Probab. Th. Rel. Fields 81, 185–188 (1989)

Probability Theory and Related Fields

Infinite-Dimensional Extension of a Theorem of Komlós

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Summary. A well-known theorem of Komlós is extended to integrable functions taking values in a reflexive Banach space.

1. Introduction

Let $(T, \mathscr{T}, \mu)$ be a finite measure space. The set of all integrable real-valued functions on T is denoted by $\mathscr{L}^1_{\mathbb{R}}$. A celebrated discovery of Komlós [11, Theorem 1a] is as follows:

Theorem (Komlós). *Suppose that* $\{f_k\} \subset \mathscr{L}^1_{\mathbb{R}}$ *is a sequence with*

$$\sup_k \int_T |f_k| \, d\mu < +\infty.$$

Then there exist $f_* \in \mathscr{L}^1_{\mathbb{R}}$ *and a subsequence* $\{f_m\}$ *of* $\{f_k\}$ *such that for every further subsequence* $\{f_{m_i}\}$ *of* $\{f_m\}$

$$\frac{1}{n} \sum_{i=1}^{n} f_{m_i}(t) \to f_*(t) \quad a.e.$$

Komlós' result has been seminal for the development of a whole class of limit results. Such results are unified by the so-called "subsequence principle". Originally this was formulated by Chatterji [8] as a heuristic principle; it received firm theoretical underpinnings by work of Aldous [1] and, more recently, Berkes and Péter [6]. Within this theory Komlós' theorem comes forward as a natural counterpart of the strong law of large numbers.

The purpose of this note is to demonstrate that Komlós' theorem can be extended to an infinite-dimensional setting in a straightforward manner. Our motivation for this extension lies in certain applications to limit theorems in the theories of differential inclusions, mathematical economics and optimal control. These will be discussed elsewhere, except for the Corollary following Theorem A. It is interesting to note that alternative tools for such applications figure in [2, 3, 4, 5], and that in fact an important technical tool used in [1] and [6] can be found in "infinite-dimensional" form in [3, 4].

2. Main Result

Let $(X, \|\cdot\|)$ be a reflexive Banach space. The set of all Bochner-integrable functions from T into X is denoted by $\mathscr{L}_X^1$. In this setting the following extension of Komlós' theorem can be given.

Theorem A. *Suppose that $\{f_k\} \subset \mathscr{L}_X^1$ is a sequence with*

$$\sup_k \int_T \|f_k\| \, d\mu < +\infty.$$

Then there exist $f_* \in \mathscr{L}_X^1$ and a subsequence $\{f_m\}$ of $\{f_k\}$ such that for every further subsequence $\{f_{m_i}\}$ of $\{f_m\}$

$$\frac{1}{n} \sum_{i=1}^{n} f_{m_i}(t) \to f_*(t) \qquad \textit{weakly in } X \quad \textit{a.e.}$$

Proof. There exists a null set N such that $f_k(T \setminus N)$ is a separable subset of $(X, \|\cdot\|)$, $k=1, 2, \ldots$ [9]. Let Y be the closed linear subspace of X which is generated by the union of all sets $f_k(T \setminus N)$, $k=1, 2, \ldots$; then $(Y, \|\cdot\|)$ is clearly a separable reflexive Banach space. Hence, the dual space Y^* of Y is certainly separable for the dual norm. Let $\{y_j^*\}$ be a countable dense subset of Y^*, and denote the duality between Y and its dual by $\langle \cdot, \cdot \rangle$, as usual. We apply an obvious (but tedious) diagonal method, based on successively applying Komlós' theorem to suitably chosen subsequences of $\{\|f_k\|\}$ and $\{\langle y_j^*, f_k \rangle\}$, $j=1, 2, \ldots$. This gives the existence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ in $\mathscr{L}_{\mathbb{R}}^1$ and a subsequence $\{f_m\}$ of $\{f_k\}$ with the following properties: For every subsequence $\{f_{m_i}\}$ of $\{f_m\}$

$$\frac{1}{n} \sum_{i=1}^{n} \|f_{m_i}(t)\| \to \varphi_0(t) \qquad \textit{a.e.} \text{ in } T \setminus N, \tag{1}$$

$$\frac{1}{n} \sum_{i=1}^{n} \langle y_j^*, f_{m_i}(t) \rangle \to \varphi_j(t), \qquad j=1, 2, \ldots \text{ a.e. in } T \setminus N. \tag{2}$$

Denote by M the union of the exceptional set for (1)–(2) and N if $\{f_m\}$ itself acts as the subsequence; of course, M is a null set. Define also $s_n := \frac{1}{n} \sum_{m=1}^{n} f_m$.

Let $t \in T \setminus M$ be arbitrary. By the triangle inequality and reflexivity of Y it folllows immediately from (1) that there exist $y_t \in Y$, $\|y_t\| \leqq \phi_0(t)$, and a subsequence $\{s_{n_p}(t)\}$ of $\{s_n(t)\}$ such that $s_{n_p}(t) \to y_t$ weakly in Y. By (2) this gives

$$\langle y_j^*, y_t \rangle = \varphi_j(t), \qquad j=1, 2, \ldots. \tag{3}$$

Since $\{y_j^*\}$ separates the points of Y, it follows from (3) that *every* limit point of $\{s_n(t)\}$ must equal y_t. Hence we conclude that for every $t \in T \setminus M$ $s_n(t)$ converges weakly to a point y_t in Y. *A fortiori* this implies that for every t in $T \setminus M$ $s_n(t) \to y_t$ weakly in X. Define $f_*(t) := y_t$ for $t \in T \setminus M$ and $f_*(t) := 0$ for $t \in M$. Then it follows elementarily from the Pettis measurability theorem that f_* is strongly measurable. Also, for every $t \in T \setminus M$ we have seen that $\|f_*(t)\| = \|y_t\| \leqq \varphi_0(t)$. Since φ_0 is integrable, it follows that f_* is Bochner-integrable. Finally, let $\{f_{m_i}\}$ be an arbitrary subsequence of $\{f_m\}$. Then the argument leading to (3) can be repeated for $s'_n := \frac{1}{n} \sum_{i=1}^{n} f_{m_i}$. Thus, there exists a null set M' such that for every $t \in M'$ the weak limit z_t of $\{s'_n(t)\}$ exists and satisfies $\langle y_j^*, z_t \rangle = \varphi_j(t)$, $j = 1, 2, \ldots$. It follows that $z_t = f_*(t)$ for almost every t. Q.E.D.

We give one immediate application of the above result; more can be found in forthcoming work of the present author. Our application concerns an influential existence result by V.L. Levin [12, Theorem 1], which runs as follows.

Corollary. *Suppose that* $I: \mathscr{L}_X^1 \to (-\infty, +\infty]$ *is a quasiconvex functional which is lower semicontinuous in measure and that* $B \subset \mathscr{L}_X^1$ *is convex, closed in measure and uniformly bounded in* $\mathscr{L}^1$*-norm. Then* I *attains its minimum on* B.

In [12] X is a reflexive Banach space, just as here. The possibility to use Komlós' theorem in the finite-dimensional case was already pointed out in a remark in [12, p. 1385]. Our present result shows that the validity of Levin's remark is actually not subject to any restriction. For more information and yet another proof of the Corollary (next to Levin's original one) we refer to [5].

An obvious modification of Theorem A is suggested by the role of weak compactness in its proof. From now on we suppose that $(X, \|\cdot\|)$ is a Banach space *which need no longer be reflexive.*

Theorem B. *Suppose that* $\{f_k\} \subset \mathscr{L}_X^1$ *is a sequence with*

$$\{f_k(t)\} \text{ is relatively weakly compact a.e.,}$$
$$\sup_k \int_T \|f_k\| \, d\mu < +\infty.$$

Then there exist $f_* \in \mathscr{L}_X^1$ *and a subsequence* $\{f_m\}$ *of* $\{f_k\}$ *such that for every further subsequence* $\{f_{m_i}\}$ *of* $\{f_m\}$

$$\frac{1}{n} \sum_{i=1}^{n} f_{m_i}(t) \to f_*(t) \qquad \text{weakly in } X \quad \text{a.e.}$$

Proof. The proof is almost the same as the one given above, so it suffices to concentrate only on the points of difference. Instead of (1) we now use the first of the above conditions (together with the Krein and Eberlein-Smulian theorems [10, 19.E, 18.A]) to prove (for t fixed) the weak convergence of some subsequence of $\{s_n(t)\}$ to some element y_t of Y. By [7, III.31] Y^* contains a countable subset $\{y_j^*\}$ which separates the points of $Y((Y, \|\cdot\|)$ is Suslin). So (3) continues to be valid. The remainder of the proof is the same as before. Q.E.D.

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Received June 6, 1987