Probab. Th. Rel. Fields 77, 307–324 (1988)

Probability Theory and Related Fields

Connectivity Properties of Mandelbrot's Percolation Process

J.T. Chayes [1,*], L. Chayes [1,**] and R. Durrett [2,***]

[1] Laboratory of Atomic and Solid State Physics, and
[2] Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853-7901, USA

Summary. In 1974, Mandelbrot introduced a process in $[0, 1]^2$ which he called "canonical curdling" and later used in this book(s) on fractals to generate self-similar random sets with Hausdorff dimension $D \in (0, 2)$. In this paper we will study the connectivity or "percolation" properties of these sets, proving all of the claims he made in Sect. 23 of the "Fractal Geometry of Nature" and a new one that he did not anticipate: There is a probability $p_c \in (0, 1)$ so that if $p < p_c$ then the set is "dustlike" i.e., the largest connected component is a point, whereas if $p \geqq p_c$ (notice the $=$) opposing sides are connected with positive probability and furthermore if we tile the plane with independent copies of the system then there is with probability one a unique unbounded connected component which intersects a positive fraction of the tiles. More succinctly put the system has a first order phase transition.

1. Introduction and Statement of Results

In this somewhat lengthy first section we describe the model, state our results, and give the easy and/or interesting parts of the proofs. The developments here are divided into three parts.

a. Definition and Hausdorff Dimension

The first step is to describe the model. Let $A_0 = [0, 1]^2$ and for $1 \leqq i, j \leqq N$ let

$$B_{ij} = \left[\frac{i-1}{N}, \frac{i}{N}\right] \times \left[\frac{j-1}{N}, \frac{i}{N}\right]$$

* Work supported by the NSF under Grant # DMR-83-14625
** Work supported by the DOE under Grant # DE-AC02-83-ER13044
*** Work supported by the NSF under Grant # DMS-85-05020

Let $\varepsilon_{ij} \in \{0, 1\}$ be independent "coin flips" with $P(\varepsilon_{ij}=1)=p$. If $\varepsilon_{ij}=1$ we say B_{ij} is occupied and we let

$$A_1 = \bigcup_{\substack{i,j \\ \varepsilon_{ij}=1}} B_{ij}$$

i.e. we keep the squares with $\varepsilon_{ij}=1$. To define A_2 we repeat the last construction (appropriately scaled) in each surviving B_{ij} or more generally, if we have constructed A_{n-1} then we let

$$B_{ij}^n = \left[\frac{i-1}{N^n}, \frac{i}{N^n}\right] \times \left[\frac{j-1}{N^n}, \frac{j}{N^n}\right], \qquad 1 \leqq i, j \leqq N^n$$

let $\varepsilon_{ij}^n \in \{0, 1\}$ be independent with $P(\varepsilon_{ij}^n = 1)=p$, and let

$$A_n = A_{n-1} \cap \Big(\bigcup_{\substack{i,j \\ \varepsilon_{ij}^n=1}} B_{ij}^n \Big)$$

$A_0, A_1, A_2, \ldots$ is a decreasing sequence of compact sets so the limit $A_\infty = \bigcap_n A_n$ exists (possibly ϕ). Mandelbrot calls A_∞ the curds and calls the complement $[0, 1]^2$-A_∞ the whey. Independent of what you call these things, the first question to be resolved is: "When is $A_\infty \neq \phi$?". Using some simple facts about branching processes it is easy to show.

(1) $A_\infty \neq \phi$ with positive probability if and only if $p > 1/N^2$.

Proof. Let Z_n be the number of squares of the form B_{ij}^n which are contained in A_n. Z_n is a branching process in which each particle has on the average $N^2 p$ offspring so if $Np^2 \leqq 1$ we have $P(Z_n > 0) \to 0$ as $n \to \infty$ and if $Np^2 > 1$ we have $P(Z_n > 0) \to \rho$ as $n \to \infty$ where ρ is positive solution of

$$((1-p)+p(1-x))^{N^2} = (1-x),$$

(see, e.g., Athreya and Ney (1972), Chap. 1).

From the results above we see that if $Np^2 \leqq 1$ then $A_n = \phi$ when n is sufficiently large so $A_\infty = \phi$. On the other hand if $Np^2 > 1$ then $P(A_n \neq \phi$ for all $n) > 0$ and since the A_n are a decreasing sequence of nonempty compact sets we have $A_\infty \neq \phi$ on $\Omega_0 \equiv \{A_n \neq \phi$ for all $n\}$.

Historically the first aspect of A_∞ which was considered was its "similarity dimension" (see "Fractal Geometry of Nature", hereafter abbreviated FGN, p. 211). To calculate this we observe that (i) if we multiply the unit square by N then on the average we have Np^2 copies of our set and (ii) if we multiply the unit cube in d-dimensions by N then we have N^d copies of it, so the "similarity dimension" of our set is

$$\frac{\log(N^2 p)}{\log N} = 2 + \frac{\log p}{N}$$

(the point of (ii) is that if we apply the last recipe to the unit cube it gives d).

The last formula is a well-known recipe for computing the Hausdorff dimension of things (e.g., the standard Cantor set has dimension $\log 2/\log 3$ because multiplying it by 3 produces 2 copies) so it is natural to try to show that the Hausdorf dimension is $2+(\log p)/\log N$. Half of this is very easy. Well-known results from branching processes (or martingale theory) imply that if Z_n = the number of B_{ij}^n contained in A_n then

$$W_n = Z_n/(N^2 p)^n \to W \quad \text{a.s.}$$

where W is a random variable with $EW=1$ and

$$\{W>0\} = \{Z_n>0 \text{ for all } n\} = \{A_\infty \neq \phi\}$$

(again this can be found in Chap. 1 of Athreya and Ney (1972)).

Since A_n can be covered by $Z_n = W_n \cdot (N^2 p)^n$ cubes with sides of length N^{-n} we see that if $\alpha = 2-(\log p)/(\log N)$ then the α-dimensional Hausdorff measure of $A_\infty \leqq W < \infty$ so the Hausdorff dimension of $A_\infty \leqq \alpha$. To get a bound in the other direction requires more work, but fortunately for us most of the work has already been done by Kahane and Peyrière (1976) in a paper titled "Sur certains martingales de Benoit Mandelbrot". Since the proof of $\dim(A_\infty)=\alpha$ is peripheral to our main concern – the connectivity properties of A_∞ – it is carried out in the appendix. For a more general result along these lines (developed independently of ours) see Mauldin and Williams (1986)

b. Basic Connectivity Properties, Existence of Phase Transition

With the random set defined and its Hausdorff dimension computed we turn our attention now to our main subject: the connectivity properties of A_∞. The first two result are essentially due to Mandelbrot (see FGN, pp. 215–216) but we use branching process arguments instead of his rule that "the co-dimensions add" (FGN, p. 213).

(1) If $p \leqq 1/N$ and x is not of the form m/N^n for some integers m and n then $P(A_\infty \cap (\{x\} \times [0, 1]) = \phi) = 1$.

Proof. The number of intervals of the form $\{x\} \times [(j-1)/N^n, j/N^n]$ contained in A_n is a branching process in which the mean number of offspring is Np.

The last result implies that if $p \leqq 1/N$ then the largest connected component is a point. By changing the value of x that we consider this result can be sharpened somewhat:

(2) If $p \leqq 1/\sqrt{N}$ then the largest connected component is a point.

Proof. We say a segment $\left[\frac{j-1}{N^n}, \frac{j}{N^n}\right] \times \{1/N\}$ is vacant if either of the two adjacent squares in the n^{th} subdivision is. The reason for this terminology can be seen in Fig. 1.1 which shows what might happen in the first two subdivisions when

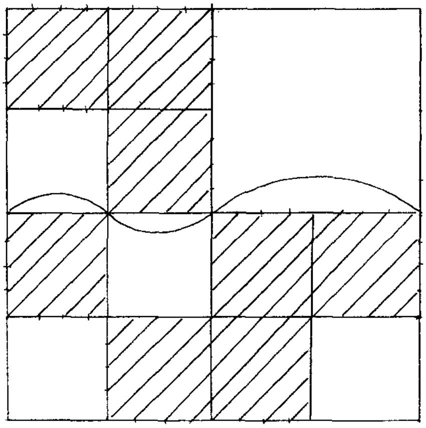


Fig. 1.1

$N=2$. After the first subdivision $[1/2, 1] \times \{1/2\}$ was vacant and after the second the whole interval is. At this point the wiggly line is not a path in the whey but eventually the squares which touch $[\frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2}]$ will become vacant and it will be.

With the last picture in mind we let Y_n be the number of occupied segments of the form $[(j-1)/N^n, j/N^n] \times \{1/N\}$. Y_n is a branching process in which each interval has $p^2 N$ offspring so if $p^2 N \leqq 1$, i.e., $p \leqq 1/\sqrt{N}$ then the branching process dies out with probability 1 and as above, if we wait a while longer there will be a path in the whey arbitrarily close to $[0, 1] \times \{1/N\}$. Repeating the last argument at heights j/N^n shows that with probability 1 there are curves in the whey arbitrarily close to all the lines $[0, 1] \times \{j/N^n\}$ and (reflecting the argument) also close to $\{j/N^n\} \times [0, 1]$ so the largest connected component is a point.

Note. When $N=2$ we have that $A_\infty \neq \phi$ for $p > 1/4$ and (2) applies when $p \leqq 1/\sqrt{2}$. A simulation of the first 8 subdivisions for that value is given in Fig. 1.2.

Having seen that A_∞ can be badly disconnected the logical next step is to ask if it can ever be connected. If we let $|A_n|$ denote the Lebesgue measure of A_n then $E|A_n| = p^n \to 0$ exponentially fast so at first this looks unlikely and in fact all three authors have thought at one time or another that the conclusion in (2) might hold for all $p < 1$. This is not the case, however, and a simple argument shows that if p is large enough then with positive probability there is a connected component which intersects all four sides of the square. After discovering our proof we noticed that the key idea appears in Mandelbrot's heuristic argument (see FGN, p. 217) so we will reverse the historical order of things and give his argument here and use it to motivate our rigorous proof.

"First consider the case in which the number of surviving squares K is non random. In this case of $N^2 - K > [N/2]$ (where $[x]$ = the largest integer $\leqq x$) there is no way that any given face between two precurd cells can fail

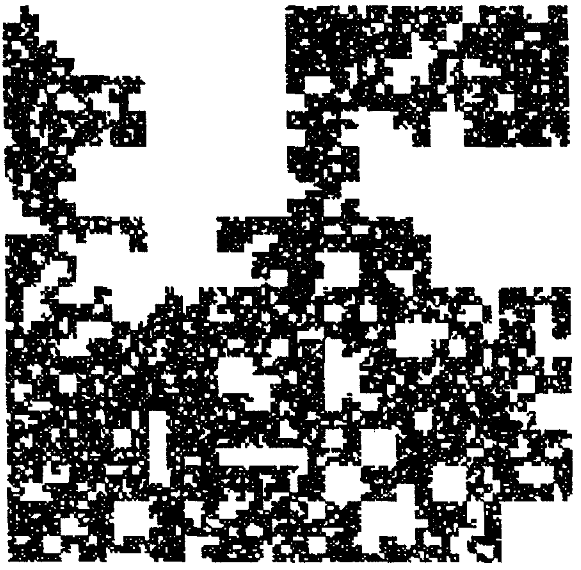
Fig. 1.2

to survive. Even if the worst happens and all the nonsurviving eddies crowd along said face, these eddies are so insufficient in number that it is sure (not almost, but absolutely) that no path becomes disconnected." Mandelbrot goes on to conclude that "With the same condition applied to unconstrained curdling, failure to percolate is no longer an impossibility but an unlikely event". While the last statement is a reasonable conclusion, it is not by mathematical standards a proof (although the bound implies on p_c is probably correct – see Fig. 1.3 for a simulation of the system with $p=8/9$) so we supply one in Sect. 2 (and invite the reader to supply the missing detail before we reveal the answer there).

To state our result and prepare for other developments below we need some definitions. Let $B_n=\{x\in A_n$: x can be connected to $\{0\}\times[0,1]$ and to $\{1\}\times[0,1]$ by paths in $A_n\}$ and let $B_\infty=\bigcap_{n=1}^{\infty} B_n$. If $x\in B_\infty$ let $C_n(x)$ be the component of B_n containing x. $C_1(x)\supset C_2(x)\supset\ldots$ are compact and connected so $\Gamma(x)=\bigcap C_n(x)$ is connected and has a nontrivial intersection with $\{0\}\times[0,1]$ and $\{1\}\times[0,1]$ (since all the $C_n(x)$ do). Let $\Omega_1=\{B_\infty\neq\phi\}$ (and recall $\Omega_0=\{A_\infty\neq\phi\}$). When Ω_1 occurs we say there is a left-to-right crossing of $[0,1]^2$. Let $p_c(N)=\inf\{p\colon P(\Omega_1)>0\}$. Our first result that we dare to call a theorem is

Theorem 1. $p_c(N)<1$ *for all* $N\geqq 2$.

As usual in arguments of this type the bound on p_c is ridiculous: we show that $p_c(3)<0.9999$ or what is a little less embarrassing: if $N=3$ and $p>0.9999$ then $P(\Omega_1)>0.999$.



P = .8888

Fig. 1.3

c. More Refined Properties

Having seen that if p is ridiculously close to 1 then the probability of a left-to-right crossing is positive and that if $p<1/\sqrt{N}$ then P(largest connected component of A_∞ = a single point) = 1 it is natural to let

$$p_d = \sup\{p: P(\text{largest connected component of } A_\infty = \text{a point}) = 1\}$$

$$p_c = \inf\{p: P(A_\infty \text{ has a left-to-right crossing}) > 0\}$$

(where d is for dustlike and c is for critical) and ask if $p_c = p_d$. This and more is proved in the next result which uses the notation introduced before Theorem 1.

Theorem 2. *Let $\Omega_1^n = \{B_n \neq \phi\}$. There is an $\varepsilon_0 > 0$ so that if $P(\Omega_1^n) \leqq \varepsilon_0$ then $P(\Omega_1) = 0$ and furthermore, the largest connected component is a point.*

The proof of Theorem 2 is based on two conclusions which are analogues of results for "ordinary" percolation. The proofs of these results are straightforward generalizations of the "classical" ones but it will take a lot of verbage to convince the reader of this and we will need to prove a second pair of result later in this section, so details are deferred to Sect. 3.

Let $\Omega_{1,K}^n$ be the event that there is a left-right crossing of $[0,1] \times [0,K]$ when independent copies of A_n are placed in each of the squares $[0,1] \times [k-1,k]$ $1 \leqq k \leqq K$.

(a) if $P(\Omega^n_{1,1}) \leqq \varepsilon$ then $P(\Omega^n_{1,K}) \leqq f_K(\varepsilon)$ where $f_K(\varepsilon) \to 0$ as $\varepsilon \to 0$ and
(b) if $P(\Omega^n_{1,2}) \leqq 0.01$ then

$$P(\Omega^{n+k}_{1,2}) \leqq \tfrac{1}{25}\exp(-N^{k-1})$$

Remark. The reader should observe that the last probability goes to 0 exponentially fast in the length of the cubes, and superexponentially fast in k.

With (a) and (b) in hand the rest is easy. If we pick ε_0 so that $f_2(\varepsilon) \leqq 0.01$ for all $\varepsilon \leqq \varepsilon_0$ and have n such that $P(\Omega^n_{1,1}) < \varepsilon_0$ then (a) and (b) imply $P(\Omega^n_{1,1}) \leqq P(\Omega^n_{1,2})$ goes to 0 as $n \to \infty$ and feeding this estimate into (a) shows $P(\Omega^n_{1,K}) \to 0$ for all $K < \infty$. The last observation implies that with probability 1 we have a "crack" (i.e., a curve from bottom to top which lies completely in the whey) in $[a, b] \times [0, 1]$ for all $a < b$ of the form $a = j/N^m$, $b = k/N^m$. Since this will also with probability 1 be true for all the reflected rectangles: $[0, 1] \times [a, b]$ it follows that the largest connected component is a point.

From Theorem 2 it follows immediately that the percolation probability $P(\Omega_1)$ is positive at p_c. To prove this, note that since $p \to P_p(\Omega^n_1)$ is continuous (it is a polynomial) and $\downarrow P_p(\Omega_1)$ as $n \to \infty$, $p \to P_p(\Omega_1)$ is upper semi-continuous. Since $p \to P_p(\Omega_1)$ is nondecreasing the last observation implies it must be right continuous on $[0, 1]$ and hence >0 at p_c i.e., there is positive probability of a left-to-right crossing when $p = p_c$.

Looking at the last result one might think "It is easy to see the source of the discontinuity above. Ω_1 is really just a sponge crossing event so the discontinuity is caused by the phenomenon in (b): if sponge crossing probabilities get too small then they go to 0". To get around this objection and have a phase transition which like other percolation processes involves the appearance of an unbounded connected component, we place an independent copy of our random set A_∞ in each square $z + [0, 1]^2$ $z \in Z^2$, call the result A'_∞, and look for percolation in A'_∞ in the usual sense: we let $\Omega_\infty = \{A'_\infty$ has an unbounded connected component$\}$ and let $p_b = \inf\{p : P(\Omega_\infty) > 0\}$.

It is clear that $p_b \geqq p_c \geqq p_d$. (This should help explain the somewhat unusual notation. To help you remember what the b stands for think of unbounded and observe that p_u is clearly unacceptable). Our last result shows that $p_b = p_c$.

Theorem 3. *If $p \geqq p_c$ then with probability 1, A'_∞ has a unique unbounded connected component.*

Comparing this with Theorem 2 shows that the system undergoes a very violent transition as we pass through $p = p_c$. When $p < p_c$ the largest connected component is a point, but when $p = p_c$ then there is a unique unbounded component. The reader should note that if we let $\Omega_\infty = \{$the unbounded component of A'_∞ touches $[0, 1]^2\}$ then $P(\Omega_\infty) > 0$ at p_c, so $p \to P_p(\Omega_\infty)$ is (like $p \to P_p(\Omega_1)$) is discontinuous at p_c.

The key to the proof of Theorem 3 is the observation that if we rescale A'_∞ by dividing by N and then flip new coins to see which squares of the form $[(i-1)/N, i/N] \times [(j-1)/N, j/N]$ are occupied then the result has the same distribution as A'_∞, so if we ignore the second step we have

$$A'_\infty/N \overset{d}{=} (A'_\infty \mid \text{all } \varepsilon^1_{ij} = 1)$$

or iterating the last result

$$A'_\infty/N^n \stackrel{d}{=} (A'_\infty \,|\, \text{all } \varepsilon_{ij}^m = 1, m \leqq n).$$

The last observation makes it easy to believe (and prove) that if $P(\Omega_1)>0$ then the probability of a left to right crossing of $[0, N^n]$ in $A'_\infty \to 1$ as $A \to \infty$. To prove this we observe that $P(\Omega_1^n) \to P(\Omega_1)$ as $n \to \infty$ so if $\varepsilon>0$ and n is large then

$$P(\Omega_1^n - \Omega_1) \leqq \varepsilon P(\Omega_1) \leqq \varepsilon P(\Omega_1^n)$$

or

$$P(\Omega_1 \,|\, \Omega_1^n) \geqq 1-\varepsilon$$

and hence

$$P(\Omega_1 \,|\, \varepsilon_{ij}^m = 1 \text{ for all } m \leqq n) \geqq 1-\varepsilon.$$

Having established that the crossing probabilities of large squares is close to 1 the result now follows from two more facts we will prove in Sect. 3. To avoid the topological nightmares which would come from trying to deal with A'_∞ directly we will consider the situation after n subdivisions and prove results which are independent of n.

Let A'_n be the set which results when we place independent copies of A_n in each square $z+[0,1]^2$, $z \in Z^2$. Let $\Omega_{J,K}^n$ be the event that there is a left-to-right crossing of $[0, J] \times [0, K]$ in A'_n.

(a′) *If* $P(\Omega_{L,L}^n) \geqq 1-\varepsilon$ then $P(\Omega_{kL,L}^n) \geqq 1-g_k(\varepsilon)$ where $g_k(\varepsilon)$ is independent of n and $\to 0$ as $\varepsilon \to 0$

(b′) If $P(\Omega_{2L,L}^n) \geqq 0.99$ then

$$P(\Omega_{2^kL,2^{k-1}L}^n) \geqq 1 - \tfrac{1}{25}\exp(-2^{k-1}).$$

The notation and the numbers in (b′) should remind the reader of (a) and (b) stated earlier. We will see in Sect. 3 that they are closely related. With (a′) and (b′) in hand the conclusion follows from a standard construction (see Smythe and Wierman (1978), Chap. 3). Let L be chosen so that $P(\Omega_{2L,L}^n) \geqq 0.99$ for all n when $p=p_c$, let $B_1=[0,2L]\times[0,L]$ $B_2=[0,2L]\times[0,4L]$ and for $j \geqq 2$ let

$$B_{2j-1} = [0, 2^jL] \times [0, 2^{j-1}L]$$

$$B_{2j} = [0, 2^jL] \times [0, 2^{j+1}L].$$

Λ_{2j-1}^n = there is a left to right crossing of B_{2j-1} in A'_n

Λ_{2j}^n = there is a top to bottom crossing of B_{2j} in A'_n

The events are chosen so that if all the crossings occur then our construction guarantees there is an infinite path in A'_n starting at some point in $\{0\} \times [0, L]$. The estimate in (b′) shows that

$$P(\Lambda_{2j-1}^{n,c}) \leqq \tfrac{1}{25}\exp(-2^{j-1})$$

$$P(\Lambda_{2j}^{n,c}) \leqq \tfrac{1}{25}\exp(-2^j)$$

(where the additional c in the superscript is for complement). Summuming over j and using the fact that $2(j-1) \leqq 2^{j-1}$ for $j \geqq 1$ and $e \geqq 2$ we get

$$\sum_{k=1}^{\infty} P(A_k^{n,c}) \leqq \sum_{j=1}^{\infty} 2 \cdot \tfrac{1}{25}(e^{-2})^{j-1} = \tfrac{2}{25}(1-e^{-2})^{-1} \leqq \tfrac{2}{25} \cdot \tfrac{4}{3} < \tfrac{1}{9}$$

From the last computation it follows immediately that $P\left(\bigcap_{k=1}^{\infty} A_k^n\right) > 8/9$ and hence with probability $\geqq 8/9$ there is an infinite path in A'_n starting at some point in $\{0\} \times [0, L]$. Letting $n \to \infty$ and taking intersections as we did before the statement of Theorem 1 it is easy to see that the same is true for $n = \infty$.

The argument above shows the existence of an unbounded component when $p \geqq p_c$. The fact that it is unique is proved in the same way as in the ordinary case and the reader is referred to Harris (1960) or p. 50 of Smythe and Wierman (1978).

Having demonstrated that Mandelbrot's model has a discontinuous transition while "ordinary" percolation has a continuous one, it is appropriate to take a moment to reflect on what caused the difference, and with this in mind we would like to observe: in "ordinary" percolation occupied and vacant crossings play essentially interchangeable roles but in Mandelbrot's model there is a fundamental asymmetry – a vacant crossing which exists at a given iteration will persist thereafter while an occupied crossing may be lost at any subsequent time.

2. Proof of Theorem 1

We will prove the result only in the case $N=3$. It will be clear from the first observation that the same proof works for any $N \geqq 3$. The case $N=2$ can be treated by comparing with $N=4$. The first observation (due to Mandelbrot and mentioned in the introduction) explains why we start with $N=3$: in this case as long as 8 of the 9 squares are occupied, then any two adjacent squares must have adjacent occupied boundary squares.

This observation motivates the following definitions. We say an outcome is good if A_1 contains at least 8 squares B_{ij}^1. We say an outcome is very good if A_1 contains at least 8 squares B_{ij}^1 which are good, i.e., contain at least 8 squares when they are subdivided. For $m \geqq 2$ we say an outcome is (very)^m good if A_1 contains at least 8 squares B_{ij}^1 in which $A_m \cap B_{ij}^1$ is (very)$^{m-1}$ good.

Let θ_m be the probability that the outcome is (very)^m good. From the recursive definition it is clear that

$$\theta_m = p^9(\theta_{m-1}^9 + 9\,\theta_{m-1}^8(1-\theta_{m-1})) + 9\,p^8(1-p)\,\theta_{m-1}^8$$

for $m \geqq 1$ and

$$\theta_0 = p^9 + 9\,p^8(1-p).$$

Here θ_0 is what we get if we let $\theta_{-1}=1$ in the previous definition, so if we let

$$\varphi(x)=p^9(9x^8-8x^9)+9p^8(1-p)x^8$$

then $\theta_m=\varphi^{m+1}(1)$ where $\varphi^{m+1}(x)=\varphi(\varphi^m(x))=\dots$ and a little thought gives:

(1) As $n\uparrow\infty$ $\varphi^n(1)\downarrow\rho=$ the largest fixed point of φ in $[0, 1]$.

With (1) in hand, the proof of Theorem 1 will be complete once we show that if p is close to 1, φ has a fixed point >0. To simplify notation let

$$\alpha=p^9 \qquad \beta=9p^8(1-p)$$

so that φ may be written as

$$\varphi(x)=(9\alpha+\beta)x^8-8\alpha x^9$$

Letting $x=1-\varepsilon$ and observing that

$$(1-\varepsilon)^k=1-k\varepsilon+\frac{k(k-1)}{1\cdot 2}\varepsilon^2-\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}\varepsilon^3+\dots$$

we see that when $(8-2)\varepsilon/3<1$

$$(1-\varepsilon)^8\geqq 1-8\varepsilon$$

and when $(9-3)\varepsilon/4<1$

$$(1-\varepsilon)^9\leqq 1-9\varepsilon+36\varepsilon^2,$$

since the last two conditions imply that the terms we have dropped alternate in sign and decrease in magnitude.

The last observation implies that for $\varepsilon<2/3$

$$\begin{aligned}\varphi(1-\varepsilon)&\geqq(9\alpha+\beta)(1-8\varepsilon)-8\alpha(1-9\varepsilon+36\varepsilon^2)\\&=(\alpha+\beta)-8\beta\varepsilon-288\alpha\varepsilon^2\end{aligned}$$

Now α, $\beta\geqq 0$ and $\alpha+\beta\leqq 1$ so if $\varepsilon<1/8$ we have

$$\varphi(1-\varepsilon)\geqq\alpha-288\varepsilon^2.$$

Setting $\varepsilon=0.001$ and $\alpha=1-0.5\varepsilon$ gives

$$\varphi(1-\varepsilon)\geqq 1-0.788\varepsilon$$

so φ has a fixed point in $[0.999, 1]$. Recalling $\alpha=p^9$ and $(1-\delta)^9\geqq 1-9\delta$ when $(9-2)\delta/3<1$ we see that if $p>0.9999$ then $P(\Omega_1)>0.999$ as claimed in the introduction.

3. Proofs of the Sponge Crossing Results

In this section we will prove (a), (b), (a′), and (b′) from Sect. 1 and therefore, complete the proofs of Theorem 2 and 3. In each case our proofs are obtained by modifying one step of the proof of the analogous statements for ordinary site percolation, so we will begin by recalling those proofs.

To prepare for our later claims the reader should observe that the variables which indicate whether the squares $[(j-1)/N^n, j/N^n] \times [(k-1)/N^n, k/N^n]$ are occupied or not are increasing functions of independent random variables and, hence, are positively correlated in the sense of Harris (see Kesten (1982), Sect. 4.1 for a statement and proof of this result), so steps which only use his inequality will generalize immediately to our settings. When all is said and done, this leaves only two places (one in each result) where the argument has to be modified.

Let $\rho_{J,K}$ be the probability there is a left to right crossing of $\{1, \dots J\} \times \{1, \dots K\}$ by open sites when sites are independently open with prob p and closed with probability $1-p$. Our first goal is to prove

(A) if $\rho_{L,L} \geqq 1-\varepsilon$ then $\rho_{kL,L} \geqq 1-h_k(\varepsilon)$ where $h_k(\varepsilon)$ is independent of L and $\to 0$ as $\varepsilon \to 0$

(B) if $\rho_{2L,L} \geqq 0.99$ then

$$\rho_{2^k L, 2^{k-1} L} \geqq 1 - \tfrac{1}{25}\exp(-2^{k-1}).$$

The hardest part of doing this is the first step: to prove (A) for some $k>1$.

$$\rho_{3L/2,L} \geqq (1-(1-\rho_{L,L})^{1/2})^3. \tag{1}$$

Proof. The first lemma explains the unusual formula in the answer.

The square root trick. Let A_1 and A_2 be increasing events. If $A = A_1 \cup A_2$ where $P(A_1) = P(A_2)$ then

$$P(A_1) \geqq 1-(1-P(A))^{1/2}$$

Proof. From set theory and Harris' inequality we get

$$\begin{aligned}(1-P(A_1))^2 &= P(A_1^c)^2 = P(A_1^c)P(A_2^c) \\ &\leqq P(A_1^c \cap A_2^c) = 1-P(A)\end{aligned}$$

so $P(A_1) \geqq 1-(1-P(A))^{1/2}$.

Remark. The reader should observe that this result only uses Harris' inequality, so the parts of the argument which use this trick also generalize immediately.

The lemma above allows us to have paths begin or end in one half of the square without dividing the probability by 2. With this and a little geometric trickery we can get the paths we want. In this part of the proof we follow Russo (1982), pp. 230–231, very closely, mostly using his notation, so we will be a little vague about the definitions and refer the reader to our picture (Fig. 3.1) or Russo's paper to figure out the precise definitions. Let E_s be the event that s is the lowest left-right crossing of $[0, L] \times [0, L]$.

Let s_r be the portion of this path from the time it last hits $\{L/2\} \times [0, L]$ until it reaches $\{L\} \times [0, L]$ (thick line in Fig. 3.1).

Let s_{rr} = the reflection of s_r through $\{L\} \times [0, L]$ (dotted line in Fig. 3.1).

Let $\mathscr{A}(s_r \cup s_{rr})$ = the points in $\left[\frac{L}{2}, \frac{3L}{2}\right] \times [0, L]$ strictly above $s_r \cup s_{rr}$.

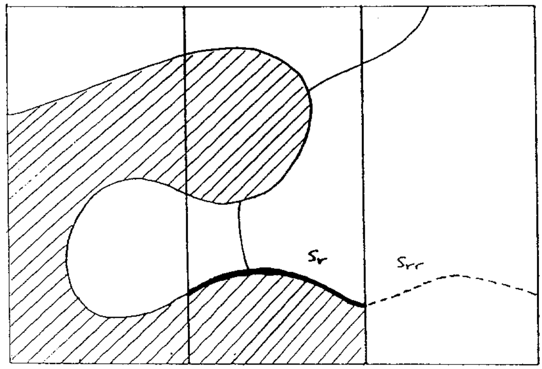


Fig. 3.1

Let $\mathscr{B}(s)$ = the points in $[0, L] \times [0, L]$ strictly below s (the shaded region in Fig. 3.1).

Let F_s be the event that there is a path starting from $[L/2, 3L/2] \times \{L\}$ and connected to s_r in $\mathscr{A}(s_r \cup s_{rr}) - \mathscr{B}(s)$ (and notice that this definition allows you to use s which consists of open sites).

Let G be the union of $E_s \cap F_s$ over all the paths s for which the first point of s_r has y coordinate $\leqq L/2$ (like the one drawn in Fig. 3.1).

Let H be the event that there is a left-to-right crossing of $[L/2, 3L/2] \times [0, L]$ which starts at a point with y-coordinate $\geqq L/2$.

We have not drawn a path of the last type in Fig. 3.1, but we invite you to do so now to convince yourself that on $G \cap H$ there is a left-to-right crossing of $[0, 3L/2] \times [0, L]$ so to prove (1) it suffices to show

$$P(G \cap H) \geqq (1 - (1 - \rho_{L,L})^{1/2})^3$$

The first step in doing this is to observe that Harris' inequality implies

$$P(G \cap H) \geqq P(G)\, P(H)$$

and using the square root trick with $A = \{$there is a crossing of $[L/2, 3L/2] \times [0, L]\}$ and $A_1 = H$ gives

$$P(H) \geqq (1 - (1 - \rho_{L,L})^{1/2})$$

To estimate $P(G)$ we write

$$P(G) = \sum_s P(E_s \cap F_s) = \sum_s P(E_s)\, P(F_s \mid E_s)$$

and observe that if F_s' is the event, that there is a path from $\left[\frac{L}{2}, \frac{3L}{2}\right] \times \{L\}$ to s_r in $\mathscr{A}(s_r \cup s_{rr})$ then

$$(*) \qquad P(F_s \mid E_s) = P(F_s) \geqq P(F_s')$$

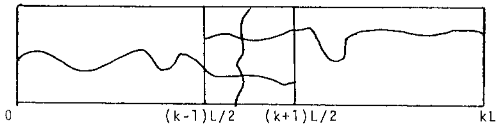


Fig. 3.2

(here we use two things (i) E_s is measurable with respect to the sites in $\bar{\mathscr{B}}(s) = \mathscr{B}(s) \cup s$, (ii) the presence of s which is open makes it easier to find the connections we want. Notice that in (i) we use independence and this is the first time we have used something more than Harris' inequality.)

With (*) established, the rest is easy because two more applications of the square root trick gives

$$P(F_s') \geqq (1-(1-\rho_{L,L})^{1/2})$$
$$\sum_s P(E_s) \geqq (1-(1-\rho_{L,L})^{1/2})$$

and putting the pieces together we have (1).

With (1) in hand the rest of the proof of (A) is easy. We only have to prove

(2) $$1-\rho_{k,L,L} \leqq 3(1-\rho_{(k+1)L/2,L}) \quad \text{for } k \geqq 1.$$

Proof. To prove this we draw a picture (Fig. 3.2) and observe that if all 3 paths exist then there is a crossing. The inequality above results from using the fact that

$$P\left(\bigcup_{i=1}^{3} A_i^c\right) \leqq \sum_{i=1}^{3} P(A_i^c)$$

and $\rho_{L,L} \geqq \rho_{(k+1)L/2,L}$ for $k \geqq 1$.

Combining (1) and (2) gives

$$\rho_{3L/2,L} \geqq (1-(1-\rho_{L,L})^{1/2})^3$$
$$1-\rho_{2L,L} \leqq 3(1-\rho_{3L/2,L})$$
$$1-\rho_{3L,L} \leqq 3(1-\rho_{2L,L})$$

etc., which gives bounds on $\rho_{kL,L}$ in terms of $\rho_{L,L}$ and completes the proof of (A).

The next two inequalities (due to Aizenman et al. (1983)) are the keys to the proof of (B):

(3a) $$1-\rho_{4L,L} \leqq 5(1-\rho_{2L,L}).$$

(3b) $$\rho_{4L,2L} \geqq 1-(1-\rho_{4L,L})^2.$$

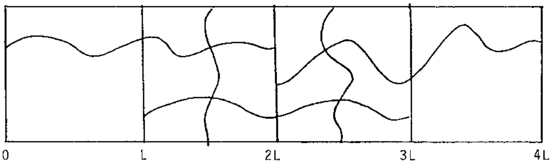


Fig. 3.3

Proof. For (3a) we draw another picture (Fig. 3.3) and observe that (i) if all five paths exist then there is a crossing and then argue as in the proof of (2).

To prove (3b) we observe that the existence of a crossing in $[1, 4L] \times [1, L]$ and $[1, 4L] \times [L+1, 2L]$ are independent events and a crossing of $[1, 4L] \times [0, 2L]$ occurs if at least one of them does. (Here we use independence for the second and final time.)

With the last two inequalities in hand the rest is just arithmetic. Combining (3a) and (3b) gives

(4) $$\rho_{4L,2L} \geqq 1 - 25(1-\rho_{2L,L})^2$$

and iteration does the rest: for if $\rho_{2L,L} = 1 - \lambda/25$ where $\lambda < 1$ then

(4) implies

$$\rho_{4L,2L} \geqq 1 - \lambda^2/25$$

$$\rho_{8L,4L} \geqq 1 - \lambda^4/25$$

and by induction that

$$\rho(2^k L, 2^{k-1} L) \geqq 1 - \tfrac{1}{25} \exp(2^{k-1} \log \lambda).$$

If we let $\lambda = 1/4$ and use the fact that $\log(1/4) < -1$ we get the inequality in (B).

Having carefully disected the independent case we turn to the proofs of (a′), (b′), (a), and (b) taking them in that order.

(a′) If $P(\Omega^n_{L,L}) \geqq 1 - \varepsilon$ then $P(\Omega^n_{kL,L}) \geqq 1 - g_k(\varepsilon)$ where $g_k(\varepsilon)$ is independent of n and $\to 0$ as $\varepsilon \to 0$.

Proof. From our discussion of the independent case it suffices to show that

(*) $$P(F_s | E_s) \geqq P(F_s)$$

(see the proof of (1) above) is valid in our setting. In the independent case, if we condition on the location of s then the sites above s have the same distribution as they did orginally (i.e., independent). In the present setting this is not true but something better happens. The presence of the path is "good news", i.e., the conditional distribution is larger than the original and the inequality we want is true.

To make the argument in the last paragraph precise we need to introduce some notation to describe the conditional distribution. The reader is encouraged

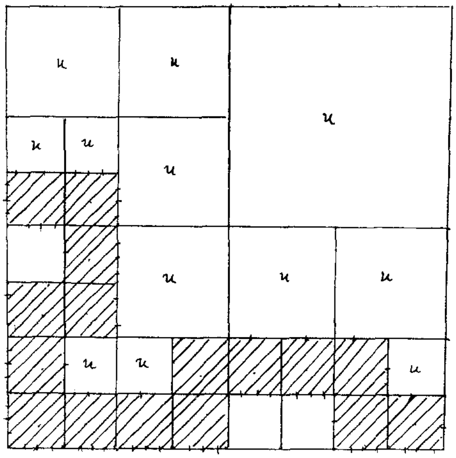


Fig. 3.4

to look at Fig. 3.4 while we do this. (In that drawing the shaded squares are occupied, blank squares are vacant (and left unsubdivided on the next level), and squares marked with u are unconditioned, i.e., we do not need to know their fate to know that the shaded set is the lowest crossing.)

Let s be the lowest left-to-right crossing of $[0, 1]^2$ in A_n ($s=a$ union of squares) when we consider squares to be adjacent if they share a side in common. Let $\mathscr{A}(s)$ be region above s, defined in the obvious way. A square of the form $[(i-1)/N^m, i/N^m] \times [(j-1)/N^m, j/N^m]$ is said to be unconditioned if it lies in $\mathscr{A}(s)$, because in this case its coin flip ε_{ij}^m is independent of the event $\{s$ is the lowest left-to-right crossing$\}$.

In addition to unconditioned squares, there are, of course, also squares which intersect s. The latter squares must be occupied, for otherwise the part of s they touch would not be, so they are our friends. The last observation shows that $(*)$ holds and completes the proof of (a') so we proceed now to the proof of:

(b') If $P(\Omega^n_{2L,L}) \geqq 0.99$ then

$$P(\Omega^n_{2^kL,\, 2^{k-1}L}) \geqq 1 - \tfrac{1}{25}\exp(-2^{k-1})$$

Proof. From our discussion of the independent case it suffices to show that (3b) is valid in this setting, but this is trivial: the existence of a crossing in $[0, 4L] \times [0, L]$ and $[0, 4L] \times [L, 2L]$ *are* independent events.

Having dealt with (a') and (b') the next item on the agenda is to prove:

(a) if $P(\Omega^n_{1,1}) \leqq \varepsilon$ then $P(\Omega^n_{1,K}) \leqq f_k(\varepsilon)$ where $f_k(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. The first step is to turn this into a problem about percolation probabilities close to 1 by looking at the vacant sites on the dual graph $G^* = (Z^2, \mathscr{E}')$, i.e., the points in the graph are Z^2 and the edges $\mathscr{E}' = \{(z, z+u)$ where $u_1^2 + u_2^2 \leqq 2\}$ i.e., in addition to nearest neighbor connections $z \to z+(1, 0), \ldots z \to z+(0, -1)$ connections to diagonal nearest neighbors $z \to z+(1, 1), \ldots z \to z+(1, -1)$ are possible.

G^* is the (graph-theoretic) dual of $G=(Z^2, \mathscr{E})$ where $\varepsilon=\{(z, z+u): u_1^2+u_2^2 \leqq 1\}$ is the usual set of edges connecting nearest neighbor sites and as is well known either there is always either an occupied left to right crossing of $\{1, \ldots J\} \times \{1, \ldots K\}$ or a vacant top to bottom crossing of the same rectangle but not both (see Sect. 2.6 and Chap. 3 of Kesten (1982) more details). From the last observation it follows that if we let $\tilde{\Omega}^n_{J,K}$ be the probability of a top-to-bottom crossing of $[0, J] \times [0, K]$ by vacant squares when we count all squares which touch as adjacent, then it is sufficient to prove

(ã) If $P(\tilde{\Omega}^n_{1,1}) \geqq 1-\varepsilon$ then $P(\tilde{\Omega}^n_{1,K}) \geqq 1-f_K(\varepsilon)$ where $f_K(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The proof of the last result is very similar to the proof of (a′). From our discussion of the independent case it suffices to show that

$$(*) \qquad P(\tilde{F}_s \mid \tilde{E}_s) \geqq P(\tilde{F}_s)$$

where the $\sim$ indicates that we are referring to vacant crossings, but otherwise considering the same events as in the proofs of (1) and (a′) above.

To prove (∗) this time we repeat the argument in the proof of (a′). The unconditioned squares are still unconditioned and the ones which touch s (now a vacant crossing) are affected by knowing the square they touch is vacant but this time we cannot conclude that the corresponding coin flip $=0$. It is a delicate matter to prove (and in general false) that conditioning on an increasing event causes the set of vacant sites to be larger than the original in the sense of stochastic monotonicity (i.e., the two sets can be constructed on the same space in such a way that one includes the other). Fortunately we do not need this here. We are interested in one decreasing event so it follows from Harris' inequality that

$$P(\tilde{F}_s \mid \tilde{E}_s) \geqq P(\tilde{F}_s)$$

and the proof of (a) is complete.

The last thing to be shown is

(b) if $P(\Omega^n_{1,2}) \leqq 0.01$ then

$$P(\Omega^{n+k}_{1,2}) \leqq \tfrac{1}{25} \exp(-N^{k-1})$$

Proof. As in the proof of (a) it suffices to prove ($\tilde{\text{b}}$) if $P(\tilde{\Omega}^n_{1,2}) \geqq 0.99$ then

$$P(\tilde{\Omega}^n_{1,2}) \geqq 1-\tfrac{1}{25} \exp(-N^{k-1})$$

Now

$$P(\tilde{\Omega}^n_{N^m, 2N^m}) = P(\tilde{\Omega}^{n+m}_{1,2} \mid \text{all } \varepsilon^k_{ij}=1 \text{ when } k \leqq m)$$

so it suffices to show

$$P(\tilde{\Omega}^n_{N^m, 2N^m}) \to 1$$

exponentially fast, but having changed our perspective so that the squares get larger, the adjacent rectangles used in the proof of (3b) *are* independent, and repeating the proof of (b′) proves ($\tilde{\text{b}}$) and hence (b).

Appendix: Computation of the Hausdorff Dimension

In this section we will complete the proof that $\dim(A_\infty)=\alpha$ by showing $\dim(A_\infty)\geqq\alpha$. As we mentioned earlier the key to doing this is found in a paper of Kahane and Peyrière who studied the original process defined by Mandelbrot in (1974) (see pp. 342–343). In that model the squares B_{ij}^n are assigned i.i.d. weights w_{ij}^n with $P(w_{ij}^n=1/p)=p$ and $P(w_{ij}^n=0)=1-p$ and we define a sequence of measures μ_n on $[0,1]^d$ with $\mu_n=u_n(x)\,dx$ where the u_n satisfy $u_0=1$ and if $x\in B_{ij}^n$

$$u_n(x)=u_{n-1}(x)\,w_{ij}^n.$$

Kahane and Peyrière studied the limiting behavior of u_n and computed the Hausdorff dimension of the support of the limit set. That as the old expression goes, is the good news, the bad news is that they proved their result only for subdivision of the unit interval so we will have to describe their results and proofs to show that in our situation there is very little difference between cutting the unit interval into $c=N^2$ pieces or the unit square into N^2 squares.

The first and most basic observation is that if we let $Y_n=\|\mu_n\|=$ the total variation of μ_n then Y_n is a nonnegative martingale so as $n\to\infty$, Y_n converges almost surely to a limit Y_∞ which has $EY_\infty\leqq EY_0=1$. Repeating the last argument in each square B_{ij}^m we see that for each i,j and m, $\mu_n(B_{ij}^m)$ converges to a limit which we call $\mu(B_{ij}^m)$ and this defines a (random) measure on $[0,1]^2$.

The first question which must be answered is: When is $\mu=0$ (i.e., $Y_\infty=0$)? The answer found by Kahane and Peyrière is

(A1) The following are equivalent:

(a) $EY_\infty=1$

(b) $EY_\infty>0$

(c) $E(w\log w)<\log(N^2)$.

In our case $E(w\log w)=\frac{1}{p}\left(\log\frac{1}{p}\right)p$ so the condition is $N^2p>1$ the necessary and sufficient condition for $\{A_\infty\neq\phi\}$ to have positive probability and a trivial necessary condition for $Y_\infty\neq 0$.

The second theorem in their paper is

(A2) Let $h>1$. One has $0<EY_\infty^h<\infty$ if and only if $E(w^h)<(N^2)^{h-1}$.

In our case $E(w^h)=(1/p)^h p=(1/p)^{h-1}$ so if $p>N^{-2}$, i.e., $1/p<N^2$ then $EY_\infty^h<\infty$ for all $h<\infty$. This fact is also confirmed by their third theorem which we will not state. We mentioned the moments of Y_∞ only so that we could apply their fourth theorem.

(A3) Suppose $E(Y_\infty\log Y_\infty)<\infty$. For each $x\in[0,1]^2$ let $I_n(x)$ be the square B_{ij}^n containing x. Then μ almost surely we have

$$\lim_{n\to\infty}\frac{\log\mu(I_n(x))}{\log|I_n(x)|}=1-E(w\log_c w)$$

where $c=N^2$ and $|I_n(x)|=$ the Lebesgue measure of $I_n(x)$.

We have stated the last result in $[0, 1]^2$ rather than in $[0, 1]$ where it was originally proved, but it is clear from the statement that it is true for subdivisions of a general measure space. The dimension only changes the

Corollary. *Let $D=2(1-E(w\log_c w))$. The measure μ is almost surely supported by a Borel set of dimension D such that each Borel set of dimension $<D$ has μ measure* 0.

It does not change the proof though: "Pour demontrer le corollaire on utilise un théorème de Billingsley (1965), pp. 136–145".

Having found the Hausdorff dimension of the support of μ the last thing to check is that it agrees with the upper bound. In our case $E(w\log_c w) = 1/p(\log_c 1/p)p$ and $\log_c x = \log x/\log(N^2)$ so

$$D=2\left(1-\frac{(-\log p)}{2\log N}\right)=2+\frac{\log p}{\log N}$$

which checks with the upper bound and we have identified the Hausdorff dimension of A_∞ as $\alpha=2+(\log p)/\log N$.

References

Aizenman, M., Chayes, J.T., Chayes, L., Fröhlich, J., Russo, L.: On a sharp transition from area to perimeter law in a system of random surfaces. Commun. Math. Phys. **92**, 19–69 (1983)

Billingsley, P.: Ergodic theory and information. New York: Wiley 1965

Harris, T.E.: A lower bound for the critical probability in a certain percolation process, Proc. Camb. Phil. Soc. **56**, 13–20 (1960)

Kahane, P., Peyrière, J.: Sur certaines martingales de Benoit Mandelbrot. Adv. Math. **22**, 131–145 (1976)

Kesten, H.: The critical probability for bond percolation on the square lattice equals 1/2. Comm. Math. Phys. **74**, 41–59 (1980)

Kesten, H.: Percolation theory for mathematicians, Boston: Birkhäuser 1982

Mandelbrot, B.: Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. J. Fluid. Mech. **62**, 331–358 (1974)

Mandelbrot, B.: The fractal geometry of nature. New York: Freeman 1983

Mauldin, R.D., Williams, S.C.: Random recursive constructions: asymptotic geometric and topological properties. Trans. Am. Math. Soc. **295**, 325–346 (1986)

Russo, L.: A note on percolation. Z. Wahrscheinlichkeitstheor. Verw. Geb. **43**, 39–48 (1978)

Russo, L.: On the critical percolation probabilities. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 229–237 (1987)

Seymour, P.D., Welsh, D.J.A.: Percolation probabilities on the square lattice. Ann. Discrete Math. **3**, 227–245 (1978)

Smythe, R.T., Wierman, J.C.: First passage percolation on the square lattice, Lect. Notes Math., vol. 671. Berlin Heidelberg New York: Springer 1978

Received July 17, 1986