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Limit Theorems for One-Dimensional Diffusions and Random Walks in Random Environments

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Summary. The limiting behavior of one-dimensional diffusion process in an asymptotically self-similar random environment is investigated through the extension of Brox's method. Similar problems are then discussed for a random walk in a random environment with the aid of optional sampling from a diffusion model; an extension of the result of Sinai is given in the case of asymptotically self-similar random environments.

Introduction

Let $\Xi=\{\xi(x),\, x\in\mathbb{Z}\}$ be a sequence in (0, 1), that is, $\Xi\in(0,1)^{\mathbb{Z}}$, and let us consider a random walk $\{X_n,\, n=0,1,\dots,P_\Xi\}$ in the environment Ξ such that

$$P_\Xi\{X_{n+1}=x+1\,|\,X_n=x\}=\xi(x),$$
$$P_\Xi\{X_{n+1}=x-1\,|\,X_n=x\}=1-\xi(x),\qquad x\in\mathbb{Z}.$$

We consider a product probability measure $Q=\prod_{x\in\mathbb{Z}} q_x$ on $(0,1)^{\mathbb{Z}}$, where q_x is a given probability measure in (0, 1) independent of x. Thus $\{\xi(x),\, x\in\mathbb{Z}, Q\}$ is a family of i.i.d. random variables. The full distribution governing $\{X_n\}$ is $\mathscr{P}=\int Q(d\Xi)P_\Xi$. Sinai [13] proved the following: if

$$E^Q\left\{\log\frac{1-\xi(x)}{\xi(x)}\right\}=0,$$

$$0<E^Q\left\{\left|\log\frac{1-\xi(x)}{\xi(x)}\right|^2\right\}=\sigma^2<\infty,$$

then $\sigma^2(\log n)^{-2}X_n$ has a limit distribution as $n\to\infty$.

Recently Brox [2] obtained a similar limit theorem for the one-dimensional diffusion process $X(t, W)$ described by the stochastic differential equation

$$dX(t) = dB(t) - (1/2)\, W'(X(t))\, dt, \qquad X(0) = 0,$$

where $W = \{W(x), x \in \mathbb{R}\}$ is a Brownian environment independent of a Brownian motion $B(t)$. Schumacher [11, 12] obtained a similar result for a considerably wider class of self-similar random environments (including symmetric stable ones) and stated, without detail of proof, that Sinai's result can be derived from the diffusion case with the aid of optional sampling.

It was known that the limit distributions for Sinai's and Brox's cases are the same, but its explicit form had been unknown until Kesten gave it in [9]; the same result as Kesten's was obtained also by Golosov as we have heard from Kesten; see also [3, 14] for the corresponding results in a similar but different model.

In [7] we discussed the following:

(A) The limiting behavior of $X(t, W)$ for a general random environment W which is asymptotically self-similar.

(B) Derivation of a result of Sinai type for a random walk from the diffusion setup.

The discussion in [7] were divided into two cases: a special case and a general case. However, in [7] the proof was given only in the special case and the results in the general case were stated without proof.

The purpose of this paper is to give full proofs to the results announced in [7] for the above problems (A) and (B) in the general case. In the special case (which still covers the case of symmetric stable environments), there exists a valley containing 0 whose bottom consists of a single point. However, in the general case the following (i) and (ii) can happen.

(i) The bottom of a valley is not a single point but a compact set.

(ii) There are no valleys containing 0.

Our definition of a valley is somewhat complicated, but still by making use of a method similar to Brox [2] we can prove our main theorems; for example, the distributions of $(\log t)^{-\alpha} X(t)$ are tight (as $t \to \infty$) where α is the exponent of the asymptotic self-similarity of the environment.

In Sect. 1 we give the definition of a valley of an environment and state our main theorems. In Sect. 2 we study some properties of valleys and in Sect. 3 prepare some estimates for an exit time from a valley of $X(t, \lambda W)$. In Sects. $4 \sim 7$ we prove our main theorems.

1. Main Results

Before stating main results we introduce some notations and definitions. Let $\mathbb{K}$ be the space of nonempty compact subsets of $\mathbb{R}$ equipped with the Hausdorff metric ρ defined by

$$\rho(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2),\ K_2 \subset U_\varepsilon(K_1)\}$$

for $K_1, K_2 \in \mathbb{K}$, where $U_\varepsilon(K)$ denotes the open ε-neighborhood of K in $\mathbb{R}$ (cf. Borsuk [1] or Nadler [10]). The space $\mathbb{K}$ is a locally compact separable metric space. Denote by $\mathbb{W}$ the space of real valued right continuous functions on $\mathbb{R}$ with left limits and vanishing at 0. The space $\mathbb{W}$ is a Polish space with the Skorohod topology. For $W \in \mathbb{W}$, we set

$$W^*(x) = W(x) \vee W(x-) \quad \text{and} \quad W_*(x) = W(x) \wedge W(x-)^1.$$

We define a space $\mathbb{W}^\#$ of environment as the set of $W \in \mathbb{W}$ satisfying

$$\overline{\lim_{x\to\infty}} \{W(x) - \inf_{[0,x]} W\} = \overline{\lim_{x\to-\infty}} \{W(x) - \inf_{[x,0]} W\} = \infty.$$

Let $W \in \mathbb{W}^\#$. Then $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ is called a *valley* of W if the following four conditions (i)–(iv) are satisfied.

(i) $\mathbb{a}, \mathbb{b}, \mathbb{c} \in \mathbb{K}$,

$$-\infty < a^- \leqq a^+ \leqq b^- \leqq b^+ \leqq c^- \leqq c^+ < \infty \quad \textit{and} \quad a^+ < c^-,$$

where $a^- = \min \mathbb{a}$, $a^+ = \max \mathbb{a}$; $b^\pm$ *and* $c^\pm$ *are defined similarly in terms of* $\mathbb{b}$ *and* $\mathbb{c}$.

(ii) *If we set* $W_{\mathbb{a}} = \max_{[a^-,a^+]} W^*$, $W_{\mathbb{b}} = \min_{[b^-,b^+]} W_*$ *and* $W_{\mathbb{c}} = \max_{[c^-,c^+]} W^*$, *then*

(1) $W_{\mathbb{a}} > W_{\mathbb{b}}$, $W_{\mathbb{c}} > W_{\mathbb{b}}$,

(2) $W_{\mathbb{b}} < W_*(x) \leqq W^*(x) < W_{\mathbb{a}}$ *for every* $x \in (a^+, b^-)$,
$W_{\mathbb{b}} < W_*(x) \leqq W^*(x) < W_{\mathbb{c}}$ *for every* $x \in (b^+, c^-)$,

(3) (a) $\mathbb{a} = \{x \in [a^-, a^+]: W^*(x) = W_{\mathbb{a}}\}$,
(b) $\mathbb{b} = \{x \in [b^-, b^+]: W_*(x) = W_{\mathbb{b}}\}$,
(c) $\mathbb{c} = \{x \in [c^-, c^+]: W^*(x) = W_{\mathbb{c}}\}$.

(iii) *If* $a^+ = b^-$, *then* $W(b^-) = W_{\mathbb{b}}$ *and* $W(a^+ -) = W_{\mathbb{a}}$.
If $b^+ = c^-$, *then* $W(b^+ -) = W_{\mathbb{b}}$ *and* $W(c^-) = W_{\mathbb{c}}$.

(iv) $H(a^-, b^+) \vee H(c^+, b^-) < (W_{\mathbb{a}} - W_{\mathbb{b}}) \wedge (W_{\mathbb{c}} - W_{\mathbb{b}})$,

where

$$H(x,y) = \begin{cases} \sup\limits_{x < x' \leqq y' < y} \{W(y') - W(x')\} & \textit{if } x < y \\ \sup\limits_{y < y' \leqq x' < x} \{W(y') - W(x')\} & \textit{if } y < x \\ 0 & \textit{if } x = y. \end{cases}$$

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[1] $W(x-) = \lim_{\varepsilon \downarrow 0} W(x-\varepsilon)$. $\sup_I W = \sup\{W(x): x \in I\}$, $\inf_I W = \inf\{W(x): x \in I\}$. $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

For a valley $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$, $D(\mathbb{V})=(W_{\mathbb{a}}-W_{\mathbb{b}}) \wedge (W_{\mathbb{c}}-W_{\mathbb{b}})$ is called the *depth* of $\mathbb{V}$ and $A(\mathbb{V})=H(a^-, b^+) \vee H(c^+, b^-)$ is called the *inner directed ascent* of $\mathbb{V}$. Throughout the paper we use the abbreviation: $D=D(\mathbb{V})$, $A=A(\mathbb{V})$, $D_i=D(\mathbb{V}_i)$, $A_i=A(\mathbb{V}_i)$, etc. It is clear that $A<D$.

Remark 1. For a valley $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ of $W \in \mathbb{W}^\#$ the following statements (i) and (ii) are easily verified.

(i) $\sup_{[b^-, b^+]} W^* \leqq W_{\mathbb{a}} \vee W_{\mathbb{c}}$.

(ii) If $\# \mathbb{b}^2 > 1$, then $\sup_{(b^-, b^+)} W^* < W_{\mathbb{a}} \wedge W_{\mathbb{c}}$;

if $\# \mathbb{a} > 1$, then $\inf_{(a^-, a^-)} W_* > W_{\mathbb{b}}$;

if $\# \mathbb{c} > 1$, then $\inf_{(c^-, c^+)} W_* > W_{\mathbb{b}}$.

A valley $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ of W is said to contain 0 if $a^+<0<c^-$. Two valleys $\mathbb{V}_1=(\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2=(\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ are said to be connected at 0 if $\mathbb{c}_1=\mathbb{a}_2$ and $a_2^-(=c_1^-) \leqq 0 \leqq a_2^+(=c_1^+)$.

Next we introduce a scaling map. For fixed $\alpha>0$ and $\lambda>0$, let τ_λ^α be the map: $\mathbb{W} \to \mathbb{W}$ defined by $(\tau_\lambda^\alpha W)(x)=\lambda^{-1} W(\lambda^\alpha x)$, $x \in \mathbb{R}$. For a probability measure ν on $\mathbb{W}$ we denote by $\tau_\lambda^\alpha \nu$ the image measure of ν under the map τ_λ^α. A measure ν is said to be self-similar with exponent $\alpha>0$ if $\tau_\lambda^\alpha \nu=\nu$ for any $\lambda>0$.

Proposition 1. (i) *Let* $W \in \mathbb{W}^\#$ *and* $r>0$. *Then either the following* (a) *and* (b) *holds:* (a) *there exists a valley* $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ *of* $\mathbb{W}$ *containing* 0 *such that* $A<r \leqq D$, (b) *there exist two valleys* $\mathbb{V}_1=(\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ *and* $\mathbb{V}_2=(\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ *of* W *connected at* 0 *such that* $A_1 \vee A_2 < r \leqq D_1 \wedge D_2$.

(ii) *Let* ν *be a self-similar probability measure on* $\mathbb{W}$ *with* $\nu(\mathbb{W}^\#)=1$. *Then there exists a subset* $\tilde{\mathbb{W}}^\#$ *of* $\mathbb{W}^\#$ *with* $\nu(\tilde{\mathbb{W}}^\#)=1$ *such that for any fixed* $W \in \tilde{\mathbb{W}}^\#$ *and* $r>0$, *either the following* (a) *or* (b) *holds:* (a) *there exists a valley* $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ *of* W *containing* 0 *with* $A<r<D$, (b) *there exist two valleys* $\mathbb{V}_1=(\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ *and* $\mathbb{V}_2=(\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ *of* W *connected at* 0 *such that* $A_1 \vee A_2 < r < D_1 \wedge D_2$.

This proposition will be proved in § 2.

Given $W \in \mathbb{W}$, we consider a stochastic differential equation

$$dX(t)=dB(t)-(1/2)\, W'(X(t))\, dt, \qquad X(0)=0, \tag{1.1}$$

where $B(t)$ is a 1-dimensional Brownian motion. The meaning of (1.1) is not clear a priori since the derivative W' does not exist in general. By a solution of (1.1) we mean a diffusion process starting at 0 with generator

$$(1/2)\, e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right)=\frac{d}{dm} \cdot \frac{d}{ds}, \tag{1.2}$$

[2] $\# A=$ the cardinality of the set A

where $s(x)=\int_0^x e^{W(x)}dy$ and $m(dx)=2e^{-W(x)}dx$. Such a diffusion can be constructed from a Brownian motion by a scale-change and time-change (see [6]). We denote the diffusion by $X(t, W)$.

We are now in a position to state our main theorems.

Theorem 1. *Let* $W\in\mathbb{W}^{\#}$, $W_\lambda\in\mathbb{W}$, $\lambda>0$ *and let us assume that* $W_\lambda\to W$[3] *as* $\lambda\to\infty$. *Then the following* (i) *and* (ii) *hold.*

(i) *If* $\mathbb{V}=(\mathbb{a},\mathbb{b},\mathbb{c})$ *is a valley of* W *containing* 0 *with* $A<r_1<r_2<D$, *then for any open set* U *including* $\mathbb{b}$

$$\lim_{\lambda\to\infty}\inf_{r\in I} P\{X(e^{\lambda r},\lambda W_\lambda)\in U\}=1,$$

where $I=[r_1,r_2]$.

(ii) *If* $\mathbb{V}_1=(\mathbb{a}_1,\mathbb{b}_1,\mathbb{c}_1)$ *and* $\mathbb{V}_2=(\mathbb{a}_2,\mathbb{b}_2,\mathbb{c}_2)$ *are valley of* W *connected at* 0 *with* $A_1\vee A_2<r_1<r_2<D_1\wedge D_2$, *then it holds that for any open set* U *including* $\mathbb{b}_1\cup\mathbb{b}_2$

$$\lim_{\lambda\to\infty}\inf_{r\in I} P\{X(e^{\lambda r},\lambda W_\lambda)\in U\}=1,$$

where $I=[r_1,r_2]$.

Theorem 2. *Let* μ *and* v *be probability measures on* $\mathbb{W}$ *and let us assume that* $v(\mathbb{W}^{\#})=1$. *If* $\tau_\lambda^\alpha\mu$ *converges to* v *as* $\lambda\to\infty$ *for some* $\alpha>0$, *then there exist Borel measurable mappings*

$$\mathbb{b}_\lambda:\ \mathbb{W}\to\mathbb{K}$$

such that

(i) *for any* $\varepsilon>0$

$$P\{\lambda^{-\alpha}X(e^\lambda, W)\in U_\varepsilon(\mathbb{b}_\lambda(W))\}\to 1$$

in probability with respect to μ *as* $\lambda\to\infty$,

(ii) *the distribution of the* $\mathbb{K}$*-valued random variable* $\mathbb{b}_\lambda$ *on* $(\mathbb{W},\mu)$ *converges as* $\lambda\to\infty$ *to that of the random variable* $\mathbb{b}(W)$ *on* $(\mathbb{W}^{\#},v)$. *Here* $\mathbb{b}(W)$ *is given as follows: when* W *has a valley* $\mathbb{V}=(\mathbb{a},\mathbb{b},\mathbb{c})$ *containing* 0 *with* $A<1<D$ *(case* (a) *in* (ii) *of Proposition 1), then* $\mathbb{b}(W)=\mathbb{b}$. *On the other hand when* W *has two valleys* $\mathbb{V}_1=(\mathbb{a}_1,\mathbb{b}_1,\mathbb{c}_1)$ *and* $\mathbb{V}_2=(\mathbb{a}_2,\mathbb{b}_2,\mathbb{c}_2)$ *connected at* 0 *with* $A_1\vee A_2<1<D_1\wedge D_2$ *(case* (b) *in* (ii) *of Proposition 1), then* $\mathbb{b}(W)=\mathbb{b}_1\cup\mathbb{b}_2$.

Remark 2. In Theorem 2, v becomes automatically self-similar with exponent $\alpha>0$.

We proceed to the random walk problem. Let $\mathbb{W}_\lambda^\alpha$ be the set of $W\in\mathbb{W}$ which are step functions flat on each interval $(n/\lambda^\alpha,(n+1)/\lambda^\alpha)$, $n\in\mathbb{Z}$. Given

[3] $W_\lambda\to W$ always means the convergence in the Skorohod topology

$W_\lambda \in \mathbb{W}_\lambda^\alpha$, we consider the solution $X^x(t, \lambda W_\lambda)$ of the stochastic differential equation

$$dX(t) = dB(t) - (1/2)\,\lambda\, W_\lambda'(X(t))\, dt, \qquad X(0) = x \in \mathbb{R}.$$

Let

$$\Gamma_0^x = \Gamma_{\lambda,0}^{\alpha,x} = \inf\{t \geqq 0\colon X^x(t, \lambda W_\lambda) \in \mathbb{Z}/\lambda^\alpha\}$$

and

$$\Gamma_n^x = \Gamma_{\lambda,n}^{\alpha,x} = \inf\left\{t \geqq 0\colon \left|X^x\left(t + \sum_{k=0}^{n-1} \Gamma_k^x, \lambda W_\lambda\right) - X^x\left(\sum_{k=0}^{n-1} \Gamma_k^x, \lambda W_\lambda\right)\right| = \lambda^{-\alpha}\right\},$$

$$n = 1, 2, \ldots.$$

We define new random variables by

$$(1.3) \qquad Y_\lambda^x(n) = Y_\lambda^x(n, \lambda W_\lambda) = X^x\left(\sum_{k=0}^{n} \Gamma_k^x, \lambda W_\lambda\right), \qquad n = 0, 1, 2, \ldots.$$

We write $Y_\lambda(n) = Y_\lambda^0(n, \lambda W_\lambda)$ and $Y(n) = Y_1^0(n, W_1)$ for simplicity.

Theorem 3. *Let $W \in \mathbb{W}^\#$, $W_\lambda \in \mathbb{W}_\lambda^\alpha$, $\lambda > 0$, and let us assume that $W_\lambda \to W$ as $\lambda \to \infty$. Then we have the following* (i) *and* (ii).

(i) *If $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ is a valley of W containing 0 with $A < r_1 < r_2 < D$, then for any open set U including $\mathbb{b}$*

$$\lim_{\lambda \to \infty} \inf_{r \in I} P\{Y_\lambda([e^{\lambda r}], \lambda W_\lambda) \in U\} = 1,$$

where $I = [r_1, r_2]$.

(ii) *If $\mathbb{V}_1 = (\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2 = (\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ are valleys of W connected at 0 with $A_1 \vee A_2 < r_1 < r_2 < D_1 \wedge D_2$, then for any open set U including $\mathbb{b}_1 \cup \mathbb{b}_2$*

$$\lim_{\lambda \to \infty} \inf_{r \in I} P\{Y_\lambda([e^{\lambda r}], \lambda W_\lambda) \in U\} = 1,$$

where $I = [r_1, r_2]$.

Theorem 4. *Let μ be a probability measure on $\mathbb{W}_1$ and let μ_n^α, $n \geqq 2$, be the image measure of μ on $\mathbb{W}_{\log n}^\alpha$ under the map $\tau_{\log n}^\alpha$. Suppose that μ_n^α converges weakly to ν as $n \to \infty$ for some $\alpha > 0$. Then there exists a sequence of Borel measurable mappings*

$$\mathbb{b}_n\colon \mathbb{W}_1 \to \mathbb{K}, \qquad n \geqq 2,$$

such that

(i) *for any $\varepsilon > 0$*

$$P\left\{\frac{Y(n, W)}{(\log n)^\alpha} \in U_\varepsilon(\mathbb{b}_n(W))\right\} \to 1$$

in probability with respect to μ as $n \to \infty$,

(ii) *the distribution of the* $\mathbb{K}$*-valued random variable* $\mathbb{b}_n$ *on* $(\mathbb{W}_1, \mu)$ *converges as* $n \to \infty$ *to that of the random variable* $\mathbb{b}(W)$ *on* $(\mathbb{W}^{\#}, \nu)$ *which is defined as in Theorem 2.*

Given $W \in \mathbb{W}_1$, we see easily that $\{Y(n, W), n \geqq 0\}$ is a Markov chain on $\mathbb{Z}$ with the transition probability

$$P\{Y(n+1, W) = x+1 \mid Y(n, W) = x\} = \xi(x),$$
$$P\{Y(n+1, W) = x-1 \mid Y(n, W) = x\} = 1 - \xi(x),$$

where

$$\xi(x) = (1 + e^{W(x) - W(x-1)})^{-1}, \qquad x \in \mathbb{Z}. \tag{1.4}$$

Therefore Theorem 4 can be rephrased as in the following theorem, which is an extension of Sinai's theorem.

Theorem 5. *Let* $\{\xi(x): x \in \mathbb{Z}\}$ *be a family of random variables with values in* $(0, 1)$. *Let* $\{Y(n, \xi), n = 0, 1, 2, \ldots\}$ *be a Markov chain on* $\mathbb{Z}$ *with the environment* $\{\xi(x), x \in \mathbb{Z}\}$. *Define* $W_\xi \in \mathbb{W}_1$ *by*

$$\begin{cases} W_\xi(x) - W_\xi(x-) = \log \dfrac{1 - \xi(x)}{\xi(x)}, & x \in \mathbb{Z} \\ W_\xi \text{ is flat on each interval } (n, n+1), & n \in \mathbb{Z}. \end{cases}$$

Let μ *be the probability distribution of* W_ξ *and let* μ_n^α *be the image measure of* μ *under the map* $\tau_{\log n}^\alpha$, $n \geqq 2$, $\alpha > 0$. *If* μ_n^α *converges to a probability measure* ν *on* $\mathbb{W}$ *with* $\nu(\mathbb{W}^{\#}) = 1$ *as* $n \to \infty$, *then there exists a sequence of Borel mappings*

$$\mathbb{b}_n : (0, 1)^{\mathbb{Z}} \to \mathbb{K}, \qquad n \geqq 2,$$

such that

(i) *for any* $\varepsilon > 0$

$$P\{(\log n)^{-\alpha} Y(n, \xi) \in U_\varepsilon(\mathbb{b}_n(\xi))\} \to 1$$

in probability with respect to μ *as* $n \to \infty$,

(ii) *the distribution of* $\mathbb{b}_n$ *converges, as* $n \to \infty$, *to that of* $\mathbb{b}(W)$, *which is defined as in Theorem 2.*

2. The Proof of Proposition 1 – On Valleys

We prepare two lemmas in order to prove Proposition 1.

Lemma 2.1. *Let* $r > 0$ *and* $W \in \mathbb{W}^{\#}$. *Then the following dichotomy occurs.*

(i) *There exists a valley* $\mathbb{V}$ *of* W *containing* 0 *such that* $A < r \leqq D$.

(ii) *There exist two valleys* $\mathbb{V}_1$ *and* $\mathbb{V}_2$ *of* W *connected at* 0 *such that* $A_1 \vee A_2 < r \leqq D_1 \wedge D_2$.

Proof. The proof is not difficult although tedius. Let

$$T_1 = \sup\{x \leqq 0:\ W^*(x) - \inf_{(x,0]} W \geqq r\},$$

$$T_2 = \inf\{x \geqq 0:\ W^*(x) - \inf_{[0,x)} W \geqq r\}.$$

In the above (and also in the below) $\inf_{(0,0]} W$ and $\inf_{[0,0)} W$ should be understood as $W_*(0)$ and 0, respectively.

We notice that $0 < T_2 < \infty$ and $-\infty < T_1 \leqq 0$ by the definition of $\mathbb{W}^\#$. Now set

$$b_1 = \inf\{0 \geqq x \geqq T_1:\ W_*(x) = \inf_{(x,0]} W\},$$

$$b_2 = \sup\{0 \leqq x \leqq T_2:\ W_*(x) = \inf_{[0,x)} W\},$$

$$V_i = W_*(b_i), \qquad i = 1, 2,$$

$$M_1 = \begin{cases} \sup\{W^*(x):\ b_1 \leqq x < 0\}, & \text{if } b_1 < 0 \\ 0, & \text{if } b_1 = 0, \end{cases}$$

$$M_2 = \begin{cases} \sup\{W^*(x):\ 0 \leqq x \leqq b_2\}, & \text{if } b_2 > 0 \\ 0, & \text{if } b_2 = 0, \end{cases}$$

$$a_1 = \begin{cases} \sup\{b_1 < x \leqq 0:\ W^*(x) = M_1\}, & \text{if } b_1 < 0 \\ 0, & \text{if } b_1 = 0, \end{cases}$$

$$a_2 = \begin{cases} \inf\{0 < x < b_2:\ W^*(x) = M_2\}, & \text{if } b_2 > 0 \\ 0, & \text{if } b_2 = 0. \end{cases}$$

Next we define the following sets:

$$\mathbb{a}(1) = \{x:\ T_1 \leqq x \leqq 0,\ W^*(x) = M_1\},$$

$$\mathbb{a}(2) = \{x:\ 0 \leqq x \leqq T_2,\ W^*(x) = M_2\},$$

$$\mathbb{b}(1) = \{x:\ 0 \geqq x \geqq T_1,\ W_*(x) = V_1\},$$

$$\mathbb{b}(2) = \{x:\ 0 \leqq x \leqq T_2,\ W_*(x) = V_2\}.$$

We notice that if $T_1 < 0$, then

(2.1) $$H(T_1, b(1)^+), H(0, b_1), H(0, b_2), H(T_2, b(2)^-) < r.$$

In fact, if $H(T_1, b(1)^+) \geqq r$, then for any positive integer n, there exist x_n and y_n such that $T_1 < x_n \leqq y_n < b(1)^+$ and

(2.2) $$W(y_n) - W(x_n) + 1/n \geqq r.$$

If $x_n, y_n \to T_1$ as $n \to \infty$, then by the right continuity of W, (2.2) implies a contradiction. Therefore we can assume that $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$. Then

$$(2.3) \qquad T_1 < x_0 \leqq y_0 \leqq b(1)^+ \quad \text{or} \quad T_1 = x_0 < y_0.$$

Thus employing (2.2) we see that

$$W^*(y_0) - V_1 \geqq W^*(y_0) - W_*(x_0) \geqq r.$$

This contradicts the definition of T_1. Thus we obtain the first inequality. In the same manner we can prove that $H(T_2, b(2)^-) < r$. Other inequalities are clear by the definition.

Now the proof is divided into the following four cases:

[I]	$T_1 < 0$ and $V_1 > V_2$
[I′]	$T_1 < 0$ and $V_1 < V_2$
[II]	$T_1 < 0$ and $V_1 = V_2$
[III]	$T_1 = 0$.

The proof in the case [I] is divided into seven subcases [I-1]–[I-7].

[I-1] If $M_2 \geqq M_1$ and $M_2 < V_1 + r$, then $\mathbb{V} = (\{T_1\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A < r \leqq D$.

Since $M_2 < V_1 + r \leqq W^*(T_1)$ and $V_2 < V_1$, we have

$$(2.4) \qquad W_{\mathbb{b}} = V_2 < W_*(x) \leqq W^*(x) < W^*(T_1) \quad \text{for } T_1 < x < b(2)^-.$$

As is easily seen,

$$(2.5) \qquad W_{\mathbb{b}} = V_2 < W_*(x) \leqq W^*(x) < W^*(T_2) \quad \text{for } b(2)^+ < x < T_2.$$

By using (2.1), we see

$$H(T_1, b_2) \leqq H(T_1, b(1)^+) \vee \sup\{M_2 - W(x) : T_1 < x < a_2\} \vee H(0, b_2) < r.$$

Since $H(T_2, b(2)^-) < r$ by (2.1), we have

$$(2.6) \qquad A = H(T_1, b_2) \vee H(T_2, b(2)^-) < r.$$

By the assumption,

$$\begin{aligned} D &= (W^*(T_2) - V_2) \wedge (W^*(T_1) - V_2) \\ &\geqq (W^*(T_2) - V_2) \wedge (W^*(T_1) - V_1) \geqq r. \end{aligned}$$

Therefore we see that the valley $\mathbb{V} = (\{T_1\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A < r \leqq D$.

[I-2] If $M_2 > M_1$, $M_2 \geqq V_1 + r$ and $a_2 > 0$ then $\mathbb{V} = (\{T_1\}, \mathbb{b}(1), \{a_2\})$ is a valley containing 0 with $A < r \leqq D$.

We notice that $a_2 \leqq b(2)^-$. We have

(2.7) $$W^*(T_1) > W^*(x) \geqq W_*(x) > W_{\mathbb{b}} = V_1 \quad \text{for } T_1 < x < b_1.$$

Since for every $x \in (0, a_2)$, $M_2 - W_*(x) < r \leqq M_2 - V_1$ by the assumption, we have

(2.8) $$W_*(x) > V_1 \quad \text{for } 0 < x < a_2.$$

Thus

(2.9) $$W_{\mathbb{b}} < W_*(x) \leqq W^*(x) < W^*(a_2) = W_{\mathbb{c}} \quad \text{for } b(1)^+ < x < a_2.$$

Clearly we have

$$H(a_2, b(1)^-) \leqq (M_2 - \inf_{[0, a_2]} W_*) \vee H(a_1, b(1)^-) < r.$$

Since $H(T_1, b(1)^+) < r$ by (2.1),

(2.10) $$A = H(T_1, b(1)^+) \vee H(a_2, b(1)^-) < r.$$

By the definition and the assumption, we have

(2.11) $$D = (W^*(T_1) - V_1) \wedge (M_2 - V_1) \geqq r.$$

Thus (2.7), (2.9), (2.10) and (2.11) imply the assertion [I-2].

[I-3] If $M_2 > M_1$, $M_2 \geqq V_1 + r$ and $a_2 = 0$, then we show that $\mathbb{V}_1 = (\{T_1\}, \mathbb{b}(1), \mathbb{a}(2))$ and $\mathbb{V}_2 = (\mathbb{a}(2), \mathbb{b}(2), \{T_2\})$ are two valleys connected at 0 with $A_1 \vee A_2 < r \leqq D_1 \wedge D_2$.

We notice that $M_2 = 0$ and $b_2 > 0$. Obviously $M_2 = 0$ and, if $b_2 = 0$, then $M_2 = V_2 = 0$. Therefore $0 = M_2 \geqq V_1 + r$ implies $V_1 < 0$. This contradicts $V_2 < V_1$. Note that $a(2)^+ \leqq b(2)^-$. Since $M_2 - W_*(x) < r$ for every $x \in (0, a(2)^+)$ and $r \leqq -V_1$ by the assumption, we have

$$W_*(x) > V_1 \quad \text{for every } x \in (0, a(2)^+).$$

Thus $\mathbb{V}_1$ satisfies the condition (2) in the definition of a valley. To prove $A_1 < r$, we observe that

$$\begin{aligned} H(a(2)^+, b_1) &\leqq H(a(2)^+, 0) \vee \sup\{W(x) - W(y) : b_1 < x < 0 < y < a(2)^+\} \vee H(0, b_1) \\ &\leqq (M_2 - \inf_{(0, a(2)^+)} W_*) \vee H(0, b_1) < r. \end{aligned}$$

Since $H(T_1, b(1)^+) < r$ by (2.1), $A_1 < r$. Therefore $\mathbb{V}_1$ is a valley with $A_1 < r \leqq D_1$. Clearly $\mathbb{V}_2$ is a valley with $A_2 < r \leqq D_2$. This is the assertion [I-3].

[I-4] If $M_2 = M_1 \geqq V_1 + r$, then $\mathbb{V}_1 = (\{T_1\}, \mathbb{b}(1), \mathbb{a}(1) \cup \mathbb{a}(2))$ and $\mathbb{V}_2 = (\mathbb{a}(1) \cup \mathbb{a}(2), \mathbb{b}(2), \{T_2\})$ are two valleys connected at 0 with $A_1 \vee A_2 < r \leqq D_1 \wedge D_2$.

Apparently $b_1 \leqq a(1)^-$, $a(2)^+ \leqq b(2)^-$ and $W_*(x) > V_1$ for every $x \in (0, a(2)^+)$ since $M_2 - W_*(x) < r \leqq M_2 - V_1$ by the assumption. Thus it is enough to show that $A_1 \vee A_2 < r \leqq D_1 \wedge D_2$.

$$\begin{aligned} &H(a(2)^+, b_1) \\ &\quad \leqq H(a(2)^+, 0) \vee \sup\{W(x) - W(y):\ b_1 < x < 0 < y < a(2)^+\} \vee H(0, b_1) \\ &\quad \leqq H(a(2)^+, 0) \vee (M_1 - \inf_{(0, a(2)^+)} W_*) \vee H(0, b_1) \\ &\quad \leqq H(a(2)^+, 0) \vee (M_2 - \inf_{(0, a(2)^+)} W_*) \vee H(0, b_1) \\ &\quad < r. \end{aligned}$$

$$\begin{aligned} &H(a(1)^-, b_2) \\ &\quad \leqq H(a(1)^-, 0) \vee \sup\{W(y) - W(x):\ a(1)^- < x < 0 < y < b_2\} \vee H(0, b_2) \\ &\quad \leqq (M_1 - \inf_{(a(1)^-, 0)} W_*) \vee (M_2 - \inf_{(a(1)^-, 0)} W_*) \vee H(0, b_2) \\ &\quad \leqq (M_1 - \inf_{(a(1)^-, 0)} W_*) \vee H(0, b_2) \\ &\quad < r. \end{aligned}$$

The above estimations together with (2.1) imply that

$$A_1 = H(T_1, b(1)^+) \vee H(a(2)^+, b_1) < r$$

and

$$A_2 = H(a(1)^-, b_2) \vee H(T_2, b(2)^-) < r.$$

Since $D_1 \wedge D_2 \geqq r$, we have obtained the assertion [I-4].

[I-5] If $M_1 > M_2$ and $M_1 < V_1 + r$, then $\mathbb{V} = (\{T_1\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A < r \leqq D$.

By the assumption and (2.1) we have

$$\begin{aligned} H(T_1, b_2) &\leqq H(T_1, b(1)^+) \vee \sup\{W(y) - W(x):\ T_1 < x \leqq y, b(1)^+ < y < b_2\} \\ &\leqq H(T_1, b(1)^+) \vee (M_1 - V_1) \vee H(0, b_2) < r \end{aligned}$$

and

$$H(T_2, b(2)^-) < r,$$

from which [I-5] follows.

[I-6] If $M_1 > M_2$, $M_1 \geqq V_1 + r$ and $a_1 < 0$, then $\mathbb{V} = (\mathbb{a}(1), \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A < r \leqq D$.

It is enough to prove that $A<r\leqq D$. As is previously done,

$$
\begin{aligned}
&H(a(1)^-, b_2)\\
&\quad\leqq H(a(1)^-,0)\vee \sup\{W(y)-W(x):\, a(1)^-<x<0<y<b_2\}\vee H(0,b_2)\\
&\quad\leqq (M_1-\inf_{(a(1)^-,0)} W_*)\vee(M_2-\inf_{(a(1)^-,0)} W_*)\vee H(0,b_2)\\
&\quad<r
\end{aligned}
$$

and

$$H(T_2, b(2)^-)<r.$$

It follows that $A=H(a(1)^-,b_2)\wedge H(T_2,b(2)^-)<r$. On the other hand

$$D=(W^*(T_2)-V_2)\wedge(M_1-V_2)\geqq r.$$

[I-7] If $M_1>M_2$, $M_1\geqq V_1+r$ and $a_1=0$, then $\mathbb{V}_1=(\{T_1\},\mathbb{b}(1),\mathbb{a}(1))$ and $\mathbb{V}_2=(\mathbb{a}(1),\mathbb{b}(2),\{T_2\})$ are two valleys connected at 0 with $A_1\vee A_2<r\leqq D_1\wedge D_2$.

Clearly $\mathbb{V}_1$ is a valley with $A_1<r\leqq D_1$ since $b(1)^+\leqq a(1)^-$. The only thing we have to show is only that $\mathbb{V}_2$ satisfies $A_2<r\leqq D_2$. We see that

$$
\begin{aligned}
H(a(1)^-,b_2)&\leqq H(a(1)^-,0)\vee(M_2-\inf_{(a(1)^-,0)} W_*)\vee H(0,b_2)\\
&\leqq H(a(1)^-,0)\vee(M_1-\inf_{(a(1)^-,0)} W_*)\vee H(0,b_2)<r
\end{aligned}
$$

and

$$H(T_2,b(1)^-)<r \qquad (\text{by } (2.1)).$$

This shows that $A_2<r$. We also have

$$D_2=(M_1-V_2)\vee(W^*(T_2)-V_2)\geqq r.$$

The proof in the case [I′] is omitted, since it is similar to that in the case [I].

Now we proceed to the case [II]. Since the proof is similar to the previous argument, we only list results.

[II-1] If $M_2\geqq M_1$ and $M_2<V_2+r$, then $\mathbb{V}=(\{T_1\},\mathbb{b}(1)\cup\mathbb{b}(2),\{T_2\})$ is a valley containing 0 wth $A<r\leqq D$.

[II-2] If $M_2>M_1$ and $M_2\geqq V_2+r$ and $a_2>0$, then $\mathbb{V}=(\{T_1\},\mathbb{b}(1),\{a_2\})$ is a valley containing 0 with $A<r\leqq D$.

[II-3] If $M_2>M_1$, $M_2\geqq V_2+r$ and $a_2=0$, then $\mathbb{V}_1=(\{T_1\},\mathbb{b}(1),\mathbb{a}(2))$ and $\mathbb{V}_2=(\mathbb{a}(2),\mathbb{b}(2),\{T_2\})$ are two valleys connected at 0 with $A_1\vee A_2<r\leqq D_1\wedge D_2$.

[II-4] If $M_1=M_2$ and $M_2\geqq V_2+r$, then $\mathbb{V}_1=(\{T_1\},\mathbb{b}(1),\mathbb{a}(1)\cup\mathbb{a}(2))$ and $\mathbb{V}_2=(\mathbb{a}(1)\cup\mathbb{a}(2),\mathbb{b}(2),\{T_2\})$ are two valleys connected at 0 with $A_1\vee A_2<r\leqq D_1\wedge D_2$.

Finally we treat the case [III]. We need more notations. Set

$$\begin{aligned}
\tilde{T} &= \sup\{x<0\colon W^*(x) - \inf_{(x,0)} W \geqq r\}, \\
\tilde{b} &= \inf\{x\colon 0 > x \geqq \tilde{T},\ W_*(x) = \inf_{(x,0)} W\}^{4}, \\
\tilde{V} &= W_*(\tilde{b}), \\
\tilde{\mathbb{b}} &= \{x\colon 0 \geqq x \geqq \tilde{T},\ W_*(x) = \tilde{V}\}, \\
\tilde{M} &= \sup\{W^*(x)\colon 0 \geqq x \geqq \tilde{b}\}, \\
\tilde{a} &= \sup\{0 \geqq x \geqq \tilde{b}\colon W^*(x) = \tilde{M}\}, \\
\tilde{\mathbb{a}} &= \{x\colon 0 \geqq x \geqq \tilde{b},\ W^*(x) = \tilde{M}\}.
\end{aligned}$$

We notice that $\tilde{T}<0$ by the definition. The proof is divided into four subcases [III-1]–[III-4].

[III-1] If $\tilde{b}=0$, then we show that $\mathbb{V}=(\{\tilde{T}\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A<r\leqq D$.

Since $\tilde{b}=0$, we have

$$(2.12)\qquad W^*(\tilde{T}) > W^*(x) \geqq W_*(x) > W(0-) \geqq r \qquad \text{for every } x \in (\tilde{T}, 0).$$

Thus what we have to notice is that $A<r\leqq D$. First we show

$$(2.13)\qquad H(\tilde{T}, 0) < r.$$

If $H(\tilde{T}, 0) \geqq r$, then for every positive integer n, there exist x_n and y_n such that $\tilde{T} < x_n \leqq y_n < 0$ and

$$(2.14)\qquad W(y_n) - W(x_n) + 1/n \geqq r.$$

We may assume that $x_n \to x_0$ and $y_n \to y_0$ as $n\to\infty$. By $W\in\mathbb{W}$, it is impossible that $x_0 = y_0 = \tilde{T}$ or 0. Hence it holds that $\tilde{T} \leqq x_0 < y_0 \leqq 0$ or $\tilde{T} < x_0 = y_0 < 0$. Then, from (2.14), we obtain

$$W^*(y_0) - W(0-) \geqq W^*(y_0) - W_*(x_0) \geqq r.$$

This contradicts the definition of $\tilde{T}$. Thus we have (2.13). By using (2.12), (2.13) and (2.1) we see

$$\begin{aligned}
A &= H(\tilde{T}, b_2) \vee H(T_2, b(2)^-) \\
&= H(\tilde{T}, 0) \vee H(0, b_2) \vee H(T_2, b(2)^-) < r.
\end{aligned}$$

Since $D = (W^*(\tilde{T}) - V_2) \wedge (W^*(T_2) - V_2) \geqq r$, the proof is established.

[III-2] If $\tilde{b}<0$ and $\tilde{M} > W(0-)$, then $\mathbb{V}=(\{\tilde{a}\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A<r\leqq D$.

[4] The infimum of the empty set is understood to be 0

[III-3] If $\tilde{b}<0$, $\tilde{M}=W(0-)$ and $\tilde{V}+r\geqq\tilde{M}$, then it is also clear that $\mathbb{V}=(\{\tilde{T}_1\}, \mathbb{b}(2), \{T_2\})$ is a valley containing 0 with $A<r\leqq D$.

[III-4] If $\tilde{b}<0$, $\tilde{M}=W(0-)$ and $\tilde{V}+r<\tilde{M}$, then $\mathbb{V}_1=(\{\tilde{T}_1\}, \tilde{\mathbb{b}}, \tilde{\mathbb{a}})$ and $\mathbb{V}_2=(\tilde{\mathbb{a}}, \mathbb{b}(2), \{T_2\})$ are two valleys connected at 0 with $A_1\vee A_2<r\leqq D_1\wedge D_2$.

Therefore we have proved Lemma 2.1 completely.

Given $W\in\mathbb{W}^{\#}$ and $r>0$, set

$$\mathbb{B}_r=\mathbb{B}_r(W)=\{\mathbb{b}\in\mathbb{K}\text{: there exists a valley } (\mathbb{a}, \mathbb{b}, \mathbb{c}) \text{ of } W \text{ with } A<r\leqq D\}.$$

Lemma 2.2. *For every $W\in\mathbb{W}^{\#}$ and $r>0$, $\mathbb{B}_r$ is locally finite in the sense that for every compact set K in $\mathbb{R}$,*

$$\#\{\mathbb{b}\in\mathbb{B}_r\text{: } \mathbb{b}\cap K\neq\emptyset\}<\infty.$$

Proof. First we notice that for every $\mathbb{b}_1, \mathbb{b}_2\in\mathbb{B}_r$, $\mathbb{b}_1\neq\mathbb{b}_2$, it holds that $[b_1^-, b_1^+]\cap[b_2^-, b_2^+]=\emptyset$. Suppose that it is not true. We can assume that $b_1^-<b_2^-\leqq b_1^+$ without loss of generality. Then clearly $W_{\mathbb{b}_1}\leqq W_{\mathbb{b}_2}$ with respect to the corresponding valleys $\mathbb{V}_1=(\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2=(\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$. On the other hand, $b_1^-\leqq a_2^+\leqq b_2^-$ and $W^*(a_2^+)-W_{\mathbb{b}_2}\geqq r$. This implies $A_1\geqq r$, which contradicts the definition of $\mathbb{B}_r$.

Thus, for any $\mathbb{b}_1, \mathbb{b}_2\in\mathbb{B}_r$ with $b_1^+<b_2^-$, we see

$$\sup_{b_1^+\leqq x\leqq b_2^-}\{W(x)-W_{\mathbb{b}_i}\}\geqq r, \qquad i=1,2. \tag{2.15}$$

This implies the local finiteness of $\mathbb{B}_r$. In fact, suppose that for a compact set $K\subset\mathbb{R}$, there exists a countable sequence of sets $\mathbb{b}_n\in\mathbb{B}_r$ such that $\mathbb{b}_n\cap K\neq\emptyset$ and $\mathbb{b}_n\neq\mathbb{b}_m$ for $n\neq m$. Then $\{b_n^+\}$ is a bounded sequence. After taking a suitable subsequence, it can be assumed that either the following (i) or (ii) holds: (i) $b_n^-\leqq b_n^+<b_{n+1}^-\leqq b_{n+1}^+$ for all $n\geqq 1$, or (ii) $b_{n+1}^+\leqq b_{n+1}^-<b_n^-\leqq b_n^+$ for all $n\geqq 1$. If (i) holds, then (2.15) implies that W has no left limit at $b=\lim_{n\to\infty} b_n^+$. This is a contradiction. If (ii) holds, then from (2.15) it follows that there exists $x_n\in(b_{n+1}^+, b_n^-)$ such that

$$W(x_n)-W_{\mathbb{b}_n}\geqq r/2. \tag{2.16}$$

On the other hand, from the definition of $W_{\mathbb{b}_n}$, there exists $y_n\in[b_{n+1}^-, b_n^+]$ such that

$$W(y_n)\leqq W_{\mathbb{b}_n}+r/4. \tag{2.17}$$

Let $b=\lim_{n\to\infty} b_n^-$. Then from (2.16) and (2.17) it follows, letting $n\to\infty$, that $W(b)\leqq W(b)-r/4$. This is again a contradiction. The proof of Lemma 2.2 is finished.

Lemma 2.1 implies (i) *of Proposition 1. So we prove* (ii) *of Proposition 1.*

Let $r>0$, $W\in\mathbb{W}^{\#}$ and let $\mathbb{V}=(\mathbb{a},\mathbb{b},\mathbb{c})$ be a valley of W with $A<r\leqq D$. We define $\mathbb{a}^{(r)}$, $\mathbb{b}^{(r)}$ and $\mathbb{c}^{(r)}$ as follows:

$$\mathbb{a}^{(r)}=\{x\colon b_1^+\leqq x\leqq b^-,\ W^*(x)=\sup_{[b_1^+,b^-]} W^*\}$$

if there exists the nearest left neighboring point $\mathbb{b}_1$ of $\mathbb{b}$ in $\mathbb{B}_r$ and $\mathbb{a}^{(r)}=\{-\infty\}$ otherwise;

$$\mathbb{b}^{(r)}=\mathbb{b};$$

$$\mathbb{c}^{(r)}=\{x\colon b^+\leqq x\leqq b_2^-,\ W^*(x)=\sup_{[b^+,b_2^-]} W^*\}$$

if there exists the nearest right neighboring point $\mathbb{b}_2$ of $\mathbb{b}$ in $\mathbb{B}_r$ and $\mathbb{c}^{(r)}=\{\infty\}$ otherwise. Then, $\mathbb{V}^{(r)}=(\mathbb{a}^{(r)},\mathbb{b}^{(r)},\mathbb{c}^{(r)})$ is a valley of W with $A<r\leqq D$.

It is easily seen that (i) if W has a valley $\mathbb{V}$ containing 0 with $A<r\leqq D$, then $\mathbb{V}^{(r)}$ is a valley containing 0 with $A^{(r)}<r\leqq D^{(r)}$, and (ii) if W has two valleys $\mathbb{V}_1$ and $\mathbb{V}_2$ connected at 0 with $A_i<r\leqq D_i$, $i=1,2$, then $\mathbb{V}_1^{(r)}$ and $\mathbb{V}_2^{(r)}$ are two valleys of W connected at 0 with $A_i^{(r)}<r\leqq D_i^{(r)}$, $i=1,2$.

We define the mappings $\Phi_i(=\Phi_{i,W})\colon(0,\infty)\to\mathbb{R}^4$, $i=1,2$, by

$$\text{(2.18)}\qquad \Phi_i\colon r\in(0,\infty)\to\begin{cases}(a^{(r)+},b^{(r)-},b^{(r)+},c^{(r)-}), & \text{if } W \text{ has a valley containing 0 such that } A<r\leqq D\\ (a_i^{(r)+},b_i^{(r)-},b_i^{(r)+},c_i^{(r)-}), & \text{if } W \text{ has two valleys connected at 0 such that } A_1\vee A_2<r\leqq D_1\wedge D_2.\end{cases}$$

Owing to Lemma 2.2, $\mathbb{B}_r^-=\{b^-\colon \mathbb{b}\in\mathbb{B}_r\}$ are locally finite sets decreasing in r. This implies that Φ_i, $i=1,2$, are step functions which have finitely many jumps in each bounded interval away from 0.

Let ν be a self-similar probability measure on $\mathbb{W}$ with exponent $\alpha>0$ and $\nu(\mathbb{W}^{\#})=1$. It is easily seen that for $r>0$ and $\lambda>0$

$$\text{(2.19)}\qquad \mathbb{V}^{(r)}(\tau_\lambda^\alpha W)=\lambda^{-\alpha}\mathbb{V}^{(\lambda r)}(W).$$

Thus we have the scaling property

$$\text{(2.20)}\qquad \{\lambda^{-\alpha}\mathbb{V}^{(\lambda r)}(W),\ r>0,\ \nu\}\stackrel{d}{=}\{\mathbb{V}^{(r)}(W),\ r>0,\ \nu\},$$

where $\stackrel{d}{=}$ means the equality in distribution.

If r is not a jump point of $\Phi_i(W)$, $W\in\mathbb{W}^{\#}$, then

$$A^{(r)}(W)<r<D^{(r)}(W)\quad\text{or}\quad A_i^{(r)}(W)<r<D_i^{(r)}(W)$$

according as W has a valley containing 0 for r or W has two valleys connected at 0 for r. Therefore, in order to complete the proof of Proposition 1(ii), it is enough to show that, for $i=1, 2$,

$$p_i(r) := \nu\{r \text{ is a jump point of } \Phi_i\} = 0.$$

Since the set of jump points of Φ_i is locally finite,

$$j_i := \inf\{r > 1: \Phi_i \text{ has a jump at } r\}$$

is strictly larger than 1. Therefore we obtain

$$p_i(r) \leqq \nu\{j_i \leqq r\} \downarrow 0 \quad \text{as } r \downarrow 1.$$

By the scaling property (2.20), $p_i(r)$ is independent of $r > 0$. Consequently $p_i(r) \equiv 0$, $i = 1, 2$.

Keeping in mind the definition of a valley, we can easily verify the following lemma.

Lemma 2.3. *Let* $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ *be a valley of* $W \in \mathbb{W}^{\#}$. *Then for any* $\varepsilon > 0$, *there exists a set of numbers*

$$\tilde{a}^+ = \tilde{a}^+(\varepsilon), \quad \dot{a}^+ = \dot{a}^+(\varepsilon), \quad \tilde{b}^\pm = \tilde{b}^\pm(\varepsilon), \quad \dot{b}^\pm = \dot{b}^\pm(\varepsilon),$$
$$\tilde{c}^- = \tilde{c}^-(\varepsilon), \quad \dot{c}^- = \dot{c}^-(\varepsilon)$$

satisfying the following:

(i) (1) $\tilde{a}^+ \leqq a^+ \leqq \dot{a}^+ \leqq \tilde{b}^- \leqq b^- \leqq \dot{b}^-$,

$\tilde{b}^+ \leqq b^+ \leqq \dot{b}^+ \leqq \tilde{c}^- \leqq c^- \leqq \dot{c}^-$,

(2) $\tilde{a}^+ < \dot{a}^+, \quad \tilde{b}^\pm < \dot{b}^\pm, \quad \tilde{c}^- < \dot{c}^-$.

(ii) $\tilde{a}^+(\varepsilon) \uparrow a^+, \quad \tilde{b}^\pm(\varepsilon) \uparrow b^\pm, \quad \tilde{c}^-(\varepsilon) \uparrow c^-$,

$\dot{a}^+(\varepsilon) \downarrow a^+, \quad \dot{b}^\pm(\varepsilon) \downarrow b^\pm, \quad \dot{c}^-(\varepsilon) \downarrow c^-, \quad$ *as* $\varepsilon \downarrow 0$.

(iii) $\inf_{(\tilde{a}^+, \dot{a}^+)} W \geqq W_{\mathbb{a}} - \varepsilon, \quad \inf_{(\tilde{c}^-, \dot{c}^-)} W \geqq W_{\mathbb{c}} - \varepsilon$.

(iv) $\sup\{W(x): x \in (\tilde{b}^-, \dot{b}^-) \cup (\tilde{b}^+, \dot{b}^+)\} \leqq W_{\mathbb{b}} + \varepsilon$.

3. Some Lemmas for Exit Times

Throughout this section, we maintain the assumption of Theorem 1, that is,

$$W \in \mathbb{W}^{\#}, \quad W_\lambda \in \mathbb{W}, \quad \lambda > 0,$$

and

$$W_\lambda \to W \quad \text{as} \quad \lambda \to \infty \quad \text{in the Skorohod topology.}$$

Thus there exist $\varepsilon_\lambda > 0$ and φ_λ, $\lambda > 0$, such that $\varepsilon_\lambda \downarrow 0$ as $\lambda \to \infty$, φ_λ is an order preserving homeomorphism on $\mathbb{R}$, and

$$\sup_{x \in \mathbb{R}} |W \circ \varphi_\lambda(x) - W_\lambda(x)| + \sup_{x \in \mathbb{R}} |\varphi_\lambda(x) - x| < \varepsilon_\lambda. \tag{3.1}$$

From now on, we fix these $\{\varepsilon_\lambda, \lambda > 0\}$ and $\{\varphi_\lambda, \lambda > 0\}$.

Let $m(dx) = 2e^{-W(x)}\, dx$ and $s(x) = \int_0^x e^{W(y)}\, dy$. Let $\{B(t), t \geqq 0\}$ be a standard one dimensional Brownian motion and set

$$L(t, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[x, x+\varepsilon]}(B(s))\, ds, \tag{3.2a}$$

$$\begin{aligned} A(t) &= \int_0^t e^{-2W(s^{-1}(B(s)))}\, ds \\ &= \int_{\mathbb{R}} e^{-2W(s^{-1}(x))} L(t, x)\, dx. \end{aligned} \tag{3.2b}$$

Then

$$X(t, W) = S^{-1}(B(A^{-1}(t))) \tag{3.3}$$

is a diffusion process starting at 0 with speed measure $m(dx)$ and scale function $s(x)$.

Now we set for $z \in \mathbb{R}$

$$X^z(t, W) = z + X(t, W^z) \tag{3.4}$$

where $W^z(y) = W(y+z) - W(z)$. Then $X^z(t, W)$ is a diffusion process with generator (1.2) starting at z.

For $a \leqq c$, let

$$\tau(a, c) = \inf\{t > 0 \colon B(t) \notin (a, c)\}$$

and

$$L(a, c, x) = L(\tau(a, c), x), \qquad x \in \mathbb{R}.$$

Then the following lemma can be proved by using the well-known scaling relation of $B(t)$ (see [2]).

Lemma 3.1. *For every $\lambda > 0$ and $a, c \in \mathbb{R}$, it holds that*

$$\{\lambda L(a, c, x), x \in \mathbb{R}\} \stackrel{d}{=} \{L(\lambda a, \lambda c, \lambda x), x \in \mathbb{R}\}.$$

Set

$$(3.5)\qquad s_\lambda(x)=\int_0^x e^{\lambda W_\lambda(y)}\,dy,$$

$$(3.6)\qquad A_\lambda(t)=\int_0^t e^{-2\lambda W_\lambda(s_\lambda^{-1}(B(s)))}\,ds$$

$$=\int_{\mathbb{R}} e^{-2\lambda W_\lambda(s_\lambda^{-1}(x))}\,L(t,x)\,dx,$$

$$(3.7)\qquad X_\lambda(t)=X(t,\lambda W_\lambda)$$

$$=s_\lambda^{-1}(B(A_\lambda^{-1}(t))),$$

$$(3.8)\qquad X_\lambda^x(t)=x+X(t,\lambda W_\lambda^x),\qquad x\in\mathbb{R},$$

$$(3.9)\qquad T_\lambda(a,c)=\inf\{t\geqq 0:\ X_\lambda(t)\notin(a,c)\},$$

$$(3.10)\qquad T_\lambda(a)=\inf\{t\geqq 0:\ X_\lambda(t)=a\},$$

$$(3.11)\qquad T_\lambda(x;a,c)=\inf\{t\geqq 0:\ X_\lambda^x(t)\notin(a,c)\},$$

$$(3.12)\qquad T_\lambda(x;a)=\inf\{t\geqq 0:\ X_\lambda^x(t)=a\}.$$

As is easily proved, we have the following lemma.

Lemma 3.2. (i) *Let $a<0<c$. Then*

$$T_\lambda(a,c)=A_\lambda(\tau(s_\lambda(a),s_\lambda(c)))$$

$$=\int_a^c e^{-\lambda W_\lambda(z)}\,L(s_\lambda(a),s_\lambda(c),s_\lambda(z))\,dz.$$

(ii) *Let $a<x<c$. Then*

$$T_\lambda(x;a,c)=\int_{a-x}^{c-x} e^{-\lambda W_\lambda^x(z)}\,L(s_\lambda(x;a-x),s_\lambda(x;c-x),s_\lambda(x;z))\,dz$$

$$\overset{d}{=}\int_a^c e^{-\lambda W_\lambda(z)}\,L(\hat{s}_\lambda(x;a),\hat{s}_\lambda(x;c),\hat{s}_\lambda(x;z))\,dz,$$

where

$$(3.13)\qquad s_\lambda(x;z)=\int_0^z e^{\lambda W_\lambda^x(y)}\,dy=e^{-\lambda W_\lambda(x)}\int_x^{x+z} e^{\lambda W_\lambda(y)}\,dy,$$

$$(3.14)\qquad \hat{s}_\lambda(x;z)=\int_x^z e^{\lambda W_\lambda(y)}\,dy.$$

Lemma 3.3. *Let $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of the environment W with inner directed ascent A and let $\delta>0$. Let F_1 and F_2 be arbitrary closed intervals included in $(a^+, b^+]$ and $[b^-, c^-)$, respectively. Then, for any sufficiently small $\varepsilon>0$,*

(i) $$\inf_{x\in F_1} P\{T_\lambda(x; \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) = T_\lambda(x; \dot{b}^+_{(\lambda)})\} \to 1,$$

(ii) $$\inf_{x\in F_1} P\{T_\lambda(x; \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) < e^{\lambda(A+\delta)}\} \to 1,$$

(iii) $$\inf_{x\in F_2} P\{T_\lambda(x; \tilde{b}^-_{(\lambda)}, \dot{c}^-_{(\lambda)}) = T_\lambda(x; \tilde{b}^-_{(\lambda)})\} \to 1,$$

(iv) $$\inf_{x\in F_2} P\{T_\lambda(x; \tilde{b}^-_{(\lambda)}, \dot{c}^-_{(\lambda)}) < e^{\lambda(A+\delta)}\} \to 1$$

as $\lambda\to\infty$, where $\tilde{a}^+_{(\lambda)} = \varphi_\lambda^{-1}(\tilde{a}^+(\varepsilon))$, $\dot{b}^+_{(\lambda)} = \varphi_\lambda^{-1}(\dot{b}^+(\varepsilon))$, etc. are defined in Lemma 2.3.

Proof. Set $F_1=[u, v]\subset(a^+, b^+]$. We may assume that u is a continuity point of W without loss of generality. We choose a positive number ε_0 so that Lemma 2.3 holds with $\varepsilon=\varepsilon_0$ and $\varepsilon_0 < W_{\mathbb{a}} - \sup_{[u,b^+)} W$. Let $\varepsilon>0$ be an arbitrary number such that $0<4\varepsilon<\varepsilon_0\wedge\delta$. Then, noticing (3.1) and (iv) in Lemma 2.3, we have

$$\begin{aligned}\sup\{W_\lambda(y)\colon y\in(u, \dot{b}^+_{(\lambda)})\} &\leqq \sup\{W\circ\varphi_\lambda(y)\colon y\in(u, \dot{b}^+_{(\lambda)})\} + \varepsilon_\lambda \\ &\leqq \sup\{W(y)\colon y\in((u), \dot{b}^+)\} + \varepsilon_\lambda \\ &\leqq W_{\mathbb{a}} - \varepsilon_0 + \varepsilon + \varepsilon_\lambda\end{aligned}$$

for sufficiently large λ by the continuity of W at u. In the same manner, using (3.1) and (iii) in Lemma 2.3, we have

$$\begin{aligned}\inf\{W_\lambda(y)\colon y\in(\tilde{a}^+_{(\lambda)}, \dot{a}^+_{(\lambda)})\} &\geqq \inf\{W\circ\varphi_\lambda(y)\colon y\in(\tilde{a}^+_{(\lambda)}, \dot{a}^+_{(\lambda)})\} - \varepsilon_\lambda \\ &\geqq \inf\{W(y)\colon y\in(\tilde{a}^+, \dot{a}^+)\} - \varepsilon_\lambda \\ &\geqq W_{\mathbb{a}} - \varepsilon - \varepsilon_\lambda.\end{aligned}$$

Thus we see that, for every $x\in F_1$,

$$\begin{aligned}&P\{T_\lambda(x; \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) < T_\lambda(x; \dot{b}^+_{(\lambda)})\} \\ &= \left(\int_x^{\dot{b}^+_{(\lambda)}} e^{\lambda W_\lambda(y)}\,dy\right)\Bigg/\left(\int_{\tilde{a}^+_{(\lambda)}}^{\dot{b}^+_{(\lambda)}} e^{\lambda W_\lambda(y)}\,dy\right) \\ &\leqq \left(\int_u^{\dot{b}^+_{(\lambda)}} e^{\lambda W_\lambda(y)}\,dy\right)\Bigg/\left(\int_{\tilde{a}^+_{(\lambda)}}^{\dot{a}^+_{(\lambda)}} e^{\lambda W_\lambda(y)}\,dy\right) \\ &\leqq \frac{(c^- - a^+)}{(\dot{a}^+ - \tilde{a}^+ - 2\varepsilon_\lambda)} e^{\lambda(2\varepsilon + 2\varepsilon_\lambda - \varepsilon_0)} \to 0 \qquad \text{as } \lambda\to\infty.\end{aligned}$$

Let us prove the formula (ii). We employ a two dimensional Bessel process $\{R(t), t \geqq 0\}$ starting at 0. Then it is known (cf. [6], p. 75) that

$$\{L(-\infty, 1, 1-t)\colon 0 \leqq t \leqq 1\} \stackrel{d}{=} \{R(t)^2\colon 0 \leqq t \leqq 1\}, \tag{3.15}$$

$$\{t R(t^{-1})\colon t>0\} \stackrel{d}{=} \{R(t)\colon t>0\}. \tag{3.16}$$

Let us fix $x \in F_1$ and set, for $\tilde{a}^+_{(\lambda)} \leqq y \leqq \dot{b}^+_{(\lambda)}$,

$$\begin{aligned} \mathscr{s}(y) &= \hat{s}_\lambda(x; y)/\hat{s}_\lambda(x; \dot{b}^+_{(\lambda)}), \\ \bar{\mathscr{s}}(y) &= 1 - \mathscr{s}(y) \\ &= \hat{s}_\lambda(x; \dot{b}^+_{(\lambda)})^{-1} \int_y^{\dot{b}^+_{(\lambda)}} e^{\lambda W_\lambda(z)}\, dz. \end{aligned}$$

Then, by (ii) in Lemma 3.2, we have

$$T_\lambda(x; \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) \stackrel{d}{=} \int_{\tilde{a}^+_{(\lambda)}}^{\dot{b}^+_{(\lambda)}} e^{-\lambda W_\lambda(y)} L(\hat{s}_\lambda(x; \tilde{a}^+_{(\lambda)}), \hat{s}_\lambda(x; \dot{b}^+_{(\lambda)}), \hat{s}_\lambda(x; y))\, dy = K_1 + K_2,$$

where K_1 and K_2 are the integrals over $(\tilde{a}^+_{(\lambda)}, x)$ and $(x, \dot{b}^+_{(\lambda)})$, respectively. By Lemma 3.1, we have

$$\begin{aligned} K_1 &\stackrel{d}{=} \hat{s}_\lambda(x; \dot{b}^+_{(\lambda)}) \int_{\tilde{a}^+_{(\lambda)}}^{x} e^{-\lambda W_\lambda(y)} L(\mathscr{s}(\tilde{a}^+_{(\lambda)}), 1, \mathscr{s}(y))\, dy \\ &\leqq \hat{s}_\lambda(x; \dot{b}^+_{(\lambda)}) \int_{\tilde{a}^+_{(\lambda)}}^{x} e^{-\lambda W_\lambda(y)} L(-\infty, 1, \mathscr{s}(y))\, dy \\ &\leqq \int_x^{\dot{b}^+_{(\lambda)}} e^{\lambda W_\lambda(z)}\, dz \int_{\tilde{a}^+_{(\lambda)}}^{x} e^{-\lambda W_\lambda(y)}\, dy \sup_{t \leqq 0} L(-\infty, 1, t) \\ &\leqq (\dot{b}^+_{(\lambda)} - \tilde{a}^+_{(\lambda)})^2 e^{\lambda(A + 2\varepsilon + 2\varepsilon_\lambda)} \sup_{t \leqq 0} L(-\infty, 1, t), \end{aligned} \tag{3.17}$$

where we used the following estimate

$$\begin{aligned} &\sup\{W_\lambda(z); x < z < \dot{b}^+_{(\lambda)}\} - \inf\{W_\lambda(y); \tilde{a}^+_{(\lambda)} < y < x\} \\ &\quad \leqq \sup\{W(z); \varphi_\lambda(x) < z < b^+\} - \inf\{W(y); a^+ < y < \varphi_\lambda(x)\} \\ &\qquad + 2\varepsilon + 2\varepsilon_\lambda \quad \text{(by (3.1) and (iii), (ii) in Lemma 2.3)} \\ &\quad \leqq A + 2\varepsilon + 2\varepsilon_\lambda. \end{aligned}$$

On the other hand

$$(3.18)\qquad K_2 \overset{d}{=} \hat{s}_\lambda(x;\dot{b}^+_{(\lambda)}) \int_x^{\dot{b}^+_{(\lambda)}} e^{-\lambda W_\lambda(y)} L(\mathfrak{a}(\tilde{a}^+_{(\lambda)}), 1, \mathfrak{a}(y))\, dy$$

$$\leqq \hat{s}_\lambda(x;\dot{b}^+_{(\lambda)}) \int_x^{\dot{b}^+_{(\lambda)}} e^{-\lambda W_\lambda(y)} L(-\infty, 1, 1-\bar{\mathfrak{a}}(y))\, dy$$

(by (3.15) and (3.16))

$$\overset{d}{=} \hat{s}_\lambda(x;\dot{b}^+_{(\lambda)}) \int_x^{\dot{b}^+_{(\lambda)}} \{e^{-\lambda W_\lambda(y)} \bar{\mathfrak{a}}(y)\}\, \bar{\mathfrak{a}}(y)\, R(\bar{\mathfrak{a}}(y)^{-1})^2\, dy$$

$$= \int_x^{\dot{b}^+_{(\lambda)}} \bar{\mathfrak{a}}(y)\, R(\bar{\mathfrak{a}}(y)^{-1})^2\, dy \int_y^{\dot{b}^+_{(\lambda)}} e^{\lambda(W_\lambda(z)-W_\lambda(y))}\, dz$$

$$\leqq (\dot{b}^+_{(\lambda)} - \tilde{a}^+_{(\lambda)})^2\, e^{\lambda(A+2\varepsilon+2\varepsilon_\lambda)}\, J_\lambda,$$

where

$$(3.19)\qquad J_\lambda = \int_x^{\dot{b}^+_{(\lambda)}} \bar{\mathfrak{a}}(y)\, R(\bar{\mathfrak{a}}(y)^{-1})^2 (\dot{b}^+_{(\lambda)} - x)^{-1}\, dy$$

and we used the estimate

$$\sup\{W_\lambda(z) - W_\lambda(y) \colon x < y \leqq z < \dot{b}^+_{(\lambda)}\}$$
$$\leqq \sup\{W(z) - W(y) \colon a^+ < y \leqq z < \dot{b}^+\} + 2\varepsilon_\lambda$$
(by (3.1) and (iv) in Lemma 2.3)
$$\leqq A + 2\varepsilon + 2\varepsilon_\lambda.$$

By easy calculation, we see for every $t > 0$

$$E\{t^2 R(t^{-1})^4\} = t^{-2} E\{|t R(t^{-1})|^4\} = t^{-2} E\{R(t)^4\} = 8,$$

and hence we have by Schwartz's inequality

$$(3.20)\qquad E\{J_\lambda^2\} \leqq \int_x^{\dot{b}^+_{(\lambda)}} E\{|\bar{\mathfrak{a}}(y)|^2 R(\bar{\mathfrak{a}}(y)^{-1})^4\} (\dot{b}^+_{(\lambda)} - x)^{-1}\, dy = 8.$$

Consequently by (3.17), (3.18), (3.20) and Chebychev's inequality, we have

$$P\{T_\lambda(x;\tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) > e^{\lambda(A+\delta)}\}$$
$$\leqq P\{K_1 > (1/2)\, e^{\lambda(A+\delta)}\} + P\{K_2 > (1/2)\, e^{\lambda(A+\delta)}\}$$
$$\leqq P\{\sup_{y \leqq 0} L(-\infty, 1, y) > (1/2)(\dot{b}^+_{(\lambda)} - \tilde{a}^+_{(\lambda)})^{-2}\, e^{\lambda(\delta - 2\varepsilon - 2\varepsilon_\lambda)}\}$$
$$+ P\{J_\lambda > (1/2)(\dot{b}^+_{(\lambda)} - \tilde{a}^+_{(\lambda)})^{-2}\, e^{\lambda(\delta - 2\varepsilon - 2\varepsilon_\lambda)}\}$$
$$\leqq P\{\sup_{y \leqq 0} L(-\infty, 1, y) > (1/2)(\dot{b}^+ - \tilde{a}^+ + 2\varepsilon_\lambda)^{-2}\, e^{\lambda(\delta - 2\varepsilon - 2\varepsilon_\lambda)}\}$$
$$+ 32(c^- - a^+ + 2\varepsilon_\lambda)^4\, e^{-2\lambda(\delta - 2\varepsilon - 2\varepsilon_\lambda)} \to 0 \quad \text{as} \quad \lambda \to \infty$$

uniformly in $x \in F_1$.

In the same way we can obtain (iii) and (iv).

Lemma 3.4. *Let $\mathbb{V}=(\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of the environment W with inner directed ascent A and depth D. Let $\delta>0$ and F be an arbitrary closed interval included in (a^+, c^-). Then, for a sufficiently small $\varepsilon>0$, it holds that, as $\lambda\to\infty$,*

$$P\{T_\lambda(\dot{b}^+_{(\lambda)};\ \tilde{a}^+_{(\lambda)},\dot{c}^-_{(\lambda)})>e^{\lambda(D-\delta)}\}\to 1,$$

$$P\{T(\tilde{b}^-_{(\lambda)};\ \tilde{a}^+_{(\lambda)},\dot{c}^-_{(\lambda)})>e^{\lambda(D-\delta)}\}\to 1,$$

and

$$\inf_{x\in F} P\{T_\lambda(x;\ \tilde{a}^+_{(\lambda)},\dot{c}^-_{(\lambda)})>e^{\lambda(D-\delta)}\}\to 1,$$

where $\tilde{a}^+_{(\lambda)}=\varphi_\lambda^{-1}(\tilde{a}+(\varepsilon))$, $\dot{b}^+_{(\lambda)}=\varphi_\lambda^{-1}(\dot{b}^+(\varepsilon))$, etc.

Proof. Setting $F_1=F\cap(a^+, b^+]$ and $F_2=F\cap[b^-, c^-)$, we take a sufficiently small $\varepsilon>0$ so that Lemma 3.3 holds. Remember that $0<4\varepsilon<\delta$. We can assume $w_{\mathbb{c}}\leqq W_{\mathbb{a}}$, so $D=W_{\mathbb{c}}-W_{\mathbb{b}}$. Noticing (3.1) and (iii) in Lemma 2.3, we have for sufficiently large λ

$$
\begin{aligned}
(3.21)\qquad & \lambda^{-1}\log \hat{s}_\lambda(\dot{b}^+_{(\lambda)};\ \dot{c}^-_{(\lambda)})\\
&\geqq \lambda^{-1}\log\int_{\tilde{c}^-_{(\lambda)}}^{\dot{c}^-_{(\lambda)}} e^{\lambda W_\lambda(y)}\,dy\\
&\geqq \lambda^{-1}\log\{(\dot{c}^- -\tilde{c}^- -2\varepsilon_\lambda)\exp[\lambda\inf\{W_\lambda(x):\ \tilde{c}^-_{(\lambda)}\leqq x\leqq \dot{c}^-_{(\lambda)}\}]\}\\
&\geqq \lambda^{-1}\log(\dot{c}^- -\tilde{c}^- -2\varepsilon_\lambda)+W_{\mathbb{c}}-\varepsilon-\varepsilon_\lambda\\
&\geqq W_{\mathbb{c}}-\varepsilon+\varepsilon'_\lambda,
\end{aligned}
$$

where

$$
\begin{aligned}
&\varepsilon'_\lambda=\lambda^{-1}(\log(\dot{c}^- -\tilde{c}^- -2\varepsilon_\lambda))\wedge(\log(\dot{a}^+ -\tilde{a}^+ -2\varepsilon_\lambda))-\varepsilon_\lambda\\
&\to 0\quad\text{as}\quad \lambda\to\infty.
\end{aligned}
$$

In the same manner, recalling $W_{\mathbb{c}}\leqq W_{\mathbb{a}}$, we have

$$(3.22)\qquad \lambda^{-1}\log|\hat{s}_\lambda(\dot{b}^+_{(\lambda)};\ \tilde{a}^+_{(\lambda)})|\geqq W_{\mathbb{c}}-\varepsilon+\varepsilon'_\lambda.$$

Set

$$H_\lambda=|\hat{s}_\lambda(\dot{b}^+_{(\lambda)};\ \tilde{a}^+_{(\lambda)})|\wedge \hat{s}_\lambda(\dot{b}^+_{(\lambda)};\ \dot{c}^-_{(\lambda)})$$

and

$$\hat{\mathfrak{s}}_\lambda(y)=\hat{s}_\lambda(\dot{b}^+_{(\lambda)};\ y)\,H_\lambda^{-1}.$$

Then (3.21) and (3.22) imply that

$$H_\lambda\geqq\exp\{\lambda(W_{\mathbb{c}}-\varepsilon+\varepsilon'_\lambda)\}.$$

Then, by Lemma 3.1 and (iv) in Lemma 2.3, we see that

$$\begin{aligned}
T^{(\lambda)} &:= T_\lambda(\dot{b}^+_{(\lambda)};\, \tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}) \\
&\overset{d}{=} H_\lambda \int_{\tilde{a}^+_{(\lambda)}}^{\dot{c}^-_{(\lambda)}} e^{-\lambda W_\lambda(y)} L(\hat{s}_\lambda(\tilde{a}^+_{(\lambda)}), \hat{s}_\lambda(\dot{c}^-_{(\lambda)}), \hat{s}_\lambda(y))\, dy \\
&\geqq H_\lambda \int_{\tilde{b}^+_{(\lambda)}}^{\dot{b}^+_{(\lambda)}} e^{-\lambda W_\lambda(y)} L(-1,1, \hat{s}_\lambda(y))\, dy \\
&\geqq (\dot{b}^+_{(\lambda)} - \tilde{b}^+_{(\lambda)})\, e^{\lambda(D-2\varepsilon+\varepsilon'_\lambda-\varepsilon_\lambda)} L_\lambda,
\end{aligned}$$

where

$$L_\lambda = \inf\{L(-1,1,y):\ y \in (\hat{s}_\lambda(\tilde{b}^+_{(\lambda)}), \hat{s}_\lambda(\dot{b}^+_{(\lambda)}))\}.$$

Thus

$$T^{(\lambda)} \geqq (\dot{b}^+ - \tilde{b}^+ - 2\varepsilon_\lambda)\, e^{\lambda(D-2\varepsilon+\varepsilon'_\lambda-\varepsilon_\lambda)} L_\lambda. \tag{3.23}$$

On the other hand, we have

$$\hat{s}_\lambda(\tilde{b}^+_{(\lambda)}) \to 0, \quad \text{as} \quad \lambda \to \infty, \quad \text{and} \quad \hat{s}_\lambda(\dot{b}^+_{(\lambda)}) = 0, \tag{3.24}$$

because

$$\begin{aligned}
|\hat{s}_\lambda(\tilde{b}^+_{(\lambda)})| &\leqq (\dot{b}^+_{(\lambda)} - \tilde{b}^+_{(\lambda)}) \exp[\lambda \sup\{W_\lambda(x):\ x \in [\tilde{b}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}]\}]\, e^{-\lambda(W_{\mathrm{c}}-\varepsilon+\varepsilon'_\lambda)} \\
&\leqq (\dot{b}^+ - \tilde{b}^+ + 2\varepsilon_\lambda) \exp\{-\lambda(W_{\mathrm{c}} - W_{\mathrm{b}} + 2\varepsilon + \varepsilon'_\lambda - \varepsilon_\lambda)\} \to 0 \quad \text{as} \quad \lambda \to \infty.
\end{aligned}$$

Therefore by (3.23) and (3.24), which implies $L_\lambda > 0$ a.e., we obtain

$$\begin{aligned}
P\{T^{(\lambda)} < e^{\lambda(D-\delta)}\} &\leqq P\{L_\lambda < (\dot{b}^+ - \tilde{b}^+ - 2\varepsilon_\lambda)^{-1} e^{-\lambda(\delta-2\varepsilon+\varepsilon'_\lambda-\varepsilon_\lambda)}\} \\
&\to 0 \quad \text{as} \quad \lambda \to \infty.
\end{aligned} \tag{3.25}$$

The second formula is obtained in the same manner. The third formula is proved as follows. Let $x \in F$. Then we can assume $x \in F_1$. Therefore

$$\begin{aligned}
&P\{T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}) > e^{\lambda(D-\delta)}\} \\
&\quad \geqq P\{T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}) > e^{\lambda(D-\delta)} \text{ and } T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) = T_\lambda(x;\, \dot{b}^+_{(\lambda)})\} \\
&\quad = P\{T'_\lambda + T_\lambda(x;\, \dot{b}^+_{(\lambda)}) > e^{\lambda(D-\delta)} \text{ and } T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) = T_\lambda(x;\, \dot{b}^+_{(\lambda)})\} \\
&\qquad (\text{where } T'_\lambda = \inf\{t > 0:\ X^x_\lambda(t + T_\lambda(x;\, \dot{b}^+_{(\lambda)})) \notin (\tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)})\}) \\
&\quad \geqq P\{T'_\lambda > e^{\lambda(D-\lambda)}\} + P\{T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{b}^+_{(\lambda)}) = T_\lambda(x;\, \dot{b}^+_{(\lambda)})\} - 1 \\
&\qquad (\text{by employing the strong Markov property}) \\
&\quad = P\{T_\lambda(\dot{b}^+_{(\lambda)};\, \tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}) > e^{\lambda(D-\delta)}\} \\
&\qquad + P\{T_\lambda(x;\, \tilde{a}^+_{(\lambda)}, \dot{b}^-_{(\lambda)}) = T_\lambda(x;\, \dot{b}^+_{(\lambda)})\} - 1 \\
&\quad \to 1
\end{aligned}$$

as $\lambda \to \infty$ by (3.25) and Lemma 3.3.

4. The Proof of Theorem 1

In this section we prove Theorem 1 by using the coupling method. We maintain also the assumption of Theorem 1 throughout this section, that is,

$$W \in \mathbb{W}^{\#}, \qquad W_\lambda \in \mathbb{W}, \qquad \lambda > 0,$$

and

$$W_\lambda \to W \quad \text{as } \lambda \to \infty \text{ in the Skorohod topology.}$$

Therefore we also have the relation (3.1) for the homeomorphism φ_λ on $\mathbb{R}$ and the positive numbers ε_λ.

Let $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of the environment W. Let ε be an arbitrary small positive number. We employ the abbreviation $\tilde{a}^+, \tilde{b}^-, \tilde{a}^+_{(\lambda)}$, etc., for $\tilde{a}^+(\varepsilon)$, $\tilde{b}^-(\varepsilon)$, $\varphi_\lambda^{-1}(\tilde{a}^+(\varepsilon))$, etc., respectively.

In addition to the diffusion process $X_\lambda^x(t)$ we also consider a reflecting diffusion process $\hat{X}_\lambda(t)$ on the interval $[\tilde{a}^+_{(\lambda)}, \tilde{c}^-_{(\lambda)}]$ with (local) generator

$$\tag{4.1} \tfrac{1}{2} e^{\lambda W_\lambda(x)} \frac{d}{dx}\left(e^{-\lambda W_\lambda(x)} \frac{d}{dx}\right)$$

and with initial distribution

$$\tag{4.2} m_\lambda(dx) = e^{-\lambda W_\lambda(x)} dx \Big/ \int_{\tilde{a}^+_{(\lambda)}}^{\tilde{c}^-_{(\lambda)}} e^{-\lambda W_\lambda(y)} dy.$$

Since m_λ is an invariant measure of the reflecting diffusion $\hat{X}_\lambda(t)$, the process $\hat{X}_\lambda(t)$ is stationary. Now we couple the processes $X_\lambda^x(t)$ and $\hat{X}_\lambda(t)$ as follows. Enlarging the basic probability space on which $X_\lambda^x(t)$ is defined, we assume that $X_\lambda^x(t)$ and $\hat{X}_\lambda(t)$ are defined on a common probability space (the enlarged probability space) in such a way that (i) $X_\lambda^x(t)$ and $\hat{X}_\lambda(t)$ move independently according to their own probability laws until they first meet each other, (ii) after the first meeting time they move together up to the (common) exit time from the open interval $(\tilde{a}^+_{(\lambda)}, \tilde{c}^-_{(\lambda)})$ and that (iii) after the common exit time they again start moving independently according to their own probability laws.

Remark. In the above (iii) we may even make the process stop at the common exit time since we do not use the process after the exit time.

It is not hard to construct such a coupling. We denote by $\mathbb{P}_\lambda$ the probability measure on the common probability space. Let

$$\sigma_\lambda^x = \inf\{t \geqq 0 \colon X_\lambda^x(t) = \hat{X}_\lambda(t)\},$$
$$T_\lambda^x = \inf\{t \geqq \sigma_\lambda^x \colon X_\lambda^x(t) \notin (\tilde{a}^+_{(\lambda)}, \tilde{c}^-_{(\lambda)})\}, \qquad x \in [\tilde{a}^+_{(\lambda)}, \tilde{c}^-_{(\lambda)}].$$

Note that ε is hidden in $\tilde{a}^+_{(\lambda)} = \varphi_\lambda^{-1}(\tilde{a}^+(\varepsilon))$, etc. Then the above (ii) means that $X_\lambda^x(t) = \hat{X}_\lambda(t)$ for $\sigma_\lambda^x \leqq t \leqq T_\lambda^x$.

First we show the following lemma for the valley $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ of the environment W.

Lemma 4.1. *Let U be an arbitrary open set including $\mathbb{b}$. Then for sufficiently small $\varepsilon>0$, it holds that*

$$m_\lambda(U)\to 1 \quad as \quad \lambda\to\infty.$$

Especially we obtain

$$m_\lambda([\tilde{b}^-_{(\lambda)}, \dot{b}^+_{(\lambda)}])\to 1 \quad as \quad \lambda\to\infty.$$

Proof. For every sufficiently small $\varepsilon>0$ it holds that, for every sufficiently large λ,

$$U\supset U_\lambda:=\varphi_\lambda^{-1}(\{x: a^-\leqq x\leqq c^+, W_{\mathbb{b}}>W(x)-2\varepsilon\}).$$

Consequently using (iv) in Lemma 2.3 with the above $\varepsilon>0$ and (3.1), we obtain

$$\begin{aligned} m_\lambda(U_\lambda^c) &\leqq \Big(\int_{[\tilde{a}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}]\setminus U_\lambda} e^{-\lambda W_\lambda(y)}\,dy\Big)\Big/\int_{\tilde{b}^-_{(\lambda)}}^{\dot{b}^-_{(\lambda)}} e^{-\lambda W_\lambda(y)}\,dy \\ &\leqq ((\dot{c}^-_{(\lambda)}-\tilde{a}^+_{(\lambda)})\,e^{-\lambda(W_{\mathbb{b}}+2\varepsilon-\varepsilon_\lambda)})/((\dot{b}^-_{(\lambda)}-\tilde{b}^-_{(\lambda)})\,e^{-\lambda(W_{\mathbb{b}}+\varepsilon+\varepsilon_\lambda)}) \\ &\leqq (\dot{c}^- -\tilde{a}^+ +2\varepsilon_\lambda)\,(\dot{b}-\tilde{b}^- -2\varepsilon_\lambda)^{-1}\,e^{-\lambda(\varepsilon-2\varepsilon_\lambda)}\to 0 \quad \text{as} \quad \lambda\to\infty. \end{aligned}$$

To prove the second assertion, it is enough to prove

$$m_\lambda((\tilde{a}^+_{(\lambda)}, \tilde{b}^-_{(\lambda)}))\to 0 \tag{4.3}$$

and

$$m_\lambda((\dot{b}^+_{(\lambda)}, \dot{c}^-_{(\lambda)}))\to 0 \quad \text{as} \quad \lambda\to\infty. \tag{4.4}$$

Notice that

$$\inf\{W_\lambda(y): \tilde{a}^+_{(\lambda)}<y<\tilde{b}^-_{(\lambda)}\}\geqq \inf\{W(y): \tilde{a}^+<y<\tilde{b}^-\}-\varepsilon_\lambda\geqq W_{\mathbb{b}}+2\eta_0-\varepsilon_\lambda$$

for some $\eta_0>0$ by the shape of the valley. Since $\tilde{b}^-<\dot{b}^-$ and $W\in\mathbb{W}^\#$, we can find $b^1, b^2\in\mathbb{R}$ such that $b^1\leqq b^-\leqq b^2$, $b^1<b^2$ and

$$\sup\{W(y): b^1<y<b^2\}\leqq W_{\mathbb{b}}+\eta_0.$$

Therefore, noticing

$$\sup\{W_\lambda(y): b^1_{(\lambda)}<y<b^2_{(\lambda)}\}\leqq \sup\{W(y): b^1<y<b^2\}+\varepsilon_\lambda\leqq W_{\mathbb{b}}+\eta_0+\varepsilon_\lambda,$$

we have

$$\begin{aligned} & m_\lambda((\tilde{a}^+_{(\lambda)}, \tilde{b}^-_{(\lambda)})) \\ &\quad\leqq \int_{\tilde{a}^+_{(\lambda)}}^{\tilde{b}^-_{(\lambda)}} e^{-\lambda W_\lambda(y)}\,dy\Big/\int_{b^1_{(\lambda)}}^{b^2_{(\lambda)}} e^{-\lambda W_\lambda(y)}\,dy \\ &\quad\leqq (\tilde{b}^- -\tilde{a}^+ +2\varepsilon_\lambda)\,(b^2-b^1-2\varepsilon_\lambda)^{-1}\,e^{-\lambda(\eta_0-2\varepsilon_\lambda)}\to 0 \quad \text{as} \quad \lambda\to\infty. \end{aligned}$$

Thus (4.3) has been proved. In the same manner, we can prove (4.4).

Lemma 4.2. *Let $\delta>0$ and let F be an arbitrary closed interval included in (a^+, c^-). Then for sufficiently small $\varepsilon>0$*

$$\inf_{x\in F} \mathbb{P}_\lambda\{\sigma_\lambda^x < e^{\lambda(A+\delta)}\} \to 1 \quad \text{as} \quad \lambda\to\infty.$$

Proof. Let us choose a sufficiently small $\varepsilon>0$ which guarantees Lemma 3.3 for $F_1 = F\cap(a^+, b^+]$, $F_2 = F\cap[b^-, c^-)$ and $\delta/2$ in place of δ in Lemma 3.3 and guarantees also Lemma 4.1. Let us set $F_1' = F_1\cap\{x\colon x\leqq \tilde{b}_{(\lambda)}^-\}$ and $F_1'' = F_1\cap\{x\colon x>\tilde{b}_{(\lambda)}^-\}$. First we discuss the case $x\in F_1'$. We have

$$\begin{aligned}
&\inf_{x\in F_1'} \mathbb{P}_\lambda\{\sigma_\lambda^x \leqq T_\lambda(x; \dot{b}_{(\lambda)}^+)\} \\
&\quad\geqq \inf_{x\in F_1'} \mathbb{P}_\lambda\{\hat{X}_\lambda(0)\in[x, \dot{b}_{(\lambda)}^+] \text{ and } \hat{X}_\lambda(T_\lambda(x; \dot{b}_{(\lambda)}^+))\in[x, \dot{b}_{(\lambda)}^+]\} \\
&\quad\geqq \inf_{x\in F_1'} 2m_\lambda([x, \dot{b}_{(\lambda)}^+]) - 1 \\
&\quad\geqq 2m_\lambda([\tilde{b}_{(\lambda)}^-, \dot{b}_{(\lambda)}^+]) - 1 \to 1 \qquad \text{as } \lambda\to\infty \text{ by Lemma 4.1.}
\end{aligned}$$

Together with Lemma 3.3, this implies

$$\begin{aligned}
&\inf_{x\in F_1'} \mathbb{P}_\lambda\{\sigma_\lambda^x < e^{\lambda(A+\delta)}\} \\
&\quad\geqq \inf_{x\in F_1'} \mathbb{P}_\lambda\{\sigma_\lambda^x \leqq T_\lambda(x; \dot{b}_{(\lambda)}^+) \text{ and } T_\lambda(x; \dot{b}_{(\lambda)}^+) < e^{\lambda(A+\delta/2)}\} \\
&\quad\geqq \inf_{x\in F_1'} \mathbb{P}_\lambda\{\sigma_\lambda^x \leqq T_\lambda(x; \dot{b}_{(\lambda)}^+)\} + \inf_{x\in F_1} P_\lambda\{T_\lambda(x; \dot{b}_{(\lambda)}^+) < e^{\lambda(A+\delta/2)}\} - 1 \\
&\quad\to 1 \quad \text{as} \quad \lambda\to\infty.
\end{aligned}$$

Next we discuss the case $x\in F_1''$. We have

$$\begin{aligned}
&\inf_{F_1''} \mathbb{P}_\lambda\{\sigma_\lambda^x < e^{\lambda(A+\delta)}\} \\
&\quad\geqq \inf_{F_1''} \mathbb{P}_\lambda\{\sigma_\lambda^x < e^{\lambda(A+\delta)} \text{ and } T_\lambda(x; \tilde{b}_{(\lambda)}^-) < e^{\lambda(A+\delta/2)}\} \\
&\quad\geqq \inf_{F_1''} \mathbb{P}_\lambda\{\sigma_\lambda'^x + T_\lambda(x; \tilde{b}_{(\lambda)}^-) < e^{\lambda(A+\delta)} \text{ and } T_\lambda(x; \tilde{b}_{(\lambda)}^-) < e^{\lambda(A+\delta/2)}\} \\
&\quad\quad \text{(by using the strong Markov Property)} \\
&\quad\geqq \mathbb{P}_\lambda\{\sigma_\lambda^{\tilde{b}_{(\lambda)}^-} < e^{\lambda(A+\delta)}(1-e^{-\lambda\delta/2})\} \\
&\quad\quad + \inf_{F_1} P\{T_\lambda(x; \tilde{b}_{(\lambda)}^-) < e^{\lambda(A+\delta/2)}\} - 1 \\
&\quad\to 1 \qquad \text{as } \lambda\to\infty \text{ (by Lemma 3.3).}
\end{aligned}$$

Here $\sigma_\lambda'^x = \inf\{t>0: X_\lambda^x(T_\lambda^x(x; \tilde{b}_{(\lambda)}^-)+t) = \hat{X}_\lambda(T_\lambda(x; \tilde{b}_{(\lambda)}^-)+t)\}$. Thus we obtain

$$\inf_{F_1} \mathbb{P}_\lambda\{\sigma_\lambda^x < e^{\lambda(A+\delta)}\} \to 1 \quad \text{as} \quad \lambda \to \infty.$$

The estimation over F_2 is verified in the same manner.

The Proof of Theorem 1 (i). Let $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of W containing 0 with $A < r_1 < r_2 < D$ and U be an open set including $\mathbb{b}$. Since $a^+ < 0 < c^-$, we can pick up a closed interval F such that $0 \in F \subset (a^+, c^-)$. We choose a sufficiently small $\varepsilon > 0$ such that for δ with $0 < \delta < (D - r_2) \wedge (r_1 - A)$ and F, (i) Lemma 3.4, 4.1 and 4.2 hold and

$$\text{(ii)} \qquad \mathbb{P}_\lambda\{\hat{T}_\lambda(\tilde{a}_{(\lambda)}^+, \dot{c}_{(\lambda)}^-) > e^{\lambda(D-\delta)}\} \to 1 \quad \text{as} \quad \lambda \to \infty,$$

where the notation $\hat{T}_\lambda(a, c)$ stands for the first exit time from (a, c) after σ_λ^x for the process $\hat{X}_\lambda(t)$. Let $r \in [r_1, r_2]$. Then

$$\begin{aligned}
&P\{X(e^{\lambda r}, \lambda W_\lambda) \in U\} \\
&\quad = \mathbb{P}_\lambda\{X_\lambda^0(e^{\lambda r}) \in U\} \\
&\quad \geqq \mathbb{P}_\lambda\{\sigma_\lambda^0 \leqq e^{\lambda r_1}, X_\lambda^0(e^{\lambda r}) \in U \text{ and } e^{\lambda r_2} \leqq T_\lambda(\tilde{a}_{(\lambda)}^+, \dot{c}_{(\lambda)}^-)\} \\
&\quad = \mathbb{P}_\lambda\{\sigma_\lambda^0 \leqq e^{\lambda r_1}, \hat{X}_\lambda(e^{\lambda r}) \in U \text{ and } e^{\lambda r_2} \leqq \hat{T}_\lambda(\tilde{a}_{(\lambda)}^+, \dot{c}_{(\lambda)}^-)\} \\
&\quad \geqq \mathbb{P}_\lambda\{\sigma_\lambda^0 \leqq e^{\lambda r_1}\} + m_\lambda(U) + \mathbb{P}_\lambda\{e^{\lambda r_2} \leqq \hat{T}_\lambda(\tilde{a}_{(\lambda)}^+, \dot{c}_{(\lambda)}^-)\} - 2 \\
&\quad\quad \to 1 \quad \text{as} \quad \lambda \to \infty \quad \text{(by Lemma 4.2, 4.1 and 3.4).}
\end{aligned}$$

To prove the latter assertion of Theorem 1, we prepare another lemma. Let us assume that the environment W has two valleys $\mathbb{V}_1 = (\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2 = (\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ connected at 0. We can pick up continuity pints $x_1 \in (a_1^+, c_1^-)$ and $x_2 \in (a_2^+, c_2^-)$ of W such that

$$\text{(4.5)} \qquad H(c_1^+, x_1) \leqq A_1 \quad \text{and} \quad H(a_2^-, x_2) \leqq A_2,$$

where $H(\cdot, \cdot)$ is the function introduced in the definition of a valley (Sect. 1).

In fact, when $b_1^- < c_1^-$, by the definition of the inner directed ascent, any continuity point x_1 in (b_1^-, c_1^-) satisfies $H(c_1^+, x_1) \leqq A_1$ since $H(c_1^+, b_1^-) \leqq A_1$.

We consider the case where $b_1^- = b_1^+ = c_1^-$. Then, since $a_1^+ < c_1^-$ and $W_{\mathbb{b}} = W(b_1^- -)$ by the definition, we can find out a point x_1 of continuity of W such that $a_1^+ < x_1 < b_1^-$ and $W_{\mathbb{b}} \leqq W(x) < W_{\mathbb{b}} + A_1/2$ for every $x \in [x_1, b_1^-]$. Therefore $H(c_1^+, x_1) \leqq A_1$.

In the same manner, we can pick up a continuity point $x_2 \in (a_2^+, c_2^-)$ which satisfies (4.5).

Hereafter we consider a pair (x_1, x_2) fixed.

Lemma 4.3. *Let* $\delta > 0$. *Then*

$$P\{T_\lambda(x_1, x_2) < e^{\lambda(A_1 \vee A_2 + \delta)}\} \to 1 \quad \textit{as} \quad \lambda \to \infty.$$

Proof. The method of this proof is almost the same as that of the formula (ii) in Lemma 3.3. We employ a two dimensional Bessel process $\{R(t)\colon t \geqq 0\}$ starting at 0 and let

$$(4.6) \qquad T_\lambda(x_1, x_2) = \int_{x_1}^{x_2} e^{-\lambda W_\lambda(y)} L(s_\lambda(x_1), s_\lambda(x_2), s_\lambda(y))\, dy = K_1' + K_2',$$

where K_1' and K_2' are the integrals over $(x_1, 0)$ and over $(0, x_2)$, respectively. Let us consider K_1'. Choose $\varepsilon > 0$ so that $4\varepsilon < \delta$ and set

$$\begin{aligned} \mathfrak{s}_1(y) &= s_\lambda(y)/|s_\lambda(x_1)|, \\ \bar{\mathfrak{s}}_1(y) &= 1 + \mathfrak{s}_1(y) \\ &= |s_\lambda(x_1)|^{-1} \int_{x_1}^{y} e^{\lambda W_\lambda(z)}\, dz. \end{aligned}$$

Then

$$\begin{aligned} (4.7) \qquad K_1' &\overset{d}{=} |s_\lambda(x_1)| \int_{x_1}^{0} e^{-\lambda W_\lambda(y)} L(-1, \mathfrak{s}_1(x_2), \mathfrak{s}_1(y))\, dy \\ &\leqq |s_\lambda(x_1)| \int_{x_1}^{0} e^{-\lambda W_\lambda(y)} L(-1, \infty, \mathfrak{s}_1(y))\, dy \\ &\quad \text{(by the symmetric property of Brownian motion)} \\ &\overset{d}{=} |s_\lambda(x_1)| \int_{x_1}^{0} e^{-\lambda W_\lambda(y)} L(-\infty, 1, -\mathfrak{s}_1(y))\, dy \\ &\quad \text{(using (3.15) and (3.16))} \\ &\overset{d}{=} |s_\lambda(x_1)| \int_{x_1}^{0} e^{-\lambda W_\lambda(y)} \bar{\mathfrak{s}}_1(y)^2 R(\bar{\mathfrak{s}}_1(y)^{-1})^2\, dy \\ &= \int_{x_1}^{0} \bar{\mathfrak{s}}_1(y) R(\bar{\mathfrak{s}}_1(y)^{-1})^2\, dy \int_{x_1}^{y} e^{\lambda(W_\lambda(z) - W_\lambda(y))}\, dz \\ &\leqq (x_2 - x_1)^2 e^{\lambda(A_1 + 2\varepsilon + 2\varepsilon_\lambda)} J_\lambda', \end{aligned}$$

where

$$J_\lambda' = \int_{x_1}^{0} \bar{\mathfrak{s}}_1(y) R(\bar{\mathfrak{s}}_1(y)^{-1})^2 (-x_1)^{-1}\, dy.$$

Here we used the estimation for sufficiently large λ

$$\begin{aligned} &\sup\{W_\lambda(z) - W_\lambda(y)\colon x_1 < z \leqq y < 0\} \\ &\quad \leqq \sup\{W_\lambda(z) - W_\lambda(y)\colon x_1 < z \leqq y < \hat{c}_{1(\lambda)}^{+}\} \\ &\quad \leqq \sup\{W(z) - W(y)\colon \varphi_\lambda(x_1) < z \leqq y < \hat{c}_1^{+}\} + 2\varepsilon_\lambda \\ &\quad \leqq A_1 + 2\varepsilon + 2\varepsilon_\lambda. \end{aligned}$$

In the same way, we obtain

$$(4.8)\qquad K_2' \leqq (x_2 - x_1)^2\, e^{\lambda(A_2 + 2\varepsilon + 2\varepsilon_\lambda)}\, J_\lambda'',$$

where

$$J_\lambda'' = \int_0^{x_2} \bar{\partial}_2(y)\, R(\bar{\partial}_2(y)^{-1})^2\, x_2^{-1}\, dy,$$

$$\bar{\partial}_2(y) = s_\lambda(x_2)^{-1} \int_y^{x_2} e^{\lambda W_\lambda(z)}\, dz.$$

By using Chebychev's inequality as in (3.20), we have

$$\begin{aligned} &P\{T_\lambda(x_1, x_2) > e^{\lambda(A_1 \vee A_2 + \delta)}\} \\ &\quad \leqq P\{K_1' > (1/2)\, e^{\lambda(A_1 \vee A_2 + \delta)}\} + P\{K_2' > (1/2)\, e^{\lambda(A_1 \vee A_2 + \delta)}\} \\ &\quad \leqq 64(x_2 - x_1)^4\, e^{-2\lambda(\delta - 2\varepsilon - 2\varepsilon_\lambda)} \to 0 \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

The Proof of Theorem 1 (ii). Let W have two valleys $\mathbb{V}_1 = (\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2 = (\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ connected at 0 with $A = A_1 \vee A_2 < D_1 \wedge D_2 = D$. Let $I = [r_1, r_2]$ be an arbitrary interval such that $A < r_1 < r_2 < D$ and U be an arbitrary open set including $\mathbb{b}_1 \cup \mathbb{b}_2$. We choose $\delta > 0$ so that $\delta < (r_1 - A) \wedge (D - r_2)$. Since x_i was chosen as an inner point of (a_i^-, c_i^-), we can find out a closed interval F_i such that $x_i \in F_i \subset (a_i^+, c_i^-)$, $i = 1, 2$. Then we have, for $r \in I$,

$$\begin{aligned} (4.9)\qquad &P\{X(e^{\lambda r}, \lambda W_\lambda) \in U\} \\ &\quad \geqq P\{X_\lambda(e^{\lambda r}) \in U,\ T_\lambda(x_1, x_2) < e^{\lambda(A+\delta)}\} \\ &\quad = P\{X_\lambda(e^{\lambda r}) \in U,\ T_\lambda(x_1, x_2) = T_\lambda(x_1) < e^{\lambda(A+\delta)}\} \\ &\qquad + P\{X_\lambda(e^{\lambda r}) \in U,\ T_\lambda(x_1, x_2) = T_\lambda(x_2) < e^{\lambda(A+\delta)}\} \\ &\qquad \text{(by the strong Markov property)} \\ &\quad = \int_0^{e^{\lambda(A+\delta)}} P\{T_\lambda(x_1, x_2) = T_\lambda(x_1) \in du\}\, P\{X_\lambda^{x_1}(e^{\lambda r} - u) \in U\} \\ &\qquad + \int_0^{e^{\lambda(A+\delta)}} P\{T_\lambda(x_1, x_2) = T_\lambda(x_2) \in du\}\, P\{X_\lambda^{x_2}(e^{\lambda r} - u) \in U\}. \end{aligned}$$

By the coupling method, we show that, for $i = 1, 2$,

$$(4.10)\qquad \inf_{u \in (0, e^{\lambda(A+\delta)})} P\{X_\lambda^{x_i}(e^{\lambda r} - u) \in U\} = 1 + o(1) \quad \text{as} \quad \lambda \to \infty.$$

For a suitable positive number ε and for each $i = 1, 2$, we consider a reflecting diffusion process $\hat{X}_{i,\lambda}(t)$ on the interval $[\tilde{a}_{i(\lambda)}^+, \dot{c}_{i(\lambda)}^-]$ with generator (4.1) and with initial distribution

$$m_{i,\lambda}(dx) = e^{-\lambda W_\lambda(x)}\, dx \Big/ \int_{\tilde{a}_{i(\lambda)}^+}^{\dot{c}_{i(\lambda)}^-} e^{-\lambda W_\lambda(z)}\, dz.$$

For each $i=1,2$, we couple $X_\lambda^{x_i}(t)$ and $\hat{X}_{i,\lambda}(t)$ in the same way as we have done for $X_\lambda^x(t)$ and $\hat{X}_\lambda(t)$. The probability measure on the common probability space is denoted by $\mathbb{P}_{i,\lambda}$. Lemma 4.1 and 4.2 are valid with $\delta>0$, $m_\lambda=m_{i,\lambda}$ and $F=F_i$, $i=1,2$. We can choose ε to guarantee also Lemma 3.4 with $\mathbb{V}_i$, δ and F_i, $i=1,2$. For each $i=1,2$, denote by $\sigma_i^{x_i}(\lambda)$ the first meeting time of $X_\lambda^{x_i}(t)$ and $\hat{X}_{i,\lambda}(t)$. Then

$$\begin{aligned}
&P\{X_\lambda^{x_i}(e^{\lambda r}-u)\in U\}\\
&\quad\geqq \mathbb{P}_{i,\lambda}\{\sigma_i^{x_i}(\lambda)<e^{\lambda(A_i+\delta)}, \hat{X}_{i,\lambda}(e^{\lambda r}-u)\in U, e^{\lambda(D-\delta)}<T_\lambda(x_i; \tilde{a}_{i(\lambda)}^+, \dot{c}_{i(\lambda)}^-)\}\\
&\quad\geqq \inf_{x\in F_i}\mathbb{P}_{i,\lambda}\{\sigma_i^x(\lambda)<e^{\lambda(A_i+\delta)}\}+m_{i,\lambda}(U)\\
&\qquad+\inf_{x\in F_i}P\{T_\lambda(x; \tilde{a}_{i(\lambda)}^-, \dot{c}_{i(\lambda)}^-)>e^{\lambda(D-\delta)}\}-2\\
&\qquad\to 1 \quad \text{as} \quad \lambda\to\infty \quad \text{(owing to Lemmas 4.2, 4.1 and 3.4).}
\end{aligned}$$

Inserting (4.10) into (4.9), we have

$$P\{X(e^{\lambda r}, \lambda W_\lambda)\in U\}\geqq P\{T_\lambda(x_1, x_2)<e^{\lambda(A_1\vee A_2+\delta)}\}+o(1) \quad \text{as} \quad \lambda\to\infty.$$

This leads to the conclusion by Lemma 4.3.

5. The Proof of Theorem 2

5.1. We Prepare Lemmas

Lemma 5.1. *Let S be a Polish space with a Borel probability measure μ. For any $\varepsilon>0$ and Borel map f from S to the space $\mathbb{K}$ defined at the beginning of* Sect. 1, *there exists a continuous map f_ε such that*

$$E\{\rho_1(f_\varepsilon, f)\}=\int \rho_1(f_\varepsilon(x), f(x))\,\mu(d\mu)<\varepsilon, \tag{5.1}$$

where $\rho_1(\cdot,\cdot)=\rho(\cdot,\cdot)\wedge 1$.

Proof. Choose a countable dense family $\{K_i,\ i=1,2,\ldots\}$ in $\mathbb{K}$ and, for each $K\in\mathbb{K}$, let $\varphi_n(K)=K_l$ where

$$l=\min\{i\colon 1\leqq i\leqq n, \rho(K_i, K)=\min_{1\leqq j\leqq n}\rho(K_j, K)\}.$$

Then clearly $\varphi_n(K)\to K$ as $n\to\infty$. Therefore, there exists an n such that

$$E\{\rho_1(\varphi_n\circ f, f)\}<\varepsilon/2. \tag{5.2}$$

We set

$$\begin{aligned}
A_i^{(n)}&=\{x\in S\colon (\varphi_n\circ f)(x)=K_i\}\\
&=\{x\in S\colon f(x)\in\varphi_n^{-1}(K_i)\}.
\end{aligned}$$

Then $A_i^{(n)}$, $1 \leqq i \leqq n$, are mutually disjoint and $\bigcup_{i=1}^{n} A_i^{(n)} = S$. Choose compact subsets $F_i^{(n)}$ of $A_i^{(n)}$ such that

$$\mu(A_i^{(n)} - F_i^{(n)}) < \varepsilon/2n, \qquad 1 \leqq i \leqq n,$$

and if we set $\delta = (1/2) \min_{1 \leqq i < j \leqq n} \theta_{ij}$ where θ_{ij} is the distance between $F_i^{(n)}$ and $F_j^{(n)}$ in S, then $\delta > 0$. Therefore we can define a continuous map f_ε by

$$f_\varepsilon(x) = \begin{cases} \{0\} & \text{if } x \notin \bigcup_{i=1}^{n} U_\delta(F_i^{(n)}) \\ \dfrac{\delta - \theta(x, F_i^{(n)})}{\delta} K_i{}^{5} & \text{if } x \in U_\delta(F_i^{(n)}), \end{cases}$$

where $U_\delta(F_i^{(n)})$ is the δ-neighborhood of $F_i^{(n)}$ and $\theta(x, K)$ is the distance between x and K. Then

$$E\{\rho_1(f_\varepsilon, \varphi_n \circ f)\} < \varepsilon/2$$

and this, combined with (5.2), implies (5.1).

Lemma 5.2. *Let S be a Polish space and let X, X_n ($n \in \mathbb{N}$) be random variables with values in S. If X_n converges to X a.s. as $n \to \infty$, then, for any $\mathbb{K}$-valued Borel function f on S, there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of $\mathbb{K}$-valued Borel functions such that $f_n(X_n)$ converges to $f(X)$ as $n \to \infty$ a.s.*

Proof. Let μ be the probability distribution of X. According to Lemma 5.1 we can find a sequence $\{g_k\}$ of continuous maps on S to $\mathbb{K}$ such that

$$E\{\rho_1(g_k(X), f(X))\} = \int \rho_1(g_k, f)\, d\mu < k^{-2}.$$

Then by the Borel-Cantelli lemma, $g_k(X)$ converges to $f(X)$ as $k \to \infty$ a.s. Since X_n converges to X as $n \to \infty$ a.s., we can choose $n_1 < n_2 < \ldots$ such that

$$P\{\sup_{n \geqq n_k} \rho_1(g_k(X_n), g_k(X)) > k^{-1}\} < k^{-2}.$$

Therefore by the Borel-Cantelli lemma

$$\sup_{n \geqq n_k} \rho_1(g_k(X_n), g_k(X)) \to 0 \qquad \text{as } k \to \infty \text{ a.s.}$$

Set

$$f_n(x) = \begin{cases} g_1(x) & \text{for } 1 \leqq n < n_2 \\ g_k(x) & \text{for } n_k \leqq n < n_{k+1},\ k = 2, 3, \ldots. \end{cases}$$

[5] The notation rK stands for $\{rx \colon x \in K\}$

Then

$$\max_{n_k \leqq n < n_{k+1}} \rho_1(f_n(X_n), f(X))$$
$$\leqq \max_{n_k \leqq n < n_{k+1}} \rho_1(g_k(X_n), g_k(X_n)) + \rho_1(g_k(X), f(X))$$
$$\leqq \sup_{n \geqq n_k} \rho_1(g_k(X_n), g_k(X)) + \rho_1(g_k(X), f(X)) \to 0 \quad \text{as} \quad k \to \infty, \text{ a.s.}$$

Lemma 5.3. *For any fixed* $\alpha > 0$, $\lambda > 0$ *and* $W \in \mathbb{W}$,

$$\{X(t, \lambda \tau_\lambda^\alpha W), t \geqq 0\} \stackrel{d}{=} \{\lambda^{-\alpha} X(\lambda^{2\alpha} t, W), t \geqq 0\}.$$

Proof. By the definition of the process we have

$$X(t, \lambda \tau_\lambda^\alpha W) \stackrel{d}{=} s_{\alpha,\lambda}^{-1}(B(A_{\alpha,\lambda}^{-1}(t))),$$

where

$$s_{\alpha,\lambda}(x) = \int_0^x e^{\lambda \tau_\lambda^\alpha W(y)} dy$$

$$A_{\alpha,\lambda}(t) = A_{\alpha,\lambda}(t, B) = \int_0^t e^{-2\lambda \tau_\lambda^\alpha W(s_{\alpha,\lambda}^{-1}(B(s)))} ds$$

and $B(t)$ is a one dimensional Brownian motion. Since

$$s_{\alpha,\lambda}(x) = \int_0^x e^{W(\lambda^\alpha y)} dy = \lambda^{-\alpha} \int_0^{\lambda^\alpha x} e^{W(y)} dy = \lambda^{-\alpha} s(\lambda^\alpha x),$$

we have $s_{\alpha,\lambda}^{-1}(x) = \lambda^{-\alpha} s^{-1}(\lambda^\alpha x)$. If we set $B_\alpha(t) = \lambda^{-\alpha} B(\lambda^{2\alpha} t)$, then $B_\alpha(t)$ is a Brownian motion and we can see that

$$s_{\alpha,\lambda}^{-1}(B(A_{\alpha,\lambda}^{-1}(t, B))) \stackrel{d}{=} s_{\alpha,\lambda}^{-1}(B_\alpha(A_{\alpha,\lambda}^{-1}(t, B_\alpha))),$$

$$A_{\alpha,\lambda}(t, B_\alpha) = \int_0^t e^{-2\lambda \tau_\lambda^\alpha W(s_{\alpha,\lambda}^{-1}(B_\alpha(s)))} ds$$
$$= \lambda^{-2\alpha} \int_0^{\lambda^{2\alpha} t} e^{-2W(s^{-1}(B(s)))} ds$$
$$= \lambda^{-2\alpha} A(\lambda^{2\alpha} t, B).$$

This implies

$$A_{\alpha,\lambda}^{-1}(t, B_\alpha) = \lambda^{-2\alpha} A^{-1}(\lambda^{2\alpha} t).$$

Therefore we obtain

$$s_{\alpha,\lambda}^{-1}(B_\alpha(A_{\alpha,\lambda}^{-1}(t, B_\alpha))) = \lambda^{-\alpha} X(\lambda^{2\alpha} t, W).$$

This shows the conclusion.

5.2. The Proof of Theorem 2. Let $m(\cdot,\cdot)$ be a metric in the space of probability distributions on $\mathbb{K}$, compatible with the weak convergence. For a Borel measurable mapping $\tilde{\mathbb{b}}$: $\mathbb{W}\to\mathbb{K}$, we write

$$(5.3)\qquad M(\tilde{\mathbb{b}})=m(\mathscr{L}(\tilde{\mathbb{b}}),\mathscr{L}(\mathbb{b})),$$

where $\mathscr{L}(\tilde{\mathbb{b}})$ is the distribution of $\mathbb{K}$-valued random variable $\tilde{\mathbb{b}}$ defined on the probability space $(\mathbb{W},\mu)$ and $\mathscr{L}(\mathbb{b})$ is that of $\mathbb{b}(\cdot)$ on $(\mathbb{W},\nu)$. Let

$$(5.4)\qquad \Phi(\lambda;\tilde{\mathbb{b}})=\int\mu(dW)\int_0^1 P\{\lambda^{-\alpha}X(e^{-\lambda},W)\notin U_\varepsilon(\tilde{\mathbb{b}}(W))\}\,d\varepsilon+M(\tilde{\mathbb{b}}).$$

Set

$$(5.5)\qquad \Phi(\lambda)=\inf\Phi(\lambda;\tilde{\mathbb{b}}),$$

where the infimum is taken over all Borel measurable mappings $\tilde{\mathbb{b}}$. Then, for each $\lambda>0$, we can choose a Borel measurable mapping $\mathbb{b}_\lambda(W)$, such that

$$(5.6)\qquad \Psi(\lambda):=\Phi(\lambda;\mathbb{b}_\lambda)\leqq\Phi(\lambda)+1/\lambda.$$

In order to complete the proof of the theorem, it is enough to prove that

$$(5.7)\qquad \lim_{\lambda\to\infty}\Psi(\lambda)=0.$$

Let $\{\lambda_n\}$ be any sequence such that $\lambda_n>0$ and $\lambda_n\to\infty$ as $n\to\infty$. Then, since $\tau^\alpha_{\lambda_n}\mu\to\nu$ weakly as $n\to\infty$, by Skorohod's realization theorem of almost sure convergence, we can find $\mathbb{W}$-valued random variables $\tilde{W}^\alpha_{\lambda_n}$ and $\tilde{W}$ defined on a suitable probability space $(\tilde{\Omega},\tilde{P})$ having the following properties:

(a) the distribution of $\tilde{W}$ and $\tilde{W}^\alpha_{\lambda_n}$ are ν and $\tau^\alpha_{\lambda_n}\mu$, respectively,

(b) $\tilde{W}^\alpha_{\lambda_n}\to\tilde{W}$ in the Skorohod topology as $n\to\infty$, $\tilde{P}$ a.s.

Since ν is a self-similar measure, Proposition 1 shows that $\tilde{W}$ has either a valley $\mathbb{V}$ containing 0 with $A<1<D$ or two valleys $\mathbb{V}_1$ and $\mathbb{V}_2$ connected at 0 with $A_1\vee A_2<1<D_1\wedge D_2$ a.s. $\tilde{P}$. Therefore by Theorem 1, for every $\varepsilon>0$ and for any sequence $\{r_n\}$ with $r_n\to1$ as $n\to\infty$, we have

$$(5.8)\qquad P\{X(e^{\lambda_n r_n},\lambda_n\tilde{W}^\alpha_{\lambda_n})\notin U_\varepsilon(\mathbb{b}(\tilde{W}))\}\to0\quad\text{as}\quad n\to\infty\ \text{a.s. }\tilde{P}.$$

On the other hand using Lemma 5.2, we can choose Borel functions $\tilde{\mathbb{b}}_n(W)$ on $\mathbb{W}$ to $\mathbb{K}$ such that

$$(5.9)\qquad \tilde{\mathbb{b}}_n(\tilde{W}^\alpha_{\lambda_n})\to\mathbb{b}(\tilde{W})\quad\text{as}\quad n\to\infty\quad\text{a.s. }\tilde{P}.$$

Thus (5.8) and (5.9) imply

$$(5.10)\qquad P\{X(e^{\lambda_n r_n},\lambda_n\tilde{W}^\alpha_{\lambda_n})\notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tilde{W}^\alpha_{\lambda_n}))\}\to0\quad\text{as}\quad n\to\infty\ \text{a.s. }\tilde{P}.$$

Since

$$(\tilde{W}^{\alpha}_{\lambda_n}, \tilde{P}) \stackrel{d}{=} (W, \tau^{\alpha}_{\lambda_n}\mu) \stackrel{d}{=} (\tau^{\alpha}_{\lambda_n} W, \mu), \tag{5.11}$$

(5.10) implies that, as $n \to \infty$,

$$P\{X(e^{\lambda_n r_n}, \lambda_n \tau^{\alpha}_{\lambda_n} W) \notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tau^{\alpha}_{\lambda_n} W))\} \to 0 \quad \text{in probability } \mu.$$

Setting $r_n = 1 - 2\alpha(\log \lambda_n)/\lambda_n$ to apply the scaling property of Lemma 5.3, we obtain

$$P\{\lambda_n^{-\alpha} X(e^{\lambda_n}, W) \notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tau^{\alpha}_{\lambda_n} W))\} \to 0 \quad \text{in probability } \mu \text{ as } n \to \infty. \tag{5.12}$$

Now (5.9), (5.11), (a) and (b) imply

$$M(\tilde{\mathbb{b}}_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.13}$$

Combining (5.12) and (5.13), we obtain

$$\Phi(\lambda_n) \leqq \Psi(\lambda_n; \tilde{\mathbb{b}}_n(\tau^{\alpha}_{\lambda_n} \cdot)) \to 0 \quad \text{as} \quad n \to \infty.$$

This proves (5.7).

6. The Proof of Theorem 3

In this section, we assume that there are $W_\lambda \in \mathbb{W}^{\alpha}_{\lambda}$ and $W \in \mathbb{W}^{\#}$ for some $\alpha > 0$ such that

$$W_\lambda \to W \quad \text{in the Skorohod topology as } \lambda \to \infty. \tag{6.1}$$

We use the same ε_λ and φ_λ as in the beginning of Sect. 3. Thus (3.1) holds. Remember that for every $a \in \mathbb{R}$, $a_{(\lambda)}$ is the abbreviation of $\varphi_\lambda^{-1}(a)$. We consider the Markov chain $\{Y^x_\lambda(n, \lambda W_\lambda)\}$ on $\lambda^{-\alpha}\mathbb{Z}$ defined by (1.3). The proof of Theorem 3 is similar to that of Theorem 1; we prepare lemmas on exit times of the Markov chain from valleys of the environment W and employ a coupling technique to complete the proof.

Now we introduce some notations. Let $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of the environment W. We set, for arbitrary small $\varepsilon > 0$,

$$\begin{aligned} \hat{a}^{+}_{\lambda} &= \min\{2k\lambda^{-\alpha}\colon 2k\lambda^{-\alpha} \geqq \tilde{a}^{+}_{(\lambda)}\}, \\ \hat{b}^{-}_{\lambda} &= \min\{2k\lambda^{-\alpha}\colon 2k\lambda^{-\alpha} \geqq \tilde{b}^{-}_{(\lambda)}\}, \\ \check{b}^{+}_{\lambda} &= \max\{2k\lambda^{-\alpha}\colon 2k\lambda^{-\alpha} \leqq \dot{b}^{+}_{(\lambda)}\}, \\ \check{c}^{-}_{\lambda} &= \max\{2k\lambda^{-\alpha}\colon 2k\lambda^{-\alpha} \leqq \dot{c}^{-}_{(\lambda)}\}, \end{aligned}$$

where k represents an integer, $\tilde{a}^{+} = \tilde{a}^{+}(\varepsilon)$, $\tilde{b}^{-} = \tilde{b}^{-}(\varepsilon)$, etc., are defined in Lemma 2.3.

We can easily see the following lemma.

Lemma 6.1. *For any $x \in \mathbb{R}$ and $\lambda>0$, $\{\lambda^{2\alpha}\Gamma_{\lambda,n}^{\alpha,x}, n=1,2,\ldots\}$ are i.i.d. random variables, for any $\lambda>0$, and their common distribution is that of the hitting time of $\{\pm 1\}$ for a Brownian motion starting at 0. The distribution has expectation 1.*

Next we set

$$\begin{aligned}
N_\lambda(l_1,l_2) &= \min\{n\in\mathbb{N}\cup\{0\}:\ Y_\lambda(n)\notin(l_1,l_2)\},\\
N_\lambda(x;l_1,l_2) &= \min\{n\in\mathbb{N}\cup\{0\}:\ Y_\lambda^x(n)\notin(l_1,l_2)\},\\
N_\lambda(l) &= \min\{n\in\mathbb{N}\cup\{0\}:\ Y_\lambda(n)=l\},\\
N_\lambda(x;l) &= \min\{n\in\mathbb{N}\cup\{0\}:\ Y_\lambda^x(n)=l\}\\
&\quad \text{for } l_1,l_2,l,x\in\lambda^{-\alpha}\mathbb{Z}.
\end{aligned}$$

Lemma 6.2. *For any closed interval $F=[u,v]$ and for sufficiently small $\varepsilon>0$ let*

$$N_{x,\lambda}=\begin{cases} N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+) & \text{if } a^+<u<v<b^+,\\ N_\lambda(x;\hat{b}_\lambda^-,\check{c}_\lambda^-) & \text{if } b^-<u<v<c^-,\\ N_\lambda(x;\hat{a}_\lambda^+,\check{c}_\lambda^-) & \text{if } a^+<u<v<c^-.\end{cases}$$

Then, for any $\eta>0$,

$$\lim_{\lambda\to\infty}\inf_{x\in F} P\{|(\lambda^{2\alpha}\Gamma_0^x+\lambda^{2\alpha}\Gamma_1^x+\ldots+\lambda^{2\alpha}\Gamma_{N_{x,\lambda}}^x)/N_{x,\lambda}-1|<\eta\}=1.$$

Proof. We exhibit the proof in the case where $N_{x,\lambda}=N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)$ and $a^+<u<v<b^+$. We have

$$\begin{aligned}
&\sup_{x\in F} P\{|(\lambda^{2\alpha}\Gamma_0^x+\ldots+\lambda^{2\alpha}\Gamma_{N_{x,\lambda}}^x)/N_{x,\lambda}-1|>\eta\}\\
&\quad\leqq \sup_{x\in F} P\{\sup_{n\geqq N_{x,\lambda}}|(\lambda^{2\alpha}\Gamma_0^x+\ldots+\lambda^{2\alpha}\Gamma_n^x)/n-1|>\eta\}\\
&\quad\leqq \sup_{x\in F} P\{N_{x,\lambda}<m\}+\sup_{x\in F} P\{\sup_{n\geqq m}|(\lambda^{2\alpha}\Gamma_0^x+\ldots+\lambda^{2\alpha}\Gamma_n^x)/n-1|>\eta\}
\end{aligned}$$

for every $m>0$. For large m, the second term is small by the law of large numbers owing to Lemma 6.1. The first term can be made arbitrarily small by taking λ large enough.

Lemma 6.3. *Let $\delta>0$ and let F_1 and F_2 be arbitrary closed intervals included in $(a^+,b^+]$ and $[b^-,c^-)$, respectively. Then, for any sufficiently small $\varepsilon>0$, it holds that*

$$\begin{aligned}
&\lim_{\lambda\to\infty}\inf_{x\in F_1} P\{N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)=N_\lambda(x;\check{b}_\lambda^+)\}=1,\\
&\lim_{\lambda\to\infty}\inf_{x\in F_1} P\{N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)<e^{\lambda(A+\delta)}\}=1,\\
&\lim_{\lambda\to\infty}\inf_{x\in F_2} P\{N_\lambda(x;\hat{b}_\lambda^-,\check{c}_\lambda^-)=N_\lambda(x;\hat{b}_\lambda^-)\}=1,\\
&\lim_{\lambda\to\infty}\inf_{x\in F_2} P\{N_\lambda(x;\hat{b}_\lambda^-,\check{c}_\lambda^-)<e^{\lambda(A+\delta)}\}=1.
\end{aligned}$$

Proof. Set $F_1=[u,v]\subset(a^+,b^+]$, we can assume that u is a continuity point of W without loss of generality. Choose $\varepsilon_0<W_a-\sup_{[u,b^+]} W$ and $\varepsilon>0$ so that $0<4\varepsilon<\varepsilon_0\wedge\delta$. Let $N_{x,\lambda}=N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)$ temporarily. Then $\Gamma_0^x+\ldots+\Gamma_{N_{x,\lambda}}^x=T_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)$. The first formula is verified by the same manner as in the proof of (i) in Lemma 3.3. Now the second one can be verified as follows:

$$\begin{aligned}
&\inf_{x\in F_1} P\{N_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)\leqq e^{\lambda(A+\delta)}\}\\
&\quad=\inf_{x\in F_1} P\{\Gamma_0^x+\ldots+\Gamma_{N_{x,\lambda}}^x\leqq(\lambda^{2\alpha}\Gamma_0^x+\ldots+\lambda^{2\alpha}\Gamma_{N_{x,\lambda}}^x)N_{x,\lambda}^{-1}\lambda^{-2\alpha}e^{\lambda(A+\delta)}\}\\
&\quad\geqq\inf_{x\in F_1} P\{|(\lambda^{2\alpha}\Gamma_0^x+\ldots+\lambda^{2\alpha}\Gamma_{N_{x,\lambda}}^x)/N_{x,\lambda}-1|<\varepsilon\}\\
&\qquad+\inf_{x\in F_1} P\{T_\lambda(x;\hat{a}_\lambda^+,\check{b}_\lambda^+)\leqq(1-\varepsilon)\lambda^{-2\alpha}e^{\lambda(A+\delta)}\}-1\\
&\quad\to 1\quad\text{as}\quad\lambda\to\infty\quad\text{(by Lemma 3.3 and 6.2).}
\end{aligned}$$

We can obtain the other formulas for F_2 in the same manner.

Similarly we can prove the following lemma by Lemma 3.4 and Lemma 6.2.

Lemma 5.4. *Let $\delta>0$ and let F be an arbitrary closed interval included in (a^+,c^-). Then for a sufficiently small $\varepsilon>0$*

$$\lim_{\lambda\to\infty}\inf_{x\in F} P\{N_\lambda(x;\hat{a}_\lambda^+,\check{c}_\lambda^-)>e^{\lambda(D-\delta)}\}=1.$$

To complete the proof of Theorem 3, we employ the coupling method again. We proceed as in Sect. 4.

In addition to $Y_\lambda^x(n)$ we consider a reflecting Markov chain $\tilde{Y}_\lambda^y(n)$ on $[\hat{a}_\lambda^+,\check{c}_\lambda^-]\cap\lambda^{-\alpha}\mathbb{Z}$ starting at y, which, so long as it is in $(\hat{a}_\lambda^+,\check{c}_\lambda^-)\cap\lambda^{-\alpha}\mathbb{Z}$, has the same transition function as $Y_\lambda^x(n)$. This reflecting Markov chain has a 2-step invariant measure m_λ given by

$$(6.2)\quad m_\lambda(\{\hat{a}_\lambda^++k\lambda^{-\alpha}\})=\begin{cases}K_0\exp\{-\lambda W_\lambda(\hat{a}_\lambda^+)\} & \text{if } k=0,\\ K_0(\exp\{-\lambda W_\lambda(\hat{a}_\lambda^++(k-1)\lambda^{-\alpha})\}\\ \quad+\exp\{-\lambda W_\lambda(\hat{a}_\lambda^++k\lambda^{-\alpha})\}) & \text{if } k \text{ is even, } 0<k<\lambda^\alpha(\check{c}_\lambda^--\hat{a}_\lambda^+),\\ K_0\exp\{-\lambda W_\lambda(\check{c}_\lambda^-\lambda^{-\alpha})\} & \text{if } k=\lambda^\alpha(\check{c}_\lambda^--\hat{a}_\lambda^+),\\ 0 & \text{if } k \text{ is odd,}\end{cases}$$

where

$$(6.3)\qquad K_0=\left(\lambda^\alpha\int_{\hat{a}_\lambda^+}^{\check{c}_\lambda^-}e^{-\lambda W_\lambda(y)}\,dy\right)^{-1}.$$

We also consider a reflecting Markov chain $\tilde{Y}_\lambda(n)$ on $[\hat{a}_\lambda^+,\check{c}_\lambda^-]\cap\lambda^{-\alpha}\mathbb{Z}$ with the same transition matrix as that of $\tilde{Y}_\lambda^y(n)$ and with initial distribution m_λ. We then couple $Y_\lambda^x(n)$ and $\tilde{Y}_\lambda(n)$ as follows: Enlarging the probability space on which $Y_\lambda^x(n)$ is defined, we assume that $Y_\lambda^x(n)$ and $\tilde{Y}_\lambda(n)$ are defined on a common probability space (the enlarged probability space) in such a way that (i) they move independently according to their own probability laws up to the first meeting time, (ii) after the first meeting time they move together up to the (common) exit time from $(\hat{a}_\lambda^+, \check{c}_\lambda^-)$ and (iii) after the exit time they again start moving independently. We denote by $\hat{\mathbb{P}}_\lambda$ the probability measure on the common probability space. Remember that the process $\tilde{Y}_\lambda$ depends on $\varepsilon>0$.

Let us define the first meeting time M_λ^x of $Y_\lambda^x(n)$ and $\tilde{Y}_\lambda(n)$ by

$$(6.4)\qquad M_\lambda^x = \inf\{n \in \mathbb{N} \cup \{0\} : Y_\lambda^x(n) = \tilde{Y}_\lambda(n)\}, \qquad x \in [\hat{a}_\lambda^+, \check{c}_\lambda^-] \cap \lambda^{-\alpha}\mathbb{Z}.$$

Before giving the proof of Theorem 3(i), we list two lemmas. The first one is proved in the same manner as in Lemma 4.1. Here we omit the proof.

Lemma 6.5. *Let U be an arbitrary open set including $\mathbb{b}$. Then for sufficiently small $\varepsilon>0$*

$$m_\lambda(U) \to 1 \quad as \quad \lambda \to \infty.$$

In particular $m_\lambda([\hat{b}_\lambda^-, \check{b}_\lambda^+]) \to 1$ as $\lambda \to \infty$.

Now we see that

$$(6.5)\qquad \lim_{\lambda\to\infty} \hat{\mathbb{P}}_\lambda(\tilde{Y}_\lambda(1) \in U) = 1.$$

In fact, by easy calculation, for $l \in (\hat{a}_\lambda^+, \check{c}_\lambda^-) \cap \lambda^{-\alpha}\mathbb{Z}$ we have

$$\begin{aligned}
&\hat{\mathbb{P}}_\lambda(\tilde{Y}_\lambda(1) = l) \\
&= \int m_\lambda(dy)\, \hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda(1) = l \mid \tilde{Y}_\lambda(0) = y\} \\
&= m_\lambda(l+\lambda^{-\alpha})\, \hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda(1) = l \mid \tilde{Y}_\lambda(0) = l+\lambda^{-\alpha}\} \\
&\quad + m_\lambda(l-\lambda^{-\alpha})\, \hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda(1) = l \mid \tilde{Y}_\lambda(0) = l-\lambda^{-\alpha}\} \\
&= \begin{cases} K_0(\exp\{-\lambda W_\lambda(l-\lambda^{-\alpha})\} + \exp\{-\lambda W_\lambda(l)\}) & \text{if } l\lambda^\alpha \text{ is odd}, \\ 0 & \text{if } l\lambda^\alpha \text{ is even}. \end{cases}
\end{aligned}$$

Here K_0 is defined in (6.3). Therefore this implies (6.5) in the same manner as in Lemma 6.5 compared with the formula (6.2).

The next lemma can be proved by using Lemma 6.3, Lemma 6.5 and almost the same argument as that of Lemma 4.2.

Lemma 6.6. *Let $\delta>0$ and let F be an arbitrary closed interval included in (a^+, c^-). Then for a sufficiently small $\varepsilon>0$*

$$\lim_{\lambda\to\infty} \inf_{x \in F \cap \mathbb{Z}_{\text{even}}^\alpha} \hat{\mathbb{P}}_\lambda\{M_\lambda^x < e^{\lambda(A+\delta)}\} = 1,$$

where $\mathbb{Z}_{\text{even}}^\alpha$ is the set of all $2k/\lambda^\alpha$, $k \in \mathbb{Z}$.

The Proof of Theorem 3(i). Let $\mathbb{V} = (\mathbb{a}, \mathbb{b}, \mathbb{c})$ be a valley of the environment W containing 0 with $A < r_1 < r_2 < D$ and U be an arbitrary open set including

$\mathbb{b}$. Then we pick up a closed interval F such that $0 \in F \subset (a^+, c^-)$. We choose a sufficiently small $\varepsilon > 0$ so that Lemma 6.5 and Lemma 6.6 hold for $\delta = \{(D - r_2) \wedge (r_1 - A)\}/2$ and for the closed interval F.

We couple $Y_\lambda^0(n)$ and $\tilde{Y}_\lambda(n)$ on $[\hat{a}_\lambda^+, \check{c}_\lambda^-]$. For $r \in [r_1, r_2]$, we have

$$\begin{aligned} &P\{Y_\lambda([e^{\lambda r}], \lambda W_\lambda) \in U\} \\ &\quad = \hat{\mathbb{P}}_\lambda\{Y_\lambda^0([e^{\lambda r}]) \in U\} \\ &\quad \geqq \hat{\mathbb{P}}_\lambda\{M_\lambda^0 \leqq e^{\lambda r_1}, \tilde{Y}_\lambda([e^{\lambda r}]) \in U \text{ and } e^{\lambda r_2} \leqq N_\lambda(\hat{a}_\lambda^+, \check{c}_\lambda^-)\} \\ &\quad \geqq \hat{\mathbb{P}}_\lambda\{M_\lambda^0 \leqq e^{\lambda r_1}\} + \hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda([e^{\lambda r}]) \in U\} \\ &\qquad + P\{e^{\lambda r_2} \leqq N_\lambda(\hat{a}_\lambda^+, \check{c}_\lambda^-)\} - 2. \end{aligned}$$

The first term goes to 1 as $\lambda \to \infty$ by Lemma 6.6 and the third term goes to 1 as $\lambda \to \infty$ by Lemma 6.4. Since

$$\begin{aligned} &\hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda([e^{\lambda r}]) \in U\} \\ &\quad = \begin{cases} m_\lambda(U) & \text{if } [e^{\lambda r}] \text{ is even,} \\ \hat{\mathbb{P}}_\lambda\{\tilde{Y}_\lambda(1) \in U\} & \text{if } [e^{\lambda r}] \text{ is odd,} \end{cases} \end{aligned}$$

the second term goes to 1 as $\lambda \to \infty$ owing to the formula (6.5) and Lemma 6.5.

Lemma 6.7. *Let* $\mathbb{V}_1 = (\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ *and* $\mathbb{V}_2 = (\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ *be two valleys connected at* 0 *and* $\delta > 0$. *Choosing* (x_1, x_2) *as in* (4.5), *we set*

$$\begin{aligned} \hat{x}_{1,\lambda} &= \min\{2k\lambda^{-\alpha}: 2k\lambda^{-\alpha} \geqq x_1\} \\ \hat{x}_{2,\lambda} &= \max\{2k\lambda^{-\alpha}: 2k\lambda^{-\alpha} \leqq x_2\}. \end{aligned}$$

Then, for every ε *in* $(0, \delta)$, *we have*

$$\lim_{\lambda \to \infty} P\{N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) < \exp\{\lambda(A_1 \vee A_2 + \delta)\}\} = 1,$$

Proof. Using an abbreviation N_λ for $N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda})$ for the time being, we see

$$\begin{aligned} &P\{N_\lambda < \exp\{\lambda(A_1 \vee A_2 + \delta)\}\} \\ &= P\{\Gamma_0^0 + \ldots + \Gamma_{N_\lambda}^0 < (\lambda^{2\alpha}\Gamma_0^0 + \ldots + \lambda^{2\alpha}\Gamma_{N_\lambda}^0) N_\lambda^{-1} \lambda^{-2\alpha} \exp\{\lambda(A_1 \vee A_2 + \delta)\}\} \\ &\geqq P\{T_\lambda(x_1, x_2) < (\lambda^{2\alpha}\Gamma_0^0 + \ldots + \lambda^{2\alpha}\Gamma_{N_\lambda}^0) N_\lambda^{-1} \lambda^{-2\alpha} \exp\{\lambda(A_1 \vee A_2 + \delta)\}\} \\ &\geqq P\{|(\lambda^{2\alpha}\Gamma_0^0 + \ldots + \lambda^{2\alpha}\Gamma_{N_\lambda}^0)/N_\lambda - 1| < \varepsilon\} \\ &\quad + P\{T_\lambda(x_1, x_2) < (1-\varepsilon)\lambda^{-2\alpha}\exp\{\lambda(A_1 \vee A_2 + \delta)\} - 1 \\ &\quad \to 1 \quad \text{as} \quad \lambda \to \infty \quad \text{by Lemma 4.3 and Lemma 6.2.} \end{aligned}$$

The Proof of Theorem 3 (ii). The proof is similar to that of theorem 1 (ii). Therefore we exhibit only its outline.

Let $\mathbb{V}_1 = (\mathbb{a}_1, \mathbb{b}_1, \mathbb{c}_1)$ and $\mathbb{V}_2 = (\mathbb{a}_2, \mathbb{b}_2, \mathbb{c}_2)$ be two valleys of W connected at 0 such that $A < r_1 < r_2 < D$, where $A = A_1 \vee A_2$, $D = D_1 \wedge D_2$. Let $I = [r_1, r_2]$

and U be an arbitrary open set including $\mathbb{b}_1 \cup \mathbb{b}_2$. We choose $\delta > 0$ so that $\delta < (r_1 - A) \wedge (D - r_2)$ and closed intervals F_i, $i = 1, 2$, such that $x_i \in F_i \subset (a_i^+, c_i^-)$. We choose a positive ε which guarantees Lemmas 6.2, 6.3, 6.4, 6.5 and 6.6 for $\mathbb{V}_i$, δ and F_i, $i = 1, 2$. We have

$$\begin{aligned} &P\{Y_\lambda([e^{\lambda r}], \lambda W_\lambda) \in U\} \\ &\quad \geqq P\{Y_\lambda([e^{\lambda r}]) \in U,\ N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) = N_\lambda(\hat{x}_{1,\lambda}) < e^{\lambda(A+\delta)}\} \\ &\qquad + P\{Y_\lambda([e^{\lambda r}]) \in U,\ N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) = N_\lambda(\hat{x}_{2,\lambda}) < e^{\lambda(A+\delta)}\} \\ &\qquad \text{(by the strong Markov property)} \\ &\quad \geqq \sum_{0 \leqq \kappa < e^{\lambda(A+\delta)}} P\{N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) = N_\lambda(\hat{x}_{1,\lambda}) = k\}\, P\{Y_\lambda^{\hat{x}_{1,\lambda}}([e^{\lambda r}] - k) \in U\} \\ &\qquad + \sum_{0 \leqslant \kappa < e^{\lambda(A+\delta)}} P\{N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) = N_\lambda(\hat{x}_{2,\lambda}) = k\}\, P\{Y_\lambda^{\hat{x}_{2,\lambda}}([e^{\lambda r}] - k) \in U\}. \end{aligned}$$

Using Lemmas 6.6, 6.4 and 6.5 and the formula (6.5), we obtain by coupling method

$$\inf_{0 \leqq k < e^{\lambda(A+\delta)}} P\{Y_\lambda^{\hat{x}_{i,\lambda}}([e^{\lambda r}] - k) \in U\} = 1 + o(1) \quad \text{as} \quad \lambda \to \infty. \tag{6.6}$$

Therefore we obtain the estimation

$$\begin{aligned} &P\{Y_\lambda([e^{\lambda r}], \lambda W_\lambda) \in U\} \\ &\quad \geqq P\{N_\lambda(\hat{x}_{1,\lambda}, \hat{x}_{2,\lambda}) < e^{\lambda(A+\delta)}\} + o(1) \quad \text{as} \quad \lambda \to \infty, \end{aligned}$$

which, combined with Lemma 6.7, implies Theorem 3 (ii).

7. The Proof of Theorem 4

We prepare the following lemma which can be proved by using the definition (1.3), Lemma 6.1 and Lemma 5.3.

Lemma 7.1. *For any fixed* $\alpha > 0$, $\lambda > 0$ *and* $W \in \mathbb{W}_1$,

$$\{Y_\lambda(n, \lambda \tau_\lambda^\alpha W,\ n = 0, 1, 2, \ldots\} \stackrel{d}{=} \{\lambda^{-\alpha} Y(n, W),\ n = 0, 1, 2, \ldots\}.$$

The Proof of Theorem 4

Since $\mu_n^\alpha \to \nu$, by Skorohod's realization theorem of almost sure convergence, there exist $\mathbb{W}_{\log n}^\alpha$-valued random variables $\tilde{W}_n$ and $\mathbb{W}$-valued random variable $\tilde{W}$ on a suitable probability space $(\tilde{\Omega}, \tilde{P})$ with the following properties:

(i) The distributions of $\tilde{W}$ and $\tilde{W}_n$ are ν and μ_n^α, respectively, $n \geqq 2$;

(ii) $\tilde{W}_n \to \tilde{W}$ in the Skorohod topology as $n \to \infty$ $\tilde{P}$-a.s.

Since ν is a self-similar measure, Proposition 1(ii) implies that, $\tilde{P}$ a.e., $\tilde{W}$ has either a valley $\mathbb{V}$ containing 0 with $A<1<D$ or has two valleys $\mathbb{V}_1$ and $\mathbb{V}_2$ connected at 0 with $A_1 \vee A_2 < 1 < D_1 \wedge D_2$. Thus Theorem 3 implies that

$$(7.1) \qquad P\{Y_{\log n}(n, \log n\, \tilde{W}_n) \notin U_\varepsilon(\mathbb{b}(\tilde{W}))\} \to 0 \quad \text{as} \quad n\to\infty \quad \tilde{P}\text{-a.s.},$$

where $\mathbb{b}(W)$ is defined in Theorem 2. Using Lemma 5.2, we see that there exist Borel mappings $\tilde{\mathbb{b}}_n$ on $\mathbb{W}$ to $\mathbb{K}$ such that

$$(7.2) \qquad \tilde{\mathbb{b}}_n(\tilde{W}_n) \to \mathbb{b}(\tilde{W}) \quad \text{as} \quad n\to\infty, \ \tilde{P}\text{-a.s.}$$

Thus

$$(7.3) \qquad P\{Y_{\log n}(n, \log n\, \tilde{W}_n) \notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tilde{W}_n))\} \to 0 \quad \text{as} \quad n\to\infty \quad \tilde{P}\text{-a.s.}$$

Since

$$(7.4) \qquad (\tilde{W}_n, \tilde{P}) \stackrel{d}{=} (W, \tau^\alpha_{\log n}\mu) \stackrel{d}{=} (\tau^\alpha_{\log n} W, \mu),$$

the formula (7.3) implies that

$$(7.5) \qquad P\{Y_{\log n}(n, \log n \cdot \tau^\alpha_{\log n} W) \notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tau^\alpha_{\log n} W))\} \to 0 \quad \text{in probability } \mu \text{ as } n\to\infty.$$

Employing the scaling relation of Lemma 7.1, we have

$$(7.6) \qquad P\{(\log n)^{-\alpha}\, Y(n, W) \notin U_\varepsilon(\tilde{\mathbb{b}}_n(\tau^\alpha_{\log n} W))\} \to 0 \quad \text{in probability } \mu \text{ as } n\to\infty.$$

Therefore, if we set

$$(7.7) \qquad \mathbb{b}_n(W) = \tilde{\mathbb{b}}_n(\tau^\alpha_{\log n} W), \qquad n=1,2,\ldots,$$

then we obtain the first assertion of the theorem. On the other hand, combining (7.2) with (i) and (ii), we see that the distribution of random variables $\mathbb{b}_n(W)$ on $(\mathbb{W}, \mu)$ converges weakly to that of $\mathbb{b}(W)$ on $(\mathbb{W}^\#, \nu)$. Thus the proof of the theorem is completed.

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