Probab. Th. Rel. Fields 77, 415–455 (1988)

Probability Theory and Related Fields

Some New Representations in Bivariate Exchangeability

Olav Kallenberg*
Mathematics ACA, 120 Mathematics Annex, Auburn University,
Auburn, AL 36849-3501, USA

Summary. Consider an array $X=(X_{ij},\ i,j\in\mathbb{N})$ of random variables, and let $U=(U_{ij})$ and $V=(V_{ij})$ be orthogonal transformations, affecting only finitely many coordinates. Say that X is separately rotatable if $UXV^T\stackrel{d}{=}X$ for arbitrary U and V, and jointly rotatable if this holds with $U=V$. Restricting U and V to the class of permutations, we get instead the property of separate or joint exchangeability. Processes on $\mathbb{R}_+^2$, $\mathbb{R}_+\times[0,1]$ or $[0,1]^2$ are said to be separately or jointly exchangeable, if the arrays of increments over arbitrary square grids have these properties. For some of the above cases, explicit representations have recently been obtained, independently, by Aldous and Hoover. The aim of the present paper is to continue the work of these authors by deriving some new representations, and by solving the associated uniqueness and continuity problems.

1. Introduction

Consider an infinite two-dimensional array of random variables $X=(X_{ij},\ i,j\in\mathbb{N})$. We shall say that X is *separately* (or row-column) *exchangeable*, if its distribution is invariant under permutations of both rows and columns, i.e., if $(X_{ij})\stackrel{d}{=}(X_{p_i q_j})$ for all permutations (p_i) and (q_j) of $\mathbb{N}$. If this condition holds with *the same* permutation for rows and columns, i.e., if $(X_{ij})\stackrel{d}{=}(X_{p_i p_j})$ for all (p_i), we shall say instead that X is *jointly* (or weakly) *exchangeable*. In the above definitions, it is clearly enough to consider permutations (p_i) such that $p_i=i$ for all but finitely many i.

Aldous [1] and Hoover [11] proved independently that an array as above is separately exchangeable iff it is distributed as

$$X_{ij}=f(\alpha,\xi_i,\eta_j,\lambda_{ij}),\qquad i,j\in\mathbb{N},\tag{1}$$

* Research supported by the Air Force Office of Scientific Research Grant No. F 49620 85C 0144

for some measurable function $f\colon [0,1]^4 \to \mathbb{R}$, where the quantities α and ξ_i, η_j, λ_{ij}, $i, j \in \mathbb{N}$, are i.i.d. random variables, uniformly distributed on $[0, 1]$ ($U(0, 1)$ for short). Hoover also showed that an array is jointly exchangeable iff it is distributed as

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \lambda_{ij}), \qquad i, j \in \mathbb{N}, \tag{2}$$

for some function f as above, where α, ξ_1, ξ_2, ... and λ_{ij}, $i<j$, are i.i.d. $U(0, 1)$, while $\lambda_{ii}=0$ and $\lambda_{ij}=\lambda_{ji}$ for all i and j. Aldous gives the same result without proof, in the special case of symmetric arrays (where $X_{ij}=X_{ji}$ and hence $f(\cdot, x, y, \cdot)=f(\cdot, y, x, \cdot)$).

Since the representation in (2) will play a basic role in this paper, we shall give a short proof in Sect. 3 below, modelled after Aldous [1, 2]. (Note incidentally that Aldous attributes certain crucial ideas in the published proof to J.F.C. Kingman. His original argument was more complicated. Hoover's unpublished proof [11] uses ideas from formal logic and non-standard analysis, and may be hard to read for most probabilists.) Notice that representation (1) follows immediately from (2), since the two representations are equivalent for $(i, j) \in (2\mathbb{N}) \times (2\mathbb{N}-1)$. This observation will often be useful in the sequel.

Aldous, in his brilliant paper [1], goes on to prove a conjecture of Dawid [5], giving the general form of a *separately rotatable* (or spherical) array. By this we mean an array X as above, such that $UXV^T \stackrel{d}{=} X$ for all linear operators U and V on $\mathbb{R}^\infty$, which transform a finite set of coordinates orthogonally while leaving the others invariant. Transformations of this type will be called *rotations* below, and for these the matrix notation above will often be convenient. The general representation theorem states that an array is separately rotatable, iff it is distributed as

$$X_{ij} = \sigma \lambda_{ij} + \sum_{k=1}^{\infty} \alpha_k \xi_{ik} \eta_{jk}, \qquad i, j \in \mathbb{N}, \tag{3}$$

for some random variables σ and $\alpha_1, \alpha_2, \ldots$ with $\sum \alpha_k^2 < \infty$, where the quantities λ_{ij}, ξ_{ik} and η_{jk} are i.i.d. $N(0, 1)$ and independent of σ and (α_k). In fact, the general array is known to be a mixture (in the distributional sense) of *dissociated* ones, where $(X_{ij}, i \vee j \leq n)$ and $(X_{ij}, i \wedge j > n)$ are independent for each n, so Aldous restricts his attention to the latter and obtains a representation (3) with constant coefficients. He also needs a moment condition for his proof. Given Aldous' work, it is not hard to supply the additional arguments needed for the general version, which is done in Sect. 4 below. Even this result will play a key role in subsequent sections.

In Sect. 5, the characterizations in (2) and (3) will be combined with some methods from Aldous' paper to yield a corresponding representation in the *jointly rotatable* case, where it is assumed that $UXU^T \stackrel{d}{=} X$ for all rotations U. For the special case of symmetric arrays, our representation becomes

$$X_{ij} = \rho \delta_{ij} + \sigma(\lambda_{ij} + \lambda_{ji}) + \sum_{k=1}^{\infty} \alpha_k (\xi_{ik} \xi_{jk} - \delta_{ij}). \qquad i, j \in \mathbb{N}, \tag{4}$$

where δ_{ij} denotes the Kronecker delta, while the λ_{ij} and ξ_{ik} are i.i.d. $N(0, 1)$ as before, and ρ, σ and $\alpha_1, \alpha_2, \ldots$ are arbitrary random variables independent of the λ_{ij} and ξ_{ik} and satisfying $\sum \alpha_k^2 < \infty$. Dawid [5] discusses the further restricted case when the finite subarrays are non-negative definite. In this case (4) simplifies to

$$X_{ij} = \rho\,\delta_{ij} + \sum_{k=1}^{\infty} \alpha_k\,\xi_{ik}\,\xi_{jk}, \qquad i, j \in \mathbb{N}, \tag{5}$$

with non-negative ρ and $\alpha_1, \alpha_2, \ldots$ satisfying $\sum \alpha_k < \infty$, as conjectured by Dawid. In fact, Dawid proves that the representation (5) is equivalent to (3) above, and so his conjecture was essentially settled already by Aldous' paper.

The last two sections are devoted to exchangeable and continuous random processes X in the plane, as introduced in Aldous [2]. Here the definition of exchangeability is stated in terms of the *increments* of X over finite rectangles I, given by

$$X(I) = X(b, d) - X(a, d) - X(b, c) + X(a, c)$$

when $I = (a, b) \times (c, d)$. We shall say that a process X on $\mathbb{R}_+^2$, $\mathbb{R}_+ \times [0, 1]$, $[0, 1] \times \mathbb{R}_+$ or $[0, 1]^2$ is *separately exchangeable*, if the array of increments of X with respect to an arbitrary rectangular grid has this property. The definition of *jointly exchangeable* processes on $\mathbb{R}_+^2$ or $[0, 1]^2$ is similar, except that we have to consider square grids emanating from the origin. For definiteness, we shall assume in both cases that $X(s, 0) \equiv X(0, t) \equiv 0$.

In Sect. 6 we show that a process on $\mathbb{R}_+^2$ is separately exchangeable and continuous iff it is distributed as

$$X_{st} = \rho\, s\, t + \sigma A_{st} + \sum_{j=1}^{\infty} (\alpha_j\, B_j(s)\, C_j(t) + \beta_j\, B_j(s)\, t + \gamma_j\, s\, C_j(t)), \tag{6}$$

for some random variables ρ, σ ands α_j, β_j, γ_j, $j \in \mathbb{N}$, with $\sum(\alpha_j^2 + \beta_j^2 + \gamma_j^2) < \infty$. Here A denotes an independent Brownian sheet, while the B_j and C_j are mutually independent Brownian motions, which are also assumed to be independent of everything else. The same representation is valid for processes on $\mathbb{R}_+ \times [0, 1]$ or $[0, 1]^2$, but now with the B_j and C_j interpreted as Brownian bridges in appropriate cases, and with the Brownian sheet A accordingly tied down. Our proof of (6) depends on the simple observation that exchangeability is equivalent to rotatability for continuous and suitably tied-down processes on $\mathbb{R}_+$. By this coincidence, the representations of rotatable arrays derived in previous sections become the basic tools to analyze exchangeable processes in higher dimensions.

In the final Sect. 7, we characterize jointly exchangeable processes on $\mathbb{R}_+^2$. For the special case of symmetric processes, our representation formula becomes

$$\begin{aligned} X_{st} = {} & \rho\, s\, t + \vartheta(s \wedge t) + \sigma(A_{st} + A_{ts}) \\ & + \sum_{j=1}^{\infty} \{\alpha_j (B_j(s)\, B_j(t) - s \wedge t) + \beta_j (s\, B_j(t) + t\, B_j(s)) + \gamma_j\, B_j(s \wedge t)\}, \end{aligned} \tag{7}$$

where ρ, ϑ, σ and the α_j, β_j and γ_j are arbitrary random variables satisfying $\sum(\alpha_j^2+\beta_j^2+\gamma_j^2)<\infty$ a.s., while A is an independent Brownian sheet and the B_j are independent Brownian motions, as before. This may be compared with Conjecture 15.20 in Aldous [2], where it is suggested that instead

$$X_{st}=\rho\, s\, t+\vartheta(s\wedge t)+\sigma A(s\wedge t,\, s\vee t)+\sum_{j=1}^{\infty}\alpha_j\, B_j(s)\, B_j(t). \tag{8}$$

Note that the centering of the product terms $B_j(s)\,B_j(t)$ is necessary for convergence in general. The missing components $\sum\beta_j s\, B_j(t)$ and $\sum\beta_j t\, B_j(s)$ represent centered drift terms in the horizontal and vertical directions respectively, themselves exchangeable, while $\vartheta(s\wedge t)+\sum\gamma_j B_j(s\wedge t)$ represents an exchangeable process along the diagonal.

We conjecture that (7) and the more general non-symmetric version below remain valid for jointly exchangeable processes on $[0,1]^2$, with A and the B_j tied down as before. We might also mention the open problem of characterizing *jointly spreadable* arrays and processes, where spreadability is defined as in [16].

Once a characterization problem has been solved, the next step becomes to examine the associated problems of uniqueness and continuity. Here the former is to identify the equivalence classes of representations giving rise to the same distribution, while the latter problem consists in describing the topology in the so defined representation space that corresponds to weak convergence for the distributions of X. This program will be carried out below for the representations in (3), (4), (6) and (7). (Note that the uniqueness problem for the representations in (1) and (2) has already been solved by Hoover [11].) We shall use the approach from the univariate discussion in [14]. Thus for each case we shall introduce a suitable set of *directing random elements*, ρ say, to be given as functions of the coefficients in the representation formula, such that convergence in distribution of ρ and X will be equivalent.

Our discussion of the main problems, as stated above, will be preceded by some general prerequisites in Sect. 2. Here we shall present some results based on the powerful section theorem (cf. Dellacherie and Meyer [6]), which will provide the technical tools to extend a representation from the dissociated to the general case. Likewise, they will yield without effort the X-measurability of the directing random elements directly from their uniqueness in the dissociated case. Throughout the paper, we shall further make frequent use of the simplifying device of *randomization*, based on the elementary Lemma 2.1 from [16]. In particular, this will enable us to proceed directly from an explicit formula for an equivalent array or process (i.e. some $X'\stackrel{d}{=}X$) to an a.s. representation of X itself. Section 2 will also contain the required background on the univariate case, as well as a brief discussion of some processes related to Brownian motion and sheet.

As for relevant literature, the lecture notes by Aldous [2] provide a broad survey of exchangeability theory. The reader is especially urged to read his Sects. 14–15, dealing with the multivariate case. Several of our arguments below have been patterned on similar passages in Aldous [1]. On such occasions, we shall often give only a brief outline, so the reader may need to consult

Aldous' paper for details. Other references on the multivariate case, not mentioned before, are the papers by Dawid [4], Hoover [12] and Lynch [17]. A referee kindly calls my attention to the paper of Dovbysh & Sudakov [7], and to a thesis by Hestir [10] 'giving results like Theorem 6.1 (in a less sophisticated way)'.

Our discussion of weak convergence and tightness for random arrays and processes presupposes some general theory on the subject, as given in Chapters 1 and 2 of Billingsley [3]. We shall further need some weak convergence theory for probabilities on measure spaces, as provided by Chapter 4 in [15]. The reason for this is that, typically, one or more of the directing random elements will turn out to be random measures on some appropriate space. Finally, we shall often need to refer to [14], not only for the basic univariate representations, but also for its elementary randomization Lemma 1.1, which will often yield immediate extensions of our weak convergence results from the dissociated to the general case.

2. Preliminaries

In this section, we shall first derive some general measure theoretic results, which will be useful in proving the main theorems of the paper. Say that $(\Omega', \mathscr{F}', \mathsf{P}')$ is an *extension* of the probability space $(\Omega, \mathscr{F}, \mathsf{P})$, if it is of the form $(\Omega \times I, \mathscr{F} \times \mathscr{B}, \mathsf{P} \times \lambda)$ for some probability space $(I, \mathscr{B}, \lambda)$, which may e.g. be taken to be the Lebesgue unit interval. Note that random elements on Ω extend immediately to Ω' with the same distribution. The procedure of constructing random elements on an extended probability space will be called *randomization.*

For easy reference, we first restate the simple Lemma 1.1 of [16]:

Lemma 2.1. *Let ξ and η be random elements in the spaces S and T respectively, where S is separable metric while T is Polish, and assume that $\xi \stackrel{d}{=} f(\eta)$ for some Borel measurable function $f: T \to S$. Then there exists, on a possibly extended probability space, some random element $\eta' \stackrel{d}{=} \eta$ satisfying $\xi = f(\eta')$ a.s.*

The next result will be needed to extend a representation formula, obtained under suitable conditioning, to the unconditional case.

Lemma 2.2. *Fix a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, a σ-field $\mathscr{G} \subset \mathscr{F}$, and three Polish spaces S, T and U. Let $\xi: \Omega \to S$, $\eta: \Omega \to U$ and $f: T \times U \to S$ be measurable mappings, and put $m_t = \mathsf{P}\{f(t, \eta) \in \cdot\}$. Assume that*

$$\mathsf{P}[\xi \in \cdot \mid \mathscr{G}] \in \{m_t;\, t \in T\} \quad a.s. \tag{1}$$

Then there exists a $\mathscr{G}$-measurable random element τ in T and an independent random element $\eta' \stackrel{d}{=} \eta$ on some extension of Ω, such that $\xi = f(\tau, \eta')$ a.s.

Proof. Let $\mathscr{S}$ and $\mathscr{T}$ denote the Borel σ-fields in S and T respectively, and conclude by Fubini's theorem that $m_t B$ is $\mathscr{T}$-measurable for every $B \in \mathscr{S}$ Writing μ for a version of $\mathsf{P}[\xi \in \cdot \mid \mathscr{G}]$, it is further seen that μB is $\mathscr{G}$-measurable for all $B \in \mathcal{S}$ Letting $B_1, B_2, \ldots \in \mathcal{S}$ be measure determining in S, we get

$$A \equiv \{(\omega, t) \in \Omega \times T: \mu_\omega = m_t\} = \bigcap_{j=1}^{\infty} \{(\omega, t): \mu_\omega B_j = m_t B_j\} \in \mathcal{G} \times \mathcal{T}$$

Note also that the projection of A on Ω has probability 1, by assumption. By the section theorem (cf. [6]), there exists some $\mathcal{G}$-measurable random element τ in T, such that $\mu = m_\tau$ a.s. Choosing by randomization some $\eta'' \stackrel{d}{=} \eta$ independent of τ, we get by Fubini's theorem

$$\mathrm{P}[\xi \in \cdot \,|\, \tau] = \mu = m_\tau = \mathrm{P}[f(\tau, \eta'') \in \cdot \,|\, \tau] \quad \text{a.s.,}$$

which shows that $(\xi, \tau) \stackrel{d}{=} (f(\tau, \eta''), \tau)$. By Lemma 2.1, there exists some random pair $(\tau', \eta') \stackrel{d}{=} (\tau, \eta'')$ on an extension of Ω, such that $\xi = f(\tau', \eta')$ and $\tau = \tau'$ a.s. Thus $\xi = f(\tau, \eta')$ a.s., and moreover η' is independent of τ, since $(\tau, \eta') \stackrel{d}{=} (\tau, \eta'')$. □

More can be said when the m_t are invariant and ergodic under a suitable class of transformations. Here we are using the terminology of Sect. 12 in Aldous [2].

Lemma 2.3. *Let the measures m_t in Lemma 2.2 be invariant and ergodic under some countable group of measurable transformations of S. Then the random measure m_τ is a.s. unique and ξ-measurable, and there is even a ξ-measurable choice of τ. Moreover, the distributions of ξ and m_τ determine each other uniquely.*

Proof. Let $\mathcal{I}_S$ be the σ-field of invariant Borel sets in S, and put $\mathcal{I} = \xi^{-1} \mathcal{I}_S \subset \mathcal{F}$ From Dynkin [8] (cf. Theorem 12.10 in [2]) it is known that $\mathrm{P}[\xi \in \cdot \,|\, \mathcal{I}]$ is a.s. ergodic, and that the integral representation of $\mathrm{P}\xi^{-1}$ over the ergodic measures is unique. Hence the random measures m_τ and $\mathrm{P}[\xi \in \cdot \,|\, \mathcal{I}]$ have the same distribution. Since the range of m is analytic, it follows that $\mathrm{P}[\xi \in \cdot \,|\, \mathcal{I}] \in \{m_t, t \in T\}$ a.s. Thus Lemma 2.2 applies with $\mathcal{G} = \mathcal{I}$, so there exists some $\mathcal{I}$-measurable random element τ' in T satisfying

$$\mathrm{P}[\xi \in \cdot \,|\, \mathcal{I}] = m_{\tau'} \quad \text{a.s.} \tag{2}$$

Let us now return to the relation

$$m_\tau = \mathrm{P}[\xi \in \cdot \,|\, \mathcal{G}] = \mathrm{P}[\xi \in \cdot \,|\, \tau]. \tag{3}$$

Here the left-hand side is a.s. ergodic, so

$$\mathrm{P}[I \,|\, \tau] \in \{0, 1\} \quad \text{a.s.,} \qquad I \in \mathcal{I},$$

and it follows easily that

$$I = \{\mathrm{P}[I \,|\, \tau] = 1\} \in \sigma(\tau) \quad \text{a.s.,} \qquad I \in \mathcal{I}.$$

This shows that $\mathcal{I} \subset \sigma(\tau)$. We now obtain from (2) and (3)

$$m_{\tau'} = \mathrm{P}[\xi \in \cdot \,|\, \mathcal{I}] = \mathrm{E}[\mathrm{P}[\xi \in \cdot \,|\, \tau] \,|\, \mathcal{I}] = \mathrm{E}[m_\tau \,|\, \mathcal{I}] \quad \text{a.s.}$$

Letting B be an arbitrary Borel set in S, we get

$$\mathsf{E}\, m_\tau B\, m_{\tau'} B = \mathsf{E}\, m_\tau B\, \mathsf{E}[m_\tau B \mid \mathscr{I}] = \mathsf{E}(\mathsf{E}[m_\tau B \mid \mathscr{I}])^2 = \mathsf{E}(m_{\tau'} B)^2,$$

and since $m_{\tau'} \overset{d}{=} m_\tau$ as above, it follows that

$$\mathsf{E}(m_\tau B - m_{\tau'} B)^2 = \mathsf{E}(m_\tau B)^2 - \mathsf{E}(m_{\tau'} B)^2 = 0.$$

This shows that $m_\tau = m_{\tau'}$ a.s., so m_τ is a.s. unique and $\mathscr{I}$-measurable. It follows in particular that $\mathsf{P}\xi^{-1}$ determines $\mathsf{P} m_\tau^{-1}$. The converse is also true, since $\mathsf{P}\xi^{-1} = \mathsf{E}\, m_\tau$. □

In the applications we have in mind, τ is the array of coefficients in the representation formula for X, and m_t is the distribution of X when $\tau = t$ is fixed. Now suppose that f is a measurable mapping from T to some space V, such that m_t and f_t determine each other uniquely. If the mappings between m_t and f_t can be shown to be measurable, a.s. $\mathsf{P}\tau^{-1}$, then the conclusion of Lemma 2.3 will remain true with m_τ replaced by $\rho = f_{\tau'}$ and ρ can serve as a directing random element for X. The following result yields the desired measurability when V is Polish.

Lemma 2.4. *Let ξ and η be random elements on some Polish probability space Ω, and taking values in the Polish spaces S and T respectively. Assume that $\xi = f(\eta)$ a.s. for some mapping $f: T \to S$. Then f can be chosen to be measurable.*

Proof. Recall that the range $A = \{(\xi, \eta)(\omega);\ \omega \in \Omega\}$ is analytic in $S \times T$. Add to S an extra point ∂. By the section theorem (cf. [6]) there exists a measurable mapping $g: T \to S \cup \{\partial\}$ with $g(\eta) \in S$ a.s., and such that

$$(g(t), t) \in A \cup (\{\partial\} \times T), \qquad t \in T.$$

This means that $(g(\eta), \eta) \in A$ a.s., so $g(\eta) = f(\eta) = \xi$ a.s. □

We need to make some further remarks on the application of the above results. First recall that the separate or joint exchangeability of a process on a continuous parameter space was defined in terms of transformations of the associated increment arrays rather than of the process itself. However, there exists in each case a countable group G of measurable transformations of the process, such that exchangeability is equivalent to invariance in distribution under G.

To see this, let us e.g. consider the case of joint exchangeability for continuous processes X on $\mathbb{R}_+^2$, the other cases being similar. We then define for fixed $h > 0$ the processes

$$Y_{ij}^h(s, t) = X((ih, ih + s) \times (jh, jh + t)), \qquad s, t \in (0, h),\ i, j \in \mathbb{N}.$$

It is easily seen that the joint exchangeability of X carries over to the array $Y^h = (Y_{ij}^h)$. Moreover, there exists some measurable mapping f_h such that $X = f_h(Y^h)$. Writing $T_p Y^h = (Y_{p_i p_j}^h)$ and $T_p^h X = f_h(T_p Y^h)$ for finite permutations p of $\mathbb{N}$, it follows that $T_p^h X \overset{d}{=} X$ for all p.

Conversely, this property implies that $T_p X^h \stackrel{d}{=} X^h$, where X^h denotes the array of increments with respect to the h-grid. Thus X is jointly exchangeable iff it is invariant in distribution under the transformations T_p^h with $h=2^{-n}$, $n\in\mathbb{N}$, and with p a finite permutation of $\mathbb{N}$. These transformations clearly form a countable group.

A second remark concerns the ergodicity of the measures m_t, required in Lemma 2.3. In our applications below, the arrays or processes X_t corresponding to m_t will have representations with constant coefficients, and so will be dissociated, when defined on $\mathbb{N}^2$ or $\mathbb{R}_+^2$. (In case of processes, this means that the associated arrays of increments are dissociated.) The desired ergodicity then follows as in the usual proof of the Hewitt-Savage 0–1 law (cf. [9]). For processes on $[0,1]^2$ or $\mathbb{R}_+\times[0,1]$, the conclusions of the lemma may instead be obtained via the transformations in Lemma 2.8 below.

We turn to the characterization of continuous and exchangeable processes on $\mathbb{R}_+$ or $[0,1]$. Recall that a one-parameter process X is *exchangeable*, if $X_0=0$ and if the increments of X over an arbitrary set of disjoint intervals of equal length form an exchangeable sequence. For continuous processes, it is clearly enough to consider intervals with dyadic endpoints. Say that an $\mathbb{R}^d$-valued process B is a *Brownian motion or bridge*, if the component processes are independent Brownian motions or bridges respectively in $\mathbb{R}$. The following result extends the one-dimensional version in [14]. Here and below, we shall use a self-explanatory matrix notation.

Lemma 2.5. *An $\mathbb{R}^d$-valued process X on $\mathbb{R}_+$ or $[0,1]$ is continuous and exchangeable, iff a.s.*

$$X_t=\alpha t+\sigma B_t, \qquad t\in\mathbb{R}_+ \text{ or } [0,1], \tag{4}$$

for some random vector α in $\mathbb{R}^d$, some random $d\times d$-matrix σ, and some $\mathbb{R}^d$-valued Brownian motion or bridge, respectively, B. Here α and $\sigma\sigma^T$ are a.s. unique and X-measurable, and their joint distribution determines that of X.

The representation (4) can be established in the same way as in the one-dimensional case, i.e., via weak convergence as in [14], or by the martingale argument in [2]. The last statement is an easy exercise in the use of Lemmas 2.3 and 2.4 above, given the fact that, in the two cases,

$$\mathsf{E}\exp(i\textstyle\int f^T dX)=\begin{cases}\mathsf{E}\exp(i\alpha^T\int f-\frac{1}{2}\int|\sigma^T f|^2);\\ \mathsf{E}\exp(i\alpha^T\int f-\frac{1}{2}\int|\sigma^T(f-\bar f)|^2),\end{cases}$$

where f is an arbitrary R^d-valued and measurable function with $|f|\in L_1\cap L_2$. (It is of course enough to consider simple step functions of this type.) Alternatively, we may obtain α and $\sigma\sigma^T$ directly as

$$\alpha=\lim_{t\to\infty} t^{-1}X_t \quad \text{or} \quad \alpha=X_1, \qquad \sigma\sigma^T=[X,X]_1 \quad \text{a.s.,}$$

where $[X,X]$ denotes the $d\times d$-matrix of mixed quadratic variations for the components of X.

Using characteristic functions as in Theorem 5.3 of [14], we may easily deduce the uniqueness of extensions (which incidentally remains true in the presence of jumps):

Corollary 2.6. *Let X be an $\mathbb{R}^d$-valued continuous and exchangeable process on $\mathbb{R}_+$ or $[0, 1]$, and let Y denote the restriction of X to some subinterval $[0, \varepsilon]$ with $\varepsilon > 0$. Then $\mathsf{P}Y^{-1}$ determines $\mathsf{P}X^{-1}$.*

We shall also need the following multi-dimensional version of Schoenberg's theorem (cf. [2, 5]). Say that an $\mathbb{R}^d$-valued random sequence $X = (X_{ij},\ i \leqq d,\ j \in \mathbb{N})$ is *rotatable*, if $XU \overset{d}{=} X$ for every rotation U. For a process X on $\mathbb{R}_+$ or $[0, 1]$ to be rotatable, we require that X be continuous in probability, and that the above property should hold for the increments over an arbitrary set of disjoint intervals of equal length.

Lemma 2.7. *An $\mathbb{R}^d$-valued random sequence $X = (X_{ij},\ i \leqq d,\ j \in \mathbb{N})$ is rotatable iff a.s.*

$$X_{ij} = \sum_{k=1}^{d} \sigma_{ik}\, \xi_{kj}, \qquad i = 1, \ldots, d,\ j \in \mathbb{N}, \tag{5}$$

for some random $d \times d$-matrix $\sigma = (\sigma_{ik})$ and some i.i.d. $N(0, 1)$ random variables ξ_{kj}, $k \leqq d$, $j \in \mathbb{N}$. Similarly, an $\mathbb{R}^d$-valued random process X on $\mathbb{R}_+$ or $[0, 1]$ is rotatable iff

$$X_t = \sigma B_t \quad a.s., \qquad t \in \mathbb{R}_+ \text{ or } [0, 1], \tag{6}$$

for some random matrix σ as above and some d-dimensional Brownian motion B. In both cases, $\sigma\sigma^T$ is a.s. unique and X-measurable, and its distribution determines that of X.

We conclude this section with an elementary discussion of some processes related to Brownian motion. First recall that a *Brownian sheet* is a centered Gaussian process X on $\mathbb{R}_+^2$ with covariance function

$$\mathsf{E}\, X_{st} X_{s't'} = (s \wedge s')(t \wedge t'), \qquad s, s', t, t' \in \mathbb{R}_+.$$

Starting from X, we may construct the further processes

$$Y_{st} = X_{st} - s X_{1t}, \qquad s \in [0, 1],\ t \in \mathbb{R}_+,$$

$$Z_{st} = X_{st} - t X_{s1} - s X_{1t} + s t X_{11} = Y_{st} - t Y_{s1}, \qquad s, t \in [0, 1],$$

with covariance functions

$$\mathsf{E}\, Y_{st} Y_{s't'} = (s \wedge s' - s s')(t \wedge t'), \qquad s, s' \in [0, 1],\ t,\ t' \in \mathbb{R}_+,$$

$$\mathsf{E}\, Z_{st} Z_{s't'} = (s \wedge s' - s s')(t \wedge t' - t t'), \qquad s, s', t, t' \in [0, 1].$$

All these processes will be referred to as *Brownian sails*. (The process Y above is also known as the Kiefer process.)

In the next result, we list some simple relationships which will be needed below. For their proofs, it suffices to compute the covariances.

Lemma 2.8. *Starting from a Brownian motion W and a Brownian sheet X, we may construct a Brownian bridge B and Brownian sails Y and Z through the formulas*

$$B(s)=(1-s)\,W\left(\frac{s}{1-s}\right), \qquad s\in[0,1],$$

$$Y(s,t)=(1-s)\,X\left(\frac{s}{1-s},t\right), \qquad s\in[0,1],\ t\in\mathbb{R}_+,$$

$$Z(s,t)=(1-t)\,Y\left(s,\frac{t}{1-t}\right)=(1-s)(1-t)\,X\left(\frac{s}{1-s},\frac{t}{1-t}\right), \qquad s,t\in[0,1]^2.$$

Conversely, W and X may be obtained from B, Y and Z through

$$W(s)=(1+s)\,B\left(\frac{s}{1+s}\right), \qquad s\in\mathbb{R}_+,$$

$$X(s,t)=(1+s)\,Y\left(\frac{s}{1+s},t\right)=(1+s)(1+t)\,Z\left(\frac{s}{1+s},\frac{t}{1+t}\right), \qquad s,t\in\mathbb{R}_+.$$

We finally state a simple consequence of Lemmas 2.5, 2.7 and 2.8, which will play an important role in Sect. 6.

Corollary 2.9. *Let X be an* $\mathbb{R}^d$*-valued, continuous and exchangeable process on* $[0,1]$ *with* $X_1=0$. *Then the process*

$$Y(t)=(1+t)\,X\left(\frac{t}{1+t}\right), \qquad t\in\mathbb{R}_+,$$

is rotatable.

3. Jointly exchangeable arrays

The purpose of this section is to give a proof, in the spirit of Aldous and Kingman (cf. [1, 2]), of the representation formula (1.2) (equation (2) of Section 1) for jointly exchangeable arrays of random variables.

Theorem 3.1. *An array* $X=(X_{ij},\ i,j\in\mathbb{N})$ *of random variables is jointly exchangeable iff*

$$X_{ij}=f(\alpha,\xi_i,\xi_j,\lambda_{ij}), \qquad i,j\in\mathbb{N}, \tag{1}$$

holds a.s. with $\lambda_{ii}=0$ *and* $\lambda_{ij}=\lambda_{ji}$, *for some measurable function* $f\colon[0,1]^4\to\mathbb{R}$ *and some i.i.d.* $U(0,1)$ *random variables* $\alpha,\xi_1,\xi_2,\ldots$ *and* λ_{ij}, $i<j$.

It is clearly equivalent to write instead of (1)

$$X_{ij}=\begin{cases} f(\alpha,\xi_i,\xi_j,\lambda_{ij}), & i<j,\\ f(\alpha,\xi_i,\xi_j,\lambda_{ji}), & i>j,\\ g(\alpha,\xi_i), & i=j,\end{cases}$$

for some measurable functions $f: [0,1]^4 \to \mathbb{R}$ and $g: [0,1]^2 \to \mathbb{R}$, and some i.i.d. $U(0,1)$ random variables α, ξ_i and λ_{ij} as above.

For the proof, we shall need some simple exercises on conditional probabilities, valid for arbitrary random variables, sequences or arrays ξ, η, ζ, ξ_1, η_1, ... Throughout this section, conditionally independent or i.i.d. will be abbreviated c.i. or c.i.i.d., respectively.

Lemma 3.2. (a) *ξ and η are c.i., given ζ, iff (ξ, ζ) and η are c.i., given ζ.*

(b) *If (ξ_1, ξ_2) and η are c.i., given ζ, then ξ_1 and η are c.i., given (ξ_2, ζ).*

(c) *If $\xi_1, \xi_2, \ldots$ are c.i.i.d. and c.i. of (η_1, η_2), given ζ, and if ζ is η_2-measurable, then $\xi_1, \xi_2, \ldots$ are c.i.i.d. and c.i. of η_1, given η_2.*

(d) *If $(\xi_1, \ldots, \xi_n)$ and ξ_{n+1} are c.i., given η, for each $n \in \mathbb{N}$, then $\xi_1, \xi_2, \ldots$ are c.i., given η.*

Lemma 3.3. *If the pairs (ξ_j, η_j), $j \in \mathbb{N}$, have the same distribution, there exists some transition kernel m, such that*

$$P[\xi_j \in \cdot \mid \eta_j] = m(\eta_j, \cdot) \quad a.s., \qquad j \in \mathbb{N}.$$

Proof of Theorem 3.1. Define $Y_{ij} = (X_{ij}, X_{ji})$, $i, j \in \mathbb{N}$, and note that the joint exchangeability of X carries over to $Y = (Y_{ij})$. By the Daniell-Kolmogorov theorem, we may extend Y to a jointly exchangeable array indexed by $\mathbb{Z}^2$. Write $A = (Y_{ij}, i \vee j \leqq 0)$ and $B = (B_1, B_2, \ldots)$, where $B_i = (Y_{ij}, j = i, 0, -1, -2, \ldots)$. For $n \in \mathbb{Z}_+$, we further define $Y^n = (Y_{ij}, i, j = 1, \ldots, n)$, $B_i^n = (Y_{ij}, j = i, 0, -1, \ldots, -n)$ and $C_i^n = (Y_{1i}, \ldots, Y_{ni})$. We shall show that

(i) For each $n \in \mathbb{Z}_+$, the pairs (C_i^n, B_i), $i > n$, are c.i.i.d. and c.i. of Y^n, given $(A, B_1, \ldots, B_n)$.

It is clearly enough to prove the corrersponding statement for the pairs (C_i^n, B_i^m), with $m \in \mathbb{N}$ arbitrarily fixed. But the latter pairs are exchangeable over $(A, B_1, \ldots, B_n, Y^n)$, and hence c.i.i.d. and c.i. of that array, given the directing random measure μ. By Lemma 3.2(c) it remains to show that μ is measurable with respect to $(A, B_1, \ldots, B_n)$, which is obvious since the extended sequence (C_i^n, B_i^m), $i = \ldots, -m-2, -m-1, 1, 2, \ldots$, is exchangeable with the same directing measure μ. Taking $n = 0$ in (i), we get in particular

(ii) $B_1, B_2, \ldots$ are c.i.i.d., given A.

Using Lemma 3.2(a, b), it is further seen from (i) that

(iii) C_{n+1}^n is c.i. of Y^n, given (A, B),

(iv) C_{n+1}^n is c.i. of B, given $(A, B_1, \ldots, B_{n+1})$.

From (iv) with $n = 1$ it is seen that Y_{12} is c.i. of B, given (A, B_1, B_2), and by the exchangeability of Y, we then get more generally

(v) Y_{ij} is c.i. of B, given (A, B_i, B_j), for all $i, j \in \mathbb{N}$ with $i \neq j$.

On the other hand, it is seen from (iii) and Lemma 3.2(d) that C_2^1 is c.i. of $(C_3^2, C_4^3, \ldots)$, given (A, B), i.e. that Y_{12} is c.i. of $(Y_{ij}, 1 \leqq i < j \neq 2)$, given (A, B). Using the exchangeability of Y and applying Lemma 3.2(d) again, we may conclude that

(vi) the elements Y_{ij}, $1 \leqq i < j$, are c.i., given (A, B).

Statements (ii), (v) and (vi) will be needed below.

We next conclude from the exchangeability of Y that (A, B_i, B_j, Y_{ij}) has the same distribution for all $i, j \in \mathbb{N}$ with $i \neq j$. Hence, by Lemma 3.3, there exists some transition kernel m between suitable spaces, such that

$$\mathsf{P}[Y_{ij}\in\cdot\,|\,A, B_i, B_j]=m(A, B_i, B_j;\cdot) \quad \text{a.s.}, \qquad i, j\in\mathbb{N} \quad \text{with} \quad i\neq j. \tag{2}$$

From the definition of Y_{ij} it is seen that, if H is an arbitrary Borel set in $\mathbb{R}^2$ and $H'=\{(x, y)\in\mathbb{R}^2; (y, x)\in H\}$, then a.s.

$$m(A, B_i, B_j; H)=m(A, B_j, B_i; H'), \qquad i, j\in\mathbb{N} \quad \text{with} \quad i\neq j. \tag{3}$$

But then (2) remains true with m replaced by the kernel

$$\bar{m}(a, b, c; H)=\tfrac{1}{2}m(a, b, c; H)+\tfrac{1}{2}m(a, c, b; H'),$$

and so we may henceforth assume that (3) holds identically.

The next step is to use the 'coding' argument of Aldous [1, 2] together with Lemma 2.1 above, to conclude from (ii) that

$$A=p(\alpha); \qquad B_i=q(\alpha, \xi_i), \quad i\in\mathbb{N}, \quad \text{a.s.}, \tag{4}$$

for some measurable functions p and q, and some i.i.d. $U(0, 1)$ random variables α and $\xi_1, \xi_2, \ldots$, defined on a possibly extended probability space. Since X_{ii} is a component of B_i, we get in particular

$$X_{ii}=g(\alpha, \xi_i) \quad \text{a.s.}, \qquad i\in\mathbb{N}, \tag{5}$$

for some measurable function g: $[0, 1]^2\to\mathbb{R}$. Writing

$$m'(a, x, y;\cdot)=m(p(a), q(a, x), q(a, y);\cdot), \qquad a, x, y\in[0, 1], \tag{6}$$

we may next introduce two measurable functions f_1, f_2: $[0, 1]^4\to\mathbb{R}$, such that whenever λ is $U(0, 1)$

$$\mathsf{P}((f_1, f_2)(a, x, y, \lambda))^{-1}=m'(a, x, y;\cdot), \qquad a, x, y\in[0, 1]. \tag{7}$$

In that case, (3) shows that also

$$\mathsf{P}((f_2, f_1)(a, y, x, \lambda))^{-1}=m'(a, x, y;\cdot), \qquad a, x, y\in[0, 1]. \tag{8}$$

Defining

$$f(a, x, y, z)=\begin{cases} f_1(a, x, y, z), & x<y,\\ f_2(a, y, x, z), & x>y,\\ g(a, x), & x=y,\end{cases}$$

we get

$$(f(a, x, y, z), f(a, y, x, z))=\begin{cases}(f_1, f_2)(a, x, y, z), & x<y,\\ (f_2, f_1)(a, y, x, z), & x>y.\end{cases}$$

If λ is $U(0, 1)$, it follows by (7) and (8) that

$$\mathsf{P}(f(a, x, y, \lambda), f(a, y, x, \lambda))^{-1}=m'(a, x, y;\cdot), \qquad a, x, y\in[0, 1], \ x\neq y. \tag{9}$$

We finally take λ'_{ij}, $i<j$, to be i.i.d. $U(0, 1)$ and independent of $\alpha, \xi_1, \xi_2, \ldots$, let $\lambda'_{ii}\equiv 0$ and $\lambda'_{ji}\equiv\lambda'_{ij}$, and define

$$X'_{ij}=f(\alpha, \xi_i, \xi_j, \lambda'_{ij}), \qquad i, j\in\mathbb{N}. \tag{10}$$

If we can show that $(X'_{ij}) \stackrel{d}{=} (X_{ij})$, it will follow by Lemma 2.1 that the original array $X=(X_{ij})$ has a similar representation.

Now (6), (9) and (10) imply that the pairs $Y'_{ij}=(X'_{ij}, X'_{ji})$ with $1 \leqq i<j$ are conditionally independent, given $\alpha, \xi_1, \xi_2, \ldots$, with conditional distributions $m(A, B_i, B_j; \cdot)$, so the same thing is true under conditioning with respect to (A, B), in agreement with properties (v), (vi) and (2) for the Y_{ij}. Hence Y and $Y'=(Y'_{ij})$ have the same conditional distribution, given (A, B), and we get $Y \stackrel{d}{=} Y'$, as desired. □

4. Separately rotatable arrays

The main purpose of this section is to remove the second moment condition, imposed by Aldous [1], to prove that separately rotatable arrays of random variables have the form (1.3), as conjectured by Dawid [5]. We shall also solve the associated uniqueness and continuity problems.

Theorem 4.1. *An array $X=(X_{ij}, i, j \in \mathbb{N})$ of random variables is separately rotatable, iff a.s.*

$$X_{ij}=\sigma \lambda_{ij}+\sum_{k=1}^{\infty} \alpha_k \xi_{ik} \eta_{jk}, \qquad i, j \in \mathbb{N}, \tag{1}$$

for some random variables $\sigma \geqq 0$ and $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq 0$ with $\sum \alpha_k^2<\infty$ a.s. and some independent set of i.i.d. $N(0,1)$ random variables $\lambda_{ij}, \xi_{ik}, \eta_{jk}$, $i, j, k \in \mathbb{N}$. Here σ and the α_k are a.s. unique and X-measurable, and they are a.s. non-random iff X is dissociated.

Proof. As before, we may extend X to a separately rotatable array indexed by $\mathbb{Z}^2$. Write $A=(X_{ij}, i \vee j \leqq 0)$, and note that $X^+=(X_{ij}, i \wedge j>0)$ remains separately rotatable under conditioning by A. Moreover, it is clear from the proof of Theorem 1.4 in Aldous [1] that X^+ is conditionally dissociated, given A. Finally, we shall prove below that $\mathsf{E}[X_{11}^2 | A]<\infty$ a.s. We may then conclude from Theorem 4.3 in Aldous [1] that X has conditionally the form (1) with constant coefficients, and the unconditional result will follow by Lemma 2.2 above.

To show that $\mathsf{E}[X_{11}^2 | A]<\infty$ a.s., let us first conclude from Lemma 2.7 above that $X_{ij}=\sigma_i \xi_{ij}$ for some random variables $\sigma_i \geqq 0$ and ξ_{ij}, where the latter are i.i.d. $N(0,1)$ for fixed i and independent of σ_i. Since σ_i is clearly A-measurable when $i \leqq 0$, and since $\mathsf{E}\, \xi_{ij}^4<\infty$, it follows that

$$\mathsf{E}[X_{ij}^4 | A]=\sigma_i^4\, \mathsf{E}[\xi_{ij}^4 | A]<\infty \quad \text{a.s.,} \qquad i \leqq 0.$$

The symmetric argument shows that also

$$\mathsf{E}[X_{ij}^4 | A]<\infty \quad \text{a.s.,} \qquad j \leqq 0. \tag{2}$$

Let us now fix $i=1$, and put $\xi_{1j}=\xi_j$ and $X_{1j}=\sigma_1 \xi_j=\eta_j$. By the conditional form of Schwarz' inequality, we get

$$\mathsf{E}[\eta_1^2|A]=\mathsf{E}\left[\frac{\xi_1^2}{\xi_{-1}^2+\ldots+\xi_{-5}^2}(\eta_{-1}^2+\ldots+\eta_{-5}^2)\middle|A\right]$$
$$\leqq\left\{\mathsf{E}\left[\left(\frac{\xi_1^2}{\xi_{-1}^2+\ldots+\xi_{-5}^2}\right)^2\middle|a\right]\mathsf{E}[(\eta_{-1}^2+\ldots+\eta_{-5}^2)^2|A]\right\}^{\frac{1}{2}}.$$

Here the second factor on the right is a.s. finite by (2), while the first one is a.s. finite since

$$\mathsf{E}\left(\frac{\xi_1^2}{\xi_{-1}^2+\ldots+\xi_{-5}^2}\right)^2=\mathsf{E}\,\xi_1^4\cdot\mathsf{E}(\xi_{-1}^2+\ldots+\xi_{-5}^2)^{-2}\lesssim\int_0^\infty r^{-4}e^{-r^2/2}r^4\,dr<\infty,$$

where $x\lesssim y$ means that $x=O(y)$. Thus $\mathsf{E}[\eta_1^2|A]<\infty$ a.s., which completes the proof of the first assertion.

In order to prove that the coefficients in (1) are a.s. unique and X-measurable, it suffices by Lemmas 2.3 and 2.4 above to assume that they are non-random. But in that case it is easily verified that

$$\mathsf{E}\exp(itX_{11})=\exp(-\tfrac{1}{2}\sigma^2t^2)\prod_{j=1}^{\infty}(1+\alpha_j^2t^2)^{-\frac{1}{2}},\qquad t\in\mathbb{R},\tag{3}$$

from which the uniqueness follows by the theory of analytic functions, or directly by differentiation.

Here we have already used the obvious fact that arrays X with constant coefficients are dissociated. Assuming conversely that X is dissociated, it is seen as in Sect. 2 that X must be ergodic. Moreover, the sequence of coefficients is clearly invariant under separate rotations of X, and hence measurable with respect to the invariant σ-field for X. Hence the coefficients are a.s. non-random in this case. □

For every separately rotatable array X as in (1), we shall define an associated *directing random measure* μ on $\mathbb{R}_+$ by

$$\mu=\sigma^2\delta_0+\sum_{j=1}^{\infty}\alpha_j^2\delta_{a_j},\tag{4}$$

where δ_x denotes the measure with a unit mass at x. Recall that $m_n\xrightarrow{w}m$ (m_n tends weakly to m) for bounded measures m_n and m on $\mathbb{R}_+$, iff $m_nf\to mf$ for every bounded continuous function f on $\mathbb{R}_+$. Here mf denotes the integral $\int f\,dm$. The corresponding notion of convergence in distribution for a.s. bounded random measures μ_n and μ on $\mathbb{R}_+$ is denoted by $\mu_n\xrightarrow{wd}\mu$. It is known that this convergence is equivalent to $\mu_nf\xrightarrow{d}\mu f$ for every bounded and continuous function f. Moreover, a sequence (μ_n) is known to be weakly tight, and hence relatively compact with respect to the above notion of convergence, iff $(\mu_n\mathbb{R}_+)$ is tight and moreover

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathsf{P}\{\mu_n(r,\infty)>\varepsilon\}=0,\qquad\varepsilon>0.\tag{5}$$

Analogous results hold for random measures on R and more general spaces. (For a complete discussion, see Chapter 4 in [15].)

For arrays of random variables, convergence in distribution is defined with respect to the usual product topology in $\mathbb{R}^\infty$. Here we shall solve the continuity problem for the representations in (1), by characterizing convergence in distribution of separately rotatable arrays in terms of their directing random measures.

Theorem 4.2. *Let the arrays $X_1, X_2, \ldots$ be separately rotatable and directed by $\mu_1, \mu_2, \ldots$ Then $X_n \xrightarrow{d}$ some X iff $\mu_n \xrightarrow{wd}$ some μ, and in that case X is separately rotatable and directed by some $\mu' \overset{d}{=} \mu$.*

Proof. If X is separately rotatable and directed by μ, then (3) and (4) yield

$$\mathsf{E}\exp(itX_{11}) = \mathsf{E}\exp\left\{-\frac{1}{2}\int \frac{\log(1+x^2t^2)}{x^2}\,\mu(dx)\right\}, \qquad t\in\mathbb{R}, \tag{6}$$

where the inner integrand on the right is defined by continuity to be equal to t^2 at $x=0$. Assume first that the μ_n are non-random with $\mu_n \xrightarrow{w}$ some μ, and note that even μ must be of the form (4). From (6) it is seen that the one-dimensional distributions of X_n converge as $n\to\infty$, with limits given by (6). This shows in particular that (X_n) is tight. If $X_n \xrightarrow{d} X$ along some subsequence, then even X will be separately rotatable and dissociated, so X must be directed by some non-random measure μ'. But then (6) holds for both μ and μ', and it follows as before that $\mu'=\mu$. Thus $X_n \xrightarrow{d} X$ along the original sequence, with X directed by μ. By Lemma 1.1 in [12], the conclusion extends immediately to the case of random directing measures μ_n, such that $\mu_n \xrightarrow{wd}$ some μ.

Assume conversely that $X_n \xrightarrow{d} X$, and suppose we can show that (μ_n) is weakly tight. If $\mu_n \xrightarrow{wd} \mu$ along some subsequence, it follows as before that $X_n \xrightarrow{d}$ some X' along the same subsequence, with X' directed by some $\mu' \overset{d}{=} \mu$. Thus X is directed by some $\mu'' \overset{d}{=} \mu$, so the distribution of μ is unique, and the convergence $\mu_n \xrightarrow{wd} \mu$ holds along the original sequence.

To see that (μ_n) is tight, conclude from the subadditivity of $\log(1+x)$ for $x \geqq 0$ that

$$\begin{aligned}\sigma^2 t^2 + \sum_{j=1}^\infty \log(1+\alpha_j^2 t^2) &\geqq \log(1+\sigma^2 t^2) + \sum_{j=1}^\infty \log(1+\alpha_j^2 t^2)\\ &\geqq \log\left(1+t^2\left(\sigma^2+\sum_{j=1}^\infty \alpha_j^2\right)\right) = \log(1+t^2\mu\mathbb{R}_+).\end{aligned}$$

Using this, we get from (3) for any $r, t>0$

$$\begin{aligned}\mathsf{E}\cos(tX_{11}) &\leqq \mathsf{E}\exp(-\tfrac{1}{2}\log(1+t^2\mu\mathbb{R}_+)) = \mathsf{E}(1+t^2\mu\mathbb{R}_+)^{-\frac{1}{2}}\\ &\leqq \mathsf{P}\{\mu\mathbb{R}_+ \leqq r\} + (1+t^2 r)^{-\frac{1}{2}}\,\mathsf{P}\{\mu\mathbb{R}_+ > r\}\\ &= 1-(1-(1+t^2r)^{-\frac{1}{2}})\,\mathsf{P}\{\mu\mathbb{R}_+ > r\}.\end{aligned}$$

Substituting X_n and μ_n for X and μ, and letting $n\to\infty$, $r\to\infty$ and $t\to 0$ in this order, we obtain

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathsf{P}\{\mu_n \mathbb{R}_+ > r\}=0. \tag{7}$$

Since $\mu_n(r,\infty)>0$ implies that $\mu_n\mathbb{R}_+ > r^2$, (7) yields in turn

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathsf{P}\{\mu_n(r,\infty)>0\}=0. \tag{8}$$

The desired tightness follows from (7) and (8). □

We shall next prove a rather straightforward extension of Theorem 4.1, which will be needed in Sect. 6.

Lemma 4.3. *Let X, Y, Z and T be arrays of random variables X_{ij}, Y_i, Z_j and T, $i,j\in\mathbb{N}$, such that*

$$(UXV,\, YV,\, UZ,\, T)\overset{d}{=}(X,\, Y,\, Z,\, T) \tag{9}$$

for all rotations U and V. Then we may write $T=\rho$ and a.s.

$$X_{ij}=\sigma\lambda_{ij}+\sum_{k=1}^{\infty}\alpha_k\,\xi_{ik}\,\eta_{jk},\qquad Y_i=\sum_{k=1}^{\infty}\beta_k\,\xi_{ik},\qquad Z_j=\sum_{k=1}^{\infty}\gamma_k\,\eta_{jk},\quad i,j\in\mathbb{N}, \tag{10}$$

for some (X, Y, Z, T)-measurable random variables ρ, σ and $\alpha_k, \beta_k, \gamma_k$, $k\in\mathbb{N}$, with $\sum(\alpha_k^2+\beta_k^2+\gamma_k^2)<\infty$ a.s., and some independent set of i.i.d. $N(0,1)$ random variables $\lambda_{ij}, \xi_{ik}, \eta_{jk}$, $i,j,k\in\mathbb{N}$. If we assume that σ, β_1, $\gamma_1\geqq 0$ and $\alpha_2\geqq\alpha_3\geqq\ldots\geqq 0=\alpha_1$, and that $\alpha_k=0$ implies $\beta_k=\gamma_k=0$ for $k\geqq 2$, then the coefficients in (10) will be a.s. unique, apart from rotations of the sequence (β_k,γ_k), $k\in\mathbb{N}$, within index sets where the α_k assume a common value.

Proof. The array (X, Y, Z, T) is separately exchangeable, so by (1.1) it has a representation

$$(X_{ij},\, Y_i,\, Z_j,\, T)=f(\alpha,\, \xi_i,\, \eta_j,\, \vartheta_{ij}),\qquad i,j\in\mathbb{N}, \tag{11}$$

for some function f and some i.i.d. $U(0,1)$ random variables α, ξ_i, η_j and ϑ_{ij}, $i,j\in\mathbb{N}$. The proof in [1] shows that α may be chosen as a 'coding' of A, a stationary extension of (X, Y, Z, T) into the index domain $\{(i,j)\colon i\vee j\leqq 0\}$. Since (9) remains conditionally valid, given A, it suffices by Lemma 2.1 above to establish the representation (10) with non-random coefficients, in the case when α is constant. In that case, (11) reduces to

$$X_{ij}=f_1(\xi_i,\eta_j,\vartheta_{ij}),\qquad Y_i=f_2(\xi_i),\qquad Z_j=f_3(\eta_j),\quad i,j\in\mathbb{N},$$

for some measurable functions $f_1\colon[0,1]^3\to\mathbb{R}$ and $f_2, f_3\colon[0,1]\to\mathbb{R}$.

Since $\mathsf{E}X_{11}^2<\infty$ by Theorem 4.1 above, we may henceforth proceed as in the proof of Theorem 4.3 in Aldous [1]. Thus we may first subtract from X_{ij} a component $\sigma\lambda_{ij}$, such that the λ_{ij} are i.i.d. $N(0,1)$ and independent of the ξ_i and η_j, while the remainder $X_{ij}-\sigma\lambda_{ij}$ is of the form $h(\xi_i,\eta_j)$. As in [1], we may further write

$$h(\xi_i,\eta_j)=\sum_{k=2}^{\infty}\alpha_k\, g_k(\xi_i)\, g_k'(\eta_j),\qquad i,j\in\mathbb{N},$$

for some constants $\alpha_k \geqq 0$ and some orthonormal sequences (g_k) and (g'_k) in $L_2[0, 1]$.

The argument in [1], p. 597, next shows that the random variables $f_2(\xi)$ and $h(\xi, y_1), \ldots, h(\xi, y_n)$ are jointly centered Gaussian for every $n \in \mathbb{N}$ and a.e. $(y_1, \ldots, y_n) \in [0, 1]^n$, whenever ξ is $U(0, 1)$. Again we may change the definition of h on a null-set in $[0, 1]^2$, to make this statement hold everywhere. By the Hilbert space argument in [1], p. 596, we may then conclude that $f_2(\xi)$ and $g_2(\xi), g_3(\xi), \ldots$ are jointly centered Gaussian. Adding another Gaussian function g_1 to the orthonormal system $g_2, g_3, \ldots$, we get an expansion $f_2(\xi) = \sum \beta_k g_k(\xi)$ for suitable constants β_k. Applying the same argument to f_3 and the g'_k, and putting $g_k(\xi_i) = \xi_{ik}$ and $g'_k(\eta_j) = \eta_{jk}$, $i, j, k \in \mathbb{N}$, we finally obtain the representation (10).

To prove the uniqueness assertion, it is enough by Lemmas 2.3 and 2.4 to consider the case of non-random coefficients. A simple computation then shows that, for any $t, u, v \in \mathbb{R}$,

$$\begin{aligned} &\mathsf{E} \exp(it X_{11} + iu Y_1 + iv Z_1) \\ &= \prod_{j=1}^{\infty} (1 + t^2 \alpha_j^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} t^2 \sigma^2 - \frac{1}{2} \sum_{j=1}^{\infty} \frac{u^2 \beta_j^2 + v^2 \gamma_j^2 + i t u v \alpha_j \beta_j \gamma_j}{1 + t^2 \alpha_j^2} \right). \end{aligned}$$

From this expression, we may obtain σ and the α_k as before by putting $u = v = 0$. Next we may divide by (3) to identify the sums

$$\sum_{j=1}^{\infty} \frac{\beta_j^2}{1 + t^2 \alpha_j^2}, \quad \sum_{j=1}^{\infty} \frac{\gamma_j^2}{1 + t^2 \alpha_j^2}, \quad \sum_{j=1}^{\infty} \frac{\alpha_j \beta_j \gamma_j}{1 + t^2 \alpha_j^2}.$$

Here we may differentiate at the origin, to construct all sums of the form

$$\sum_{j=0}^{\infty} \beta_j^2 \alpha_j^{2k}, \quad \sum_{j=0}^{\infty} \gamma_j^2 \alpha_j^{2k}, \quad \sum_{j=0}^{\infty} \beta_j \gamma_j \alpha_j^{2k+1}, \quad k \in \mathbb{Z}_+. \tag{12}$$

If $\alpha_2 > 0$, we may finally divide by α_2^{2k} and let $k \to \infty$ to obtain the sums

$$\sum_{j \in J} \beta_j^2, \quad \sum_{j \in J} \gamma_j^2, \quad \sum_{j \in J} \beta_j \gamma_j, \tag{13}$$

where $J = \{j \in \mathbb{N}: \alpha_j = \alpha_2\}$. Subtracting the corresponding sums from (12) and continuing recursively, we may construct all sums as in (13) with $J = \{j \in \mathbb{N}: \alpha_j = x\}$, $x > 0$, and finally also β_1^2 and γ_1^2. □

5. Jointly Rotatable Arrays

The aim of this section is to characterize the class of jointly rotatable arrays of random variables, and to solve the corresponding uniqueness and continuity problems in the special case of symmetric arrays, for which the explicit representation was given in (1.4).

Theorem 5.1. *An array $X=(X_{ij}, i, j\in\mathbb{N})$ of random variables is jointly rotatable, iff a.s.*

$$X_{ij}=\rho\,\delta_{ij}+\sigma\,\lambda_{ij}+\sigma'\,\lambda_{ji}+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\alpha_{kl}(\xi_{ik}\,\xi_{jl}-\delta_{ij}\,\delta_{kl}),\qquad i,j\in\mathbb{N},\tag{1}$$

for some random variables ρ, σ, σ' and α_{kl}, $k,l\in\mathbb{N}$, with $\alpha_{kl}+\alpha_{lk}=0$ for $k\neq l$ and $\sum\sum\alpha_{kl}^2<\infty$ a.s., and some independent set of i.i.d. $N(0,1)$ random variables λ_{ij} and ξ_{ik}, $i,j,k\in\mathbb{N}$. The random variables ρ, $(\sigma\pm\sigma')^2$ and $\sum\sum\alpha_{kl}^2$ are a.s. unique, as are the α_{kk} apart from order. Moreover, the coefficients in (1) can be chosen to be X-measurable, and they may further be taken to be non-random iff X is dissociated.

Note that the double sum in (1) converges in probability and that the limit is a.s. independent of the order of summation. To see this, reduce by conditioning to the case of constant coefficients. In this case the series converges in L_2, since the products $\xi_{ik}\,\xi_{jl}$ are orthonormal for k, $l\in\mathbb{N}$ when $i\neq j$, and for $k<l$ when $i=j$. Furthermore, the variables ξ_{ik}^2-1 are i.i.d. with zero mean and finite variance. Note also that the double sum reduces to $\sum\alpha_{kk}(\xi_{ik}^2-1)$ when $i=j$ and to $\sum\sum\alpha_{kl}\,\xi_{ik}\,\xi_{jl}$ when $i\neq j$.

When X is symmetric, we may write $X_{ij}=\frac{1}{2}(X_{ij}+X_{ji})$ to see that (1) holds with $\sigma=\sigma'$ and with $\alpha_{kl}=0$ for $k\neq l$. Thus (1.4) holds in this case with $\alpha_k=\alpha_{kk}$.

Proof. To prove that arrays as in (1) are jointly rotatable, we may clearly assume that the coefficients are non-random. By independence, we may then treat the arrays δ_{ij} and $\lambda_{ij}\pm\lambda_{ji}$ and the double sum separately. For δ_{ij} the result is well-known from linear algebra, and for the double sum it follows easily from Lemma 2.7 when the summation is finite, and then in general by approximation in L_2. In case of $\lambda_{ij}\pm\lambda_{ji}$, notice that the arrays $2^{\frac{1}{2}}(\xi_i\,\xi_j-\delta_{ij})$ and $\xi_i\,\eta_j-\xi_j\,\eta_i$ have mean zero and the same covariances, when the ξ_i and η_i are i.i.d. $N(0,1)$. By the multivariate central limit theorem, it follows that $\lambda_{ij}\pm\lambda_{ji}$ can be approximated in distribution by jointly rotatable arrays of the form

$$X_{ij}=\left(\frac{2}{n}\right)^{\frac{1}{2}}\sum_{k=1}^{n}(\xi_{ik}\,\xi_{jk}-\delta_{ij}),\qquad i,j\in\mathbb{N},$$

and

$$X_{ij}=n^{-\frac{1}{2}}\sum_{k=1}^{n}(\xi_{ik}\,\eta_{jk}-\xi_{jk}\,\eta_{ik}),\qquad i,j\in\mathbb{N},$$

respectively, where the ξ_{ik} and η_{ik} are i.i.d. $N(0,1)$. This shows that the arrays $\lambda_{ij}\pm\lambda_{ji}$ are jointly rotatable, and hence completes the proof of the sufficiency part.

Our next aim is to establish the representation (1) for an arbitrary jointly rotatable array X. Since rotatability is stronger than exchangeability, we get by Theorem 3.1 a representation of the form

$$X_{ij}=f(\alpha,\xi_i,\xi_j,\eta_{ij}),\qquad i,j\in\mathbb{N},\tag{2}$$

with $\eta_{ij}\equiv\eta_{ji}$ and $\eta_{ii}\equiv 0$, for some measurable function $f\colon[0,1]^4\to\mathbb{R}$ and some i.i.d. $U(0,1)$ random variables α, ξ_1, ξ_2, ... and η_{ij}, $i<j$. Rotating by U, we

get an array UXU^T with the same distribution, and hence with a representation

$$(UXU^T)_{ij}=f(\alpha', \xi'_i, \xi'_j, \eta'_{ij}), \qquad i,j\in\mathbb{N}, \tag{3}$$

for some α', (ξ'_i) and (η'_{ij}) as above. Here we may assume that $\alpha'=\alpha$, and that $\xi'_j=\xi_j$ for those indices j which are not affected by U. Indeed, we get these relations automatically, if we use the coding construction of Sect. 3, based on a stationary extension X' of X, and on the corresponding extension $UX'U^T$ of UXU^T.

Under these conditions, X remains jointly rotatable, conditionally on α. By Lemma 2.2, it is enough to prove that almost every conditional distribution agrees with the distribution of (1), for some non-random choice of coefficients. Now (2) shows that X is conditionally of the form

$$X_{ij}=f(\xi_i, \xi_j, \eta_{ij}), \qquad i,j\in\mathbb{N}, \tag{4}$$

for some measurable function $f: [0,1]^3\to\mathbb{R}$, with the same ξ_i and η_{ij} as before. Again, the representation of a rotated array UXU^T may be assumed to use the same variables ξ_j, for indices which are not affected by U. To simplify the writing, we shall henceforth consider a fixed conditional distribution satisfying these assumptions, and suppress the conditioning from our language and notation.

Next we note that the restriction of X to the index set $I=(2\mathbb{N})\times(2\mathbb{N}-1)$ is separately rotatable and dissociated. Hence Theorem 4.1 shows that $\mathsf{E}X_{ij}^2<\infty$ for $i\neq j$, so the arguments in Sect. 4 of Aldous [1] apply, and we get for $(i,j)\in I$ a decomposition

$$X_{ij}=g(\xi_i, \xi_j, \eta_{ij})+h(\xi_i, \xi_j) \quad \text{a.s.}, \tag{5}$$

where the variables $g(\xi_i, \xi_j, \eta_{ij})$, $(i,j)\in I$, are i.i.d. centered Gaussian, while

$$h(x, y)=\sum_{k=1}^{\infty}\alpha_k\, g_k(x)\, g'_k(y) \qquad \text{in } L_2([0,1]^2) \tag{6}$$

for some constants $\alpha_1, \alpha_2, \ldots$ with $\sum\alpha_k^2<\infty$, and some orthonormal sequences $g_1, g_2, \ldots$ and $g'_1, g'_2, \ldots$ in $L_2[0,1]$. Moreover,

$$h(\xi_i, \xi_j)=\mathsf{E}[X_{ij}\,|\,\xi_i, \xi_j] \quad \text{a.s.} \tag{7}$$

for all $(i,j)\in I$, and hence by symmetry whenever $i\neq j$. Comparing (4) and (5), it is clear that we can choose

$$f(x, y, z)=g(x, y, z)+h(x, y), \qquad x, y, z\in[0,1] \quad \text{with} \quad z\neq 0, \tag{8}$$

so even (5) extends to arbitrary $i\neq j$.

To analyze g, we note that the array $Y=aX+bX^T=(aX_{ij}+bX_{ji})$ is jointly rotatable for fixed $a, b\in R$, since for any rotation U,

$$UYU^T=U(aX+bX^T)U^T=aUXU^T+b(UXU^T)^T\stackrel{d}{=}aX+bX^T=Y.$$

Moreover, even Y has a representation (4) in terms of the same random variables ξ_i and η_{ij}, since in fact

$$Y_{ij}=aX_{ij}+bX_{ji}=af(\xi_i,\xi_j,\eta_{ij})+bf(\xi_j,\xi_i,\eta_{ij}),\qquad i,j\in\mathbb{N}.$$

Thus the above arguments apply to Y as well, and show that the variables

$$\begin{aligned}a\,g(\xi_i,\xi_j,\eta_{ij})&+b\,g(\xi_j,\xi_i,\eta_{ij})\\ &=a(X_{ij}-\mathsf{E}[X_{ij}|\xi_i,\xi_j])+b(X_{ji}-\mathsf{E}[X_{ji}|\xi_i,\xi_j])\\ &=Y_{ij}-\mathsf{E}[Y_{ij}|\xi_i,\xi_j]\end{aligned}$$

are i.i.d. centered Gaussian for $(i,j)\in I$. By Corollary 3.13 of Aldous [1], they must then be independent of $\xi_1,\xi_2,\ldots$ Since a and b were arbitrary, it follows that each of the pairs

$$(g(\xi_i,\xi_j,\eta_{ij}),\,g(\xi_j,\xi_i,\eta_{ij})),\qquad i<j, \tag{9}$$

is bivariate centered Gaussian and independent of $\xi_1,\xi_2,\ldots$ But then it must also be independent of the other pairs in (9), which means that all these pairs are i.i.d. centered Gaussian and independent of $\xi_1,\xi_2,\ldots$ We now put

$$s^2=\mathsf{E}(g(\xi_i,\xi_j,\eta_{ij}))^2,\qquad r s^2=\mathsf{E}\,g(\xi_i,\xi_j,\eta_{ij})\,g(\xi_j,\xi_i,\eta_{ij}),$$

and define

$$\sigma,\sigma'=\frac{s}{2}((1+r)^{\frac{1}{2}}\pm(1-r)^{\frac{1}{2}}).$$

Letting λ_{ij}, $i\neq j$, be i.i.d. $N(0,1)$ and independent of the ξ_i, it is easy to check that the array $(\sigma\lambda_{ij}+\sigma'\lambda_{ji},\ i\neq j)$ has the same distribution as $(g(\xi_i,\xi_j,\eta_{ij}),\ i\neq j)$. By Lemma 2.1, we may then redefine the λ_{ij} such that

$$g(\xi_i,\xi_j,\eta_{ij})=\sigma\lambda_{ij}+\sigma'\lambda_{ji}\quad\text{a.s.},\qquad i\neq j. \tag{10}$$

We shall next examine the functions g_k and g'_k occurring in (6). Let us then fix a rotation $U=(u_{ik})$, write $X'=(X'_{ij})=UXU^T$, and note that X' has a representation (4) with ξ_i replaced by ξ'_i. Let us further denote the shell-σ-field (cf. [1, 2]) of X by $\mathscr{S}$, and note that $\mathscr{S}$ is also the shell-σ-field of X'. Fix indices $i<j$ such that U only affects components number $1,\ldots,j-1$. Combining (7) with Lemma 3.7 of Aldous [1], we get

$$\begin{aligned}h(\xi'_i,\xi'_j)&=\mathsf{E}[X'_{ij}|\xi'_i,\xi'_j]=\mathsf{E}[X'_{ij}|\mathscr{S}]=\sum_k u_{ik}\,\mathsf{E}[X_{kj}|\mathscr{S}]\\ &=\sum_k u_{ik}\,\mathsf{E}[X_{kj}|\xi_k,\xi_j]=\sum_k u_{ik}\,h(\xi_k,\xi_j).\end{aligned}$$

Assuming that $\xi'_j=\xi_j$ and using (6), we hence obtain

$$\sum_n \alpha_n\,g'_n(\xi_j)(g_n(\xi'_i)-\sum_k u_{ik}\,g_n(\xi_k))=0\quad\text{a.s.}$$

By Fubini's theorem, the same relation holds a.s. for almost every realization $(x_1,\ldots,x_{j-1},x'_i)$ of $(\xi_1,\ldots,\xi_{j-1},\xi'_i)$. Since $g'_1,g'_2,\ldots$ are orthogonal, it follows that

$$g_n(\xi_i') = \sum_k u_{ik}\, g_n(\xi_k) \quad \text{a.s.,} \qquad i, n \in \mathbb{N}. \tag{11}$$

Interchanging the roles of rows and columns, we get in the same way

$$g_n'(\xi_i') = \sum_k u_{ik}\, g_n'(\xi_k) \quad \text{a.s.,} \qquad i, n \in \mathbb{N}. \tag{12}$$

The next step is to replace the sequences (g_k) and (g_k') by a single orthonormal sequence. Let us then introduce the Hilbert space H in $L_2[0, 1]$ spanned by $g_1, g_2, \ldots$ and $g_1', g_2', \ldots$, and note that

$$h^+ u(x) = \int (h(x, y) + h(y, x))\, u(y)\, dy, \qquad u \in H,$$

defines a compact and self-adjoint operator on H. Thus h^+ has a complete orthonormal sequence of eigenfunctions $h_1, h_2, \ldots \in H$. In particular, we get an expansion

$$h(x, y) = \sum_i \sum_j \alpha_{ij}\, h_i(x)\, h_j(y) \qquad \text{in } L_2([0, 1]^2), \tag{13}$$

where $\sum\sum \alpha_{ij}^2 = \sum \alpha_j^2 < \infty$. Moreover,

$$h(x, y) + h(y, x) = \sum_i \sum_j (\alpha_{ij} + \alpha_{ji})\, h_i(x)\, h_j(y),$$

so $\alpha_{ij} + \alpha_{ji} = 0$ for $i \neq j$. From (11) and (12) it is further seen that

$$h_n(\xi_i') = \sum_k u_{ik}\, h_n(\xi_k) \quad \text{a.s.,} \qquad i, n \in \mathbb{N}. \tag{14}$$

It follows in particular that the array $(h_n(\xi_k))$ is rotatable in k, and since the ξ_k are further independent, we may conclude from Lemma 2.7 that $h_1(\xi), h_2(\xi), \ldots$ are i.i.d. $N(0, 1)$. This proves the representation (1) for $i \neq j$, with $\xi_{ik} = h_k(\xi_i)$.

To extend (1) to the diagonal, we put

$$Y_{ij} = \sum_k \sum_l \alpha_{kl} (\xi_{ik}\, \xi_{jl} - \delta_{ij}\, \delta_{kl}), \qquad Z_{ij} = X_{ij} - Y_{ij}, \quad i, j \in \mathbb{N},$$

and conclude from (14) that (Y, Z) is jointly rotatable. To determine the distribution of Z_{11}, we put $Z' = UZU^T$, where the rotation $U = (u_{ik})$ is such that $u_{11} = u_{21} = u_{22} = -u_{12} = 2^{-\frac{1}{2}}$, and compute

$$Z_{12}' = \tfrac{1}{2}(Z_{11} - Z_{22} + Z_{12} - Z_{21}) \stackrel{d}{=} Z_{12}. \tag{15}$$

Here the variables Z_{11}, Z_{22} and $Z_{12} Z_{21}$ are independent, while Z_{12} and $Z_{12} - Z_{21}$ are Gaussian, so it follows from Cramér's factorization theorem (cf. [9]) that Z_{11} is $N(\rho, s^2)$ for some ρ and s. Computing the variances in (15) yields by (10)

$$\tfrac{1}{4}(2s^2 + 2(\sigma - \sigma')^2) = \sigma^2 + \sigma'^2,$$

so $s^2=(\sigma+\sigma')^2$. Thus we may extend (10) by writing

$$Z_{ij}=\rho\,\delta_{ij}+\sigma\,\lambda_{ij}+\sigma'\,\lambda_{ji},\qquad i,j\in\mathbb{N},$$

where the λ_{ij} are i.i.d. $N(0,1)$.

It remains to prove that Y and Z are independent. To see this, choose for each $n\in\mathbb{N}$ a rotation $U=(u_{ik})$ with $u_{11}=\ldots=u_{n1}=n^{-\frac{1}{2}}$. Writing $Z'=UZU^T$, so that $Z=U^TZ'U$, we get

$$\begin{aligned}Z_{11}&=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}Z'_{ij}=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\delta_{ij})\,Z'_{ij}+\frac{1}{n}\sum_{i=1}^{n}(Z'_{ii}-\rho)+\rho\\&=S_n+T_n+\rho.\end{aligned}$$

Here $ET_n^2=n^{-1}(\sigma+\sigma')^2\to 0$, so $S_n+\rho\xrightarrow{P}Z_{11}$. Writing $\Xi=(\xi_{ik})$, it is further seen from (14) that S_n is independent of $\Xi'=U\Xi$ and hence of $\Xi=U^T\Xi'$. Hence Z_{11} is independent of Ξ, and the same thing is true for each Z_{ii}. Since the sequences $(Z_{ii},\xi_{i1},\xi_{i2},\ldots)$, $i\in\mathbb{N}$, are mutually independent, it follows that the whole diagonal $(Z_{11},Z_{22},\ldots)$ is independent of Ξ, and hence of Y. The independence of Y and Z now follows, since the non-diagonal part of Z was shown before to be independent of $(\xi_1,\xi_2,\ldots)$, and hence of the diagonal plus Y. This completes the proof of the representation (1).

If the coefficients in (1) are non-random, then X is clearly dissociated. Conversely, a dissociated array X is not affected by conditioning on A, and as the above proof shows, a representation exists in the conditional situation where the coefficients are non-random. In the general case, it is seen from Lemma 2.3 that the coefficients can be chosen to be X-measurable. It remains to prove that ρ, $(\sigma\pm\sigma')^2$, $\sum\sum\alpha_{kl}^2$ and the α_{kk} are a.s. unique, and by Lemmas 2.3 and 2.4 it is then enough to consider the dissociated case. The uniqueness of ρ, $(\sigma+\sigma')^2$ and the α_{kk} is then obtained from the formula

$$\mathsf{E}\exp(tX_{11})=\exp(\rho\,t+\tfrac{1}{2}(\sigma+\sigma')^2\,t^2)\prod_{k=1}^{\infty}(1-2t\,\alpha_{kk})^{-\frac{1}{2}}\exp(-t\,\alpha_{kk}),\tag{16}$$

valid for small t, while the uniqueness of $(\sigma-\sigma')^2$ and $\sum\sum\alpha_{kl}^2$ follows by applying Theorem 4.1 to the restriction of $X-X^T$ or X respectively to $I=(2\mathbb{N})\times(2\mathbb{N}-1)$. □

In the symmetric case, i.e. when

$$X_{ij}=\rho\,\delta_{ij}+\sigma(\lambda_{ij}+\lambda_{ji})+\sum_{k=1}^{\infty}\alpha_k(\xi_{ik}\,\xi_{jk}-\delta_{ij}),\qquad i,j\in\mathbb{N},\tag{17}$$

we may associate with X the *directing random elements* ρ and μ, where

$$\mu=2\,\sigma^2\,\delta_0+\sum_{k=1}^{\infty}\alpha_k^2\,\delta_{\alpha_k}.\tag{18}$$

Note that μ is a random measure on $\mathbb{R}$ in this case, since the α_k may be both positive and negative. In the space $\mathcal{M}(\mathbb{R})$ of bounded measures on $\mathbb{R}$, we define weak convergence $\xrightarrow{w}$ as before, and write $\xrightarrow{wd}$ for the corresponding notion of convergence in distribution. The same notation will be used for convergence with respect to the associated product topology on $\mathbb{R} \times \mathcal{M}(\mathbb{R})$.

Theorem 5.2. *Let $X_1, X_2, \ldots$ be symmetric and jointly rotatable arrays directed by (ρ_n, μ_n), $n \in \mathbb{N}$. Then $X_n \xrightarrow{d}$ some X iff $(\rho_n, \mu_n) \xrightarrow{wd}$ some (ρ, μ), and in that case X is symmetric, jointly rotatable, and directed by some $(\rho', \mu') \stackrel{d}{=} (\rho, \mu)$.*

Proof. In the symmetric dissociated case, formula (16) becomes

$$\mathsf{E}\exp(tX_{11}) = \exp\left\{\rho t + 2\sigma^2 t^2 - \tfrac{1}{2}\sum_{j=1}^{\infty} [\log(1-2t\alpha_j) + 2t\alpha_j]\right\}$$

$$= \exp\left\{\rho t - \frac{1}{2}\int \frac{\log(1-2tx)+2tx}{x^2}\,\mu(dx)\right\}, \tag{19}$$

where it is assumed that $|t| < \frac{1}{2}(\max|\alpha_j|)^{-1}$. Here the integrand on the right is defined by continuity to be $-2t^2$ at $x=0$. Note also that the restriction of X to $I = (2\mathbb{N}) \times (2\mathbb{N}-1)$ is separately rotatable and directed by the measure

$$\mu' = 2\sigma^2 \delta_0 + \sum_{j=1}^{\infty} \alpha_j^2 \delta_{|\alpha_j|}. \tag{20}$$

Let us first assume that the directing pairs (ρ_n, μ_n) are non-random, and that $\rho_n \to \rho$ while $\mu_n \xrightarrow{w} \mu$, where μ must again be of the form (18). Then the measures μ'_n in (20) will converge along with μ_n, so Theorem 4.2 shows that the non-diagonal elements of X_n form tight sequences. As for the diagonal elements, we get even convergence in distribution, with the limits satisfying (19). This is because $\max|\alpha_j|$ stays bounded by the weak convergence of μ_n. We may thus conclude that (X_n) is tight, with every limiting array X satisfying (19). Since even the limits are dissociated, symmetric and jointly rotatable, ρ and μ must be the directing elements of X, so the limiting law is unique, and we have in fact convergence $X_n \xrightarrow{d} X$. As before, the result in this direction extends immediately to the non-dissociated case.

For the result in the opposite direction, it is enough as before to show that $X_n \xrightarrow{d} X$ implies tightness of the sequences (ρ_n) and (μ_n). Considering the restrictions to the index set I and using Theorem 4.2, it is seen that the associated sequence (μ'_n) is tight, which clearly implies tightness of (μ_n). From the first part of the proof we may then conclude that the reduced arrays $X_n - \rho_n(\delta_{ij})$ form a tight sequence, and since X_n converges by assumption, the desired tightness of (ρ_n) follows by subtraction. □

6. Separately Exchangeable Processes

In this section, we shall prove the representation (1.6) of separately exchangeable and continuous processes X on $\mathbb{R}_+^2$, $\mathbb{R}_+ \times [0, 1]$ or $[0, 1]^2$, and we shall further solve the corresponding uniqueness and continuity problems. Recall from Sect. 2 the definition and elementary properties of Brownian sails. Say that a process X is *dissociated*, if its increment arrays have this property.

Theorem 6.1. *A process X on $I=\mathbb{R}_+^2$, $\mathbb{R}_+ \times [0, 1]$ or $[0, 1]^2$ is continuous and separately exchangeable, iff a.s.*

$$X_{st}=\rho\, s\, t+\sigma A_{st}+\sum_{j=1}^{\infty}(\alpha_j B_j(s)\, C_j(t)+\beta_j B_j(s)\, t+\gamma_j s\, C_j(t)), \qquad (s, t)\in I, \qquad (1)$$

for some random variables ρ, σ and $\alpha_j, \beta_j, \gamma_j$, $j\in\mathbb{N}$, with $\sum(\alpha_j^2+\beta_j^2+\gamma_j^2)<\infty$ a.s., some independent Brownian sail A, and some independent sequences (B_j) and (C_j) of i.i.d. Brownian motions or bridges. The coefficients in (1) *may be chosen to be X-measurable, and if $I=\mathbb{R}_+^2$, they may further be taken to be non-random iff X is dissociated.*

First of all we need to show that the right-hand side of (1) defines a continuous process:

Lemma 6.2. *The series in* (1) *converges a.s. uniformly on bounded sets.*

Proof. By Fubini's theorem, we may take the α_j, β_j and γ_j to be non-random. By Lemma 2.8 we may further assume that $I=\mathbb{R}_+^2$, so that the B_j and C_j are Brownian motions. By an obvious scaling argument, it is enough to prove a.s. uniform convergence within the unit square.

For this purpose, put $\xi_j=B_j(1)$ and $\eta_j=C_j(1)$, $j\in\mathbb{N}$, and decompose the sum S in (1) into three parts $T+U+V$, corresponding to the decomposition of each term. Let us first assume that these sums are finite. By Doob's inequality, we obtain for U

$$\mathsf{E}\sup_{s,t\leq 1}|U(s, t)|^2\lesssim \mathsf{E}\,|\sum_j \beta_j\, \xi_j|^2=\sum_j \beta_j^2$$

and similarly for V. (Here $x\lesssim y$ means $x=O(y)$, as before.) In case of T, we may use Doob's and Schwarz' inequalities, as well as the scaling and symmetry properties of Brownian motion, to obtain for fixed $s\leq 1$

$$\begin{aligned}\mathsf{E}\sup_{t\leq 1}|T(s, t)|^4 &\lesssim s^2\, \mathsf{E}\left|\sum_j \alpha_j\, \xi_j\, \eta_j\right|^4=s^2\, \mathsf{E}\left|\sum_j \alpha_j^2\, \xi_j^2\, \eta_j^2\right|^2\\ &\leq s^2\, \mathsf{E}\sum_j \alpha_j^2\, \xi_j^4 \sum_k \alpha_k^2\, \eta_k^4\\ &=s^2\left(\mathsf{E}\sum_j \alpha_j^2\, \xi_j^4\right)^2\lesssim s^2\left(\sum_j \alpha_j^2\right)^2.\end{aligned}$$

Thus

$$\mathsf{E}\sup_{k\leq 2^n}\, \sup_{t\leq 1}|T(k\,2^{-n}, t)-T((k-1)\,2^{-n}, t)|^4\lesssim 2^{-n}\left(\sum_j \alpha_j^2\right)^2,$$

so by Minkowski's inequality and a.s. continuity,

$$\left\|\sup_{s,t\leq 1}|T(s,t)|\right\|_4 \leq \left\|\sum_{n\in N}\sup_{k\leq 2^n}\sup_{t\leq 1}|T(k2^{-n},t)-T((k-1)2^{-n},t)|\right\|_4$$

$$\lesssim\left(\sum_j \alpha_j^2\right)^{\frac{1}{2}}\sum_{n\in\mathbb{N}}2^{-n/4}\lesssim\left(\sum_j \alpha_j^2\right)^{\frac{1}{2}}.$$

Summarizing these results, we get

$$\mathsf{E}\sup_{s,t\leq 1}|S(s,t)|^2\lesssim\sum_j(\alpha_j^2+\beta_j^2+\gamma_j^2). \tag{2}$$

Returning to the case of infinite sums, let S_n denote the n-th partial sum, and conclude from (2) that, for $m\leq n$,

$$\mathsf{E}\sup_{s,t\leq 1}|S_m(s,t)-S_n(s,t)|^2\lesssim\sum_{j=m+1}^{n}(\alpha_j^2+\beta_j^2+\gamma_j^2).$$

By a standard argument, there must then exist some continuous process S on $[0,1]^2$, such that

$$\sup_{s,t\leq 1}|S_n(s,t)-S(s,t)|\to 0 \quad \text{a.s.}, \tag{3}$$

as $n\to\infty$ along some suitable subsequence. Hence (2) extends to infinite sums by Fatou's lemma, and we get in particular

$$\lim_{n\to\infty}\mathsf{E}\sup_{s,t\leq 1}|S_n(s,t)-S(s,t)|^2=0.$$

Since the terms of S are independent, we may finally invoke a result in Itô and Nisio [13], to conclude that (3) remains true along the original sequence. □

To prove the necessity of (1), we shall need two further lemmas, both exhibiting exchangeability preserving transformations.

Lemma 6.3. *Let the real valued process X on $[0,1]^2$ be separately exchangeable. Then so is the $\mathbb{R}^4$-valued process*

$$Y(s,t)=(X(s,t),\ sX(1,t),\ tX(s,1),\ stX(1,1)),\qquad s,t\in[0,1].$$

Proof. By the definition of exchangeability for continuous parameter processes, it is enough to prove the corresponding statement in the discrete case. Let us thus assume that $X=(X_{ij},\ i,j\in\{1,\dots,n\})$ is a separately exchangeable array of random variables, and write

$$Y_{ij}=(X_{ij},X_{.j},X_{i.},X_{..}),\qquad i,j\in\{1,\dots,n\},$$

where the dots indicate summation over the corresponding indices. It is then required to show that $(Y_{p_iq_j})\overset{d}{=}(Y_{ij})$ for arbitrary permutations (p_i) and (q_j) of $(1,\dots,n)$. But this follows immediately from the fact that

$$Y_{p_i q_j} = (X_{p_i q_j}, X_{\cdot q_j}, X_{p_i \cdot}, X_{\cdot\cdot})$$
$$= (\tilde{X}_{ij}, \tilde{X}_{\cdot j}, \tilde{X}_{i \cdot}, \tilde{X}_{\cdot\cdot}), \quad i, j \in \{1, \ldots, n\},$$

where $\tilde{X} = (X_{p_i q_j})$. □

Lemma 6.4. *Fix $I = \mathbb{R}_+$ or $[0, 1]$, and let X be a continuous and separately exchangeable process on $I \times \mathbb{R}_+$. Then the process*

$$Y(s, t) = (1-t)\, X\left(s, \frac{t}{1-t}\right), \quad s \in I, \; t \in [0, 1),$$

has a continuous and separately exchangeable extension to $I \times [0, 1]$.

Proof. From Lemmas 2.5 and 2.7 it is seen that Y is separately exchangeable on $I \times [0, 1)$, and this property is clearly shared by a possible continuous extension of Y to $I \times [0, 1]$. It is thus enough to show that such an extension exists. By scaling, we may then assume that $I = [0, 1]$, in which case it is equivalent to show that Y is a.s. uniformly continuous.

To see this, let W_h and W'_h, $h > 0$, denote the moduli of continuity of Y on $[0, 1] \times [0, 1)$ and $[0, 1] \times [0, \frac{1}{2}]$, respectively, and let W''_h, $h > 0$, be the corresponding modulus for the restriction of $Y(s, t) - Y(s, \frac{1}{2})$ to $[0, 1] \times [\frac{1}{2}, 1)$. Then $W \leqq W' + W''$, and from the exchangeability of Y it is further seen that $W'' \stackrel{d}{=} W'$. Since Y is a.s. uniformly continuous on $[0, 1] \times [0, \frac{1}{2}]$, it follows that $W_h \to 0$ a.s. as $h \to 0$, which means that Y is a.s. uniformly continuous even on $[0, 1] \times [0, 1)$. □

Proof of Theorem 6.1. Let X be given by (1). Then X is continuous by Lemma 6.2. To see that X is also separately exchangeable, we may clearly take the coefficients in (1) to be non-random, and by independence it is then enough to consider separately the individual terms of the form $\rho s t$, σA_{st} or $\alpha B_s C_t + \beta B_s t + \gamma s C_t$. For the first and last of these the result is obvious, and for the second one it follows easily by Lemmas 2.5 and 2.8.

Suppose conversely that the process X on $I = \mathbb{R}_+^2$, $\mathbb{R}_+ \times [0, 1]$ or $[0, 1]^2$ is continuous and separately exchangeable. In order to prove the representation in (1), it suffices by Lemmas 2.8 and 6.4 to take $I = [0, 1]^2$. In this case we may define

$$\begin{aligned} &X'_{s1} = X_{s1} - s X_{11}, \qquad X'_{1t} = X_{1t} - t X_{11}, \\ &X''_{st} = X_{st} - s X_{1t} - t X_{s1} + s t X_{11}, \qquad s, t \in [0, 1], \end{aligned} \tag{4}$$

and conclude from Lemma 6.3 that the $\mathbb{R}^4$-valued process

$$(X''_{st}, s X'_{1t}, t X'_{s1}, s t X_{11}), \quad s, t \in [0, 1],$$

is separately exchangeable as well. Equivalently, the process (X''_{st}, X'_{1t}) is conditionally exchangeable in t, given (X'_{s1}, X_{11}), while (X''_{st}, X'_{s1}) is conditionally exchangeable in s, given (X'_{1t}, X_{11}). Note also that $X''_{s1} = X''_{1t} = X'_{11} = 0$.

Let us next define the processes

$$Y'_{s.} = (1+s)\, X'\left(\frac{s}{1+s}, 1\right), \qquad Y'_{.t} = (1+t)\, X'\left(1, \frac{t}{1+t}\right),$$
$$Y''_{st} = (1+s)(1+t)\, X''\left(\frac{s}{1+s}, \frac{t}{1+t}\right), \qquad s, t \in \mathbb{R}_+, \tag{5}$$

and conclude from Corollary 2.9 that the pair $(Y''_{st}, Y'_{.t})$ is conditionally rotatable in t, given $(Y'_{s.}, X_{11})$, while $(Y''_{st}, Y'_{s.})$ is conditionally rotatable in s, given $(Y'_{.t}, X_{11})$. In terms of the increments of the process

$$(Y''_{st}, s\,Y'_{.t}, t\,Y'_{s.}, s\,t\,X_{11}), \qquad s, t \in \mathbb{R}_+, \tag{6}$$

this is precisely the hypothesis of Lemma 4.3, so on every fixed square lattice, we get a representation of the form

$$X_{11} = \rho, \qquad Y''_{st} = \sigma A'_{st} + \sum_{j=1}^{\infty} \alpha_j B'_j(s)\, C'_j(t),$$
$$Y'_{s1} = \sum_{j=1}^{\infty} \beta_j B'_j(s), \qquad Y'_{.t} = \sum_{j=1}^{\infty} \gamma_j C'_j(t), \qquad s, t \in \mathbb{R}_+, \tag{7}$$

for some X-measurable random variables ρ, σ and α_j, β_j, γ_j as in (1), some independent Brownian sheet A', and some independent set of i.i.d. Brownian motions B'_j and C'_j, $j \in \mathbb{N}$.

Halving the grid size yields a similar representation (7), and by the uniqueness part of Lemma 4.3, we may take the coefficients to be the same. Continuing recursively, it follows that the finite-dimensional distributions of (6) for dyadic s and t are the same as for the processes in (7). This result extends by continuity to arbitrary $s, t \in \mathbb{R}_+$. By Lemma 2.1, we may then assume that (7) holds a.s. for all s and t.

From (7) it is seen that the process

$$Y_{st} = Y''_{st} + s\,Y'_{.t} + t\,Y'_{s.} + s\,t\,X_{11}, \qquad s, t \in \mathbb{R}_+,$$

can be represented by the right-hand side of (1), but with A, (B_j) and (C_j) replaced by A', (B'_j) and (C'_j). Moreover, we get from (4) and (5)

$$(1-s)(1-t)\, Y\left(\frac{s}{1-s}, \frac{t}{1-t}\right) = X''_{st} + s\,X'_{1t} + t\,X'_{s1} + s\,t\,X_{11}, \qquad s, t \in [0, 1).$$

Hence (1) holds with

$$A_{st} = (1-s)(1-t)\, A'\left(\frac{s}{1-s}, \frac{t}{1-t}\right),$$
$$B_j(s) = (1-s)\, B'_j\left(\frac{s}{1-s}\right), \qquad C_j(t) = (1-t)\, C'_j\left(\frac{t}{1-t}\right), \quad j \in \mathbb{N}, \; s, t \in [0, 1),$$

which have the desired distributions by Lemma 2.8.

It remains to show that the coefficients in (1) can be taken to be non-random, whenever X is dissociated. One way of seeing this is to extend a fixed increment array for X to the index set $\mathbb{Z}^2$, and notice that the subarrays indexed by $\mathbb{N}^2$ and $(-\mathbb{N})^2$ are independent. As seen above, both determine measurably the coefficients in (1), to the extent described by Lemma 4.3. Indeed, under the stated conventions, the coefficients ρ, σ and the α_j, as well as the rotational invariants in (4.13) are all uniquely determined. Thus all these parameters are independent of themselves and hence a.s. non-random. In this case, there is clearly even a non-random choice of the β_j and γ_j. $\square$

To every process X as in (1), we shall associate the *directing random elements* ρ and $\mu=(\mu_1, \ldots, \mu_4)$, where the μ_k are a.s. bounded random measures on $\mathbb{R}_+$, given under the normalizing condition $\alpha_j \geqq 0$, $j \in \mathbb{N}$, by

$$\mu_1=\sigma^2\,\delta_0+\sum_{j=1}^{\infty}\alpha_j^2\,\delta_{\alpha_j}, \qquad \mu_2=\sum_{j=1}^{\infty}\beta_j^2\,\delta_{\alpha_j},$$

$$\mu_3=\sum_{j=1}^{\infty}\gamma_j^2\,\delta_{\alpha_j}, \qquad \mu_4=\sum_{j=1}^{\infty}\alpha_j^2(\beta_j^2+\gamma_j^2)\,\delta_{\alpha_j}.$$

As before, $\xrightarrow{w}$ denotes weak convergence in the space $\mathscr{M}(\mathbb{R}_+)$ of bounded measures on $\mathbb{R}_+$, while $\xrightarrow{wd}$ denotes convergence in distribution with respect to the associated weak topology. The same notation will be used for convergence in the product spaces $(\mathscr{M}(\mathbb{R}_+))^4$ or $\mathbb{R}\times(\mathscr{M}(\mathbb{R}_+))^4$, when endowed with the corresponding product topologies.

On the other hand, the processes X in (1) will be considered as random elements in the space $C(I)$ of continuous functions on $I=\mathbb{R}_+^2$, $\mathbb{R}_+\times[0,1]$ or $[0,1]^2$, and here the associated topology is taken to be that of uniform convergence on bounded sets. Convergence in distribution with respect to this topology will be denoted by $\xrightarrow{d}$, and we shall write $\xrightarrow{fd}$ for convergence of the finite-dimensional distributions. Note in particular that $X_n \xrightarrow{d} X$ for random elements in $C(\mathbb{R}_+^2)$ or $C(\mathbb{R}_+\times[0,1])$, iff convergence holds for the restrictions to an arbitrary rectangle $[0,a]\times[0,b]$. Thus the theory reduces in both cases to that of $C([0,1]^2)$, for which most results in Chapter 2 of Billingsley [3] remain valid with obvious changes. In particular, a sequence (X_n) of random elements in $C([0,1]^2)$ is tight, iff $(X_n(0))$ is tight and moreover

$$\lim_{h\to 0}\sup_n \mathsf{P}\{w(X_n,h)>\varepsilon\}=0, \qquad \varepsilon>0, \tag{8}$$

where $w(f,\cdot)$ denotes the modulus of continuity of the function f.

The following theorem justifies the above terminology for ρ and μ, and solves the uniqueness and continuity problems for the representation in (1).

Theorem 6.5. *The directing random elements ρ and μ of a continuous and separately exchangeable process X on $\mathbb{R}_+^2$, $\mathbb{R}_+\times[0,1]$ or $[0,1]^2$ are a.s. unique measurable functions of X, and the distributions of (ρ,μ) and X determine each other uniquely. If $X_1, X_2, \ldots$ are processes as above and directed by (ρ_n,μ_n), $n\in\mathbb{N}$, then the statements*

(i) $X_n \xrightarrow{d}$ *some* X, (ii) $X_n \xrightarrow{fd}$ *some* X, (iii) $(\rho_n, \mu_n) \xrightarrow{wd}$ *some* (ρ, μ) *are equivalent and imply that* X *is separately exchangeable and directed by* (ρ, μ).

In order to apply the tightness criterion (8) to the processes in (1), we shall need a bound for the modulus of continuity in a special case. Recall that $f \lesssim g$ means $f = O(g)$.

Lemma 6.6. *Let* X *be given by* (1) *with vanishing* ρ, σ *and* β_j, γ_j, $j \in \mathbb{N}$, *and let* X' *denote the restriction of* X *to* $[0,1]^2$. *Then*

$$\mathsf{E}\,|w(X', h)|^2 \lesssim h^{\frac{1}{2}}\, \mathsf{E} \sum_{j=1}^{\infty} \alpha_j^2, \qquad h \in [0,1]. \tag{9}$$

Proof. By Fubini's theorem, it is enough to consider the case of non-random α_j. Let us first assume that X is defined on $\mathbb{R}_+^2$. Proceeding as in the proof of Lemma 6.2, we get with s, s', t restricted to $[0,1]$

$$\left\| \sup_{|s-s'| \leqq 2^{-m}} \sup_t |X(s,t) - X(s',t)| \right\|_4$$

$$\lesssim \left\| \sum_{n \geqq m} \sup_{k \leqq 2^n} \sup_t |X(k 2^{-n}, t) - X((k-1) 2^{-n}, t)| \right\|_4$$

$$\lesssim \left(\sum_j \alpha_j^2 \right)^{\frac{1}{2}} \sum_{n \geqq m} 2^{-n/4} \lesssim 2^{-m/4} \left(\sum_j \alpha_j^2 \right)^{\frac{1}{2}}.$$

The symmetric argument yields the same estimate with s and t interchanged, and (9) follows by combination.

If X is instead defined on $\mathbb{R}_+ \times [0,1]$, the transformations in Lemma 2.7 yield the above estimates for the restrictions of X to $[0,1] \times [0, \frac{1}{2}]$ and $[0,1] \times [\frac{1}{2}, 1]$, from which (9) is obtained by combination. Similarly, (9) follows for processes on $[0,1]^2$ from the estimates obtained via Lemma 2.8 for the restrictions to the squares $I \times J$ with $I, J = [0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. □

We shall also need the following simple result about convergence of measures. Recall that a sequence of Radon measures μ_n on some topological space converges *vaguely* to μ (written $\mu_n \xrightarrow{v} \mu$), if $\mu_n f \to \mu f$ for every continuous function f with compact support.

Lemma 6.7. *Let* (μ_n) *be a weakly tight sequence of bounded measures on* $\mathbb{R}_+$, *and assume that* $\mu_n \xrightarrow{v} \mu$ *on* $(0, \infty)$. *Then* $\mu_n f \to \mu f$ *for every bounded continuous function* $f: \mathbb{R}_+ \to \mathbb{R}$ *with* $f(0) = 0$.

Proof. The tightness implies that (μ_n) is weakly relatively compact, so it is enough to consider weakly convergent subsequences. But if $\mu_n \xrightarrow{w} \mu'$, then $\mu' = \mu$ on $(0, \infty)$, and therefore $\mu_n f \to \mu' f = \mu f$. □

Proof of Theorem 6.5. To prove the first assertion, it suffices by Lemmas 2.3 and 2.4 to consider the case of non-random coefficients. The uniqueness of ρ and μ then follows as in the proof of Theorem 6.1 from the uniqueness part of Lemma 4.3. Conversely, μ determines the coefficients in (1) to the extent described by that lemma. Thus it remains to show that rotations of the type mentioned there do not affect the distribution of X. Let us then assume that

$$X_{st}=\sum_{j=1}^{n}(\alpha_j B_j(s)\,C_j(t)+\beta_j B_j(s)\,t+\gamma_j s\,C_j(t)), \tag{10}$$

and that, in matrix notation, $\beta'=U\beta$ and $\gamma'=U\gamma$ for some rotation (orthogonal matrix) U. Then

$$\begin{aligned}X_{st}&=\alpha B_s^T C_t+t\beta^T B_s+s\gamma^T C_t\\&=\alpha B_s^T U^T U C_t+t\beta'^T U B_s+s\gamma'^T U C_t\\&=\alpha B_s'^T C_t'+t\beta'^T B_s'+s\gamma'^T C_t',\end{aligned}$$

with $B_s'=UB_s$ and $C_t'=UC_t$. Since clearly $(B', C')\stackrel{d}{=}(B, C)$, this shows that X has a second representation as in (10) with (β', γ') in place of (β, γ). Thus both pairs yield the same distribution, as asserted.

Let us next consider sequences of processes X_n directed by (ρ_n, μ_n), and show that (iii) implies (i). By the continuity of the mappings in Lemma 2.8, it is then enough to consider processes on $[0, 1]^2$, and by Lemma 1.1 in [14] we may further take the ρ_n and μ_n to be non-random. If $(\rho_n, \mu_n)\xrightarrow{w}\text{some}(\rho, \mu)$, then the sequences of parameters ρ, σ, $\sum\alpha_j^2$, $\sum\beta_j^2$ and $\sum\gamma_j^2$ for these processes are clearly bounded, so it is seen from (1) and Lemma 6.6 that (X_n) is tight. If $X_n\xrightarrow{d}X'$ along some subsequence, then X' will also be separately exchangeable, say with directing pair (ρ', μ'). Here ρ' and μ' must also be non-random. In fact, this would be obvious for processes on $\mathbb{R}_+^2$, since X' would then be dissociated like all the X_n. For processes on $[0, 1]^2$ it then follows by the mappings in Lemma 2.8.

It remains to prove that $(\rho', \mu')=(\rho, \mu)$, since (i) will then hold by the uniqueness result above, with X as a process directed by (ρ, μ). To identify (ρ', μ'), let us drop the subscripts of X_n, ρ_n, μ_n, ... for convenience, and write

$$U=2X(\tfrac{1}{2}, 1)-X(1, 1), \qquad V=2X(1, \tfrac{1}{2})-X(1, 1),$$
$$T=4X(\tfrac{1}{2}, \tfrac{1}{2})-2X(\tfrac{1}{2}, 1)-2X(1, \tfrac{1}{2})+X(1, 1).$$

Using the transformations in Lemma 2.8, it is seen as in case of Lemma 4.3 that

$$\begin{aligned}&\mathsf{E}\exp(ir\rho+itT+iuU+ivV)\\&=\exp\left[ir\rho-\frac{1}{2}t^2\sigma^2-\frac{1}{2}\sum_{j=1}^{\infty}\log(1+t^2\alpha_j^2)-\frac{1}{2}\sum_{j=1}^{\infty}\frac{u^2\beta_j^2+v^2\gamma_j^2+ituv\alpha_j\beta_j\gamma_j}{1+t^2\alpha_j^2}\right]\\&=\exp\left[ir\rho-\frac{1}{2}\int\frac{\log(1+t^2x^2)}{x^2}\,\mu_1(dx)-\frac{1}{2}\int\frac{(u^2\mu_2+v^2\mu_3)(dx)}{1+t^2x^2}\right.\\&\quad\left.+\frac{ituv}{4}\int\frac{(x^2\mu_2+x^2\mu_3-\mu_4)(dx)}{x(1+t^2x^2)}\right].\end{aligned} \tag{11}$$

Here the exponent on the left is continuous in X, while the one on the right is continuous in (ρ, μ) by Lemma 6.7. The same relation must then hold in the limit as $n \to \infty$, i.e. for the process X' and the pair (ρ, μ). Since this relation is also true with (ρ', μ') in place of (ρ, μ), it follows as in case of Lemma 4.3 that indeed $(\rho', \mu') = (\rho, \mu)$.

Since (i) trivially implies (ii), it remains to show that (ii) implies (iii). By Lemma 2.8, we may then restrict our attention to processes on $\mathbb{R}_+^2$. Assuming (ii), it is enough, as in case of Theorem 4.2, to show that the sequence of pairs (ρ_n, μ_n) is weakly tight. To see this, drop the subscript n as before, and write $X_{11} = \rho + T + U + V$, where U and V denote the sums in (1) with coefficient arrays (β_j) and (γ_j) respectively. Proceeding as in (11), we get

$$\mathsf{E} \exp(i t X_{11}) = \mathsf{E} \exp\left[i t \rho - \frac{1}{2} \int \frac{\log(1+t^2 x^2)}{x^2} \mu_1(dx) - \frac{t^2}{2} \int \frac{(\mu_2+\mu_3)(dx)}{1+t^2 x^2} \right.$$
$$\left. + \frac{i t^3}{4} \int \frac{(x^2 \mu_2 + x^2 \mu_3 - \mu_4)(dx)}{x(1+t^2 x^2)} \right],$$

so

$$|\mathsf{E} \exp(i t X_{11})| \leqq \mathsf{E} \exp\left[-\frac{1}{2} \int \frac{\log(1+t^2 x^2)}{x^2} \mu_1(dx) \right],$$

and it follows as in case of Theorem 4.2 that the sequence of random measures μ_1 is weakly tight.

This implies in particular tightness of the variables T above, so even the sequence of variables $\rho + U + V$ must be tight. Now

$$\mathsf{E} \exp[i t(\rho + U + V)] = \mathsf{E} \exp\left[i t \rho - \frac{t^2}{2} (\mu_2 + \mu_3) \mathbb{R} \right],$$

so for any $c > 0$ we get

$$\begin{aligned} |\mathsf{E} \exp[i t(\rho + U + V)]| &\leqq \mathsf{E} \exp\left[-\frac{t^2}{2} (\mu_2 + \mu_3) \mathbb{R} \right] \\ &\leqq \mathsf{P}\{(\mu_2+\mu_3)\mathbb{R} \leqq c\} + e^{-t^2 c/2} \mathsf{P}\{(\mu_2+\mu_3)\mathbb{R} > c\} \\ &= 1 - (1 - e^{-t^2 c/2}) \mathsf{P}\{(\mu_2+\mu_3)\mathbb{R} > c\}, \end{aligned}$$

which shows as before that the sequence of random variables $(\mu_2+\mu_3)\mathbb{R}$ is tight. Thus the random measures $\mu_2+\mu_3$ form a vaguely tight sequence, and since clearly $\mu_4(dx) \leqq 2x^2(\mu_2+\mu_3)(dx)$, the same thing must be true for the measures μ_4. Since μ_2, μ_3 and μ_4 are all zero outside the support of μ_1, the above conclusions extend immediately to weak tightness. This proves the desired tightness of the sequence (μ_n).

From this it follows in particular that the sequence of random variables $T + U + V$ is tight, so the same thing must be true for ρ. Thus even (ρ_n) is tight, as well as the sequence of pairs (ρ_n, μ_n). □

7. Jointly exchangeable processes

Here we shall characterize the class of jointly exchangeable and continuous processes X on $\mathbb{R}_+^2$, and we shall further solve the corresponding uniqueness and continuity problems, in the special case of symmetric processes X, for which the representation reduces to (1.7).

Theorem 7.1. *A process X on $\mathbb{R}_+^2$ is continuous and jointly exchangeable, iff a.s.*

$$X_{st} = \rho\, s\, t + \sigma\, A_{st} + \sigma'\, A_{ts} + \vartheta(s \wedge t) + \sum_i \sum_j \alpha_{ij}(B_i(s)\, B_j(t) - \delta_{ij}(s \wedge t))$$
$$+ \sum_j (\beta_j\, t\, B_j(s) + \beta'_j\, s\, B_j(t) + \gamma_j\, B_j(s \wedge t)), \qquad s, t \in \mathbb{R}_+, \tag{1}$$

for some random variables ρ, σ, σ', ϑ and $\alpha_{ij}, \beta_j, \beta'_j, \gamma_j$, $i, j \in \mathbb{N}$, with $\alpha_{ij} + \alpha_{ji} = 0$ for $i \neq j$, and such that $\sum\sum \alpha_{ij}^2 < \infty$ and $\sum(\beta_j^2 + \beta_j'^2 + \gamma_j^2) < \infty$, some independent Brownian sheet A, and some independent sequence (B_j) of i.i.d. Brownian motions. The random variables ρ, ϑ, $(\sigma \pm \sigma')^2$, $\sum\sum \alpha_{ij}^2$, $\sum \beta_j^2$, $\sum \beta_j'^2$, $\sum \gamma_j^2$, $\sum \beta_j \beta'_j$, $\sum \beta_j \gamma_j$ and $\sum \beta'_j \gamma_j$ are a.s. unique, as are the α_{jj} apart from order. Moreover, the coefficients in (1) *can be chosen to be X-measurable, and they may further be taken to be non-random iff X is dissociated.*

First we need to examine the convergence of the series in (1).

Lemma 7.2. *The series in* (1) *converge in probability with respect to the uniform metric on every compact set, and the limit is a.s. independent of the order of summation. If $\alpha_{ij} = 0$ for $i \neq j$, then the convergence is even a.s.*

Proof. It is clearly enough to consider the case of non-random coefficients. The last term in (1) can be treated as in case of Lemma 6.2, so we need only consider the double sum, S say. By a scaling argument, it is further enough to consider convergence within the unit square. We shall prove below that

$$\mathsf{E} \sup_{s,t \leq 1} S_{st}^2 \lesssim \sum_i \sum_j \alpha_{ij}^2, \tag{2}$$

provided the summation is finite. In the general case, we may then obtain the desired convergence and uniqueness of the limit by applying (2) to differences between partial sums. Note that (2) extends to the limit in this case. If $\alpha_{ij} = 0$ for $i \neq j$, then the terms will be independent, so we may use [13] as before to strengthen the conclusion to a.s. convergence.

To prove (2), take $s \leq t$, and note that

$$S_{st} = \sum_i \alpha_{ii}(B_i^2(s) - s) + \sum_i \sum_j \alpha_{ij}\, B_i(s)(B_j(t) - B_j(s)) = T_s + U_{st},$$

since $\alpha_{ij} + \alpha_{ji} = 0$ for $i \neq j$. Write $\alpha_i = (\sum_j \alpha_{ij}^2)^{\frac{1}{2}}$, and let ξ_i and η_j be i.i.d. $N(0, 1)$ random variables. By Doob's inequality, we get

$$\mathsf{E} \sup_{s \leq 1} T_s^2 \lesssim \mathsf{E}\, T_1^2 = \sum_i \alpha_{ii}^2\, \mathsf{E}(\xi_i^2 - 1)^2 \lesssim \sum_i \alpha_{ii}^2.$$

Using Doob's and Schwarz' inequalities plus the symmetry of $N(0, 1)$, we further obtain for a fixed $s \in [0, 1]$

$$\begin{aligned}\mathsf{E} \sup_{t \in [s,1]} U_{st}^4 &\lesssim \mathsf{E} U_{s1}^4 \lesssim s^2\, \mathsf{E} \left| \sum_i \xi_i \sum_j \alpha_{ij} \eta_j \right|^4 \\ &\lesssim s^2\, \mathsf{E} \left| \sum_i \xi_i^2 \Big(\sum_j \alpha_{ij} \eta_j \Big)^2 \right|^2 \\ &\lesssim s^2\, \mathsf{E} \sum_i \alpha_i^2\, \xi_i^4 \sum_i \alpha_i^{-2} \Big(\sum_j \alpha_{ij} \eta_j \Big)^4 \\ &= s^2 \sum_i \alpha_i^2\, \mathsf{E}\, \xi_i^4 \sum_i \alpha_i^{-2}\, \mathsf{E} \Big(\sum_j \alpha_{ij} \eta_j \Big)^4 \\ &\lesssim s^2 \Big(\sum_i \alpha_i^2 \Big)^2 = s^2 \Big(\sum_i \sum_j \alpha_{ij}^2 \Big)^2 .\end{aligned}$$

The proof may now be completed as in case of Lemma 6.2. □

Proof of Theorem 7.1. A process X as in (1) is a.s. continuous by Lemma 7.2. To see that X is also jointly exchangeable, it suffices by the same lemma to consider the case of finite sums. We may further take the coefficients in (1) to be non-random, and consider separately the three terms $\rho s t$, $\sigma A_{st} + \sigma' A_{ts}$, $\vartheta(s \wedge t)$, and the remainder of X. For the first and third of these, the joint exchangeability is obvious, and for the second it follows from the joint rotatability of the corresponding terms in Theorem 5.1. Finally, the result for the remaining expression in (1) follows easily by the exchangeability of Brownian motion. This establishes the sufficiency of the representation (1).

Suppose conversely that X is continuous and jointly exchangeable. Our first aim is to reduce the discussion to the case when X is dissociated. Let us then denote by X_n the array of increments of X with respect to the square grid of size 2^{-n}. Note that the sequence of arrays X_n is *consistent*, in the sense that an element in X_m is the sum of the corresponding elements in X_n whenever $m \leqq n$. By Kolmogorov's theorem, we may extend each X_n to the index set $\mathbb{Z}^2$, in such a way that the consistency and the joint exchangeability are both preserved. Let $A_1, A_2, \ldots$ be the restrictions to $(-\mathbb{Z}_+)^2$ of these extended arrays. From the discussion in Sect. 3 it is clear that X_n is conditionally jointly exchangeable and dissociated, given A_n. The same thing is then true for all X_m with $m \leqq n$. Fixing m and letting $n \to \infty$, it follows by martingale theory that X_m is conditionally jointly exchangeable and dissociated, given all the A_n. Since m was arbitrary, we get the same property for X. By Lemma 2.2, it is then enough to show, in the dissociated case, that X has a representation as in (1) with constant coefficients. We may thus assume from now on that X is dissociated.

In that case, it is seen from Theorem 3.1 that any fixed increment array (X_{ij}) as above has a representation

$$X_{ij} = f(\xi_i, \xi_j, \chi_{ij}) \quad \text{a.s.}, \qquad i, j \in \mathbb{N}, \tag{3}$$

for some measurable function f, where the variables $\xi_1, \xi_2, \ldots$ and χ_{ij}, $i<j$, are i.i.d. $U(0, 1)$, and moreover $\chi_{ii} \equiv 0$ while $\chi_{ij} \equiv \chi_{ji}$. On the other hand, the

increments of X within squares indexed by $I=(2\mathbb{N})\times(2\mathbb{N}-1)$ combine in an obvious way to form a continuous, separately exchangeable and dissociated process on $\mathbb{R}_+^2$, so by Theorem 6.1 we have on I another representation

$$X_{ij}=\rho+\sigma\lambda_{ij}+\sum_{k=1}^{\infty}(\alpha_k\xi_{ik}\eta_{jk}+\beta_k\xi_{ik}+\gamma_k\eta_{jk})\quad \text{a.s.},\qquad (i,j)\in I, \tag{4}$$

where ρ, σ and the α_k, β_k, γ_k are constants, while the λ_{ij}, ξ_{ik} and η_{jk} are i.i.d. $N(0,1)$. We need to show that we can choose

$$\lambda_{ij}=g(\xi_i,\xi_j,\chi_{ij}),\qquad \xi_{ik}=g_k(\xi_i),\qquad \eta_{jk}=g'_k(\xi_j), \tag{5}$$

for some functions g, g_k and g'_k. In that case, (4) determines the functional dependence in (3) for $i\neq j$, so (4) remains valid with the λ_{ij}, ξ_{ik} and η_{jk} given by (5), for all pairs (i,j) with $i\neq j$.

To prove (5), we shall need some relations between (3) and (4). First note that

$$\rho=\mathsf{E}X_{ij}=\mathsf{E}f(\xi_i,\xi_j,\chi_{ij})=f(\cdot,\cdot,\cdot),$$

where the dots on the right indicate integration with respect to the corresponding variables. Applying the law of large numbers to both (3) and (4), it is further seen that a.s.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}X_{i,2j-1}=\rho+\sum_{k=1}^{\infty}\beta_k\xi_{ik}=f(\xi_i,\cdot,\cdot),\qquad i\in 2\mathbb{N},$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_{2i,j}=\rho+\sum_{k=1}^{\infty}\gamma_k\eta_{jk}=f(\cdot,\xi_j,\cdot),\qquad j\in 2\mathbb{N}-1.$$

Combining these relations with (3) and (4), we get for $(i,j)\in I$

$$\sigma\lambda_{ij}+\sum_{k=1}^{\infty}\alpha_k\xi_{ik}\eta_{jk}=f(\xi_i,\xi_j,\chi_{ij})-f(\xi_i,\cdot,\cdot)-f(\cdot,\xi_j,\cdot)+f(\cdot,\cdot,\cdot),$$

$$\sum_{k=1}^{\infty}\beta_k\xi_{ik}=f(\xi_i,\cdot,\cdot)-f(\cdot,\cdot,\cdot),$$

$$\sum_{k=1}^{\infty}\gamma_k\eta_{jk}=f(\cdot,\xi_j,\cdot)-f(\cdot,\cdot,\cdot).$$

The set of arrays on the left (together with ρ) is clearly separately rotatable in the sense of Lemma 4.3, so from the proof of that result it is seen that

$$\sigma\lambda_{ij}+\sum_{k=1}^{\infty}\alpha_k\xi_{ik}\eta_{jk}=\sigma' g(\xi_i,\xi_j,\chi_{ij})+\sum_{k=1}^{\infty}\alpha'_k g_k(\xi_i)\,g'_k(\xi_j),$$

$$\sum_{k=1}^{\infty}\beta_k\xi_{ik}=\sum_{k=1}^{\infty}\beta'_k g_k(\xi_i),\qquad \sum_{k=1}^{\infty}\gamma_k\eta_{ik}=\sum_{k=1}^{\infty}\gamma'_k g'_k(\xi_j),$$

for some constants σ', α'_k, β'_k, γ'_k and functions g, g_k, g'_k, where the latter are such that the random variables on the left and right have the same distributional properties. This shows that (4) and (5) are simultaneously true, but possibly with some new set of coefficients ρ, σ, α_k, β_k, γ_k and random variables λ_{ij}, ξ_{ik}, η_{jk}, all with the same properties as before. As already pointed out, the result extends immediately to arbitrary (i, j) with $i \neq j$.

Applying the same argument to the array $(aX_{ij}+bX_{ji})$ for arbitrary a and b, and proceeding as in the proof of Theorem 5.1, it may next be seen that

$$\sigma \lambda_{ij} = \sigma' \lambda'_{ij} + \sigma'' \lambda'_{ji}, \qquad i \neq j,$$

for some constants σ', σ'' and some i.i.d. $N(0, 1)$ random variables λ'_{ij}, $i \neq j$, independent of (ξ_j). Moreover, the ξ_{jk} and η_{jk} are seen as before to be jointly Gaussian for fixed j, so we may again use the spectral theorem, to obtain a representation for $i \neq j$ of the form

$$\sum_{k=1}^{\infty} (\alpha_k \xi_{ik} \eta_{jk} + \beta_k \xi_{ik} + \gamma_k \eta_{jk}) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha'_{kl} \xi'_{ik} \xi'_{jl} + \sum_{k=1}^{\infty} (\beta'_k \xi'_{ik} + \gamma'_k \xi'_{jk}),$$

where the coefficients on the right satisfy $\alpha'_{kl} + \alpha'_{lk} = 0$ for $k \neq l$, while the variables ξ'_{jk} are i.i.d. and $N(0, 1)$. For convenience, we may change the notation and assume from now on that

$$X_{ij} = \rho + \sigma_+ \lambda_{ij} + \sigma_- \lambda_{ji} + \sum_k \sum_l \alpha_{kl} \xi_{ik} \xi_{jl} + \sum_k \beta_k \xi_{ik} + \sum_l \gamma_l \xi_{jl}, \qquad i \neq j,$$

where the λ_{ij} and ξ_{ik} are i.i.d. $N(0, 1)$ random variables, while ρ, σ_+, σ_- and the α_{kl}, β_k and γ_l are constants satisfying $\alpha_{kl} + \alpha_{lk} = 0$ for $k \neq l$, and moreover

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha_{kl}^2 < \infty, \qquad \sum_{k=1}^{\infty} \beta_k^2 < \infty, \qquad \sum_{l=1}^{\infty} \gamma_l^2 < \infty.$$

Note that this agrees with (1) if we put $\sigma_+ = \sigma$, $\sigma_- = \sigma'$ and $\gamma_j = \beta'_j$.

Halving the grid size, we get a similar representation for the corresponding increments X'_{ij}, say with coefficients ρ', σ'_+, σ'_-, α'_{kl}, β'_k, γ'_k. Hence the original increments X_{ij} have another representation of the form

$$X_{ij} = 4\rho' + 2\sigma'_+ \lambda'_{ij} + 2\sigma'_- \lambda'_{ji} + 2 \sum_k \sum_l \alpha'_{kl} \xi'_{ik} \xi'_{jl} + 2^{3/2} \sum_k (\beta'_k \xi'_{ik} + \gamma'_k \xi'_{jk}),$$

where the random variables on the right are again i.i.d. $N(0, 1)$. Equating the expectations yields $\rho = 4\rho'$, and by applying the law of large numbers as before, we further obtain a.s.

$$\sigma_+ \lambda_{ij} + \sigma_- \lambda_{ji} + \sum_k \sum_l \alpha_{kl} \xi_{ik} \xi_{jl} = 2\sigma'_+ \lambda'_{ij} + 2\sigma'_- \lambda'_{ji} + 2 \sum_k \sum_l \alpha'_{kl} \xi'_{ik} \xi'_{jl},$$

$$\sum_k (\beta_k \xi_{ik} + \gamma_k \xi_{jk}) = 2^{3/2} \sum_k (\beta'_k \xi'_{ik} + \gamma'_k \xi'_{jk}).$$

Thus the X'_{ij} have the same joint distribution for $i \neq j$ as the variables

$$\rho' + \sigma'_+ \lambda'_{ij} + \sigma'_- \lambda'_{ji} + \sum_k \sum_l \alpha'_{kl} \xi'_{ik} \xi'_{jl} + \sum_k (\beta'_k \xi'_{ik} + \gamma'_k \xi'_{jk})$$

$$= \frac{\rho}{4} + \frac{1}{2} (\sigma_+ \lambda_{ij} + \sigma_- \lambda_{ji} + \sum_k \sum_l \alpha_{kl} \xi_{ik} \xi_{jl}) + 2^{-3/2} \sum_k (\beta_k \xi_{ik} + \gamma_k \xi_{jk}),$$

in full agreement with (1). Continuing recursively, and using the fact that both X and the process in (1) are continuous, it follows that the entire set of increments outside the diagonal is distributed as in (1). By Lemma 2.1, we may thus construct a Brownian sheet A and an independent sequence of i.i.d. Brownian motions B_j, such that the increments of the two processes in (1) agree a.s. outside the diagonal.

To extend this result to the diagonal, we write

$$Y(s, t) = \rho\, s\, t + \sigma A_{st} + \sigma' A_{ts} + \sum_i \sum_j \alpha_{ij} (B_i(s) B_j(t) - (s \wedge t) \delta_{ij})$$
$$+ \sum_j (\beta_j t B_j(s) + \beta'_j s B_j(t)), \qquad s, t \geqq 0. \tag{6}$$

Let us further write $X^{(n)} = (X^{(n)}_{ij})$ for the increment array of X with respect to a grid of size 2^{-n}, and put

$$Y_n(s, t) = \sum_i \sum_j 1\{i 2^{-n} \leqq s,\ j 2^{-n} \leqq t,\ i \neq j\} X^{(n)}_{ij}, \qquad s, t \geqq 0,\ n \in \mathbb{N}.$$

For fixed dyadic s and t and for large enough n, we get with ξ $N(0, 1)$

$$\mathsf{E}(Y(s, t) - Y_n(s, t))^2 = (s \wedge t) \{2^{-2n} \rho^2 + 2^{-n} (\sigma + \sigma')^2$$
$$+ 2^{-n} \mathsf{E}(\xi^2 - 1)^2 \sum_i \sum_j \alpha_{ij}^2 + 2^{-2n} \sum_j (\beta_j + \beta'_j)^2\} \to 0,$$

so $Y_n(s, t) \xrightarrow{P} Y(s, t)$ for dyadic s and t. Note in particular that Y is measurably determined by X and independent of the choice of representation.

Let us next define $Z = X - Y$ and $Z_n = X - Y_n$, and note that $Z_n \xrightarrow{P} Z$ at dyadic points. Since moreover

$$Z_n(i 2^{-n}, j 2^{-n}) = Z_n((i \wedge j) 2^{-n}, (i \wedge j) 2^{-n}), \qquad i, j \in \mathbb{N},$$

we get the same relation for Z, so there must exist some continuous process U with $U_0 = 0$, and such that

$$Z(s, t) = Z(s \wedge t, s \wedge t) = U(s \wedge t), \qquad s, t \geqq 0. \tag{7}$$

From the joint exchangeability of $X^{(n)}$, it is further seen that $(X, Y_n)^{(m)}$ is jointly exchangeable for $m = n$, and hence also for $m < n$. Letting $n \to \infty$ for fixed m, we may conclude that $(X, Y)^{(m)}$ is jointly exchangeable. The same thing will then be true for the $\mathbb{R}^2$-valued process (Y, Z) on $\mathbb{R}^2_+$.

Writing $U_i^{(n)} = U(i2^{-n}) - U((i-1)2^{-n})$, we get from (7)

$$Z_{ij}^{(n)} = U_i^{(n)} \delta_{ij}, \qquad i, j \in \mathbb{N},$$

which shows that even the process $(Y(s,t), sU(t))$, $s, t \geqq 0$, is jointly exchangeable. Proceeding as for X above, we may then obtain a representation as in (6) for each component, in terms of a common sequence of Brownian motions B_j. Using the law of large numbers as before, it is seen that the constants σ, σ' and all the α_{ij} and β_j must vanish in the formula for $sU(t)$. Thus we get, jointly with (6), a representation of the form

$$sU(t) = \vartheta st + \sum_{j=1}^{\infty} \gamma_j s B_j(t), \qquad s, t \geqq 0,$$

where ϑ and the γ_j are constants with $\sum \gamma_j^2 < \infty$. By (7) it follows that

$$Z(s,t) = \vartheta(s \wedge t) + \sum_{j=1}^{\infty} \gamma_j B_j(s \wedge t), \qquad s, t \geqq 0, \tag{8}$$

and adding this to (6) yields (1).

In view of the results in Sect. 2, it remains only to prove the uniqueness assertions. Then recall that the diagonal process (8) is measurably determined by X, and that the processes $\sum \beta_j B_j(s)$ and $\sum \beta'_j B_j(t)$ can be measurably recovered through the law of large numbers. All these processes form together a mixed Brownian motion in $\mathbb{R}^3$ with drift $(\vartheta, 0, 0)$ and mixed quadratic variations $\sum \gamma_j^2$, $\sum \beta_j^2$, $\sum \beta_j'^2$, $\sum \gamma_j \beta_j$, $\sum \gamma_j \beta'_j$ and $\sum \beta_j \beta'_j$, so these quantities are a.s. unique. Subtracting the corresponding terms from (1), we end up with a jointly rotatable process, for which the a.s. uniqueness of the parameters ρ, $(\sigma \pm \sigma')^2$ and $\sum\sum \alpha_{ij}^2$ as well as of the sequence (α_{jj}) follows by Theorem 5.1. □

When X is symmetric, the representation (1) simplifies to (1.7), i.e. we have $\sigma' = \sigma$, $\beta'_j \equiv \beta_j$ and $\alpha_{ij} \equiv \alpha_j \delta_{ij}$. In this case, we may introduce the *directing random elements* ρ, ϑ and $\mu = (\mu_1, \ldots, \mu_4)$, where the μ_j are a.s. bounded random measures on $\mathbb{R}$, given by

$$\mu_1 = 2\sigma^2 + \sum_j \alpha_j^2 \delta_{\alpha_j}, \qquad \mu_2 = \sum_j \beta_j^2 \delta_{\alpha_j},$$

$$\mu_3 = \sum_j \gamma_j^2 \delta_{\alpha_j}, \qquad \mu_4 = \sum_j (\beta_j + \gamma_j)^2 \delta_{\alpha_j}. \tag{9}$$

The uniqueness and continuity problems for the representation (1.7) have the following solutions in terms of the triple (ρ, ϑ, μ).

Theorem 7.3. *The directing random elements ρ, ϑ and μ of a symmetric, continuous and jointly exchangeable process X on $\mathbb{R}_+^2$ are a.s. unique and X-measurable, and the distributions of (ρ, ϑ, μ) and X determine each other uniquely. If X_1, $X_2, \ldots$ are processes as above directed by $(\rho_n, \vartheta_n, \mu_n)$, $n \in \mathbb{N}$, then the statements*

(i) $X_n \xrightarrow{d}$ *some* X, (ii) $X_n \xrightarrow{fd}$ *some* X, (iii) $(\rho_n, \vartheta_n, \mu_n) \xrightarrow{wd}$ *some* (ρ, ϑ, μ) *are equivalent and imply that even X is such as above and directed by (ρ, ϑ, μ).*

For the proof, we shall need a tightness criterion for the processes X in (1.7), regarded as random elements in $C(\mathbb{R}_+^2)$.

Lemma 7.4. *Let $B_1, B_2, \ldots$ be independent Brownian motions, and put*

$$X_n(s,t)=\sum_{j=1}^{\infty}\alpha_{nj}(B_j(s)\,B_j(t)-s\wedge t), \qquad s,t\in\mathbb{R}_+,\ n\in\mathbb{N},$$

where the α_{nj} are non-random. Then (X_n) is tight if

$$\sup_n \sum_{j=1}^{\infty}\alpha_{nj}^2<\infty.$$

Proof. For $s, t\geqq 0$ we write

$$\begin{aligned} Y_n(s,t)&=X_n(s,s+t)\\ &=\sum_{j=1}^{\infty}\alpha_{nj}(B_j^2(s)-s)+\sum_{j=1}^{\infty}\alpha_{nj}\,B_j(s)(B_j(s+t)-B_j(s))\\ &=T_n(s)+U_n(s,t).\end{aligned}$$

Proceeding as in the proofs of Lemmas 7.2 and 6.6, we get

$$\mathsf{E}\,|w(U_n,h)|^2\lesssim h^{\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_{nj}^2,$$

which shows that (U_n) is tight.

As for (T_n), we write $M_j(t)=B_j^2(t)-t$, and note that $dM_j=2B_j\,dB_j$ by Itô's formula. Since the martingales M_j are further orthogonal, we obtain for T_n the quadratic variation process

$$[T_n,T_n]_t=\sum_{j=1}^{\infty}\alpha_{nj}^2[M_j,M_j]_t=4\sum_{j=1}^{\infty}\alpha_{nj}^2\int_0^t B_j^2(s)\,ds.$$

Hence

$$\mathsf{E}\sup_{t\leqq 1}[T_n,T_n]_{t-h}^t\leqq 4h\sum_{j=1}^{\infty}\alpha_{nj}^2\,\mathsf{E}\sup_{s\leqq 1}B_j^2(s)\lesssim h\sum_{j=1}^{\infty}\alpha_{nj}^2,$$

which shows that the sequence $([T_n,T_n])$ is tight. Since $T_n=W_n\circ[T_n,T_n]$ for some Brownian motions W_n, it follows that even (T_n) is tight.

The above results combine to show that (Y_n) is tight. The tightness of (X_n) then follows from the fact that $X_n=Y_n\circ\varphi$, where φ denotes the continuous mapping

$$\varphi(s,t)=(s\wedge t,\ s\vee t-s\wedge t), \qquad s,t\geqq 0. \quad \square$$

Proof of Theorem 7.3. Let X be directed by (ρ,ϑ,μ), and note that ρ and ϑ are a.s. unique and X-measurable by Theorem 7.1. To prove the same thing for μ, it suffices as before to consider processes (1.7) with constant coefficients.

By the proof of Theorem 7.1, we can construct the processes

$$
\begin{aligned}
T(s,t) &= \sigma(A_{st}+A_{ts}) + \sum_{j=1}^{\infty} \alpha_j (B_j(s)\, B_j(t) - s \wedge t), \\
U(t) &= \sum_{j=1}^{\infty} \beta_j\, B_j(t), \qquad V(t) = \sum_{j=1}^{\infty} \gamma_j\, B_j(t), \quad s, t \geqq 0,
\end{aligned} \tag{10}
$$

as measurable functions of X. A simple computation further shows that, for $|t|$ sufficiently small,

$$
\begin{aligned}
&\mathsf{E}\exp(t\,T_{11} + i u\, U_1 + i v\, V_1) \\
&= \exp\left(2t^2\sigma^2 - \frac{1}{2}\sum_{j=1}^{\infty}\left\{\log(1-2t\alpha_j) + 2t\alpha_j + \frac{u^2\beta_j^2 + v^2\gamma_j^2 + 2uv\beta_j\gamma_j}{1-2t\alpha_j}\right\}\right) \\
&= \exp\left\{-\frac{1}{2}\int \frac{\log(1-2tx)+2tx}{x^2}\,\mu_1(dx)\right. \\
&\quad \left. -\frac{1}{2}\int \frac{(u^2\mu_2 + v^2\mu_3 + uv(\mu_4 - \mu_2 - \mu_3))(dx)}{1-2tx}\right\},
\end{aligned} \tag{11}
$$

where the first integrand is defined by continuity to be $-2t^2$ at $x=0$. Putting $u=v=0$ yields the uniqueness of μ_1. Using a recursive argument as in the proof of Lemma 4.3, it may next shown that the sums

$$
\sum_{j\in J}\beta_j^2, \qquad \sum_{j\in J}\gamma_j^2, \qquad \sum_{j\in J}\beta_j\gamma_j
$$

are unique for all index sets J of the form $\{j\in\mathbb{N}\colon \alpha_j = x\}$. From these we may easily construct the measures μ_2, μ_3 and μ_4.

Conversely, these four measures determine the parameters σ^2 and α_j, β_j, γ_j, apart from order and from rotations of the sequence of pairs (β_j, γ_j) within groups of indices where the α_j assume a common value. As in case of Theorem 6.5, it is clear that such rotations do not affect the distribution of X. Thus (ρ, ϑ, μ) and $\mathsf{P}X^{-1}$ determine each other uniquely. The uniqueness part of the theorem now follows by Lemmas 2.3 and 2.4.

Let us next consider a sequence of processes X_n directed by $(\rho_n, \vartheta_n, \mu_n)$, $n\in\mathbb{N}$. To prove that (iii) implies (i), we may assume as before that the ρ_n, ϑ_n and μ_n are non-random. From (iii) it then follows by Lemma 7.4 that the corresponding sequence of triples (T_n, U_n, V_n), as defined by (10), is tight. Moreover, these triples are clearly jointly rotatable and dissociated in the obvious sense, so the same thing must be true for any limiting triple (T, U, V). The proof of Theorem 7.1 then shows that even the latter must be of the form (10), say with coefficients σ' and α'_j, β'_j, γ'_j, so (11) must hold for (T, U, V) with the associated measure μ'. But (11) is also true with the limiting measure μ, as may be seen by proceeding to the limit in formula (11) for (T_n, U_n, V_n). As above, we may then conclude that $\mu' = \mu$, so (T, U, V) is uniquely distributed, and we have in fact convergence $(T_n, U_n, V_n) \xrightarrow{d} (T, U, V)$. Thus (i) holds by continuity with X directed by (ρ, ϑ, μ).

To complete the proof, it remains to show, as in case of Theorem 6.5, that (ii) implies tightness of the sequence of directing triples $(\rho_n, \vartheta_n, \mu_n)$. For this purpose, consider first the increments of X over a square grid outside the diagonal, and conclude as in case of Theorem 6.5 that the sequence of triples (ρ, μ_1, μ_2) is tight. Using the implication (iii) ⇒(i), we may next conclude that the sequence of processes

$$\rho s t + \sum_{j=1}^{\infty} \{\alpha_j (B_j(s) B_j(t) - s \wedge t) + \beta_j (s B_j(t) + t B_j(s))\}, \qquad s, t \geqq 0,$$

is tight, and by subtraction from X we get tightness of the variables Z_{st} in (8) for fixed s and t. Taking $c > 0$ and writing

$$\begin{aligned} |\mathsf{E} \exp(i t Z_{11})| &= \left|\mathsf{E} \exp(i t \vartheta - \tfrac{1}{2} t^2 \textstyle\sum \gamma_j^2)\right| \leqq \mathsf{E} \exp\left(-\tfrac{1}{2} t^2 \textstyle\sum \gamma_j^2\right) \\ &\leqq 1 - (1 - e^{-\frac{1}{2} t^2 c})\, \mathsf{P}\{\textstyle\sum \gamma_j^2 > c\}, \end{aligned}$$

it follows easily that the sequence of sums $\sum \gamma_j^2$ is tight. The same thing must then be true for the measures μ_3, since their supports are contained in those for μ_1. The formula $\mu_4 \leqq 2(\mu_2 + \mu_3)$ shows that even the measures μ_4 are tight. From the result for $\sum \gamma_j^2$ it is further seen that the processes $\sum \gamma_j B_j(s \wedge t)$ form a tight sequence, and subtracting these from (8), we get the same result for $\vartheta(s \wedge t)$ at every fixed (s, t), and hence also for the variables ϑ. □

Acknowledgements. This work was done under the ideal working conditions provided by the Center for Stochastic Processes, Chapel Hill. I am grateful to faculty and staff at the Statistics Department, UNC, for their great hospitality during my stay. I would also like to thank a referee for a very careful reading of the manuscript.

References

1. Aldous, D.J.: Representations for partially exchangeable arrays of random variables. J. Multivariate Anal. **11**, 581–598 (1981)
2. Aldous, D.J.: Exchangeability and related topics. In: Hennequin, P.L. (ed.) École d'été de probabilités de Saint-Flour XIII-1983. (Lect. Notes Math., vol. 1117, pp. 1–198) Berlin Heidelberg New York: Springer 1985
3. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
4. Dawid, A.P.: Spherical matrix distributions and a multivariate model. J. R. Stat. Soc. Ser. B **39**, 254–261 (1977)
5. Dawid, A.P.: Extendibility of spherical matrix distributions. J. Multivariate Anal. **8**, 559–566 (1978)
6. Dellacherie, C., Meyer, P.A.: Probabilités et potentiel, chap. I–IV. Paris: Hermann 1975
7. Dovbysh, L.N., Sudakov, V.N.: Gram – de Finetti matrices (in Russian with English summary). Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **119**, 77–86, 238, 244–245 (1982). (See Math. Reviews 84d: 60006.)
8. Dynkin, E.B.: Sufficient statistics and extreme points. Ann. Probab. **6**, 705–730 (1978)
9. Feller, W.: An introduction to probability theory and its applications, vol. 2, 2nd edn. New York: Wiley 1971
10. Hestir, K.: Thesis, Statistics Department, University of California Berkeley

11. Hoover, D.N.: Relations on probability spaces and arrays of random variables. Preprint, Princeton 1979
12. Hoover, D.N.: Row-column exchangeability and a generalized model for probability. In: Koch, G., Spizzichino, F. (eds.) Exchangeability in probability and statistics (pp. 281–291) Amsterdam: North-Holland 1982
13. Itô, K., Nisio, M.: On the convergence of sums of independent Banach space valued random variables. Osaka J. Math. **5**, 35–48 (1968)
14. Kallenberg, O.: Canonical representations and convergence criteria for processes with interchangeable increments. Z. Wahrscheinlichkeitstheor. Verw. Geb. **27**, 23–36 (1973)
15. Kallenberg, O.: Random measures, 4th edn. Berlin London: Akademie-Verlag and Academic Press 1986
16. Kallenberg, O.: Spreading and predictable sampling in exchangeable sequences and processes. Ann. Probab. **26** (in press)
17. Lynch, J.: Canonical row-column-exchangeable arrays. J. Multivariate Anal. **15**, 135–140 (1984)

Received July 29, 1986; received in revised form October 10, 1987