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Laws of Large Numbers for Semimartingales with Applications to Stochastic Regression*

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Summary. Strong laws of large numbers for matrix-normalised vector-valued local martingales are established. The results are derived from strong laws for positive local submartingales and purely discontinuous local martingales and a Borel-Cantelli-type lemma for local martingales of finite variation. The multivariate strong laws are applied to study strong consistency of estimates in stochastic linear regression models.

1. Introduction and Notations

The general aim of this paper is to establish strong laws of large numbers for vector-valued local martingales. This classical problem has received little attention since it seems to require only a straightforward extension from the one-dimensional case. However, norming by scalars is not appropriate in higher dimensions because any such sequence of scalars must have the same order of magnitude as the maximum of all the componentwise one-dimensional norming scalars. To remedy these difficulties we use matrices for normalisation of vector-valued martingales. Typically this is needed in proving strong consistency of some estimator of a vector parameter (see e.g., Anderson and Taylor [1], Lai and Wei [17], Novikov [26], Christopeit [4]).

Section 2 contains auxiliary results. At first the asymptotic behaviour of positive local semimartingales is described. Then a strong law of large numbers for purely discontinous local martingales and a Borel-Cantelli-type lemma for locally integrable increasing processes are proved.

Section 3 is devoted to the statement and proof of the main results. We consider a vector-valued local martingale N and a matrix-valued right continuous increasing predictable process Γ. Sufficient conditions, in terms of Γ, which guarantee convergence to zero of $\Gamma^{-1}N$ are obtained. They improve those of Melnikov [23] (see also Kaufmann [16]).

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Finally Sect. 4 is concerned with applications of the multivariate strong laws to study strong consistency of parameter estimates in stochastic linear regression models for semimartingales. We strengthen the theorem of Lai and Wei [17]. In particular we prove that sufficiently fast convergence of the minimal eigenvalue of Γ to infinity (and no restriction on the behaviour of the maximal eigenvalue of Γ) guarantees the strong consistency of the least squares estimate.

Let us fix some terminology and notations. Let $(\Omega, \mathbf{F}, P)$ be a probability space equipped with a filtration $(\mathbf{F}_t : t \geqq 0)$ satisfying the usual conditions and $\mathbf{F}_{0-} = \mathbf{F}_0$. A real-valued process X based on $(\Omega, \mathbf{F}, P)$ is said to be a semimartingale (with respect to the family $(\mathbf{F}_t)$) if it admits a decomposition

$$X_t = X_0 + M_t + A_t; \qquad t \geqq 0 \tag{1}$$

where X_0 is a $\mathbf{F}_0$-measurable random variable, $M = (M_t : t \geqq 0)$ is a cadlag $(\mathbf{F}_t)$-local martingale, $M_0 = 0$, $A = (A_t : t \geqq 0)$ is a cadlag $(\mathbf{F}_t)$-adapted process of finite variation, $A_0 = 0$. The semimartingale X is said to be special if there exists a decomposition of type (1) such that A is $(\mathbf{F}_t)$-predictable. Decomposition (1) with A predictable is unique. It is called canonical. A special semimartingale X such that the process A in its canonical decomposition is decreasing (resp. increasing) is a local supermartingale (resp. submartingale).

If X is a special semimartingale based on $(\Omega, \mathbf{F}, (\mathbf{F}_t), P)$ we denote by μ^X the random measure

$$\mu^X(\omega, dx, dt) = \sum_s I_{\{\Delta X_s \neq 0\}} \varepsilon_{(\Delta X_s, s)}(dx, dt),$$

where ε_a is the Dirac measure concentrated at the point a. We write ν^X for the predictable compensator of μ^X (cf. e.g. Jacod [13]). We define the increasing processes $W_p(X)$ and $V_p(X)$, $1 \leqq p \leqq 2$, by

$$W_p(X)_t = \sum_{0 < s \leqq t} \min(|\Delta X_s|^2, |\Delta X_s|^p) = \int_{(R - \{0\}) \times]0, t]} \min(|x|^2, |x|^p)\, d\mu^X,$$

and

$$V_p(X)_t = \sum_{0 < s \leqq t} |\Delta X_s|^p = \int_{(R - \{0\}) \times]0, t]} |x|^p\, d\mu^X,$$

respectively for $t > 0$ puting zero for $t = 0$. For details on semimartingales see Jacod [13] and Dellacherie and Meyer [7].

If X is real valued process based on $(\Omega, \mathbf{F}, P)$ and if $a \in R \cup \{-\infty, \infty\}$, then $\{X_{\infty -} = a\}$ denotes the set of those elementary events ω in Ω for which $X_t(\omega)$ converges to a when t tends to ∞. Moreover $\{X \rightarrow\}$ stands for $\bigcup_{a \in R} \{X_{\infty -} = a\}$.

A function $g : [0, \infty[\rightarrow]0, \infty[$ is said to belong to the class $\mathbf{G}$ if it is continuous increasing and such that

$$\int_0^\infty g^{-1}(u)\, du < \infty. \qquad \text{For } g \in \mathbf{G} \text{ let } G(x) = \int_x^\infty g^{-1}(u)\, du,\ x \geqq 0.$$

We use the somewhat misleading but convenient notation writing g^{-1} for the function $1/g$ and also $g^{-1}(A)$ for the process $1/g(A)$.

2. Auxiliary results

2.1. Asymptotic Behaviour of Positive local Submartingales

The almost sure asymptotic behaviour of semimartingales and associated laws of large numbers have been extensively studied (see recent papers by Lépingle [20], Kabanov, Liptser and Shiryaev [14], Lenglart [19]. Liptser [21]). We investigate the behaviour of positive local submartingales.

Our first technical lemma is in the spirit of Neveu [25] p. 168 (see also Dubins and Freedman [8] and Kallenberg [15]).

Lemma 1. *Let X be a positive local submartingale and $X=X_0+M+A$ its canonical decomposition. Let g belong to* $\mathbf{G}$. *Then the process $Z=g^{-1}(A)X+G(A)$ is a positive local supermartingale. Moreover the local martingale part of the canonical decomposition of Z is nothing but the stochastic integral $g^{-1}(A)\cdot M$.*

Proof. Since the process $g(A)$ is predictable increasing, by use of the integration by parts formula ([7]) p. 343) we get

$$g^{-1}(A)X_t=g^{-1}(0)X_0+\int_{]0,t]} g^{-1}(A)_s\,dM_s+\int_{]0,t]} g^{-1}(A)_s\,dA_s$$
$$-\int_{]0,t]} X_{s-}g^{-1}(A)_{s-}g^{-1}(A)_s\,dg(A)_s. \qquad (2)$$

The function G is convex and its derivative $G'=-g^{-1}$ is continous increasing. Therefore since for every $t>0$

$$\int_{]0,t]} (G'(A_s)-G'(A_{s-}))\,dA_s=\sum_{0<s\leqq t} (G'(A_s)-G'(A_{s-}))\,\Delta A_s$$

the Ito formula ([7] p. 353) provides

$$G(A_t)=G(0)-\int_{]0,t]} g^{-1}(A_s)\,dA_s+\sum_{0<s\leqq t} (G(A_s)-G(A_{s-})+g^{-1}(A_s)\,\Delta A_s). \qquad (3)$$

Taking into account identities (2) and (3) and setting $C_0=0$,

$$C_t=\int_{]0,t]} Xg^{-1}(A)_{s-}g^{-1}(A)_s\,dg(A)_s+\sum_{0<s\leqq t}\int_{A_{s-}}^{A_s} (g^{-1}(u)\quad -g^{-1}(A_s))\,du,\ t>0$$

we get

$$Z_t=Z_0+g^{-1}(A)\cdot M_t-C_t.$$

Since C is predictable increasing, $C_0=0$, then Z is a positive local supermartingale with $g^{-1}(A)\cdot M$ and C as terms of its canonical decomposition.

Now we are able to prove the following result. Note that the first assertion is well-known (see e.g. [7], and also [22]).

Lemma 2. *Let X be a positive local submartingale with $X_0>0$ a.s. and let $X=X_0+M+A$ be the canonical decomposition of X. Then we have*

(i) $$\{A_{\infty-}<\infty\}=\{X\to\}\cap\{M\to\} \text{ a.s.,}$$

(ii) for every $g\in\mathbf{G}$

$$\{A_{\infty-}=\infty\}=\{g^{-1}(A)\,X_{\infty-}=0\}\cap\{g^{-1}(A)\,M_{\infty-}=0\} \text{ a.s..}$$

Proof. From Lemma 1 we get that $Z_0+g^{-1}(A)\cdot M$ is a positive local martingale. Then we have

$$P\{g^{-1}(A)\cdot M\to\}=1 \tag{4}$$

and since $G(A)$ is a positive decreasing process we have also

$$P\{g^{-1}(A)\,X\to\}=1. \tag{5}$$

Assertion (i) follows immediately from (5). Applying the Kronecker lemma for local martingales (cf. e.g. [20]) and equality (4) we obtain

$$\{A_{\infty-}=\infty\}\subset\{g^{-1}(A)\,M_{\infty-}=0\} \text{ a.s. .} \tag{6}$$

Moreover, defining $a(t)=\inf\{s: A_s>t\}$, noting that $t\leqq A_{a(t)}$ (cf. [7] p. 131) and using the Lebesgue lemma on the transformation of Stieltjes integrals (cf. [7] p. 132]) we get

$$\int_{]0,t]} g^{-1}(A_s)\,dA_s\leqq\int_0^\infty g^{-1}(A_{a(t)})\,dt\leqq\int_0^\infty g^{-1}(t)\,dt=G(0).$$

So $P\{g^{-1}(A)\cdot A\to)=1$ and applying the usual Kronecker lemma we obtain

$$\{A_{\infty-}=\infty\}\subset\{g^{-1}(A)\,A_{\infty-}=0\} \text{ a.s. .} \tag{7}$$

Therefore taking into account (6) and (7) we see that the inclusion $\subset$ in (ii) holds. Finally, from the assumption $X_0>0$ a.s. and the inclusion

$$\{g^{-1}(A)\,X_{\infty-}=0\}\cap\{g^{-1}(A)\,M_{\infty-}=0\}\subset\{(g^{-1}(A)\,X_0)_{\infty-}=0\} \text{ a.s.}$$

we get that assertion (ii) holds.

Remark 1. Note that it is possible to derive the strong law of large numbers for locally square integrable local martingales (cf. e.g. [20]) from assertion (ii) in Lemma 2. More generally let N be a local martingale which is locally in $L^p(p\geqq 1)$ and let $A^{(p)}$ denote the predictable increasing process of the canonical decomposition of the positive local submartingale $|N|^p$ (e.g., $A^{(2)}=\langle N\rangle$, predictable quadratic variation of N). Lemma 2 (ii) asserts that for every $g\in\mathbf{G}$ we have

$$\{A^{(p)}_{\infty-}=\infty\}\subset\{g^{-1/p}(A^{(p)})\,N_{\infty-}=0\} \text{ a.s. .}$$

2.2. A Law of Large Numbers for Purely Discontinuous Local Martingales

While the strong law of large numbers for locally square integrable local martingales is well understood (see e.g. Lépingle [20], Liptser [21]) there is a much less definite theory in the case of local martingales which are locally in L^p. Discrete time martingales were studied by Chow [6] (see also Stout [29], Elton [9]). Laws concerning stochastic integrals with respect to a stable motion were recently established by Rosiński and Woyczyński [27]. We deal with purely discontinuous local martingales.

Let, for a cadlag special semimartingale X, the increasing process $W_p(X)$ and $V_p(X)$, $1 \leqq p \leqq 2$, be defined as in Sect. 1. Note that the process $W_1(X)$ is locally integrable.

Lemma 3. *Let M, $M_0=0$, be a purely discontinous local martingale and let A, $A_0=1$, be a predictable increasing process. Choose $p \in [1, 2]$ for which $W_p(M)$ is locally integrable and denote by $\tilde{W}_p(M)$ the predictable dual projection of $W_p(M)$. Then*

$$\{A_{\infty-}=\infty, \int_0^\infty A^{-p}\, d\tilde{W}_p(M) < \infty\} \subset \{A^{-1} M_{\infty-}=0\} \text{ a.s.}.$$

Proof. The stochastic integral $m = A^{-1} \cdot M$ is a local martingale. Moreover

$$W_1(m)_t = \int_{(R-\{0\}) \times]0,t]} \min(|A^{-1}x|^2, |A^{-1}x|)\, d\mu^M$$

and

$$\tilde{W}_1(m)_t = \int_{(R-\{0\}) \times]0,t]} \min(|A^{-1}x|^2, |A^{-1}x|)\, d\nu^M.$$

But for $1 \leqq p \leqq 2$ and $A \geqq 1$ we have

$$\min(|A^{-1}x|^2, |A^{-1}x|) \leqq \min(|A^{-1}x|^2, |A^{-1}x|^p) \leqq A^{-p} \min(|x|^2, |x|^p).$$

This implies that

$$\tilde{W}_1(m)_t \leqq \int_{(R-\{0\}) \times]0,t]} A^{-p} \min(|x|^2, |x|^p)\, d\nu^M = \int_{]0,t]} A^{-p}\, d\tilde{W}_p(M).$$

Now since $\{\tilde{W}_1(m)_{\infty-} < \infty\} \subset \{m \rightarrow\}$ a.s. (cf. Jacod [13]) the Kronecker lemma for stochastic integrals (cf. Lépingle [20]) finishes the proof.

Remark 2. Note that when M is a locally square integrable purely discontinuous local martingale one can choose $p=2$. Then the process $\tilde{W}_2(M)$ is nothing but the predictable quadratic variation $\langle M \rangle$ of M. Note also that Lemma 3 extends to continuous time a result by Chow [6].

Lemma 4. *Let B, $B_0=0$ be a cadlag $(\mathbf{F}_t)$-adapted process of locally integrable variation and let $\tilde{B}$ be its predictable dual projection. Moreover, let $p \in [1, 2]$*

be such that $V_p(B)$ *is locally integrable and let* A, $A_0=1$, *be a predictable increasing process. Then*

$$\{A_{\infty-}=\infty,\ \int_0^\infty A^{-p}\,d\tilde{V}_p(B)<\infty\}\subset\{A^{-1}(B-\tilde{B})_{\infty-}=0\}\text{ a.s.}.$$

Proof. First note that $M=B-\tilde{B}$ is a purely discontinuous local martingale and that $\min(|\Delta M|^2, |\Delta M|^p)\leqq|\Delta M|^p\leqq 2(|\Delta B|^p+|\Delta\tilde{B}|^p)$. Consequently $W_p(M)$ is locally integrable and by Lemma 3 it is sufficient to prove that $\tilde{W}_p(M)$ is a.s. absolutely continuous with respect to $\tilde{V}_p(B)$ with a bounded density. But the locally integrable process $2V_p(B)+2V_p(\tilde{B})-W_p(M)$ is increasing and so is its predictable dual projection $2\tilde{V}_p(B)+2V_p(\tilde{B})-\tilde{W}_p(M)$. This implies that $\tilde{W}_p(M)$ is a.s. absolutely continuous with respect to $\tilde{V}_p(B)+V_p(\tilde{B})$ with a bounded density. Finally, since for every $t>0$

$$|\Delta\tilde{B}_t|^p=|E(\Delta B_t|\,\mathbf{F}_{t-})|^p\leqq E(|\Delta B_t|^p|\,\mathbf{F}_{t-})\text{ a.s.}$$

i.e., $\Delta V_p(\tilde{B})\leqq\Delta\tilde{V}_p(B)$ a.s., $V_p(\tilde{B})$ is a.s. absolutely continuous with respect to $\tilde{V}_p(B)$ with a bounded density and so is $\tilde{W}_p(M)$.

Remark 3. Let B be a real process which is adapted to the family $(\mathbf{F}_t)$, zero at 0, increasing, purely discontinuous and such that $\Delta B\leqq 1$. Since $\Delta V_2(B)=|\Delta B|^2\leqq\Delta B=\Delta V_1(B)$ and $\tilde{V}_1(B)=\tilde{B}$, then

$$\{A_{\infty-}=\infty,\ \int_0^\infty A^{-2}\,d\tilde{B}<\infty\}\subset\{A^{-1}(B-\tilde{B})_{\infty-}=0\}\text{ a.s.}$$

where A is a predictable increasing process, $A_0=1$. In particular one can take $A=1+\tilde{B}$.

2.3. *A Borel-Cantelli Lemma*

Dubins and Freedman [8] proved a theorem which sharpens and unifies two results by P. Lévy: his conditional form of a Borel-Cantelli lemma and his martingale strong law of large numbers. Various generatlizations of this result were later obtained by Freedman [10], Lépingle [20], Chen [3], Liptser [21], Hill [12] and recently by Bouzar [2]. In this section another version of Lévy's conditional form of the Borel-Cantelli lemma is established.

Let B be an adapted increasing cad process which is locally integrable, $B_0=0$, and let $\tilde{B}$ be its predictable dual projection. Moreover let A be a predictable increasing process, $A_0=1$, and f be a positive increasing function. Assume that one of the following conditions (a) and (b) hold:

(a) $A\geqq\tilde{B}$ and $f\in\mathbf{G}$

(b) For some $p \in [1, 2]$ $\tilde{V}_p(B)$ is a.s. absolutely continuous with respect to A and $\sup_{t>0} h^{-1}(A_t)\, d\tilde{V}_p(B)_t/dA_t < \infty$ a.s., where h is a positive increasing function such that $h^{-1} f^p \in \mathbf{G}$.

Then from Lemmas 2 and 4 respectively, it comes that

$$\{A_{\infty-} = \infty\} \subset \{f^{-1}(A)(B - \tilde{B})_{\infty-} = 0\} \text{ a.s.}. \tag{8}$$

The following result provides another set of sufficient conditions for (8) to hold.

Lemma 5. *Let B and $\tilde{B}$ be an adapted increasing locally integrable process and its predictable dual projection, respectively. Let A be a predictable increasing process such that $A \geqq \tilde{B}$ and f and h be positive increasing functions. If $h^{-1} f^2$ belongs to $\mathbf{G}$ and $E \sup_{t \geqq 0} h^{-1}(A_t)\, \Delta B_t < \infty$, then (8) holds.*

Proof. Let us define the local martingale $m = (m_t;\, t \geqq 0)$ by

$$m_0 = 0,\, m_t = \int_{]0,t]} f^{-1}(A_s)\, d(B - \tilde{B})_s,\, t > 0.$$

The optional quadratic variation process $[m] = ([m]_t;\, t \geqq 0)$ of m is given by

$$[m]_0 = 0,\, [m]_t = \sum_{0 < s \leqq t} f^{-2}(A)_s (\Delta B - \Delta\tilde{B})_s^2,\, t > 0.$$

Therefore we have

$$\begin{aligned}[m]_{\infty-} &\leqq 2 \sum_{t>0} f^{-2}(A)_t (\Delta B_t)^2 + 2 \sum_{t>0} f^{-2}(A)_t (\Delta\tilde{B}_t)^2 \\ &\leqq 2\, (\sup_{t>0} h^{-1}(A)\, \Delta B_t) \sum_{t>0} g^{-1}(A)_t\, \Delta B_t + 2\, (\sup_{t>0} h^{-1}(A)\, \Delta\tilde{B}_t) \sum_{t>0} g^{-1}(A)_t \Delta\tilde{B}_t,\end{aligned}$$

where g stands for $h^{-1} f^2$. Since $A \geqq \tilde{B}$ and g^{-1} decreases we get

$$[m]_{\infty-} \leqq 2\, (\sup_{t>0} h^{-1}(A)\, \Delta B_t) \int_0^\infty g^{-1}(\tilde{B})_t\, dB_t + 2\, (\sup_{t>0} h^{-1}(A)\, \Delta\tilde{B}_t) \int_0^\infty g^{-1}(\tilde{B})_t\, d\tilde{B}_t.$$

Setting $j(t) = \inf\{s: \tilde{B}_s > t\}$ we also have

$$\int_0^\infty g^{-1}(\tilde{B})_t\, d\tilde{B}_t \leqq \int_0^\infty g^{-1}(\tilde{B}_{j(t)})\, dt \leqq \int_0^\infty g^{-1}(t)\, dt = G(0). \tag{9}$$

Consequently we get

$$[m]_{\infty-}^{1/2} \leqq \sup_{t>0} h^{-1}(A)\, \Delta B_t + \int_0^\infty g^{-1}(\tilde{B})_t\, dB_t + (2\, G(0))^{1/2} (\sup_{t>0} h^{-1}(A)\, \Delta\tilde{B}_t)^{1/2}. \tag{10}$$

Moreover, since $\tilde{B}$ is the compensator of B, we have

$$E \int_0^\infty g^{-1}(\tilde{B})_t dB_t = E \int_0^\infty g^{-1}(\tilde{B})_t d\tilde{B}_t \leqq G(0) \tag{11}$$

where the last inequality comes from (9).

Since $h^{-1}(A)\,\Delta\tilde{B}$ is the predictable projection of $h^{-1}(A)\,\Delta B$, from Lemma 6 below, we obtain

$$E\,(\sup_{t>0} h^{-1}(A_t)\,\Delta\tilde{B}_t)^{1/2} \leqq 3(E \sup_{t>0} h^{-1}(A_t)\,\Delta B_t)^{1/2}. \tag{12}$$

Taking into account (10), (11) and (12) we obtain the following inequality

$$E[m]_{\infty-}^{1/2} \leqq 4(E \sup_{t>0} h^{-1}(A_t)\,\Delta B_t + G(0)).$$

Therefore, since $|\Delta m_t|^2 \leqq [m]_{\infty-} < \infty$ for $t>0$, we get

$$E \sup_{t>0} |\Delta m|_t < E[m]_\infty^{1/2} < \infty.$$

Hence (cf. [13] p. 168) we have $P\{m \to\} = 1$ and the use of the Kronecker lemma for local martingales (cf. [20]) leads to the assertion in the lemma.

Lemma 6. *Let Y be a positive measurable process and let ${}^pY$ be its predictable projection. If $E \sup_{t>0} Y_t < \infty$ then*

$$E\,(\sup_{t>0} {}^pY_t)^{1/2} \leqq 3(E \sup_{t>0} Y_t)^{1/2}.$$

Proof. Note that for every bounded predictable stopping time T

$$E\,{}^pY_T = EE(Y_T | \mathbf{F}_{T-}) = EY_T \leqq E \sup_{t \geqq 0} Y_t.$$

Hence, considering the constant $E \sup_{t \geqq 0} Y_t$ to be a positive increasing predictable process, from domination inequalities (cf. [7] p. 198), it follows that for any n

$$E\,(\sup_{0 \leqq t \leqq n} {}^pY_t)^{1/2} \leqq 3(E \sup_{t \geqq 0} Y_t)^{1/2}.$$

Then, letting $n \to \infty$, we get the statement in the lemma.

Remark 4. Note that choosing $A = \tilde{B}$, $g(t) = 1 + t$ and $h(t) = 1$ in Lemma 5 we obtain the following well-known Borel-Cantelli type result (cf. [20]):

$$\text{if } E \sup_{t>0} \Delta B_t < \infty \quad \text{then } \{\tilde{B}_{\infty-} = \infty\} \subset \{\tilde{B}^{-1} B_{\infty-} = 1\} \text{ a.s.}.$$

3. Laws of Large Numbers for Vector-Valued Local Martingales

In the present section N stands for a R^n-valued local martingale, $N_0=0$, and Γ denotes a positive symmetric $n\times n$ matrix valued cad increasing predictable process such that Γ_0 is non singular. The problem of the a.s. convergence to zero of N normalized from the left by the inverse of Γ (i.e., $\Gamma^{-1}N$) is investigated on the basis of the results from the previous sections. Such a problem arises in proving strong consistency of vector parameter estimates in stochastic linear regression models (see Section 4 below).

Throughout, for the vector (matrix) x, the symbols $|x|$ and x^* stand for the Euclidian norm and the transpose of x respectively. Moreover, $\operatorname{tr} x$, $\det x$, $\lambda_1(x)$ and $\lambda_n(x)$ stand for the trace, the determinant, the smallest eigenvalue and the largest eigenvalue of a symmetric $n\times n$ matrix x respectively. Finally I denotes the $n\times n$ identity matrix.

Lemma 7. *Let N and Γ be as above and let $[N]$ be the optional tensor quadratic variation of N. Then*

(i) *the semimartingale decomposition of $N^*\Gamma^{-1}N$ is given by*

$$N^*\Gamma^{-1}N_t=2\int_{]0,t]} N_{s-}^*\,\Gamma_s^{-1}\,dN_s+\operatorname{tr}\int_{]0,t]}\Gamma_s^{-1}\,d[N]_s \\ -\Big(\int_{]0,t]} N_{s-}^*\,\Gamma_s^{-1}\,d\Gamma_s\,\Gamma_s^{-1}N_{s-}+\sum_{0<s\leqq t}|\Gamma_{s-}^{-1/2}\,\Delta\Gamma_s\,\Gamma_s^{-1}N_{s-}|^2\Big), \tag{13}$$

(ii) *the predictable process* $\log\det\Gamma-\operatorname{tr}\int_{]0,.]}\Gamma_s^{-1}\,d\Gamma_s$ *is increasing and*

$$\log\det\Gamma_t-\operatorname{tr}\int_{]0,t]}\Gamma_s^{-1}\,d\Gamma_s=\sum_{0<s\leqq t}(\Delta\log\det\Gamma_s-\operatorname{tr}\Gamma_s^{-1}\Delta\Gamma_s). \tag{14}$$

Proof. Assertion (i) follows from a straightforward application of stochastic integration rules (cf. [7]). Applying the Ito formula to the function $x\to\log\det x$ and using the fact that $\frac{\partial}{\partial x}\log\det x=x^{-1}$, one gets (14) in assertion (ii). Now, since eigenvalues of $\Gamma^{-1}\Delta\Gamma$ belong to $[0,1]$, it comes that $\det(I-\Gamma^{-1}\Delta\Gamma)\leqq\exp(-\operatorname{tr}\Gamma^{-1}\Delta\Gamma)$ and also that $\operatorname{tr}\Gamma^{-1}\Delta\Gamma\leqq\Delta\log\det\Gamma$. Therefore, using (14), we obtain the first statement in (ii).

Now let N be locally square integrable and let $\langle N\rangle$ be its predictable tensor quadratic variation i.e. $\langle N\rangle=[\tilde{N}]$. The following conditions (C) and (C*) will play a central role in the next statements:

(C) (resp. (C*)) there exists a positive finite a.s. random variable ξ such that the process $\xi\Gamma-\langle N\rangle$ is positive a.s. (resp. is positive and moreover has increasing paths a.s.).

We are able to prove the following consequence of Lemma 2.

Theorem 1. *Let N and Γ be as above and let condition* (C) *be satisfied. Then, for every function $g\in\mathbf{G}$, the following holds:*

$$\{\lambda_1(\Gamma)_{\infty-}=\infty,\ \sup_{t\geqq 0}\lambda_1^{-1}(\Gamma_t)g(\log(1+\lambda_n(\Gamma_t)))<\infty\}\subset\{\Gamma^{-1}N_{\infty-}=0\}\ \text{a.s.}\,.$$

Proof. Consider the semimartingale $X=N^*(I+\langle N\rangle)^{-1}N$. Applying (13) and (14) it is easy to see that X is a positive special semimartingale and that the predictable process appearing in its canonical decomposition can be written as $\log\det(I+\langle N\rangle)-D$, where D is a predictable increasing process, $D_0=0$. Therefore $X+D$ is a positive local submartingale and Lemma 2 asserts that

$$\{\lambda_n(\langle N\rangle)_{\infty-}<\infty\}\subset\{X\rightarrow\}\text{ a.s.} \tag{15}$$

and

$$\{\lambda_n(\langle N\rangle)_{\infty-}=\infty)\subset(g^{-1}(\log(1+\lambda_n(\langle N\rangle)))\,X_{\infty-}=0\}\text{ a.s.} \tag{16}$$

for every $g\in\mathbf{G}$.

Define the stopping time $T=\inf\{t>0:\lambda_1(\Gamma_t)\geqq 1\}$ and note that, on the set $\{\lambda_1(\Gamma)_{\infty-}=\infty\}$ for $t\geqq T$

$$N^*\Gamma^{-2}N_t\leqq\operatorname{tr}\Gamma^{-2}(I+\langle N\rangle)\,X_t\leqq\eta\,\lambda_1^{-1}(\Gamma)\,X_t$$

where $\eta=2n\max(1,\xi)$. Then also

$$N^*\Gamma^{-2}N_t\leqq\eta\,(\sup_{t\geqq 0}R_t)\,g^{-1}(\log(1+\lambda_n(\langle N\rangle)))\,X_t$$
$$g(\log(1+\lambda_n(\langle N\rangle_t)))\,g^{-1}(\log(1+\lambda_n(\Gamma_t)))$$

where $R_t=\lambda_1^{-1}(\Gamma_t)\,g(\log(1+\lambda_n(\Gamma_t)))$. Therefore the assertion follows from (15) and (16).

Using Lemmas 4, 5 and 7 we are able to prove the following statement.

Theorem 2. *Let N and Γ be as above and let condition* (C*) *be satisfied. Then*

$$\{\lambda_1(\Gamma)_{\infty-}=\infty,\ \int_0^\infty\operatorname{tr}\Gamma_t^{-1}\operatorname{tr}(\Gamma_t^{-1}d\Gamma_t)<\infty\}\subset\{\Gamma^{-1}N_{\infty-}=0\}\text{ a.s.}. \tag{17}$$

Moreover (17) *can be strengthened into*

$$\{\lambda_1(\Gamma)_{\infty-}=\infty,\ (\operatorname{tr}\Gamma^{-1}\operatorname{tr}\int_{]0,.]}\Gamma_t^{-1}d\Gamma_t)_{\infty-}=0\}\subset\{\Gamma^{-1}N_{\infty-}=0\}\text{ a.s.} \tag{18}$$

if in addition to (C*) *one of the two following conditions* (D) *and* (E) *holds*:

(D) *for $B=\operatorname{tr}\int_{]0,.]}\Gamma_t^{-1}d[N]_t$ and some $p>1$, the process $V_p(B)$ is locally integrable and $\tilde V_p(B)$ is* a.s. *absolutely continuous with respect to* $\operatorname{tr}\int_{]0,.]}\Gamma_t^{-1}d\Gamma_t$ *with a bounded density.*

(E) *for some $p\in[1,2[$, $E\sup_{t\geqq 0}\lambda_1^{-p}(\Gamma)|\Delta N|_t^2<\infty$.*

Proof. Note that the stochastic integral $m=N^*\Gamma^{-1}\cdot N$ is a locally square integrable local martingale and that $\sup_{t\geqq 0}(m_t-(2\xi)^{-1}\langle m\rangle_t)=\alpha<\infty$ a.s..

Hence (13) leads to

$$N^*\Gamma^{-1}N_t\leqq 2\alpha+\operatorname{tr}\int_{]0,t]}\Gamma_s^{-1}d[N]_s.$$

Moreover since

$$\left\{\int_0^\infty \operatorname{tr}\Gamma_t^{-1}\operatorname{tr}(\Gamma_t^{-1}\,d\Gamma_t)<\infty\right\}\subset\left\{\int_0^\infty \operatorname{tr}\Gamma_t^{-1}\operatorname{tr}(\Gamma_t^{-1}\,d[N]_t)<\infty\right\}\text{ a.s.}$$

and

$$N^*\Gamma^{-2}N_t\leqq \operatorname{tr}\Gamma_t^{-1}(2\alpha+\operatorname{tr}\int_{]0,t]}\Gamma_s^{-1}\,d[N]_s) \tag{19}$$

the Kronecker lemma gives (17).

Now assume that (D) holds together with (C*). From (19) it follows that

$$N^*\Gamma^{-2}N_t\leqq \operatorname{tr}\Gamma_t^{-1}(2\alpha+\xi\operatorname{tr}\int_{]0,t]}\Gamma_s^{-1}\,d\Gamma_s+B_t-\tilde{B}_t). \tag{20}$$

But, on the set appearing in the left hand side of (18)

$$\int_0^\infty(\operatorname{tr}\Gamma_t^{-1})^p\,d\tilde{V}_p(B)_t\leqq(\operatorname{tr}\Gamma_0^{-1})^p\,\tilde{V}_p(B)_S$$

$$+C\int_S^\infty\Big(\int_{]0,t]}\operatorname{tr}\Gamma_s^{-1}\,d\Gamma_s\Big)^{-p}\operatorname{tr}(\Gamma_t^{-1}\,d\Gamma_t) \tag{21}$$

where S is such that for $t\geqq S$

$$\operatorname{tr}\Gamma_t^{-1}\leqq\Big(\int_{]0,t]}\operatorname{tr}\Gamma_s^{-1}\,d\Gamma_s\Big)^{-1}.$$

Then, since $p>1$, the integral in the right hand side of (21) is finite. This combined with Lemma 4 and (20) shows that (18) holds.

Finally, if (C*) and (E) are satisfied, applying Lemma 5 to $\operatorname{tr}\Gamma^{-1}(B-\tilde{B})$ in (20), (18) follows easily.

4. Applications to Stochastic Regression

In recent years several authors investigated the strong consistency of least squares estimates in stochastic multiple linear regression models. For contributions see e.g. Anderson and Taylor [1], Christopeit and Helmes [5], Lai and Wei [17] and Solo [28] in discrete time and Novikov [26], Christopeit [4], Le Breton and Musiela [18], Melnikov [23] in continuous time models.

The m-dimensional response process Y of a continuous time linear regression model is a special semimartingale with a canonical decomposition of the form:

$$Y_t=Y_0+M_t+\int_{]0,t]}H_s^*\,\theta\,d\gamma_s,\quad t>0. \tag{22}$$

Here $\theta=(\theta_1, \ldots, \theta_n)^*$ is an unknown parameter in R^n, $H=(H_s, s \geqq 0)$ is an observed $n \times m$ matrix valued locally bounded predictable design process and $\gamma=(\gamma_s, s \geqq 0)$ is a given predictable cad increasing "weight" process in R_+. Moreover $M=(M_s, s \geqq 0)$, $M_0=0$, is a R^m-valued cadlag local martingale standing for an unobservable error process.

Note that model (22) covers in particular the usual discrete time model as well as the continuous time Ito and Skorohod equations (see e.g. [4] and Example 2 below).

For $t>0$ such that the matrix

$$A_t = \int_{]0,t]} HH_s^* \, d\gamma_s \tag{23}$$

is nonsingular the least squares estimate θ_t of θ based on the observation $(Y_s: 0<s \leqq t)$ is given by

$$\theta_t = A_t^{-1} \int_{]0,t]} H_s \, dY_s. \tag{24}$$

Then for such a $t>0$

$$\theta_t - \theta = A_t^{-1} N_t \tag{25}$$

where $N=(N_t; t \geqq 0)$ is a R^n-valued local martingale given by

$$N_t = \int_{]0,t]} H_s \, dM_s, \qquad t>0, \qquad N_0=0. \tag{26}$$

Consequently, (25) shows that the strong consistency of θ_t is equivalent to the a.s. convergence to zero of $A^{-1}N$. Theorems 1 and 2 lead to the following statement (compare with [26], Theorem 1).

Corollary 1. *Let the local martingale M in* (22) *be locally square integrable. Assume that*:

(F) $\operatorname{tr}\langle M\rangle$ *is absolutely continuous with respect to γ and*

$$\sup_{t \geqq 0} (d \operatorname{tr}\langle M\rangle / d\gamma)_t < \infty \text{ a.s.}.$$

Let A be as in (23), *then the estimate θ_t of θ given in* (24) *satisfies*

$$\{\lambda_1(A)_{\infty-} = \infty, \sup_{t \geqq 0} (1+\lambda_1(A_t))^{-1} g(\log(1+\lambda_n(A_t))) < \infty\} \subset \{\theta_{\infty-} = \theta\} \text{ a.s.} \tag{27}$$

for any $g \in \mathbf{G}$, and also

$$\{\lambda_1(A)_{\infty-} = \infty, \int_0^\infty \operatorname{tr}(I+A_t)^{-1} \operatorname{tr}((I+A_t)^{-1} dA_t) < \infty\} \subset \{\theta_{\infty-} = \theta\} \text{ a.s.}. \tag{28}$$

Proof. Let us set $\Gamma = I + A$ and look at $\Gamma^{-1}N$ where N is given by (26). First note that a.s. $\langle M\rangle$ is absolutely continuous with respect to $\operatorname{tr}\langle M\rangle$ and the density $d\langle M\rangle / d \operatorname{tr}\langle M\rangle$ is positive symmetric with trace equal to 1 (cf. Metivier

[24] p. 141). Therefore $d\langle M\rangle\, d\,\mathrm{tr}\langle M\rangle \leqq I$ and since $\langle N\rangle_t = \int_{]0,t]} H_s\, d\langle M\rangle_s H_s^*$ and $\sup_{t\geqq 0}(d\,\mathrm{tr}\langle M\rangle/d\lambda)_t = \xi < \infty$ a.s. we get that $\xi A - \langle N\rangle$ is a positive (raw) increasing process and so is $\xi\Gamma - \langle N\rangle$. Then condition (C*) is satisfied for N and Γ. Finally, since $|\Gamma^{-1}N_t|^2 \leqq 4n|\Gamma^{-1}N_t|^2$ for t such that $\lambda_1(A_t)\geqq 1$, the statement in the theorem follows from Theorem 1 and the first assertion in Theorem 2.

Corollary 2. *Let the local martingale M in* (22) *be locally in L^{2p} for some $p>1$. Assume that condition* (*F*) *holds and also*

(G) $$\tilde{V}_{2p}(M), \quad \text{where } V_{2p}(M) = \sum_{0<s\leqq} |\Delta M|_s^{2p}, \quad \textit{is a.s.}$$

absolutely continuous with respect to γ with

(H) $$\sup_{t\geqq 0}(\mathrm{tr}\, H^*(I+A)^{-1}H)_t^{p-1}(d\tilde{V}_{2p}(M)/d\gamma)_t < \infty \text{a.s.}$$

where A is given by (23).

Then the estimate θ_t of θ given by (24) *satisfies*

$$\{\lambda_1(A)_{\infty-} = \infty, (\mathrm{tr}\,(I+A)^{-1} \int_{]0,.]} \mathrm{tr}\,((I+A)^{-1}dA))_{\infty-} = 0\} \subset \{\theta_{\infty-} = \theta\} \text{ a.s.} \tag{29}$$

and also

$$\{\lambda_1(A)_{\infty-} = \infty, (1+\lambda_1(A))^{-1}\log(1+\lambda_n))_{\infty-} = 0\} \subset \{\theta_{\infty-} = \theta\} \text{ a.s.}. \tag{30}$$

Proof. Taking into account the second assertion in Theorem 2 (14) and the proof of Corollary 1, it remains to show that (D) is satisfied for

$$B_t = \mathrm{tr} \int_{]0,t]} \Gamma_s^{-1} H_s\, d[M]_s H_s^*, \; B_0 = 0, \quad \text{and } \Gamma = I + A.$$

But, since

$$\Delta B = \Delta M^* H^* \Gamma^{-1} H \Delta M \leqq \mathrm{tr}\, H^* \Gamma^{-1} H |\Delta M|^2,$$

$\tilde{V}_p(B)$ is a.s. absolutely continuous with respect to

$$\int_{]0,.]} (\mathrm{tr}\, H^* \Gamma^{-1} H)_t^p \, \mathrm{d}\tilde{V}_{2p}(M)_t.$$

Therefore, from (F), (G) and (H), we get that $\tilde{V}_p(B)$ is also a.s. absolutely continuous with respect to $\mathrm{tr} \int_{]0,.]} \Gamma_t^{-1} d\Gamma_t$ with a bounded density. This means that (D) holds and finishes the proof.

Let us now look at some examples.

Example 1. Let the local martingale M in model (22) be continuous. Then assumptions (G) and (H) in Corollary 3 are both satisfied. Therefore $\theta_{\infty-} = \theta$

a.s. under (F), $\lambda_1(\Lambda)_{\infty-}=\infty$ a.s. and $\lambda_1(\Lambda)=o(\log(1+\lambda_n(\Lambda)))$ a.s.. This could have been obtained directly from (14) and the assertion in Theorem 5 saying that (18) holds under (E) since if M is continuous then N given by (26) is continuous too and (E) is satisfied. Note that for instance (F) holds in the case when in (22) $\lambda=\operatorname{tr}\langle M\rangle$ (see also [18]).

Example 2. The usual discrete time stochastic multiple linear regression model can be embedded into model (22) by setting

$$M_t=\sum_{0<k\leqq t}\Delta M_k,\ t>0,\ M_0=0,\ \text{and}\ \gamma_t=[t],\ t\geqq 0 \qquad \text{(see e.g., [4]).}$$

Let $\Lambda_k=\sum_{j=1}^{k}H_jH_j^*$. Corollary 1 states that if

$$\text{(F')} \qquad \sup_{k\geqq 1}E(|\Delta M_k|^2|\mathbf{F}_{k-1})<\infty \text{ a.s.}$$

then

$$\{\lambda_1(\Lambda)_{\infty-}=\infty,\ \sum_{k=1}^{\infty}\operatorname{tr}(I+\Lambda)_k^{-1}\operatorname{tr}(I+\Lambda)_k^{-1}H_kH_k^*<\infty\}\subset\{\theta_{\infty-}=\theta\}\text{ a.s..} \tag{31}$$

Since $\operatorname{tr}H^*(I+\Lambda)^{-1}H_k=\operatorname{tr}(I+\Lambda)^{-1}\Delta\Lambda_k\leqq n$, Corollary 2 states that if (F′) is strengthened into

$$\text{(H')} \qquad \sup_{k\geqq 1}E(|\Delta M_k|^{2p}|\mathbf{F}_{k-1})<\infty \text{ a.s. for some } p>1$$

then (31) can be strengthened into

$$\{\lambda_1(\Lambda)_{\infty-}=\infty,(\operatorname{tr}(I+\Lambda)^{-1}\sum_{1\leqq j\leqq}\operatorname{tr}(I+\Lambda_j)^{-1}H_jH_j^*)_{\infty-}=0\}\subset\{\theta_{\infty-}=\theta\}\text{ a.s..} \tag{32}$$

Finally, note that since

$$\operatorname{tr}(I+\Lambda)_k^{-1}H_kH_k^*\leqq n\min(1,\Delta\log\det(I+\Lambda)_k,\operatorname{tr}(I+\Lambda)_k^{-1}\operatorname{tr}H_kH_k^*)$$

then under assumption (F′) (resp. (H')) (31) (resp. 32)) sharpens Corollary 3 (resp. Theorem 1) of Lai and Wei [17]. In particular, under (F′), if either $\lambda_1(\Lambda)_k\geqq ck^\alpha$ a.s. for some $\alpha>1$ or $\lambda_1(\Lambda)_k\geqq ck^\alpha$ a.s. for some $\alpha>1/2$ and $\sup_{k\geqq 1}\operatorname{tr}H_kH_k^*<\infty$ a.s., then θ_k is strongly consistent by (31). Moreover, under (H'), if either

$\lambda_1(A)_k \geqq ck(\log k)^\alpha$ a.s. for some $\alpha > 0$ or $\lambda_1(A)_k \geqq c\sqrt{k}(\log k)^\alpha$ a.s. for some $\alpha > 0$ and $\sup_{k \geqq 1} \operatorname{tr} H_k H_k^* < \infty$, a.s., then θ_k converges a.s. to θ by (32).

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