial deterministic limit, $\int \varphi(x)\, \rho(t, x)\, dx$, and ρ, the limiting spin density, satisfies

$$\frac{\partial \rho}{\partial t} = \operatorname{div}[D(\rho) \operatorname{grad} \rho(t, x)], \qquad \rho(0, x) = \rho_0(x), \tag{1.6}$$

where D: $\mathbb{R} \to \mathbb{R}$ is strictly positive and bounded.

The first attempt to derive the macroscopic equations of hydrodynamics from statistical mechanics is due to Morrey [18], a more general formulation of the problem goes back to Dobrushin [8], see [3, 4, 9] and the early papers [17, 22, 23] on the subject. A survey of some mathematical and physical ideas and results is presented in [6]. The evolution (1.3)–(1.4) is a lattice version of the time dependent Ginzburg-Landau equation, sometimes it is called a Cahn-Hilliard theory, see Spohn [26] for some further references on the physics literature. This paper is a continuation of the project outlined in [12, 13]. The proof of [14] is simplified and extended to arbitrary dimensions, we prove a law of large numbers for continuous functionals of the conserved field. The deterministic (zero temperature) case was discussed in [11]. Some related results on the one-dimensional, general zero range model were announced by Rost [24]. An extension of the law of large numbers to the one-dimensional continuum Ginzburg-Landau equation was obtained by Funaki [15]. Equilibrium fluctuations (the related central limit problem) were described by Spohn [26]. In a recent paper [16] Guo, Papanicolau and Varadhan [16] propose a new approach to the law of large numbers. This method is restricted to finite volumes, but it is fairly general in other respects. For example, the convexity of the self-potential V implying attractivity of the evolution is not needed. Donsker and Varadhan [10] treat the associated large deviation problem by similar methods.

This work was completed during my stay at Center for Mathematical Sciences Research, Rutgers University. I am indebted to Joel L. Lebowitz for stimulating discussions and kind hospitality there.

2. Main Result

It is very convenient to embed our configuration space into a space of functions on $\mathbb{R}^d$, then the configurations will be interpreted as step functions. The topology of this space will be specified by a family of weighted $\mathbb{L}^2$-norms, $|\cdot|_r$, $r\in\mathbb{R}$. The weight functions, θ_r: $\mathbb{R}^d\to(0,1]$ are defined as follows. Let θ: $[0,\infty)\to(0,1]$ be a nonincreasing and twice continuously differentiable function such that $\theta(u)=\frac{1}{2}e^{2-u}$ if $u\geqq 2$, $\theta(u)=1$ if $u\leqq 1$, while $\theta(u)\geqq e^{-u}$ and $0\leqq -\theta'(u)\leqq\theta(u)\leqq\frac{1}{2}e^{2-u}$ for all $u\geqq 0$. Now $\theta_r(x)=[\theta(|x|)]^r$ for $x\in\mathbb{R}^d$ and $r\in\mathbb{R}$, and

$$|\sigma|_r^2=\int\theta_r(x)\,|\sigma(x)|^2\,dx \tag{2.1}$$

whenever $\sigma:\mathbb{R}^d\to\mathbb{R}$ is measurable. The associated scalar product will be denoted as $\langle\cdot,\cdot\rangle_r$, the same notation will be used for vector valued functions. Introduce now $\mathbb{L}_r^2(\mathbb{R}^d)$ as the real Hilbert space of locally integrable σ: $\mathbb{R}^d\to\mathbb{R}$ with norm $|\cdot|_r$. Since $\theta_r\geqq\theta_s$ if $r\leqq s$, we have $\mathbb{L}_s^2\subset\mathbb{L}_r^2$ in this case. Since $\theta_{-r}=1/\theta_r$, the spaces $\mathbb{L}_r^2$ and $\mathbb{L}_{-r}^2$ are the duals of each other with respect to the scalar product $\langle\cdot,\cdot\rangle_0$ of $\mathbb{L}_0^2(\mathbb{R}^d)=\mathbb{L}^2(\mathbb{R}^d)$. Define now $\mathbb{L}_e^2$ as the locally convex space with seminorms $|\cdot|_r$, $r>0$. This simply means that

$$\mathbb{L}_e^2=\bigcap_{r>0}\mathbb{L}_r^2(\mathbb{R}^d), \tag{2.2}$$

and $\sigma_n\to\sigma$ in the (strong) topology of $\mathbb{L}_e^2$ iff $|\sigma_n-\sigma|_r\to 0$ for each $r>0$. The dual space of $\mathbb{L}_e^2$ is just

$$\mathbb{L}_{-e}^2=\bigcup_{r>0}\mathbb{L}_{-r}^2(\mathbb{R}^d), \tag{2.3}$$

the elements of $\mathbb{L}_{-e}^2$ are represented as

$$\varphi=\varphi(\sigma)=\int\varphi(x)\,\sigma(x)\,dx,\qquad\sigma\in\mathbb{L}_e^2. \tag{2.4}$$

The weak topology of $\mathbb{L}_e^2$ is given by a fundamental system of the neighborhoods of $0\in\mathbb{L}_e^2$, namely

$$U_\gamma(\varphi_1,\varphi_2,\dots,\varphi_n)=[\sigma\in\mathbb{L}_e^2\colon |\varphi_k(\sigma)|<\gamma,\ k=1,2,\dots,n], \tag{2.5}$$

where $\gamma>0$ and $\varphi_k\in\mathbb{L}_{-e}^2$. A subset B of $\mathbb{L}_e^2$ is called a ball if

$$B=[\sigma\in\mathbb{L}_e^2\colon |\sigma|_r\leqq b_r\text{ for all }r>0] \tag{2.6}$$

with some $b_r<\infty$. It is easy to check that $\mathbb{L}_e^2$ is a reflexive space, and its balls are weakly compact. If $\Sigma\subset\mathbb{L}_e^2$ then $\mathbb{C}_s(\Sigma)$ and $\mathbb{C}_w(\Sigma)$ denote the spaces of strongly (weakly) continuous and bounded maps of Σ into $\mathbb{R}$. The spaces of weakly differentiable $\lambda\in\mathbb{L}_e^2$ and $\varphi\in\mathbb{L}_{-e}^2$ with partial derivatives belonging to $\mathbb{L}_e^2$ and $\mathbb{L}_{-e}^2$, respectively, will be denoted by $\mathbb{H}_e^1$ and $\mathbb{H}_{-e}^1$.

The embedding of our process into $\mathbb{L}_e^2$ is based on the following correspondence between sequences and step functions. Let $C_\varepsilon(\varepsilon k)$ denote the Dirichlet cell of $\varepsilon\mathbb{Z}^d$ centered at εk, $\varepsilon>0$, $k\in\mathbb{Z}^d$. More exactly, if x_i and k_i denote the coordinates of x and k, then

$$C_\varepsilon(\varepsilon k)=[x\in\mathbb{R}^d\colon -\varepsilon/2\leqq x_i-\varepsilon k_i<\varepsilon/2,\ i=1,2,\dots,d]. \tag{2.7}$$

Moreover, let $C_\varepsilon(x)=C_\varepsilon(\varepsilon k)$ if $x\in C_\varepsilon(\varepsilon k)$, and define Ω_ε as the space of $\omega\in\mathbb{L}_e^2$ such that $\omega(x)=\omega(y)$ whenever $x\in C_\varepsilon(y)$. For each $\varepsilon>0$ and $\omega\in\mathbb{L}_e^2$ there is a step function, $\omega^\varepsilon=I_\varepsilon\,\omega$ defined by

$$I_\varepsilon\,\omega(x)=\varepsilon^{-d}\int_{C_\varepsilon(x)}\omega(y)\,dy, \tag{2.8}$$

that is, $\Omega_\varepsilon=I_\varepsilon\,\mathbb{L}_e^2$. On the other hand, if $\omega=(\omega_k)_{k\in\mathbb{Z}^d}$ is a sequence indexed by $\mathbb{Z}^d$, then $\omega^\varepsilon(x)=\omega_k$ iff $x\in C_\varepsilon(\varepsilon k)$ defines a step function $\omega^\varepsilon\in\Omega_\varepsilon$ for each $\varepsilon>0$, provided that

$$|\omega|_r=\Big[\sum_{k\in\mathbb{Z}^d} I_1\,\theta_r(k)\,\omega_k^2\Big]^{1/2}<\infty \tag{2.9}$$

for each $r>0$. The space of such sequences will be denoted by Ω, and the evolution law (1.3) will be considered in this configuration space. Since Ω and Ω_1 are essentially identical, we are not going to make a sharp distinction between them.

The existence problem for (1.3) is almost trivial. Let $b_k=b_k(\omega)$ denote the drift of (1.3), and consider $b=(b_k(\omega))_{k\in\mathbb{Z}^d}$ as a map on Ω. It is easy to check that for each $r\in\mathbb{R}$ we have

$$|b(\omega)|_r\leqq L_r\,|\omega|_r, \tag{2.10}$$

$$|b(\omega)-b(\bar\omega)|_r\leqq L_r\,|\omega-\bar\omega|_r \tag{2.11}$$

for all ω, $\bar\omega\in\Omega$ with some $L_r<\infty$. Since the martingale part of (1.3) does not depend on the configuration, the easiest iteration procedure can be used to construct continuous solutions in Ω, and the very same methods yield smooth dependence on initial data. Although no stochastic calculus is needed at this point, for convenience we refer to the abstract results of Chapter VII in [5]. Let $\omega(t,\sigma)$ denote the solution to (1.3) with initial condition $\omega(0,\sigma)=\sigma\in\Omega$. The solution is a continuous process in Ω for each $\sigma\in\Omega$, and we have some constants, K_r such that for $r>0$

$$\mathbb{E}[|\omega(t,\sigma)|_r^2]\leqq(|\sigma|_r^2+K_r)\exp(K_r t), \tag{2.12}$$

see Theorem VII.2.1 in [5]. Moreover, the solution is a differentiable function of the initial data, and its derivatives satisfy the linearized equation associated with (1.3). For measurable and bounded $f\colon\Omega\to\mathbb{R}$ we define $\mathbb{P}^t f=\mathbb{P}^t f(\sigma)=\mathbb{E}[f(\omega(t,\sigma))]$. If f is a bounded cylinder function with continuous and bounded first and second derivatives, then $\mathbb{P}^t f$ is twice differentiable, and

$$\mathbb{P}^t f(\sigma)=f(\sigma)+\int_0^t \mathbb{G}\,\mathbb{P}^s f(\sigma)\,ds, \tag{2.13}$$

where $\mathbb{G}$ denotes the formal generator of $\mathbb{P}^t$,

$$\mathbb{G}f=\tfrac{1}{2}\sum_k\sum_{|j-k|=1} e^{H_j}\,\partial_j[e^{-H_j}(\partial_k f-\partial_j f)], \tag{2.14}$$

and ∂_k denotes differentiation with respect to σ_k; see Theorems VII.3.3 and VII.4.1 in [5].

Since (1.3) is a conservation law, we expect that $\mathbb{P}^t$ has a whole family of stationary measures. Let μ_λ denote the Gibbs state on Ω with energy $H_\lambda = H - \sum \lambda_k \sigma_k$, where $\lambda \in \Omega$ is arbitrary. This means that the conditional densities of μ_λ are specified as

$$\mu_\lambda(d\sigma_k | \sigma_j, j \neq k) = Z_k^{-1} \exp[-H_k(\sigma) + \lambda_k \sigma_k] \, d\sigma_k, \tag{2.15}$$

where Z_k is a normalizing factor depending on λ_k and σ_j with $|j-k|=1$. It is easy to verify that if $\lambda \in \Omega$ then μ_λ is unique, and $\mu_\lambda(\Omega)=1$, see [14] with some further references. In particular, if $\lambda_k = v$ for all $k \in \mathbb{Z}^d$, then a distinguishing notation, $\mu_v^0 = \mu_\lambda$ will be used. Observe that (2.14) implies

$$\begin{aligned} &\int f_1(\sigma)\, \mathbb{G} f_2(\sigma)\, \mu_v^0(d\sigma) \\ &\quad -\tfrac{1}{4} \sum_k \sum_{|j-k|=1} \int (\partial_j f_1 - \partial_k f_1)(\partial_j f_2 - \partial_k f_2)\, d\mu_v^0 \end{aligned} \tag{2.16}$$

for smooth cylinder functions. This Dirichlet form contains the main information on $\mathbb{P}^t$. For example, we obtain $\int \mathbb{G} f \, d\mu_\lambda = 0$ for all smooth cylinder functions, whenever λ is a harmonic sequence, i.e.

$$\lambda_k = \frac{1}{2d} \sum_{|j-k|=1} \lambda_j \qquad \text{for all } k \in \mathbb{Z}^d. \tag{2.17}$$

It is not difficult to show that every Gibbs state of this type is in fact a stationary measure of $\mathbb{P}^t$. The converse statement is more difficult to prove, the free energy method seems to be applicable. A direct calculation yields $\lambda_k = \int \partial_k H(\sigma)\, \mu_\lambda(d\sigma)$ suggesting the following expression for the diffusion coefficient D of the limiting Eq. (1.6), see [6, 13]. Indeed, if $\lambda = v'$ is a constant, then the mean spin equals the derivative, F', of the equilibrium free energy, F as a function of the chemical potential v. For brevity we only define $F': \mathbb{R} \to \mathbb{R}$ by

$$F'(v) = \int \sigma_k \, \mu_v^0(d\sigma). \tag{2.18}$$

Since μ_v^0 is translation invariant, F' does not depend on k. We shall show that F' is strictly increasing, thus

$$D(u) = 1/2 F''(v) \qquad \text{if } u = F'(v) \tag{2.19}$$

defines D for all $u \in \mathbb{R}$. Moreover, D is continuous, and $0 < c \leqq D(u) \leqq c_2 < \infty$ with some constants, consequently (1.6) is uniquely solved in the following sense, see Sect. 7. For every $\sigma \in \mathbb{H}_e^1$ there exists a continuous trajectory, $\rho_t \in \mathbb{L}_e^2$, $t \geqq 0$ such that $\rho_0 = \sigma$ and for each twice continuously differentiable $\varphi: \mathbb{R}^d \to \mathbb{R}$ with compact support we have

$$\int \varphi(x)\, \rho_t(x)\, dx = \int \varphi(x)\, \sigma(x)\, dx + \int_0^t \int E(\rho_s(x))\, \Delta \varphi(x)\, dx\, ds, \tag{2.20}$$

where E denotes the inverse function of F', that is, $E' = D$.

The problem of hydrodynamic limit can be formulated in terms of a family of Markov processes, ω_t^ε, $t \geqq 0$, $\varepsilon > 0$; $\omega_t^\varepsilon \in \Omega_\varepsilon$ is defined by $\omega_t^\varepsilon(x) = \omega_k(t/\varepsilon^2)$ if $x \in C_\varepsilon(\varepsilon k)$, where $\omega = (\omega_k(t))_{k \in \mathbb{Z}^d}$ is a strong solution to (1.4). The initial value of ω^ε will usually be denoted by $\sigma = \omega_0^\varepsilon$, the rescaled spin field becomes $\varphi(\omega_t^\varepsilon)$, and the associated Markov semigroup is defined as

$$\mathbb{P}_\varepsilon^t g(\sigma) = \mathbb{E}[g(\omega_t^\varepsilon) \mid \omega_0^\varepsilon = \sigma], \quad \sigma \in \Omega_\varepsilon, \quad g \in \mathbb{C}_s(\Omega_\varepsilon). \tag{2.21}$$

In the traditional formulation of the problem the initial configuration is random, it is distributed by a family of local equilibrium states, $\mu_{\lambda,\varepsilon}$, see [6]. The initial randomness is certainly necessary in the case of conservative (Hamiltonian) systems, this assumption is relevant also for the Guo-Papanicolau-Varadhan approach [16]. Due to the convexity of V, our model has very good ergodic properties: it seems that local equilibrium is established after time ε^{-2}, even if the initial distribution is deterministic, but we are not in a position to give an exact meaning to this conjecture. We can control the relaxation of our system to a local equilibrium only at a level of the law of large numbers. The following presentation of the main result is a little bit stronger than that proposed by Funaki [15].

Theorem 1. *Suppose that ω_0^ε converges weakly in $\mathbb{L}_e^2$ to some $\rho_0 \in \mathbb{H}_e^1$ as $\varepsilon \to 0$, then $\varphi(\omega_t^\varepsilon)$ converges in probability to $\varphi(\rho_t)$ for each $t > 0$ and $\varphi \in \mathbb{H}_{-e}^1$, where ρ_t is specified as the weak solution to* (1.6) *with initial configuration ρ_0.* □

One of the most crucial features of our dynamics is the weakly continuous dependence of solutions on initial data, this property will be used to reduce Theorem 1 to the case of random initial configurations, and then to solve the problem. The simplest choice of the local equilibrium distributions is the following one. For $\varepsilon > 0$ and $\lambda \in \mathbb{L}_e^2$ let $\mu_{\lambda,\varepsilon} = \mu_{\lambda_\varepsilon}$, where $\lambda^\varepsilon \in \Omega$ is defined by $\lambda_k^\varepsilon = I_\varepsilon \lambda(\varepsilon k)$ Remember that $\mu_{\lambda,\varepsilon}$ is originally defined on Ω, but we can project it to Ω_ε by means of the usual correspondence $\sigma \to \sigma^\varepsilon$ between Ω and Ω_ε, i.e. $\sigma^\varepsilon(x) = \sigma_k$ if $x \in C_\varepsilon(\varepsilon k)$. For notational convenience we assume that $\mu_{\lambda,\varepsilon}$ is defined on the Borel field of the whole space $\mathbb{L}_e^2$; of course, $\mu_{\lambda,\varepsilon}(\Omega_\varepsilon) = 1$. Observe that $\mu_{\lambda,\varepsilon}$ satisfies a law of large numbers. We shall show that if $\varphi \in \mathbb{L}_{-e}^2$, then

$$\int \varphi(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) \xrightarrow[\varepsilon \to 0]{} \int \varphi(x)\, \rho_0(x)\, dx, \tag{2.22}$$

where $\rho_0(x) = F'(\lambda(x))$, and $\varphi(\sigma)$ converges in probability to $\varphi(\rho_0)$ as $\varepsilon \to 0$. This statement is trivial if $\alpha = 0$, because $\mu_{\lambda,\varepsilon}$ is a product measure in this particular case. If $\alpha > 0$ then the convexity of V implies that $\mu_{\lambda,\varepsilon}$ belongs to Dobrushin's uniqueness domain, thus it satisfies an exponential decay of correlations, see [17]. Therefore $\mu_{\lambda,\varepsilon}$ converges to the δ-measure concentrated on in the sense that

$$\int g(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) \xrightarrow[\varepsilon \to 0]{} g(\rho_0) \quad \text{if } g \in \mathbb{C}_w(\mathbb{L}_e^2). \tag{2.23}$$

Notice that (2.23) fails to hold for strongly continuous functions. The weak topology of $\mathbb{L}_e^2$ plays a fundamental role in the proofs, and this role is not a technical one: the law of large numbers is formulated in a natural way in terms of this weak topology. The second version of the main result maintains that (2.23) remains in force for positive times, too.

Theorem 2. *If $g \in \mathbb{C}_w(\mathbb{L}_e^2)$ and $\lambda \in \mathbb{H}_e^1$ then*

$$\lim_{\varepsilon \to 0} \int \mathbb{P}_\varepsilon^t \, g(\sigma) \, \mu_{\lambda,\varepsilon}(d\sigma) = g(\rho_t),$$

where ρ_t is the weak solution to (1.6) *with initial condition $\rho_0 = \rho_0(x) = F'(\lambda(x))$.* □

The ideas of the proof are outlined in the next section. Following [12] we prove Theorem 2 first for a more restricted class of functions, $\mathbb{D}_L(\mathbb{L}_e^2)$ to be defined in Sect. 7. The next step is the tightness of the family of time evolved measures, $\mu_{\lambda,\varepsilon} \mathbb{P}_\varepsilon^t$, whence the general case follows by the Stone-Weierstrass theorem. Finally, we derive Theorem 1 from Theorem 2 by means of the weakly continuous dependence of solutions on initial data, cf. Funaki [15].

The methods of this paper work under some more general conditions. The case of an external driving force, and a reaction term will be discussed in a forthcoming paper with Christian Maes. Systems with random, or macroscopically inhomogeneous conductivities can also be treated. The case of configuration dependent conductivities, that means correlated currents, is more difficult, but not absolutely hopeless.

3. On the Ideas of the Proof

The framework is fairly general, we want to expose the underlying particular structure, too. For each $\varepsilon > 0$ we are given a Markov process $\mathbb{P}_\varepsilon^t$, and a family of initial distributions, $\mu_{\lambda,\varepsilon}$, $\lambda \in \Lambda = \mathbb{H}_e^1$. Let $\mathbb{G}_\varepsilon$ denote the formal generator of $\mathbb{P}_\varepsilon^t$, and consider the class $\mathbb{C}_0(\mathbb{L}_e^2) \subset \mathbb{C}_w(\mathbb{L}_e^2)$ consisting of functions of type $f(\sigma) = h(\varphi_1(\sigma), \varphi_2(\sigma), \ldots, \varphi_n(\sigma))$, where $h\colon \mathbb{R}^n \to \mathbb{R}$ and $\varphi_k\colon \mathbb{R}^d \to \mathbb{R}$ are twice continuously differentiable with compact supports. We want to have a statement of the following kind. For each $g \in \mathbb{C}_0(\mathbb{L}_e^2)$ and $\lambda \in \Lambda$ there exists a limit

$$\int \mathbb{P}_\varepsilon^t \, g(\sigma) \, \mu_{\lambda,\varepsilon}(d\sigma) \xrightarrow[\varepsilon \to 0]{} \tilde{\mathbb{P}}^t \, \tilde{g}(\lambda), \tag{3.1}$$

and the limit is specified as follows. The map $g \to \tilde{g}$ is given by

$$\tilde{g}(\lambda) = \lim_{\varepsilon \to 0} \int g(\sigma) \, \mu_{\lambda,\varepsilon}(d\sigma), \tag{3.2}$$

while $\tilde{\mathbb{P}}^t$ is a semigroup in $\mathbb{C}_w(\Lambda)$ specified by its formal generator

$$\tilde{\mathbb{G}} \tilde{f}(\lambda) = \lim_{\varepsilon \to 0} \int \mathbb{G}_\varepsilon f(\sigma) \, \mu_{\lambda,\varepsilon}(d\sigma). \tag{3.3}$$

Of course, this loosely formulated statement is too general, there is no theorem claiming that (3.2) and (3.3) imply (3.1). Nevertheless, we can go a little bit further. We summarize below the main steps of a proof we are going to materialize in the case of the Ginzburg-Landau model (1.3).

Introduce the resolvent, $\mathbb{R}_{z,\varepsilon}$ for $z>0$, $g\in\mathbb{C}_0(\mathbb{L}^2_e)$ and $\sigma\in\Omega_\varepsilon$,

$$\mathbb{R}_{z,\varepsilon}\, g(\sigma)=\int_0^\infty e^{-zt}\,\mathbb{P}^t_\varepsilon\, g(\sigma)\,dt. \tag{3.4}$$

From (2.13) we obtain that $f_\varepsilon=\mathbb{R}_{z,\varepsilon}\,g$ satisfies

$$g(\sigma)=z f_\varepsilon(\sigma)-\mathbb{G}_\varepsilon f_\varepsilon(\sigma), \qquad \sigma\in\Omega_\varepsilon \tag{3.5}$$

the resolvent equation, whence

$$\int g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=z\int f_\varepsilon(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)-\int \mathbb{G}_\varepsilon f_\varepsilon(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma). \tag{3.6}$$

Imagine now that we can use a compactness argument to pass to a limiting resolvent equation via (3.2) and (3.3) along some subsequence $\varepsilon_n\to 0$. This means that each term of (3.6) tends to the corresponding term of

$$\tilde{g}(\lambda)=z\tilde{f}(\lambda)-\tilde{\mathbb{G}}\tilde{f}(\lambda) \tag{3.7}$$

for each $\lambda\in\Lambda$ and $z\geqq z_0$, where $z_0>0$ depends only on g. It is very relevant here that the subsequence may depend on g and z, but it does not depend on λ. Let $\mathbb{R}_z\,\tilde{g}=\int_0^\infty e^{-zt}\,\tilde{\mathbb{P}}^t\,\tilde{g}\,dt$, then

$$z\,\mathbb{R}_z\,\tilde{g}-\mathbb{G}\mathbb{R}_z\,\tilde{g}=\tilde{g}, \tag{3.8}$$

i.e., $\mathbb{R}_z g$ solves (3.7). Therefore, if $\tilde{h}=\tilde{f}-\mathbb{R}_z\,\tilde{g}$, then $\partial_t\tilde{\mathbb{P}}^t\,\tilde{h}=\tilde{\mathbb{P}}^t\,\tilde{\mathbb{G}}\,\tilde{h}=z\,\tilde{\mathbb{P}}\,\tilde{h}$, whence $\tilde{\mathbb{P}}^t\,\tilde{h}=\tilde{h}\,e^{zt}$. Since $\tilde{\mathbb{P}}^t$ is a contraction semigroup, and $z>0$, this is possible only if $\tilde{h}=0$, consequently

$$\lim_{\varepsilon\to 0}\int_0^\infty e^{-zt}\int \mathbb{P}^t_\varepsilon\, g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)\,dt=\mathbb{R}_z\,\tilde{g}(\lambda) \tag{3.9}$$

for all $\lambda\in\Lambda$, $g\in\mathbb{C}_0(\mathbb{L}^2_e)$ and $z\geqq z_0$.

Suppose now that the family $h_\varepsilon(t)=\int\mathbb{P}^t_\varepsilon\, g\,d\mu_{\lambda,\varepsilon}$ is equicontinuous at each $t>0$ as $\varepsilon\to 0$, at least if λ and g are fixed. Since h_ε is uniformly bounded, by the Arzela-Ascoli criterion of compactness we can select a subsequence $\varepsilon_n\to 0$ in such a way that h_{ε_n} converges uniformly on compact intervals of time to some limit, h_0. Thus, for $z>z_0$ we have

$$\lim_{n\to\infty}\int_0^\infty e^{-zt}\,h_{\varepsilon_n}(t)\,dt=\int_0^\infty e^{-zt}\,h_0(t)\,dt=\mathbb{R}_z\,\tilde{g}, \tag{3.10}$$

whence $h_0(t)=\tilde{\mathbb{P}}^t\,\tilde{g}$ for all $t>0$ by the uniqueness of the Laplace transform, which implies (3.1). Now we can approximate weakly continuous functions by smooth functions of class $\mathbb{C}_0(\mathbb{L}_e^2)$ via the Stone-Weierstrass theorem on the balls of $\mathbb{L}_e^2$, thus we can extend (3.1) to $g \in \mathbb{C}_w(\mathbb{L}_e^2)$ by means of the tightness of the time evolved measure $\mu_{\lambda,\varepsilon}\mathbb{P}_\varepsilon^t$. This follows by an a priori bound for the mean value of $|\omega_t^\varepsilon|_r^2$ with some $r>0$.

The crucial step of the proof is certainly the compactness argument resulting in (3.7). Since we need the convergence of each term of (3.6) simultaneously for all $\lambda \in \Lambda$, we have to prove an Arzela-Ascoli criterion for f_ε as a function of the initial configuration. Unfortunately, the strongly compact sets of $\mathbb{L}_e^2$ are not rich enough to carry the initial distributions, $\mu_{\lambda,\varepsilon}$, thus we have to show that some time averages, like $f_\varepsilon = \mathbb{R}_{z,\varepsilon} g$ are weakly equicontinuous functions of the initial data, $\sigma \in \mathbb{L}_e^2$. The very same continuity property allows us to conclude Theorem 1 from (3.1), cf. Funaki [15]. The proof of this compactness criterion reduces to a study of the linearized system of (1.3); its parabolic structure is the key of our investigations. Let us remark that (1.3) is attractive in a very strong sense, at least if $\alpha=0$. Indeed, if $\sigma_k \leqq \bar\sigma_k$ for all k, then $\omega_k(t,\sigma) \leqq \omega_k(t,\bar\sigma)$ for all k and $t>0$. This strong attractivity breaks down if $\alpha>0$, thus we believe that the parabolicity of the linearized system is a useful, general condition for continuous spin systems.

Now we are in a position to enter into some further, more technical details. Taking into account (2.16), the Dirichlet form of $\mathbb{G}$ we can simplify (3.3) in a very radical way. We obtain that

$$\int \mathbb{G} f \, d\mu_\lambda = -\tfrac{1}{4} \sum_{k \in \mathbb{Z}^d} \sum_{|j-k|=1} \int (\lambda_j - \lambda_k)(\partial_j f - \partial_k f)\, d\mu_\lambda. \tag{3.11}$$

This fundamental identity scales into

$$\int \mathbb{G}_\varepsilon f \, d\mu_{\lambda,\varepsilon} = -\tfrac{1}{2} \iint \langle \nabla_\varepsilon \lambda^\varepsilon(x), \nabla_\varepsilon \mathbb{D} f(x,\sigma)\rangle \, dx \, \mu_{\lambda,\varepsilon}(d\sigma), \tag{3.12}$$

where $\lambda^\varepsilon = I_\varepsilon \lambda$, ∇_ε denotes the lattice approximation of the gradient of a function, $\langle \cdot,\cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$, and $\mathbb{D} f$ is the functional derivative of f. More exactly, let $e_1, e_2, \ldots, e_d$ denote the unit vectors of our system of coordinates in $\mathbb{R}^d$, then

$$\nabla_\varepsilon \sigma(x) = \frac{1}{\varepsilon} \sum_{i=1}^d e_i [\sigma(x+\varepsilon e_i) - \sigma(x)] \tag{3.13}$$

if $\sigma: \mathbb{R}^d \to \mathbb{R}$ is a scalar, while

$$\nabla_\varepsilon^* \varphi(x) = \frac{1}{\varepsilon} \sum_{i=1}^d [\varphi_i(x-\varepsilon e_i) - \varphi_i(x)] \tag{3.14}$$

if $\varphi = e_1\varphi_1 + e_2\varphi_2 + \ldots + e_d\varphi_d$ is a vector field. Notice that ∇_ε^* is a lattice approximation to $-\mathrm{div}$, and $\Delta_\varepsilon = -\nabla_\varepsilon^* \nabla_\varepsilon$, that is

$$\Delta_\varepsilon \varphi(x) = \varepsilon^{-2} \sum_{i=1}^d [\varphi(x+\varepsilon e_i) - 2\varphi(x) + \varphi(x-\varepsilon e_i)]. \tag{3.15}$$

We are using the following notation of functional (variational) derivatives. Let $\Sigma \subset \mathbb{L}_e^2$ denote a convex set and define $\mathbb{D}_s(\Sigma)$ as the space of $f\colon \Sigma \to \mathbb{R}$ such that if $\sigma, \bar{\sigma} \in \Sigma$ and $\delta = \sigma - \bar{\sigma}$, then

$$f(\sigma) - f(\bar{\sigma}) = \int_0^1 \int \delta(x)\, \mathbb{D} f(x, \bar{\sigma} + q\delta)\, dx\, dq, \tag{3.16}$$

where $\mathbb{D} f\colon \Sigma \to \mathbb{L}_{-e}^2$ is strongly continuous. If Σ is dense in $\mathbb{L}_e^2$ then $\mathbb{D} f$ is uniquely defined. If $\Sigma = \Omega_\varepsilon$, as in (3.12), then $\mathbb{D} f(x, \sigma) = \mathbb{D} f(y, \sigma)$ if $x \in C_\varepsilon(y)$ is a natural convention we accept, that is, $\mathbb{D} f(x, \sigma) = \varepsilon^{-d}\, \partial f(\sigma)/\partial \sigma(x)$ in this case.

Now we are in a position to identify the limit (3.1). In view of (2.25), F' defines a one-to-one map of $\mathbb{L}_e^2$ onto itself by $F'(\lambda)(x) = F'(\lambda(x))$, and the law of large numbers, see (2.23), implies that $\tilde{g}(\lambda) = g(F'(\lambda))$ if $g \in \mathbb{C}_w(\mathbb{L}_e^2)$. Similarily, if $\lambda \in \mathbb{H}_e^1$ and $f \in \mathbb{C}_0(\mathbb{L}_e^2)$, then from (3.12) by the chain rule we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int \mathbb{G}_\varepsilon f\, d\mu_{\lambda,\varepsilon} &= -\tfrac{1}{2} \int \langle \operatorname{grad} \lambda(x), \operatorname{grad} \mathbb{D} f(x, F'(\lambda)) \rangle\, dx \\ &= -\int \left\langle \operatorname{grad} \lambda(x), \operatorname{grad} \frac{\mathbb{D}\tilde{f}(x, \lambda)}{2F''(\lambda(x))} \right\rangle dx. \end{aligned} \tag{3.17}$$

This means that $\tilde{\mathbb{P}}^t\, \tilde{g}(\lambda) = \tilde{g}(\lambda_t)$, where λ_t is a continuous trajectory in $\mathbb{L}_e^2$ satisfying

$$2F''(\lambda_t(x))\, \partial_t \lambda_t(x) = \Delta \lambda_t(x), \qquad \lambda_0 = \lambda \tag{3.18}$$

in a weak sense. In view of the correspondence $\rho = F'(\lambda)$, the Eqs. (1.6) and (3.18) are equivalent. Let us remark that (3.1)–(3.3) extend to some $g \notin \mathbb{C}_w(\mathbb{L}_e^2)$, but $\tilde{g} \neq g(F'(\lambda))$ in such cases, see [13]. Of course, both (2.23) and the solvability of (3.18) need a proof, but the crucial step is certainly the compactness argument allowing us to pass from (3.6) via (3.12) to (3.7). The basic ideas are exposed as follows.

In the spirit of passing from (3.11) to (3.12), we interpret the rescaled process ω_t^ε as a trajectory in Ω_ε satisfying

$$d\omega_t^\varepsilon = \frac{1}{2} \Delta_\varepsilon V'(\omega_t^\varepsilon)\, dt - \frac{\alpha}{2} \varepsilon^2 \Delta_\varepsilon^2 \omega_t^\varepsilon\, dt + \nabla_\varepsilon^* dw_t^\varepsilon, \qquad \omega_0^\varepsilon = \sigma \in \Omega_\varepsilon, \tag{3.19}$$

where $w_t^\varepsilon(x) = \varepsilon \sum e_{j-k}\, w_{kj}(t/\varepsilon^2)$ if $x \in C_\varepsilon(\varepsilon k)$, the sum being for $j = k + e_i$, $i = 1, 2, \ldots, d$. This means that the current, see (1.3) is considered as a vector. The linearized equation (first variational system) of (3.19) is obtained by differentiating both sides of (3.19) with respect to some parameter of the initial configuration. We get

$$\partial_t u_t(y) = \frac{1}{2} \Delta_\varepsilon [a_t(y)\, u_t(y)] - \frac{\alpha}{2} \Delta_\varepsilon^2 u_t(y), \tag{3.20}$$

where $\partial_t = \partial/\partial t$, $a_t(y) = V''(\omega_t^\varepsilon(y))$. Let $p_a = p_a(s, x; t, y)$, $0 \leqq s \leqq t$, $x, y \in \mathbb{R}^d$ denote the fundamental solution to (3.20), that is, $u_t = p_a(s, x; t, \cdot)$ satisfies (3.20) for $t \geqq s$ with initial condition $p_a(s, x; s, y) = \varepsilon^{-d}$ if $y \in C_\varepsilon(x)$, and $p_a(s, x; s, y) = 0$ if

$y \notin C_\varepsilon(x)$. Since p_a is just the functional derivative of $\omega_t^\varepsilon(y)$ with respect to ω_s^ε, by the chain rule we obtain a representation for $\mathbb{D}\mathbb{R}_{z,\varepsilon}$ in terms of p_a, namely if $f_\varepsilon = \mathbb{R}_{z,\varepsilon} g$, $z>0$, then

$$\mathbb{D} f_\varepsilon(x, \sigma) = \mathbb{E}[Y_z^\varepsilon(0, x, \sigma, g) \,|\, \omega_0^\varepsilon = \sigma], \quad \sigma \in \Omega_\varepsilon, \quad g \in \mathbb{D}_b(\Omega_\varepsilon), \quad x \in \mathbb{R}^d, \tag{3.21}$$

where

$$Y_z^\varepsilon(s, x, \sigma, g) = \int_s^\infty e^{zs-zt} \int p_a(s, x; t, y)\, \mathbb{D} g(y, \omega_t^\varepsilon)\, dy\, dt. \tag{3.22}$$

In view of (3.16) and (3.22), the compactness criteria we need for f_ε and $\nabla_\varepsilon \mathbb{D} f_\varepsilon$ can be reduced to some regularity properties of the fundamental solution, p_a. The very same estimates are sufficient to prove the equicontinuity of $\int \mathbb{P}_\varepsilon^t g \, d\mu_{\lambda,\varepsilon}$ as a function of $t>0$. In the next two sections we prove some a priori bounds for Y_z^ε that do not depend on $\varepsilon>0$ and the Wiener processes w_t^ε, as well. Since (3.20) is a parabolic equation, we do not need too much information on its coefficient, measurability of a_t and the obvious bound $|1-a_t(y)| \leqq c < 1$ will do. The methods of [12–14] are simplified by using the backward equation associated with (3.20) and the perturbative treatment of the problem will be combined with interpolation of operators in a sense of M. Riesz and Stein.

4. The Energy Inequality

In this section we are going to develop $\mathbb{L}_r^2$-estimates for a backward equation associated with (3.20). Suppose that we are given two continuous trajectories in Ω_ε, a_s and h_s, $s \geqq 0$, such that $|a_s(x) - 1| \leqq c$ for every x and s with some $c<1$; $z>0$, $\alpha \geqq 0$, $\varepsilon>0$ are arbitrary constants. We are interested in the following equation,

$$-\partial_s u_s + z u_s = \frac{1}{2} a_s \Delta_\varepsilon u_s - \frac{\alpha}{2} \varepsilon^2 \Delta_\varepsilon^2 u_s + h_s, \tag{4.1}$$

the reason is very simple. Our fundamental quantity, Y_z^ε satisfies (4.1) with $a_s(x) = V''(\omega_s^\varepsilon(x))$ and $h_s(x) = \mathbb{D} g(x, \omega_s^\varepsilon)$.

We start with some elementary vector calculus on the lattice. Remember that if u and v are vector fields, then

$$\langle u, v \rangle_r = \int \theta_r(x) \langle u(x), v(x) \rangle \, dx, \tag{4.2}$$

and $|u|_r^2 = \langle u, u \rangle_r$. In the forthcoming calculations the following properties of our weight function θ_r will be exploited. Since $-\theta'(x) \leqq \theta(x)$ for every $x \geqq 0$, we have

$$|\operatorname{grad} \theta_r(x)| \leqq |r| \, \theta_r(x), \tag{4.3}$$

$$\theta_r(y) \leqq \theta_r(x) \exp(|r x - r y|) \tag{4.4}$$

for every $x, y \in \mathbb{R}^d$ and $r \in \mathbb{R}$, whence

$$|\theta_r(x) - \theta_r(y)| \leqq |rx - ry| \exp(|rx - ry|)\, \theta_{r/2}(x)\, \theta_{r/2}(y). \tag{4.5}$$

The energy inequality for (4.1) can be localized by means of the following three lemmas.

Lemma 1. *If $|\varepsilon r| \leqq 1$ and $|u|_r < \infty$, $|v|_r < \infty$ with some $r \in \mathbb{R}$, where $u: \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, while $v \in \mathbb{L}_r^2$ is a scalar, then for every $b > 0$ we have*

$$|\langle u, \nabla_\varepsilon v\rangle_r - \langle \nabla_\varepsilon^* u, v\rangle_r| \leqq \frac{1}{b} |u|_r^2 + 3\, d\, b\, r^2 |v|_r^2. \quad \square$$

Proof. Let $u_1, u_2, \ldots, u_d$ denote the coordinates of u. Since ∇_ε^* is the formal adjoint of ∇_ε with respect to the usual scalar product $\langle \cdot, \cdot \rangle_0$, an easy calculation yields

$$\begin{aligned}\langle u, \nabla_\varepsilon v\rangle_r &= \int \nabla_\varepsilon^* [\theta_r(x)\, u(x)]\, v(x)\, dx \\ &= \langle \nabla_\varepsilon^* u, v\rangle_r + \frac{1}{\varepsilon} \sum_{i=1}^d \int [\theta_r(x - \varepsilon e_i) - \theta_r(x)]\, u_i(x)\, v(x)\, dx,\end{aligned}$$

whence by (4.3), (4.5) and by the Schwarz inequality we obtain

$$|\langle u, \nabla_\varepsilon v\rangle_r - \langle \nabla_\varepsilon^* u, v\rangle_r| \leqq 3r \sum_{i=1}^d |u_i|_r\, |v|_r,$$

which completes the proof as $2uv \leqq u^2/b + b v^2$. $\square$

Lemma 2. *If $|\varepsilon r| \leqq 1$ and $u, v \in \mathbb{L}_r^2(\mathbb{R}^d)$, then for every $b > 0$,*

$$\langle u, \Delta_\varepsilon v\rangle_r \leqq \langle \Delta_\varepsilon u, v\rangle_r + \frac{1}{b} |u|_r^2 + 9\, b (r\, d/\varepsilon)^2 |v|_r^2. \quad \square$$

Proof. Since Δ_ε is a symmetric operator, we have

$$\begin{aligned}\langle u, \Delta_\varepsilon v\rangle_r &= \int \Delta_\varepsilon [\theta_r(x)\, u(x)]\, v(x)\, dx \\ &= \langle \Delta_\varepsilon u, v\rangle_r + \varepsilon^{-2} \sum_{i=1}^d \int [\theta_r(x + \varepsilon e_i) - \theta_r(x)]\, u(x + \varepsilon e_i)\, v(x)\, dx \\ &\quad + \varepsilon^{-2} \sum_{i=1}^d \int [\theta_r(x - \varepsilon e_i) - \theta_r(x)]\, u(x - \varepsilon e_i)\, v(x)\, dx,\end{aligned}$$

whence by (4.3), (4.5) and by the Schwarz inequality we get

$$\langle u, \Delta_\varepsilon v\rangle_r \leqq \langle \Delta_\varepsilon u, v\rangle_r + 6\, d\, |r|\, \varepsilon^{-1} |u|_r\, |v|_r,$$

which implies the statement by a direct calculation. $\square$

Lemma 3. *Let u and v be vector fields such that $|u|_r<\infty$, $|v|_r<\infty$, and $|\varepsilon r|\leqq 1$, then for each $b>0$ we have*

$$\langle u, \nabla_\varepsilon \nabla_\varepsilon^* v\rangle_r \leqq \langle \nabla_\varepsilon \nabla_\varepsilon^* u, v\rangle_r + \frac{1}{b}|u|_r^2 + 36\, b(r\,d/\varepsilon)^2\, |v|_r^2. \quad \square$$

Proof. In a very similar way as above, we obtain

$$\begin{aligned}\langle u, \nabla_\varepsilon \nabla_\varepsilon^* v\rangle_r &= \int \langle \nabla_\varepsilon \nabla_\varepsilon^* [\theta_r(x)\, u(x)], v(x)\rangle\, dx\\
&= \varepsilon^{-2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int \theta_r(x+\varepsilon e_j-\varepsilon e_i)\, u_i(x+\varepsilon e_j-\varepsilon e_i)\, v_j(x)\, dx\\
&\quad - \varepsilon^{-2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int \theta_r(x-\varepsilon e_i)\, u_i(x-\varepsilon e_i)\, v_j(x)\, dx\\
&\quad - \varepsilon^{-2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int \theta_r(x+\varepsilon e_j)\, u_i(x+\varepsilon e_j)\, v_j(x)\, dx\\
&\quad + \varepsilon^{-2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int \theta_r(x)\, u_i(x)\, v_j(x)\, dx\\
&\leqq \langle \nabla_\varepsilon \nabla_\varepsilon^* u, v\rangle + 12\, \frac{r}{\varepsilon} \sum_{i=1}^{d} \sum_{j=1}^{d} |u|_i\, |v|_j\end{aligned}$$

where u_i and v_j are the coordinates of u and v, respectively, which completes the proof. $\square$

Now we define the current of a configuration, that is, an inverse operator of the discrete divergence.

Lemma 4. *To every $\sigma \in \Omega_\varepsilon$ there corresponds a vector field, $v=\mathbb{K}_\varepsilon \sigma$, such that the coordinates of v belong to Ω_ε, $-\nabla_\varepsilon^* \mathbb{K}_\varepsilon \sigma = \sigma$ is an identity, and $|\mathbb{K}_\varepsilon \sigma|_r \leqq K_r |\sigma|_{r/2d}$ if $r>0$, $|r\varepsilon|\leqq 1$, where K_r depends only on r and d.* $\square$

Proof. In view of the additivity of $\mathbb{K}_\varepsilon$, we may, and do assume that σ vanishes outside of an octant, say

$$0_\varepsilon^+ = [x \in \mathbb{R}^d \colon \langle x, e_q\rangle \geqq -\varepsilon/2,\ q=1, 2, \ldots, d].$$

We define v in terms of a directed random walk, X_n, on $\mathbb{Z}^d$ being a Markov chain such that $P[X_{n+1}=j \mid X_n=k]=1/d$ if $j=k+e_q$, $q=1, 2, \ldots, d$, all other jumps are excluded. Let $i, k \in \mathbb{Z}^d$, $j=k+e_q$ and denote $p_i(k, j)$ denote the probability that our walk, started from i, hits both k and j; if (k, j) is a negatively oriented bond, i.e. $j=k-e_q$, then $p_i(k, j) = -p_i(j, k)$ by convention. Observe that

$$\sum_{j: |j-k|=1} p_i(k, j) = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i=k \end{cases},$$

therefore $\mathbb{K}_\varepsilon \sigma = v$ defined by

$$v(x) = \varepsilon \sum_{q=1}^{d} e_q \sum_{i \in \mathbb{Z}^d} \sigma(\varepsilon i)\, p_i(k, k+e_q) \qquad \text{if } x \in C_\varepsilon(\varepsilon k)$$

satisfies $-\nabla_\varepsilon^* v = \sigma$. On the other hand, $p_i(k,j) = 0$ if $k - i \notin 0_\varepsilon^+$, and if $\varphi(k)$ denotes the sum of the coordinate of $k \in \mathbb{Z}^d$, then

$$\sum_{q=1}^{d} \sum_{\varphi(k)=m} p_i(k, k+e_q) = 1 \qquad \text{for } \varphi(i) \leqq m,$$

while

$$\sum_{q=1}^{d} \sum_{i \in 0_\varepsilon^+} p_i(k, k+e_q) \leqq 1 + \varphi(k),$$

thus the Cauchy inequality implies the statement by a direct calculation. □

The main information on the dynamics is contained in

Lemma 5. *Suppose that $a \in \Omega_\varepsilon$, $|a(x) - 1| \leqq c < 1$ for all x, and let*

$$\mathbb{L}_a v = -z v - \frac{1}{2} \nabla_\varepsilon (a \nabla_\varepsilon^* v) + \frac{\alpha}{2} \varepsilon^2 \nabla_\varepsilon \Delta_\varepsilon \nabla_\varepsilon^* v,$$

where $z \geqq 0$, $\alpha \geqq 0$, $\varepsilon > 0$ and $v: \mathbb{R}^d \to \mathbb{R}^d$ is a vector field. There exists a constant, C, depending only on c, d, α such that

$$2\langle v, \mathbb{L}_a v + \nabla_\varepsilon h\rangle_r + 2z\,|v|_r^2 + c(1-c)\,|\nabla_\varepsilon^* v|_r^2 \leqq 2Cr^2\,|v|_r^2 + C\,|h|_r^2$$

whenever $h \in \Omega_\varepsilon$ and $|\varepsilon r| \leqq 1$. □

Proof. Lemma 1 and Lemma 3 imply that

$$\begin{aligned}
&2\langle v, \mathbb{L}_a v + \nabla_\varepsilon h\rangle_r + 2z\,|v|_r^2 \\
&\quad \leqq -(1-c)\,|\nabla_\varepsilon^* v|_r^2 - \alpha\varepsilon^2\,|\nabla_\varepsilon \nabla_\varepsilon^* v|_r^2 + 2\langle \nabla^* \mathrm{v}, h\rangle \\
&\qquad + (1+c)\, b_1^{-1}\,|v|_r^2 + 3r^2(1+c)\, d b_1\,|\nabla_\varepsilon^* v|_r^2 + b_2^{-1}\,|v|_r^2 \\
&\qquad + 3dr^2 b_2\,|h|_r^2 + \alpha\varepsilon^2 b_3^{-1}\,|v|_r^2 + 36\alpha d^2 r^2 b_3\,|\nabla_\varepsilon \nabla_\varepsilon^* v|_r^2 \\
&\quad \leqq [(1+c)\, b_1^{-1} + 2b_2^{-1} + \alpha\varepsilon^2 b_3^{-1}]\,|v|_r^2 + [b_4^{-1} + 6dr^2 b_2]\,|h|_r^2 \\
&\qquad + [3(1+c)\, dr^2 b_1 + b_4 - (1-c)]\,|\nabla_\varepsilon^* v|_r^2 \\
&\qquad + [36\alpha d^2 r^2 b_3 - \alpha\varepsilon^2]\,|\nabla_\varepsilon \nabla_\varepsilon^* v|_r^2,
\end{aligned}$$

where b_1, b_2, b_3, b_4 are arbitrary positive numbers. The statement follows by setting $b_4 = (1-c)^2/2$, $b_3 = (\varepsilon/6rd)^2$, $b_2 = 1/r^2$ and $b_1 = (1-c)^2[6(1+c)\,dr^2]^{-1}$. □

Suppose now that u_s satisfies (4.1) for $s \leqq T \leqq \infty$, and $z \geqq z' \geqq z_r = C r^2$, $|\varepsilon r| \leqq 1$, then Lemma 5 implies immediately that

$$|\nabla_\varepsilon u_0|_r^2 + c(1-c) \int_0^T \exp(2 z_r s - 2 z' s)\, |\Delta_\varepsilon u_s|_r^2\, ds$$
$$\leqq \exp(2 z_r T - 2 z' T)\, |\nabla_\varepsilon u_T|_r^2 + C \int_0^T \exp(2 z_r t - 2 z' t)\, |h_t|_r^2\, dt. \tag{4.6}$$

Multiplying (4.1) by $2 u_s$, from Lemma 2 we obtain

$$-\partial_s |u_s|_r^2 + 2z\, |u_s|_r^2 \leqq (1+c)\, |u_s|_r\, |\Delta_\varepsilon u_s|_r + 2\, |u|_r\, |h_s|_r$$
$$-\alpha \varepsilon^2\, |\Delta_\varepsilon u_s|_r^2 + \frac{\alpha}{b} \varepsilon^2\, |u_s|_r^2 + 9 b \alpha (r d)^2\, |\Delta_\varepsilon u_s|_r^2,$$

whence by (4.6)

$$|u_0|_r^2 \leqq \exp(2 z_r T - 2 z' T) [|u_T|_r^2 + |\nabla_\varepsilon u_T|_r^2]$$
$$+ 2C \int_0^T \exp(2 z_r t - 2 z' t)\, |h_t|_r^2\, dt. \tag{4.7}$$

(4.6) and (4.7) yield a priori bounds for f_ε, $\mathbb{D} f_\varepsilon$, $\nabla_\varepsilon \mathbb{D} f_\varepsilon$ and $\partial_t \int \mathbb{P}_\varepsilon^t g\, d\mu_{\lambda,\varepsilon}$. The weak equicontinuity of f_ε is based on

Lemma 6. *Let B denote a ball of $\mathbb{L}_e^2$. For every B, and for each $\beta > 0$, $r > 0$ and $K < \infty$ there exists a weak neighborhood of 0 in $\mathbb{L}_e^2$, $U_\gamma(\varphi_1, \varphi_2, \ldots, \varphi_n)$, such that $|\varphi(\delta)| < \beta$ whenever $\varphi \in \Omega_\varepsilon$, $|\varphi|_{-r} + |\nabla_\varepsilon \varphi|_{-r} \leqq K$ and $\delta \in B \cap U_\gamma(\varphi_1, \varphi_2, \ldots, \varphi_n)$.* □

Proof. Let $H_r(K)$ denote the set of $\varphi \in \mathbb{L}_{-r}^2$ such that $\varphi \in \Omega_\varepsilon$ for some $\varepsilon > 0$ and $|\nabla_\varepsilon \varphi|_{-r} + |\varphi|_{-r} \leqq K$. Since $|I_\varepsilon \sigma|_r \leqq |\sigma|_r$, this definition makes sense. In view of the F. Riesz criterion of compactness, the set $H_r(K)$ is precompact in the strong topology of $\mathbb{L}^2(\mathbb{R}^d)$, and hence also in that of $\mathbb{L}_{-r/2}^2(\mathbb{R}^d)$. Indeed, estimating $\int \theta_{-r/2}(x)\, \varphi^2(x)\, dx$ separately in the ball $[x \in \mathbb{R}^d \colon |x| \leqq 2n/r]$, and outside of it, since $e^{-u} \leqq \theta(u) \leqq 5 e^{-u}$, we obtain that

$$|\varphi|_{-r/2}^2 \leqq e^n\, |\varphi|_0^2 + 5^{r/2} e^{-n}\, |\varphi|_{-r}^2 \leqq 2K\, 5^{r/4}\, |\varphi|_0$$

if $\varphi \in H_r(K)$. Therefore, for each $\gamma' > 0$ we can select a finite sequence, $\varphi_1, \varphi_2, \ldots, \varphi_n$ from $\mathbb{L}_{-r/2}^2$ in such a way that $|\varphi - \varphi_k|_{-r/2} < \gamma'$ for each $\varphi \in H_r(K)$ with some $k = 1, 2, \ldots, n$. Let $\delta \in B \cap U_{\gamma'}$, then

$$|\varphi(\delta)| \leqq |\varphi(\delta) - \varphi_k(\delta)| + |\varphi_k(\delta)| \leqq \gamma' + \gamma' \sup_{\sigma \in B} |\sigma|_{-r/2},$$

which completes the proof. □

To prove the equicontinuity of $\nabla_\varepsilon \mathbb{D} f_\varepsilon$, we have to compare two solutions, u_s and $\bar{u}_s$ to (4.1) with different a_s, h_s and $\bar{a}_s$, $\bar{h}_s$ corresponding to different initial configurations, σ and $\bar{\sigma}$. Since

$$a_s\, \Delta_\varepsilon\, u_s - \bar{a}_s\, \Delta_\varepsilon\, \bar{u}_s = (a_s - \bar{a}_s)\, \Delta_\varepsilon\, u_s + \bar{a}_s\, \Delta_\varepsilon (u_s - u_s),$$

from (4.6)

$$|\nabla_\varepsilon u_0 - \nabla_\varepsilon u_0|_r^2 \leqq C \int_0^\infty e^{-\tilde{z}t}\, |h_t - \bar{h}_t|_r^2\, ds + C \int_0^\infty e^{-\tilde{z}s}\, |(a_s - \bar{a}_s)\, \Delta_\varepsilon\, u_s|_r^2\, ds \tag{4.8}$$

where $0 \leqq \tilde{z} \leqq 2z - 2z_r$ and $|\varepsilon r| \leqq 1$. For the second term we need an $\mathbb{L}^q$-bound with some $q > 2$.

5. Singular Integrals and Interpolation

In this section we investigate an integral operator, $P_{a,z}$,

$$P_{a,z}\, h(s, x) = \int_0^\infty e^{zs - zt} \int p_a(s, x; t, y)\, h(t, y)\, dy\, dt, \tag{5.1}$$

where $z > 0$ and p_a denotes the fundamental solution of (3.20). Remember that $P_{a,z}$ depends also on $\varepsilon > 0$, we need bounds that do not depend on ε. Let $\mathbb{R}^d_+ = [0, \infty) \times \mathbb{R}^d$, and introduce $\mathbb{L}^q_r(\mathbb{R}^d_+)$ as the space of locally integrable h: $\mathbb{R}^d_+ \to \mathbb{R}$ with norm

$$|h|_{r,q} = \left[\int_0^\infty \int \theta_r(y)\, |h(t, y)|^q\, dy\, dt\right]^{1/q}. \tag{5.2}$$

We want to prove that $\Delta_\varepsilon P_{a,z}$ is a bounded map of $\mathbb{L}^q_r(\mathbb{R}^d_+)$ into itself, at least if $q > 2$ is small enough.

Lemma 7. *For each $q \geqq 2$ we have a constant, C_q such that $C_q \to 2$ as $q \to 2$ and*

$$|\Delta_\varepsilon P_{1,0}\, h|_{0,q} \leqq C_q\, |h|_{0,q} \quad \textit{if } h \in \mathbb{L}^q_0(\mathbb{R}^d_+). \quad \Box$$

Proof. The main problem is that of the boundedness of $\Delta_\varepsilon P_{1,0}$, that is the point where singular integrals enter the stage, see the Appendix of [29]. Following this approach, the statement has been verified if $d = 1$, see Lemma 5 in [14], but p_a factorizes if $a = 1$:

$$p_1(s, x; t, y) = \prod_{i=1}^d J_\varepsilon(s, x_i; t, y_i),$$

where x_i and y_i denote the coordinates of x, $y \in \mathbb{R}^d$, and J_ε is the associated one-dimensional kernel. Since the functions of type

$$h(t, y) = \psi(t) \prod_{i=1}^d \varphi(y_i)$$

are obviously dense in $\mathbb{L}_0^q(\mathbb{R}_+^d)$, the above factorization property reduces the statement to the one-dimensional case by a direct calculation.

In view of the Riesz-Thorin interpolation theorem, see [28], the proof is completed by showing that $C_2=2$ may be assumed. Since $P_{1,0}h$ satisfies (4.1) with $a_s=1$ and $z=0$, we have

$$-\partial_s |\nabla_\varepsilon u_s|_0^2 = -|\Delta_\varepsilon u_s|_0^2 - \alpha\,\varepsilon^2 |\nabla_\varepsilon \Delta_\varepsilon u_s|_0^2 - 2\langle \Delta_\varepsilon u_s, h_s\rangle_0,$$

whence $|\Delta_\varepsilon u|_{0,2} \leqq 2|h|_{0,2}$ by the Schwarz inequality. □

Now we are in a position to prove the main tool of the compactness argument.

Lemma 8. *Suppose that a_s is a continuous trajectory in Ω_ε satisfying $|a_s(x)-1| \leqq c<1$, and let C and C_q, $q \geqq 2$ denote the constants of Lemma 5 and Lemma 7, respectively. If $c\,C_q<2$ then we have a constant, M_q, depending only on q and c such that*

$$|\Delta_\varepsilon P_{a,z} h|_{r,q} \leqq M_q |h|_{r,q} \quad \textit{for } h \in \mathbb{L}_r^q(\mathbb{R}_+^d), \quad z \geqq C r^2, \quad |\varepsilon r| \leqq 1. \quad \square$$

Proof. The bounds (4.6), (4.7) imply that (4.1) is uniquely solved in $\mathbb{L}_r^2$ whenever $h \in \mathbb{L}_r^2(\mathbb{R}_+^d)$, $|r\varepsilon| \leqq 1$, $z \geqq C r^2$, and the solution is given by $u_s(x) = P_{a,z} h(s,x)$. Since $\Omega_\varepsilon \cap \mathbb{L}_r^q(\mathbb{R}^d)$ consists of bounded functions if $r \leqq 0$, this uniqueness result extends to all $q \geqq 2$ and $r \leqq 0$. In particular, as $a_s = 1 + (a_s - 1)$, we have

$$P_{a,z} h = P_{1,z} h + \tfrac{1}{2} P_{1,z}(a-1)\,\Delta_\varepsilon P_{a,z} h, \tag{5.3}$$

where $a-1 = a_s(x)-1$ is acting as a multiplication operator, whence

$$|P_{a,z} h|_{r,q} \leqq C_q |h|_{r,q} + \tfrac{1}{2} c\, C_q |P_{a,z} h|_{r,q}, \tag{5.4}$$

at least if $r=z=0$, see Lemma 7. Letting q go to two, we obtain the statement in this particular case, the general case follows by interpolation. Indeed, replacing $dt\,dx$ by $e^{-zt}\,dt\,\theta_r(x)\,dx$, and using the Stein interpolation theorem, see Theorem 4.1 in [27] or Theorem 2.11 in [26], we can extend Lemma 7 to all $z \geqq 0$ and $r \in \mathbb{R}$ with the same C_q, which completes the proof. □

Now we are in a position to start the proof of Theorem 2.

6. The Compactness Criteria

This section summarizes the final information on the microscopic dynamics. Just as before, ω_t^ε and $\bar{\omega}_t^\varepsilon$ are different strong solutions to (3.19) with identical Wiener trajectories, $a_t = a_t(y) = V''(\omega_t^\varepsilon(y))$, $|a_t(y)-1| \leqq c < 1$, p_a denotes the fundamental solution to (4.1), and K_r and C are some universal constants, see Lemma 4 and Lemma 5.

We shall prove Theorem 2 first for some functions of class $\mathbb{D}_L$. Let $\Sigma \subset \mathbb{L}_e^2$ be convex, and denote $\mathbb{D}_L(\Sigma)$ the space of $g \in \mathbb{D}_s(\Sigma)$ for which we have some $r>0$ and $q>2$ such that

$$\|g\| = \sup_{\sigma\in\Sigma} |g(\sigma)| < \infty, \qquad \|\mathbb{D} g\|_{-r} = \sup_{\sigma\in\Sigma} |\mathbb{D} g(\cdot, \sigma)|_{-r} < \infty, \tag{6.1}$$

$$L_r(\mathbb{D} g) = \sup_{\sigma,\bar\sigma\in\Sigma} |\mathbb{D} g(\cdot,\sigma) - \mathbb{D} g(\cdot,\bar\sigma)|_{-r} (|\sigma-\bar\sigma|_r)^{-1} < \infty, \tag{6.2}$$

$$\|\mathbb{D} g\|_{-r,q} = \sup_{\sigma\in\Sigma} [\int \theta_{-r}(x)\, |\mathbb{D} g(x,\sigma)|^q\, dx]^{1/q} < \infty. \tag{6.3}$$

Theorem 1 can be extended to functions of class $\mathbb{D}'_w(\Sigma)$ defined as the set of weakly continuous $g \in \mathbb{D}_s(\Sigma)$ such that besides (6.1), with the same $r>0$, we have

$$\|\nabla_\varepsilon \mathbb{D} g\|_{-r} = \sup_{\sigma\in\Sigma} |\nabla_\varepsilon \mathbb{D} g(\cdot,\sigma)|_{-r} < \infty. \tag{6.4}$$

Finally, let $g_t^\varepsilon(\sigma) = g(\omega_t^\varepsilon)$ if $\omega_0^\varepsilon = \sigma$, and

$$X_z^\varepsilon(\sigma, g) = \int_0^\infty e^{-zt}\, g(\omega_t^\varepsilon)\, dt, \tag{6.5}$$

then $\mathbb{D} g_t^\varepsilon(x,\sigma) = \int p_a(0, x; t, y)\, \mathbb{D} g(y, \omega_t^\varepsilon)\, dy$, and $Y_z^\varepsilon(0, x, \sigma, g) = \mathbb{D} X_z^\varepsilon(x, \sigma, g)$, see (3.22).

Lemma 9. *Let $r<0$, $|r\varepsilon| \leqq 1$, $z > Cr^2$ and $g \in \mathbb{D}_s(\Omega_\varepsilon)$, then*
(i) $\|\mathbb{D} g_t^\varepsilon\|_r^2 + \|\nabla_\varepsilon \mathbb{D} g_t^\varepsilon\|_r^2 \leqq \exp(Cr^2 t)[\|\mathbb{D} g\|_r^2 + 2\|\nabla_\varepsilon \mathbb{D} g\|_r^2]$,
(ii) $\|\mathbb{D} X_z^\varepsilon(\cdot,\cdot,g)\|_r^2 + \|\nabla_\varepsilon \mathbb{D} X_z^\varepsilon(\cdot,\cdot,g)\|_r^2 \leqq C(z - Cr^2)^{-1} \|\mathbb{D} g\|_r^2$. □

Proof. Both statements are direct consequences of (4.6) and (4.7). In the case of (i) we choose $z=0$, $T=t$ and $h=0$, while $T=\infty$ and $h_t(y) = \mathbb{D} g(y, \omega_t^\varepsilon)$ for (ii). □

The following lemma is based on Lemma 4.

Lemma 10. *Let $r>0$, $r\varepsilon \leqq 1$, $z \geqq Cr^2$ and $g \in \mathbb{D}_s(\Omega_\varepsilon)$, then*

(i) $c(1-c) \int_0^\infty e^{-\tilde z t}\, \mathbb{E}[|\omega_t^\varepsilon|_r^2 \,|\, \omega_0^\varepsilon = \sigma]\, dt \leqq K_r^2 |\sigma|_{r/2d}^2 + \dfrac{5}{\tilde z}(2d/r)^d$,

(ii) $c(1-c) \int_0^\infty e^{-\tilde z t}\, |\omega_t^\varepsilon - \bar\omega_t^\varepsilon|_r^2\, dt \leqq K_r^2 |\omega^\varepsilon - \bar\omega^\varepsilon|_{r/2d}^2$,

(iii) $c(1-c) \int_0^\infty e^{-\tilde z t}\, |\int \mathbb{D} g_t^\varepsilon(x,\sigma)\, \Delta_\varepsilon \lambda(x)\, dx|_r^2\, dt \leqq |\nabla_\varepsilon \lambda|_r^2\, \|\mathbb{D} g\|_{-r}^2$. □

Proof. Introduce the current, v_t^ε of ω_t^ε by $v_0^\varepsilon = \mathbb{K}_\varepsilon \sigma$ and

$$d v_t^\varepsilon(y) = \frac{1}{2} \nabla_\varepsilon [\hat a_t(y)\, \omega_t^\varepsilon(y)]\, dt - \frac{\alpha}{2} \varepsilon^2 \nabla_\varepsilon \Delta_\varepsilon \omega_t^\varepsilon(y)\, dt - d w_t^\varepsilon(y),$$

where $\hat{a}_t(y)=[V'(\omega_t^\varepsilon(y))-V'(0)]/\omega_t^\varepsilon(y)$, thus $\omega_t^\varepsilon=-\nabla_\varepsilon^* v_t$, and the drift of $v_t^\varepsilon \exp(-zt)$ turns out to be $\mathbb{L}_{\hat{a}} v_t^\varepsilon$, thus the Ito lemma, Lemma 5 and Lemma 4 imply (i) by a direct calculation. The proof of (ii) is quite similar, but even simpler; the current of $(\omega_t^\varepsilon-\bar{\omega}_t^\varepsilon)\, e^{-zt}$ satisfies $\partial_t v=\mathbb{L}_{\tilde{a}} v$ with

$$\tilde{a}_t(y)=[V'(\omega_t^\varepsilon(y))-V'(\bar{\omega}_t^\varepsilon(y))]\,[\omega_t^\varepsilon(y)-\bar{\omega}_t^\varepsilon(y)]^{-1}.$$

To prove (iii), let

$$u_t(y)=\int p_a(0,x;t,y)\,\Delta_\varepsilon\lambda(x)\,dx,$$

and let v_t denote the solution to $\partial_t v_t=\mathbb{L}_a v_t$ with initial condition $v_0=\nabla_\varepsilon\lambda$, then $-\nabla_\varepsilon^* v_t=u_t\, e^{-zt}$, whence by Lemma 5

$$c(1-c)\int_0^\infty e^{-\bar{z}t}\,|u_t|_r^2\,dt\leqq|\nabla_\varepsilon\lambda|_r^2,$$

which completes the proof of (iii) by the Schwarz inequality. □

The problems of weak equicontinuity reduce to Lemma 6; remember that C_q denotes the constant of Lemma 7.

Lemma 11. *For each $\beta>0$, $r<0$, $K<\infty$, $q>2$ such that $cC_q<2$, and for every ball $B\subset\mathbb{L}_e^2$ there exists a weak neighborhood $U=U_\gamma(\varphi_1,\varphi_2,\ldots,\varphi_n)$ of $0\in\mathbb{L}_e^2$ such that if $|r\varepsilon q|\leqq q-2$, $z\geqq 2Cr^2$, $g\in\mathbb{D}_s(\Omega_\varepsilon)$, $\sigma,\bar{\sigma}\in\Omega_\varepsilon$, $\delta=\sigma-\bar{\sigma}$, then $\delta\in B\cap U$ implies*

(i) $|X_z^\varepsilon(\sigma,g)-X_z^\varepsilon(\bar{\sigma},g)|<\beta$ *whenever* $\|\mathbb{D}g\|_r\leqq K$,

(ii) $|g_t^\varepsilon(\sigma)-g_t^\varepsilon(\bar{\sigma})|<\beta$ *if* $t+\|\mathbb{D}g\|_r+\|\nabla_\varepsilon\mathbb{D}g\|_r\leqq K$,

(iii) $|\mathbb{D}X_z^\varepsilon(\cdot,\sigma,g)-\mathbb{D}X_z^\varepsilon(\cdot,\bar{\sigma},g)|_r^2+|\nabla_\varepsilon\mathbb{D}X_z^\varepsilon(\cdot,\sigma,g)-\nabla_\varepsilon\mathbb{D}X_z^\varepsilon(\cdot,\bar{\sigma},g)|_r^2<\beta$ *whenever* $\|\mathbb{D}g\|_r+L_r(g)+\|\mathbb{D}g\|_{qr,q}\leqq K$. □

Proof. In view of the definition of functional derivatives, we have

$$X_z^\varepsilon(\sigma,g)-X_z^\varepsilon(\bar{\sigma},g)=\int\delta(x)\int_0^1\mathbb{D}X_z^\varepsilon(x,\bar{\sigma}+u\delta,g)\,du\,dx,$$

thus (ii) of Lemma 9 implies the conditions of Lemma 6, which proves (i). Similarily,

$$g_t^\varepsilon(\sigma)-g_t^\varepsilon(\bar{\sigma})=\int\delta(x)\int_0^1\mathbb{D}g_t^\varepsilon(x,\bar{\sigma}+u\delta)\,du\,dx,$$

whence (ii) follows in the same way. The proof of (iii) is a little bit more involved. Using a decomposition $a\Delta_\varepsilon u-\bar{a}\Delta_\varepsilon\bar{u}=(a-\bar{a})\,\Delta_\varepsilon u+\bar{a}\,\Delta_\varepsilon(u-\bar{u})$, from (4.6) and (4.7) we obtain that the left hand side of (iii) is bounded by $3CJ_1+(3C/4)J_2$, where

$$J_1=\int_0^\infty e^{-\bar{z}t}\,|\mathbb{D}g(\cdot,\omega_t^\varepsilon)-\mathbb{D}g(\cdot,\bar{\omega}_t^\varepsilon)|_r^2\,dt,$$

$$J_2=\int_0^\infty e^{-\bar{z}s}\int\theta_r(x)\,|a_s(x)-\bar{a}_s(x)|^2\,[\Delta_\varepsilon P_{a,z}\,h(s,x)]^2\,dx\,ds,$$

$\tilde{z}=2z-2Cr^2$, $a_s(x)=V''(\omega_s(x))$, $\bar{a}_s(x)=V''(\bar{\omega}_s(x))$, $h(t,y)=\mathbb{D}g(y,\omega_t^\varepsilon)$. In view of (6.2), the estimation of J_1 reduces to that of J_3,

$$J_3=\int_0^\infty e^{-zt}\,|\omega_t^\varepsilon-\bar{\omega}_t^\varepsilon|_{-r}^2\,dt$$

$$=\int\delta(x)\int_0^\infty e^{-zt}\int p_{\tilde{a}}(0,x;t,y)\,k(t,y)\,dy\,dt\,dx,$$

where $k(t,y)=\theta_{-r}(x)[\omega_t^\varepsilon(y)-\bar{\omega}_t^\varepsilon(y)]$, $r<0$, and $\tilde{a}$ is the same as in the proof of Lemma 10. From (ii) of Lemma 10 we obtain that k is uniformly bounded in $\mathbb{L}^2_{-r}(\mathbb{R}^d_+)$, (4.6) and (4.7) imply the conditions of Lemma 6, that is, J_3 is small if $\delta\in B\cap U$.

The second term, J_2 can be estimated by means of Lemma 7 and Lemma 8. Using $\theta_r(x)=\theta_{2r}(x)\,\theta_{-r}(x)$ and the Hölder inequality with powers $q/(q-2)$ and $q/2$, we can bound J_2 by a product of two factors such that the second factor is controlled by Lemma 8, while the first one turns out to be a power, $1-2/q$, of J_4,

$$J_4=\int_0^\infty e^{-\tilde{z}t}\int\theta_{r'}(y)\,|a_t(y)-\bar{a}_t(y)|^{2q/q-2}\,dy\,dt$$

$$=\int\delta(x)\int_0^\infty e^{-\tilde{z}t}\int p_{\tilde{a}}(0,x;t,y)\,h(t,y)\,dy\,dt\,dx,$$

where $\tilde{a}$ and $\tilde{z}$ are the same as above, $r'=-r\,q/(q-2)>0$, and

$$h(t,y)=\theta_{r'}(y)\,|V''(\omega_t^\varepsilon(y))-V''(\bar{\omega}_t^\varepsilon(y))|^{2q/q-2}\,[\omega_t^\varepsilon(y)-\bar{\omega}_t^\varepsilon(y)]^{-1}.$$

Since V''' is bounded by assumption, this h is uniformly bounded in $\mathbb{L}^q_{r'}(\mathbb{R}^d_+)$, thus the proof of (iii) can be completed by repeating the argument above. □

The results of this section are sufficient to select a uniformly convergent subsequence from f_ε, $\mathbb{D}f_\varepsilon$ and from $\nabla_\varepsilon\mathbb{D}f_\varepsilon$, too.

7. The Initial Distribution and the Limiting Equation

The a priori bounds summarized in the previous section allow us to select convergent subsequences from f_ε and $\nabla_\varepsilon\mathbb{D}f_\varepsilon$ as $\varepsilon\to 0$, thus (3.2), the law of large numbers for $\mu_{\lambda,\varepsilon}$, is sufficient to pass to the limiting resolvent Eq. (3.7). The final identification of limit points of the resolvent, see (3.8) and (3.9), reduces then to the existence of the semigroup $\tilde{\mathbb{P}}^t$ defined by the limiting hydrodynamic Eq. (1.6).

The study of the inhomogeneous Gibbs states μ_λ will be based on an auxiliary dynamics having μ_λ as its only stationary state. For each $\lambda\in\Omega$ and $\sigma\in\Omega$ we define a Markov process $R(t)$ in Ω as the strong solution to

$$dR_k=\tfrac{1}{2}(\lambda_k-\partial_k H(R))\,dt+dw_k,\qquad R_k(0)=\sigma_k,\quad k\in\mathbb{Z}^d, \tag{7.1}$$

where w_k, $k \in \mathbb{Z}^d$ is a family of independent standard Wiener processes. Solutions to (7.1) can be constructed in the same, well known way, as to (1.4), see [5]. It is more remarkable that (7.1) is just the stochastic gradient dynamics associated with the energy function $H_\lambda = H(\sigma) - \sum \lambda_k \sigma_k$, thus μ_λ is a stationary measure of (7.1), provided that μ_λ does exist as a Gibbs state on Ω.

Since V is convex, it is quite easy to derive a priori bounds for (7.1). Using $2z(x-2z+y) \leqq x^2 - 2z^2 + y^2$ we obtain

$$\begin{aligned} dR_k^2 &\leqq \lambda_k R_k\, dt - (1-c) R_k^2\, dt + \frac{\alpha}{2} \sum_{|j-k|=1} (R_j^2 - R_k^2)\, dt + dt + 2R_k\, dw_k \\ &\leqq (2-2c)^{-2} \lambda_k^2\, dt - [c(1-c) + \alpha d]\, R_k^2\, dt \\ &\quad + \frac{\alpha}{2} \sum_{|j-k|=1} R_j^2\, dt + dt + 2R_k\, dw_k. \end{aligned} \tag{7.2}$$

We also have a fairly effective coupling for (7.1). Let $\bar{R}$ denote another solution with initial configuration $\bar{\sigma} \in \Omega$ and profile $\bar{\lambda} \in \Omega$, while the Wiener trajectories are the very same for both realizations. Like above, we obtain that $\delta_k = (R_k - \bar{R}_k)^2$ satisfies

$$d\delta_k/dt \leqq (2-2c)^{-2} (\lambda_k - \bar{\lambda}_k)^2 - [c(1-c) + \alpha d]\, \delta_k + \frac{\alpha}{2} \sum_{|j-k|=1} \delta_j. \tag{7.3}$$

In a stationary regime the left hand sides of (7.2) and (7.3) vanish in the mean, thus the following lemma can be used.

Lemma 12. *Suppose that* $0 \leqq b < 1/2d$, *and* $u, v \in \Omega$ *satisfy*

$$\text{(i)} \qquad u_k \leqq v_k + b \sum_{|j-k|=1} u_j \quad \textit{for } k \in \mathbb{Z}^d,$$

then $u_k \leqq \sum_{j \in \mathbb{Z}^d} J_{k-j} v_j$ *for all* k, *where* $J_k \geqq 0$ *vanishes exponentially as* $|k| \to \infty$; J *is given by* (7.4). □

Proof. Since both u_k and v_k increase slower than any exponential rate, iterating (i) infinitely many times we obtain (ii) with

$$J_k = (2\pi)^{-d} \int_{C_{2\pi}(0)} \left[1 - 2b \sum_{i=1}^{d} \cos\langle \omega, e_i \rangle\right]^{-1} \cos\langle \omega, k \rangle\, d\omega. \tag{7.4}$$

The positivity of J is a direct consequence of its iterative construction, an exponential bound follows by integrating by parts. □

All the information we need about μ_λ is summarized in

Proposition 1. *For every $\lambda \in \Omega$ there exists a unique Gibbs state, μ_λ, specified by* (2.15) *as a Borel probability on Ω such that*

(i) $$\int \sigma_k^2 \,\mu_\lambda(d\sigma) \leqq K \sum_{|j-k|=1} J_{k-j}(1+\lambda_j^2),$$

where K depends only on c, α, d, and $b=\alpha[2c(1-c)+2\alpha d]^{-1}$. For λ, $\bar{\lambda} \in \Omega$ we have a joint distribution, $\mu_{\lambda,\bar{\lambda}}$ on $\Omega \times \Omega$ such that

(ii) $$\iint (\sigma_k - \bar{\sigma}_k)^2 \,\mu_{\lambda,\bar{\lambda}}(d\sigma, d\bar{\sigma}) \leqq K \sum_{|j-k|=1} J_{k-j}(\lambda_j - \bar{\lambda}_j)^2.$$

Consequently, if $\rho_k^\lambda = \int \sigma_k \,\mu_\lambda(d\sigma)$, then

(iii) $$|\rho_k^\lambda - F'(\lambda_k)|^2 \leqq K \sum_{|j-k|=1} J_{k-j}(\lambda_k - \lambda_j)^2,$$

(iv) $$\int \Big[\sum_{k\in\mathbb{Z}^d} \varphi_k(\sigma_k - \rho_k^\lambda)\Big]^2 \mu_\lambda(d\sigma) \leqq K \sum_{k\in\mathbb{Z}^d} \varphi_k^2.$$

Finally, F' admits a Lipschitz continuous derivative, F'' is bounded, and it is bounded away from zero. $\square$

Proof. The basic idea is easy: since V is convex, each μ_λ, $\lambda \in \Omega$ belongs to the domain of Dobrushin uniqueness, cf. [7, 18]. In fact, we can follow a completely elementary argument. Suppose first that μ_λ has been defined already as a Gibbs state on Ω, and let $\mathbb{G}^\lambda$ denote the generator of our auxiliary process, (7.1). The associated Dirichlet form can be written as

$$\int f_1 \,\mathbb{G}^\lambda f_2 \,d\mu_\lambda = -\tfrac{1}{2} \sum_{k\in\mathbb{Z}^d} \int (\partial_k f_1)\, \partial_k f_2 \,d\mu_\lambda. \tag{7.5}$$

Therefore, μ_λ is a stationary state of (7.1), thus (i) follows from (7.2) by Lemma 12. To prove (ii) we construct $\mu_{\lambda,\bar{\lambda}}$ first as a stationary state of the coupled evolution, then (ii) follows from (7.3) via Lemma 12. Since μ_λ is certainly well defined if λ equals a constant outside of a finite volume, (i) and (ii) extend by continuity to all $\lambda \in \Omega$ involving also the existence and uniqueness of μ_λ.

Suppose now that $\varphi\colon \Omega \to \mathbb{R}$ vanishes at an exponential rate, and let $\varphi(\sigma) = \sum \varphi_k \sigma_k$, $F_\varphi(v) = \log \int \exp(v\,\varphi(\sigma))\,\mu_\lambda(d\sigma)$, then

$$F'_\varphi(v) = \int \varphi(\sigma) \exp(v\,\varphi(\sigma) - F_\varphi(v))\,d\mu_\lambda = \int \varphi(\sigma)\,d\mu_{\lambda+v\varphi}, \tag{7.6}$$

$$F_\varphi(0) = \int \Big[\sum \varphi_k(\sigma_k - \rho_k^\lambda)\Big]^2 \mu_\lambda(d\sigma), \tag{7.7}$$

thus (ii) implies (iii), (iv), the uniform Lipschitz continuity of F' and the Lipschitz continuity of F'' by a direct calculation. To obtain a lower bound for F'', let

$\bar{R}_n = \sum (R_k - F'(v))$, $\partial \bar{H}_n = \sum (\partial_k H - v)$, where both sums are over $C_{2n}(0) \cap \mathbb{Z}^d$, and observe that

$$F''(v) = \lim_{n \to \infty} (2n+1)^{-d} \int [\sum_{k \in C_{2n}(0) \mathbb{Z}^d} (\sigma_k - F'(v))]^2 \mu_v^0(d\sigma), \tag{7.8}$$

$$d\bar{R}_n^2 = -\bar{R}_n \partial \bar{H}_n \, dt + \sum_{k \in C_{2n}(0) \mathbb{Z}^d} 2 R_n \, dw_k + (2n+1)^d \, dt. \tag{7.9}$$

Starting (7.1) with initial distribution μ_v^0, we obtain that $\int \bar{R}_n \partial \bar{H}_n \, d\mu_v^0 = (1+2n)^d$. Integrating by parts, we obtain an identity

$$\int (\partial_k H - v)(\partial_j H - v) \, d\mu_v^0 = \begin{cases} 0 & \text{if } |j-k| > 1 \\ 2\alpha d + \int V''(\sigma_k) \, d\mu_v^0 & \text{if } j = k \\ -\alpha & \text{if } |j-k| = 1, \end{cases} \tag{7.10}$$

consequently

$$\begin{aligned} \int \bar{R}_n^2 \, d\mu_v^0 &\geqq 2 \int \bar{R}_n \partial \bar{H}_n \, d\mu_v^0 - \int \bar{H}_n^2 \, d\mu_v^0 \\ &\geqq 2(2n+1)^d - (1+c+2\alpha d)(2n+1)^d + 2\alpha d (2n-1)^d, \end{aligned}$$

which proves $F''(v) \geqq 1 - c$. □

Now we turn to the existence problem of the limiting semigroup. In fact, we have two equations,

$$\partial_t \rho_t = \operatorname{div} [D(\rho_t) \operatorname{grad} \rho_t], \tag{7.11}$$

$$2F''(\lambda_t) \, \partial_t \lambda_t = \Delta \lambda_t, \tag{7.12}$$

they are related to each other by $\rho_t(x) = F'(\lambda_t(x))$ and $D(u) = 1/2F''(v)$ if $u = F'(v)$. Suppose that ρ_t and λ_t are classical solutions, and $0 < c_1 \leqq D \leqq c_2$, then

$$\begin{aligned} \partial_t |\rho_t|_r^2 &\leqq -2c_1 |\operatorname{grad} \rho_t|_r^2 + 2rc_2 |\rho_t|_r |\operatorname{grad} \rho_t|_r \\ &\leqq -c_1 |\operatorname{grad} \rho_t|^2 + (rc_2/c_1)^2 |\rho_t|_r^2, \end{aligned}$$

whence for all $r \geqq 0$ we obtain an energy inequality,

$$|\rho_t|_r^2 + c_1 \int_0^t \exp(c_3 t - c_3 s) |\operatorname{grad} \rho_s|_r^2 \, ds \leqq \exp(c_3 t) |\rho_0|_r^2, \tag{7.13}$$

where $c_3 = (rc_2/c_1)^2$. The equation for $\operatorname{grad} \lambda_t$ is also self-adjoint, thus we have a second energy inequality, namely

$$|\operatorname{grad} \lambda_t|_r^2 + c_1 \int_0^t \exp(c_3 t - c_3 s) |\Delta \lambda_s|_r^2 \, ds \leqq \exp(c_3 t) |\operatorname{grad} \lambda_0|_r^2. \tag{7.14}$$

Proposition 2. *For each $\lambda \in \mathbb{H}_e^1$ there exists a continuous trajectory, $\rho_t \in \mathbb{L}_e^2$, $t \geqq 0$ satisfying* (2.20) *with initial condition $\rho_0 = \sigma \in \mathbb{H}_e^1$ given by $\sigma(x) = F'(\lambda(x))$. Moreover, if $\lambda_t = \lambda_t(x) = E(\rho_t(x))$, where E is the inverse function of F', then $\lambda_t \in \mathbb{H}_e^1$, $|\operatorname{grad} \lambda_t|_r^2 \leqq |\operatorname{grad} \lambda_0|_r^2 \exp(c_3 t)$ for all $t \geqq 0$, $r \geqq 0$, and there is no other weak solution having such properties.* □

Proof. For smooth initial data one can construct classical solutions by means of a Galerkin approximation, see [19]. If λ_0 varies in a bounded set of $\mathbb{H}_e^1$ then (7.13), (7.14) and the Riesz criterion of compactness show that ρ_t remains in a strongly compact set of $\mathbb{L}_e^2$ for finite time. From (7.14) we see that ρ_t is an equicontinuous function of time, thus the Arzela-Ascoli theorem implies the existence of weak solutions satisfying the desired $\mathbb{H}_e^1$-bound, see (7.13) and (7.14).

Suppose now that ρ_t and $\bar{\rho}_t$ are weak solutions in the above sense, and $\rho_0=\bar{\rho}_0$. Let $\delta_t=\rho_t-\bar{\rho}_t$, and introduce ω_t by

$$\omega_t=\tfrac{1}{2}\int_0^t \operatorname{grad}[E(\rho_s)-E(\bar{\rho}_s)]\,ds. \tag{7.15}$$

Observe that $\operatorname{div}\omega_t=\delta_t$ in the weak sense. Indeed, for smooth φ

$$\int\langle\operatorname{grad}\varphi,\omega_t\rangle\,dx=\tfrac{1}{2}\int_0^t\int\langle\operatorname{grad}\varphi,\operatorname{grad}[E(\rho_s)-E(\bar{\rho}_s)]\rangle\,dx\,ds$$

$$=-\tfrac{1}{2}\int_0^t\int[\Delta\varphi][E(\rho_s)-E(\bar{\rho}_s)]\,dx\,ds=-\int\varphi(x)\,\delta_t(x)\,dx,$$

see (2.20), consequently

$$\begin{aligned}\partial_t\,|\omega_t|_r^2&=\int\theta_r(x)\langle\omega_t,\operatorname{grad}[E(\rho_t)-E(\bar{\rho}_t)]\rangle\,dx\\&\leqq-2c_1\,|\delta_t|_r^2+2c_2 r\,|\omega_t|_r\,|\delta_t|_r\leqq(c_3/2)\,|\omega_t|_r^2,\end{aligned}$$

whence $\omega_t=\delta_t=0$ by the Gronwall lemma. □

Now we are in a position to complete the proof of the main result. Consider first the scaled distributions $\mu_{\lambda,\varepsilon}$ corresponding to μ_λ with $\lambda_k=I_\varepsilon\lambda(\varepsilon k)$. (iii) of Proposition 1 implies (2.22), while (iv) and the Markov inequality show that $\varphi(\sigma)$ converges also in probability to $\int\varphi(x)\,\rho_0(x)\,dx$. In view of (i), $\mu_{\lambda,\varepsilon}$ is tight in the weak topology of $\mathbb{L}_e^2$, thus (2.23) extends from $\mathbb{C}_0(\mathbb{L}_e^2)$ even to $\mathbb{C}_w(\Sigma)$ by the Stone-Weierstrass approximation theorem, where Σ is a union of the increasing balls $B_n\subset\mathbb{L}_e^2$ such that $\mu_{\lambda,\varepsilon}(B_n)\to 1$ uniformly in $\varepsilon>0$ as $n\to\infty$; thus we have (3.2), (3.3), and we can also pass to (3.7). The uniqueness of the limiting resolvent Eq. (3.7) follows from the existence of $\mathbb{P}^t$ as a strongly continuous contraction semigroup in $\mathbb{C}_w(\mathbb{H}_e^1)$, see Proposition 2. Some more technical details of the proof are given in the next section.

8. Proof of Theorems 1 and 2

First we prove a variant of Theorem 2.

Theorem 3. *Suppose that* $\lambda\in\mathbb{H}_e^1$ *and* $g^\varepsilon\in\mathbb{D}_L(\Omega_\varepsilon)$ *for each* $\varepsilon>0$ *such that the norms* (6.1)–(6.3) *remain bounded as* $\varepsilon\to 0$. *If we have some* $g\in\mathbb{D}_s(\mathbb{L}_e^2)$ *such that*

(i) $$\lim_{\varepsilon\to 0}\int g^\varepsilon(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=\tilde{g}(\lambda)\quad \textit{for } \lambda\in\mathbb{H}_e^1,$$

and $\rho_t \in \mathbb{H}_e^1$, $t \geqq 0$ *solves* (2.20) *with* $\rho_0(x) = F'(\lambda(x))$, *then*

(ii) $$\lim_{\varepsilon \to 0} \int \mathbb{P}_\varepsilon^t \, g^\varepsilon(\sigma) \, \mu_{\lambda,\varepsilon}(d\sigma) = \tilde{g}(\lambda_t),$$

where $\lambda_t \in \mathbb{L}_e^2$ *is defined by* $\rho_t(x) = F'(\lambda_t(x))$. □

Proof. In view of (i) of Proposition 1, for every increasing sequence of balls B_n of $\mathbb{L}_e^2$ we have another sequence of balls, $\bar{B}_n \subset \mathbb{L}_e^2$ such that if $\Sigma = \cup B_n$, $\bar{\Sigma} = \cup \bar{B}_n$, then $\lambda \in \Sigma$ implies $\mu_{\lambda,\varepsilon}(\bar{\Sigma}) = 1$ for all $\varepsilon > 0$. The convention $g^\varepsilon(\sigma) = g^\varepsilon(I_\varepsilon \sigma)$ extends g^ε to $\mathbb{L}_e^2$ in a trivial way. Consider the functions $f_\varepsilon \colon \mathbb{L}_e^2 \times (0, \infty) \to \mathbb{R}$,

$$f_\varepsilon(\sigma, z) = \int_0^\infty e^{-zt} \, \mathbb{P}_\varepsilon^t \, g^\varepsilon(\sigma) \, dt,$$

$\mathbb{D} f_\varepsilon \colon \mathbb{L}_e^2 \times (0, \infty) \to \mathbb{L}_{-e}^2$ and $\nabla_\varepsilon \mathbb{D} f_\varepsilon \colon \mathbb{L}_e^2 \times (0, \infty) \to \mathbb{L}_{-e}^2$. Lemma 9 and Lemma 11 imply that each of them is a bounded and equicontinuous function of $(\sigma, z) \in B \times [z_1, z_2]$ if B is a ball of $\mathbb{L}_e^2$, and $0 < z_1 < z_2 < \infty$; therefore the Arzela-Ascoli theorem applies. We can select a subsequence, $\varepsilon_n \to 0$ such that $f_\varepsilon(\sigma, z) \to f(\sigma, z)$, $\mathbb{D} f_\varepsilon(\cdot, \sigma, z) \to h_1(\cdot, \sigma, z)$, $\nabla_\varepsilon \mathbb{D} f_\varepsilon(\cdot, \sigma, z) \to h_2(\cdot, \sigma, z)$ along ε_n for each $\sigma \in \bar{\Sigma}$ and $z > 0$, and the convergence is uniform on compacts of type $\bar{B}_m \times [z_1, z_2]$; $\mathbb{D} f_\varepsilon$ and $\nabla_\varepsilon \mathbb{D} f_\varepsilon$ converge in the topology of $\mathbb{L}_{-e}^2$. Moreover, f, h_1 and h_2 are continuous on each compact $\bar{B}_m \times [z_1, z_2]$, consequently the definition of functional derivatives implies that $f \in \mathbb{D}_s(\bar{\Sigma})$ and $h_1 = \mathbb{D} f$. Similarily, as $\mathbb{D} f_\varepsilon$ and $\nabla_\varepsilon \mathbb{D} f_\varepsilon$ converge simultaneously, we see that h_1 is weakly differentiable, and $h_2 = \operatorname{grad} h_1$. Since $\nabla_\varepsilon \lambda$ converges strongly to $\operatorname{grad} \lambda$, Proposition 1 yields

$$\lim_{n \to \infty} \int f_{\varepsilon_n}(\sigma, z) \, \mu_{\lambda,\varepsilon_n}(d\sigma) = f(\rho_0, z) \qquad \text{if } z > 0, \quad \lambda \in \Sigma, \tag{8.1}$$

where $\rho_0(x) = F'(\lambda(x))$, while

$$\lim_{n \to \infty} \iint \langle \nabla_{\varepsilon_n} \lambda(x), \nabla_{\varepsilon_n} \mathbb{D} f_{\varepsilon_n}(x, \sigma, z) \rangle \, dx \, \mu_{\lambda,\varepsilon_n}(d\sigma)$$
$$= \int \langle \operatorname{grad} \lambda(x), \operatorname{grad} \mathbb{D} f(x, \rho_0, z) \rangle \, dx \tag{8.2}$$

for all $z > 0$ and $\lambda \in \Sigma \cap \mathbb{H}_e^1$. This means that we can pass to the limiting resolvent Eq. (3.7), thus the uniqueness of solutions to (7.3)–(7.4) implies that

$$f(\rho_0, z) = \int_0^\infty e^{-zt} \, \tilde{g}(\lambda_t) \, dt;$$

thus (iii) of Lemma 10 implies the equicontinuity of $\int \mathbb{P}_\varepsilon^t g^\varepsilon \, d\mu_{\lambda,\varepsilon}$ as a function of time, which completes the proof. □

To conclude Theorem 2 from Theorem 3, we have to show that the family of time evolved measures, $\mu_{\lambda,\varepsilon} \mathbb{P}_\varepsilon^t$, $\varepsilon > 0$, is a tight one.

Proof of Theorem 2. Let $h(t)=\int \mathbb{P}_\varepsilon^t\,|\sigma|_r^2\,\mu_{\lambda,\varepsilon}(d\sigma)$, then

$$h'(t)=2\iiint(\Delta_\varepsilon\,\lambda(x))\,p_a(0,x;t,y)\,\theta_r(y)\,\omega_t^\varepsilon(y)\,dy\,dx\,d\mu_{\lambda,\varepsilon},$$

where $a(t,y)=V''(\omega_t^\varepsilon(y))$. Lemma 10, Proposition 1 and the Schwarz inequality imply that if $r>0$ and $r\varepsilon\leqq 1$ then

$$\int_t^{t+1}[h(t)+(h'(t))^2]\,dt\leqq C_r(\lambda,t),\qquad t\geqq 0,$$

where the bound does not depend on $\varepsilon>0$. Therefore, again by the Schwarz inequality, $h(t)\leqq h(s)+C_r^{1/2}$ if $t<s\leqq t+1$, thus

$$\int \mathbb{E}[|\omega_t^\varepsilon|_r^2\,|\,\omega_0^\varepsilon=\sigma]\,\mu_{\lambda,\varepsilon}(d\sigma)\leqq C_r(\lambda,t)+C_r^{1/2}(\lambda,t). \tag{8.3}$$

This means that we can select a ball, $B_{\lambda,t}$ such that $\mu_{\lambda,\varepsilon}\mathbb{P}_\varepsilon^t(B_{\lambda,t})$ is arbitrarily close to one for all $\varepsilon>0$, which completes the proof by the Stone-Weierstrass theorem. □

The following statement is slightly stronger than Theorem 1.

Theorem 4. *Let $g\in\mathbb{D}_w'(\mathbb{L}_e^2)$ and suppose that $\omega_0^\varepsilon\in\Omega_\varepsilon$ converges weakly on $\mathbb{L}_e^2$ to some $\rho_0\in\mathbb{H}_e^1$, then $\mathbb{P}_\varepsilon^t\,g(\omega_0^\varepsilon)\to g(\rho_t)$ as $\varepsilon\to 0$, for each $t>0$, where ρ_t solves* (2.20) *with initial condition ρ_0.* □

Proof. In view of (ii) of Lemma 11, for every ball B, and for each $\beta>0$ there exist a $\gamma>0$ and a finite sequence, $\varphi_1,\varphi_2,\ldots,\varphi_n$ from $\mathbb{L}_{-e}^2$ such that

$$|\mathbb{P}_\varepsilon^t\,g(\sigma)-\mathbb{P}_\varepsilon^t\,g(I_\varepsilon\,\rho_0)|<\beta \text{ if } \sigma\in\Omega_\varepsilon,\ \sigma-\rho_0\in B,\ |\varphi_k(\sigma)-\varphi_k(\rho_0)|<\gamma \tag{8.4}$$

for $k=1,2,\ldots,n$. Suppose first that $\rho_0=F'(\lambda)$, and σ is distributed by $\mu_{\lambda,\varepsilon}$; by the law of large numbers, see Proposition 1, it follows that

$$\lim_{\varepsilon\to 0}\left[\int \mathbb{P}_\varepsilon^t\,g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)-\mathbb{P}_\varepsilon^t\,g(I_\varepsilon\,\rho_0)\right]=0. \tag{8.5}$$

On the other hand, if $\sigma=\omega_0^\varepsilon$, then $\varphi(\omega_0^\varepsilon)\to\varphi(\rho_0)$ *for each* $\varphi\in\mathbb{L}_{-e}^2$, thus ω_0^ε is bounded in $\mathbb{L}_e^2$, consequently

$$\lim_{\varepsilon\to 0}[\mathbb{P}_\varepsilon^t\,g(\omega_0^\varepsilon)-\mathbb{P}_\varepsilon^t\,g(I_\varepsilon\,\rho_0)]=0, \tag{8.6}$$

which reduces the problem to Theorem 2. □

Remark. The weak equicontinuity property expressed by (8.4) allows us to extend both Theorem 2 and Theorem 4 to all initial data, $\rho_0 = F'(\lambda)$ such that (2.20) is uniquely solved. It is not really interesting, but it seems to be nontrivial to decide if $g \in \mathbb{D}'_w$ could be replaced by $g \in \mathbb{C}_w$ in Theorem 4.

References

1. Aronson, D.G.: Bounds for the fundamental solution of a parabolic equation. Bull. Am. Math. Soc. **73**, 890–896 (1967)
2. Aronson, D.G., Serrin, J.: Local behaviour of solutions of quasilinear parabolic equations. Arch. Rational Mech. Anal. **25**, 81–122 (1967)
3. Boldrighini, C., Dobrushin, R.L., Suhov, Yu.M.: The hydrodynamic limit of a degenerate model of statistical physics. Usp. Mat. Nauk. **35**.4.152 (1980 short communication, in Russian)
4. Boldrighini, C., Dobrushin, R.L., Suhov, Yu.M.: One-dimensional hard rod caricature of hydrodynamics. J. Stat. Phys. **31**, 577–616 (1983)
5. Daleckii, Yu.L., Fomin, S.V.: Measures and differential equations in infinite dimensional spaces (in Russian). Moscow: Nauka 1983
6. De Masi, A., Ianiro, N., Pellegrinotti, S., Presutti, E.: A survey of the hydrodynamical behaviour of many-particle systems. In: Lebowitz, J.L., Montroll, E.W. (eds.) Nonequilibrium phenomena II: from stochastics to hydrodynamics. Stud. Stat. Mech., pp. 123–294. Amsterdam: North-Holland 1984
7. Dobrushin, R.L.: Prescribing a system of random variables by the help of conditional distributions. Theory Probab. Appl. **15**, 469–497 (1970)
8. Dobrushin, R.L.: On the derivation of the equations of hydrodynamics. Lecture, Budapest 1978
9. Dobrushin, R.L., Siegmund-Schultze, R.: The hydrodynamic limit for systems of particles with independent evolution. Math. Nachr. **105**, 199–244 (1982)
10. Donsker, M.D., Varadhan, S.R.S.: In preparation (1987)
11. Fritz, J.: Local stability and hydrodynamical limit of Spitzer's lattice model. Commun. Math. Phys. **86**, 363–373 (1983)
12. Fritz, J.: The Euler equation for the stochastic dynamics of a one-dimensional continuous spin system. Heidelberg: Preprint 1986
13. Fritz, J.: On the hydrodynamic limit of a scalar Ginzburg-Landau lattice model: The resolvent approach. In: IMA Volumes in mathematics, vol. 9. Hydrodynamic behaviour and interacting particle systems. New York Berlin Heidelberg: Springer 1987
14. Fritz, J.: On the hydrodynamic limit of a one-dimensional Ginzburg-Landau lattice model: The a priori bounds. J. Stat. Phys. **47**, 551–572 (1987)
15. Funaki, T.: Derivation of the hydrodynamical equation for a 1-dimensional Ginzburg-Landau model. Nagoya University: Preprint 1987
16. Guo, M.Z., Papanicolau, G.C., Varadhan, S.R.S.: In preparation (1987)
17. Kipnis, C., Marchioro, C., Presutti, E.: Heat flow in an exactly solvable model. J. Stat. Phys. **27**, 65–74 (1982)
18. Künsch, H.-R.: Decay of correlations under Dobrushin's uniqueness condition. Commun. Math. Phys. **84**, 20–222 (1982)
19. Ladyzenskaya, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and quasilinear equations of parabolic type. AMS Translations of Mathematical Monographs, No. 23. Providence: AMS 1968
20. Morrey, C.: On the derivation of the equations of hydrodynamics from statistical mechanics. Commun. Pure. Appl. Math. **8**, 279–327 (1955)
21. Papanicolau, G.C., Varadhan, S.R.S.: Boundary value problems with rapidly oscillating coefficients. In: Fritz, J., Lebowitz, J.L., Szász, D. (eds.) Random fields, vol. II, pp. 835–853. Amsterdam: Bólyai Mathematical Society and North Holland 1981
22. Presutti, E., Spohn, H.: Hydrodynamics of the voter model. Ann. Probab. **11**, 867–875 (1983)
23. Rost, H.: Non-equilibrium behaviour of a many particle system. Density profile and local equilibrium. Z. Wahrscheinlichkeitstheor. Verw. Geb. **58**, 41–55 (1981)
24. Rost, H.: The Euler equation for the one-dimensional zero range process. Lecture, Minneapolis 1986

25. Sinai, Ya.G.: Evolution of local equilibrium Gibbs states and the Euler equation. The one-dimensional case, (in Russian) (1986)
26. Spohn, H.: Equilibrium fluctuations for some stochastic particle systems. In: Fritz, J., Jaffe, A., Szász, D. (eds.) Statistical physics and dynamical systems, pp. 67–81. Basel Boston: Birkhäuser 1985
27. Stein, E.M., Weiss, G.: Interpolation of operators with change of measures. Trans. Am. Math. Soc. **87**, 159–172 (1958)
28. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean spaces. Princeton: Princeton University Press 1971
29. Stroock, D., Varadhan, S.R.S.: Multidimensional diffusion processes. Berlin Heidelberg New York: Springer 1979
30. Yoshida, K.: Functional analysis. Berlin Heidelberg New York: Springer 1980

Received September 12, 1987; in revised form August 1, 1988