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Probability Theory and Related Fields

Brownian Motion on the Sierpinski Gasket

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Summary. We construct a "Brownian motion" taking values in the Sierpinski gasket, a fractal subset of $\mathbb{R}^2$, and study its properties. This is a diffusion process characterized by local isotropy and homogeneity properties. We show, for example, that the process has a continuous symmetric transition density, $p_t(x, y)$, with respect to an appropriate Hausdorff measure and obtain estimates on $p_t(x, y)$.

1. Introduction and Statement of Results

The Sierpinski gasket (introduced in Sierpinski (1915)) is a "fractal" subset of $\mathbb{R}^2$. Let A_0 be a (closed, convex) triangle of unit side. Let A_1 be the set obtained from A_0 by deleting the open convex triangle whose vertices are the midpoints of the edges of A_0. Thus A_1 consists of 3 closed convex triangles with side $\frac{1}{2}$. Repeating this procedure one obtains successively $A_2, A_3, \ldots$ (Fig. 1 shows A_2 and A_3). Let $A = \bigcap_{n=0}^{\infty} A_n$: this is the (bounded) Sierpinski gasket.

The following construction, which builds an unbounded set up from a sequence of graphs, is, however, more convenient for our purposes.

Let $a_0 = 0$, $a_1 = (1, 0)$ and $a_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, let $F_0 = \{a_0, a_1, a_2\}$ be the vertices of an equilateral triangle in the plane of side one, and let J_0 be the closed convex equilateral triangle with vertices F_0. Define inductively

$$F_{n+1} = F_n \cup (2^n a_1 + F_n) \cup (2^n a_2 + F_n), \qquad n = 0, 1, \ldots .$$

(Here, and throughout the paper we use the notation $y + A = \{y + x : x \in A\}$ and $\lambda A = \{\lambda x : x \in A\}$.) Thus $F_n \subset 2^n J_0$. Now let

$$G'_0 = \bigcup_{n=0}^{\infty} F_n$$

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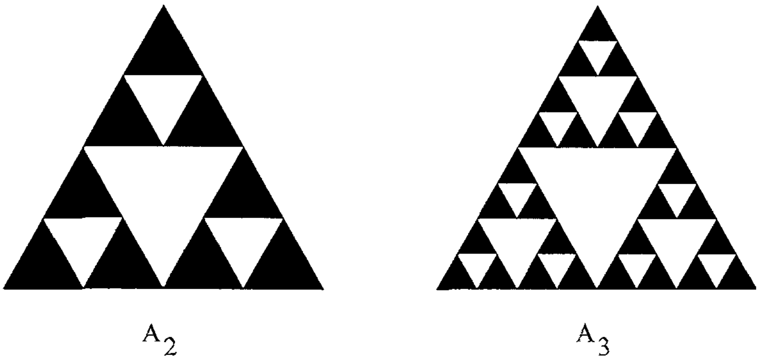

Fig. 1

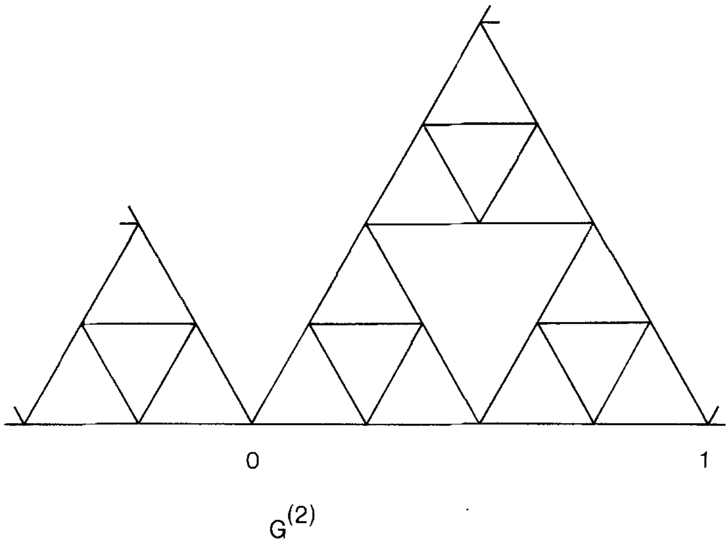

Fig. 2

and let G_0 be G'_0 together with its reflection in the y-axis. Let

$$G_n = 2^{-n} G_0, \quad n \in \mathbb{Z}, \qquad G_\infty = \bigcup_{n=0}^{\infty} G_n, \qquad G_{-\infty} = \{0\},$$

and let $G = cl(G_\infty)$: G is the Sierpinski gasket. It is easily seen that G is a closed connected subset of $\mathbb{R}^2$. In the inductive definition of F_{n+1} we have always chosen to translate F_n to the east and north-east. This does give $2G = G$ but the results of this paper remain valid for gaskets in which a general choice of directions of translation are made at each stage.

Let $G^{(0)}$ be the graph with vertices G_0 and with an edge between x and y in G_0 if and only if $|x-y|=1$ and the line segment joining x and y is contained in G. For $n \in Z$ let $G^{(n)} = 2^{-n} G^{(0)}$. See Fig. 2.

Each vertex in $G^{(0)}$ has valency 4: that is, is joined by an edge to four other vertices in $G^{(0)}$.

Notation. For $x \in G_n$, let $N_n(x)$ denote the four neighbours of x in $G^{(n)}$.

A G_n-triangle is the closed set of points in G that lie inside an equilateral triangle which is a translation of $2^{-n} J_0$ and whose vertices are three neighbouring points in $G^{(n)}$. Let $\mathcal{T}_n$ denote the set of G_n-triangles.

Let μ_n denote the measure on G which assigns mass $(\frac{2}{3}) 3^{-n}$ to each point in G_n. An easy induction argument shows that if $\Delta \in \mathcal{T}_m$,

$$\text{(1.1)} \qquad \operatorname{card}(G_n \cap \Delta) = \tfrac{1}{2}(3^{n-m+1} + 3)$$

so that $\lim_{n \to \infty} \mu_n(\Delta) = 3^{-m}$.

The following lemma summarizes the required measure-theoretic properties of G.

Notation. $d_f = \log 3/\log 2 = 1.58496\ldots$

Lemma 1.1. (a) *There is a unique measure μ on $\{\mathbb{R}^2, \mathcal{B} \pm (\mathbb{R}^2))$ supported on G such that $\mu(\Delta_n) = 3^{-n}$ for all $\Delta_n \in \mathcal{T}_n$, $n \in \mathbb{Z}$.*

(b) *$\{\mu_n\}$ converges to μ in the vague topology: that is*

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu \qquad \textit{for all } f \in C_K(G).$$

(c) *μ is a multiple of the Hausdorff x^{d_f}-measure on G. Thus, in particular, G has Hausdorff dimension d_f.*

Proof. (a) and (c) follow from the Carathéodory extension theorem and Eggleston (1953).

(b) is easy to check using (1.1). □

Our aim in this paper is to construct and study a diffusion process, X, on G, which arises as the limit of simple random walks on the graphs $G^{(n)}$. The reasons for our interest are:

(1) Mathematical. Mathematically rigorous constructions of X have been given by Goldstein (1987) and Kusuoka (1987). Although X is a diffusion on $\mathbb{R}^2$ it lives on a set $G \subset \mathbb{R}^2$ with Lebesgue measure zero. As we will see, it has very different properties from the well-known diffusions obtained by the differential equation approach as in Stroock-Varadhan (1979). Like the increasing diffusion studied by Salisburg (1986), this shows that the theory of higher dimensional diffusions remains a wilderness (Breiman (1968)).

(2) Applications in mathematical physics. There are now a large number of papers in the physics literature dealing with random walks and diffusions on fractals. See Rammal and Toulouse (1982) for an introduction and some physical motivation. This literature suggests that much of the behaviour of a random walk on a fractal F is captured by two constants associated with F. The first, d_f, is the *fractal dimension* of F (for regular fractals like G, this is

the Hausdorff dimension of F), while the second, denoted d_s, is called the *spectral dimension* of F. d_s is defined in terms of the "density of states", that is, the asymptotic frequency of the large eigenvalues of the Laplacian on a bounded region of F (for G, the Laplacian is the infinitesimal generator of X). A third quantity, denoted d_w, and called the *dimension of the walk*, is given by

$$d_w = \frac{2\,d_f}{d_s}.$$

Notation. Throughout this paper let

$$d_s = 2\log 3/\log 5 = 1.36521\ldots, \qquad d_w = 2\,d_f/d_s = \log 5/\log 2 = 2.32193\ldots$$
$$\gamma = 1/d_w = \log 2/\log 5 = 0.43068\ldots$$

We will see that these constants, as defined here, do have the meaning associated with them in the physics literature.

The following stopping times play an important role in the study of G-valued processes.

Notation. Let $X(t)$ be a continuous time or discrete time G-valued process. For $m \in \mathbb{Z}$, set

$$T^m_{-1}(X) = 0, \qquad T^m(X) \equiv T^m_0(X) = \inf\{t \geqq 0 \colon X(t) \in G_m\},$$
$$T^{-\infty}(x) = \inf\{t \geqq 0 \colon X(t) = 0\},$$
$$T^m_{i+1}(x) = \inf\{t > T^m_i(X) \colon X(t) \in G_m - \{X(T^m_i(X))\}\}, \qquad i \geqq 0,$$
$$W^m_i(X) = T^m_i(X) - T^m_{i-1}(X), \qquad i \geqq 1.$$

We will write T^m_i and W^m_i if the process X is clear from the context.

Note that if $X(t)$ is continuous, then $Y_i = X(T^m_i(X))$, $i \geqq 0$ is a path in $G^{(m)}$: that is, $Y_i \in G_m$, and $Y_{i+1} \in N_m(Y_i)$ for each $i \geqq 0$.

Definition. A simple random walk on G_n is a G_n-valued Markov chain $\{Y_r \colon r = 0, 1, \ldots\}$ with transition probabilities

$$P(Y_{r+1} = y \mid Y_r = x) = \begin{cases} \frac{1}{4} & \text{if } y \in N_n(x) \\ 0 & \text{otherwise.} \end{cases}$$

Let Y_n be a simple random walk on G_0, starting at 0. If $m \leqq 0$ and $\tilde{Y}_i = 2^m Y_{T^m_i}$, then Y is again a simple random walk on G_0. This property of Y, called decimation invariance by Goldstein (1987), is crucial to the study of processes on G. It is related to the following property of G. Let $f(t)$ be a continuous path in G with $f(0) \in G \cap J_0$. Then f can only leave $G \cap J_0$ through one of the three vertices of J_0. In the terminology of the physics literature, G has "finite ramification". Thus, for a continuous or discrete time process on G, the times of successive hits on G_n provide a natural collection of renewal times, and the behaviour of the process can be recovered from its values at these times.

The speed at which Y moves across G_0 is given by

Lemma 1.2. $E(T^m_1(Y)) = 5^{|m|}$, $m \leqq 0$.

(See Lemma 2.2: a simple calculation shows $E(T_1^{-1})=5$ and the above result then follows from decimation invariance).

This suggests we consider the processes

$$X^{(n)}(t)=2^{-n}Y_{[5^n t]} \qquad t \geqq 0,\ n \in \mathbb{Z}^+. \tag{1.2}$$

Theorem 1.3 (Goldstein (1987), Kusuoka (1987), Theorem 2.8 below). *The processes $X^{(n)}$ converge weakly to a process X. X is a continuous non-constant G-valued process, starting at zero.*

Here the weak convergence is on the space of cadlag G-valued paths with the Skorokhod J_1 topology. In fact, the proof in Sect. 2 gives an almost sure construction of X as the uniform limit of a sequence of nested random walks. It is motivated by the construction of Brownian motion given in Knight (1981, p. 10). The construction can easily be adapted to define X with $X_0=x\in G_\infty$. Let P^x denote the law of X, starting at $x\in G_\infty$, on the canonical space of paths $\Omega=C(\mathbb{R}_+, G)$. A coupling argument in Sect. 2 shows that $x\to P^x$ is uniformly continuous on G_∞ and hence has a unique continuous extension to G. In fact P^x is the weak limit of $\{X^{(n)}\}$, providing we introduce initial points $X^{(n)}(0)=x_n\to x\,(x_n\in G_n)$ (see Kusuoka (1987) or Theorem 2.14 below).

We now summarize the main properties of X derived in this paper. Assume $X_t(\omega)=\omega(t)$ is defined on the canonical space Ω and write E^x for expectation with respect to P^x. The transition semigroup P_t and resolvent U_λ are defined by

$$P_t f(x)=E^x f(X_t), \qquad U_\lambda f(x)=E^x \int_0^\infty e^{-\lambda s} f(X_s)\, ds, \qquad f\in bB(G).$$

Theorem 1.4 (Goldstein (1986), Kusuoka (1987), Theorem 2.15, 2.21 below). (a) *X is a continuous, strong Markov process with state space G.*

(b) *X is a Feller process: that is P_t maps $C_b(G)$ into $C_b(G)$ for each $t\geqq 0$.*

(c) *X is μ-symmetric: that is,*

$$\int g(x)\, P_t f(x)\, \mu(dx)=\int P_t g(x) f(x)\, \mu(dx) \qquad \textit{for all } f, g\in C_K(G).$$

Our derivation of Theorems 1.3 and 1.4 were carried out independently of the work of Goldstein (1987) and Kusuoka (1987), but we acknowledge the priority of their results. Our construction of X is similar in spirit to the almost sure construction of Kusuoka but there are several technical differences between the two proofs.

The set G does not have very many global isometries. It is invariant under reflection in the y-axis, and, for each $n\in\mathbb{Z}$ we have $2^n G=G$. There is, however, a large class of local isometries, arising from the fact that every G_n-triangle is a translation of $(2^{-n}J_0)\cap G$, and invariant under rotation by $2\pi/3$, and reflection in an axis perpendicular to one of its edges.

It is clear that the random walks $X^{(n)}$ are locally invariant with respect to these local isometries, and this property is inherited by X. In Theorem 8.1 we prove that this invariance, relative to a suitable class of local isometries, determines X in the class of diffusions on G up to a linear change of time. By analogy with Brownian motion on $\mathbb{R}^d$, which, up to a linear time change is the unique translation and rotation invariant diffusion of $\mathbb{R}^d$, we call X *Brownian motion on G.*

The key to many properties of X are good estimates on P_t. Here is our main result. The proofs are given in Sect. 7.

Theorem 1.5. *There is a function* $p_t(x, y)$, $(t, x, y) \in (0, \infty) \times G \times G$ *such that*

(a) p_t *is the transition density of* X *with respect to* μ, *i.e.,*

$$P_t f(x) = \int f(y)\, p_t(x, y)\, \mu(dy) \qquad \text{for all } x \in G,\ t > 0,\ f \in C_b(G).$$

(b) $p_t(x, y) = p_t(y, x)$ *for all* $(x, y) \in G \times G$, $t > 0$.

(c) $(t, x, y) \to p_t(x, y)$ *is jointly continuous on* $(0, \infty) \times G \times G$.

In fact,

(1.3) $|p_t(x, y) - p_t(x', y')| \leqq c_{1.1} t^{-1} |(x, y) - (x', y')|^{d_w - d_f}$ *for all* $t > 0$, (x, y), $(x', y') \in G \times G$.

(d) *For each* (x, y), $t \to p_t(x, y)$ *is* C^∞ *on* $(0, \infty)$, *and for each* $k \in \mathbb{N}$

$$(t, x, y) \to \frac{\partial^k}{\partial t^k} p_t(x, y)$$

is jointly continuous, and for t *fixed is Hölder continuous of index* $d_w - d_f$ *in* (x, y).

(e) *There are constants* $c_{1.2}, \ldots, c_{1.5}$ *such that*

(1.4) $$c_{1.2} t^{-d_s/2} \exp\{-c_{1.3}(|x-y|\, t^{-1/d_w})^{d_w/(d_w-1)}\} \leqq p_t(x, y)$$
$$\leqq c_{1.4} t^{-d_s/2} \exp\{-c_{1.5}(|x-y|\, t^{-1/d_w})^{d_w/(d_w-1)}\} \text{ for all } t > 0,\ (x, y) \in G \times G.$$

Remarks. 1. In the above theorem and throughout this paper $c_{i.1}, c_{i.2}, \ldots$ denote fixed constants in $(0, \infty)$ introduced in Sect. i. c_1, c_2 will represent constants whose value may change from line to line in a proof. $|x-y|$ denotes the Euclidean distance between x and y in G or in $G \times G$.

2. $d_w - d_f = 0.73697\ldots$, $d_s/2 = 0.68261\ldots$, $d_w/(d_w - 1) = 1.75647\ldots$

3. The transition density of standard Brownian motion on $\mathbb{R}^d$ satisfies (1.4) with $d_s = d (= d_f)$ and $d_w = 2$.

4. It is perhaps interesting to note that there has been some confusion in the physics literature over the power of $|x-y|\, t^{-\gamma}$ that appears in the exponential in (1.4) (see Banaver and Willemson (1984), Guyer (1985), O'Shaughnessy and Procaccia (1985)).

5. (c) and (e) imply that X is a strong Feller process.

6. Integrating (1.4) we have that (see Theorem 4.3)

(1.5) $$c_{1.6} \exp\{-c_{1.7}(\delta t^{-\gamma})^{d_w/(d_w-1)}\} \leqq P^x(|X_t - x| > \delta)$$
$$\leqq c_{1.8} \exp\{-c_{1.9}(\delta t^{-\gamma})^{d_w/(d_w-1)}\} \qquad \forall \delta,\ t > 0,\ x \in G.$$

Thus the tail of the distribution of $|X_t - x|$ is thinner than an exponential but fatter than that of a Gaussian law.

Corollary 1.6 (see Corollary 4.4). *There are constants* $c_{1.10}$, $c_{1.11}$ *such that* $c_{1.10}\, t^{2/d_w} \leqq E^x |X_t - x|^2 \leqq c_{1.11}\, t^{2/d_w}$ *for all* $x \in G$, $t > 0$.

Since $2/d_w \doteq 0.86135 < 1$, we see that the mean square displacement is sublinear in t. This is called sub-diffusive behaviour in the physics literature.

Kesten (1986) studies random walks on percolation clusters, and proves that, in two different cases, the random walk on the incipient infinite cluster has subdiffusive behaviour. It is proved there that this subdiffusive behaviour is due to the random walk spending most of its time in the "dangling ends". For the Sierpinski gasket there are no "dangling ends", and the subdiffusive behaviour must have a different cause.

It is now easy to use (1.4) or (1.5) to deduce various sample path properties of X.

Corollary 1.7. *There exist constants* $c_{1.12}$, $c_{1.13}$ *such that for all* $x \in G$

(1.6) $$c_{1.12} \leqq \lim_{\delta \downarrow 0} \sup_{\substack{0 \leqq s \leqq t \leqq T \\ |s-t| \leqq \delta}} |X_s - X_t| / (|s-t|^\gamma (\log 1/|s-t|)^{1-\gamma}) \leqq c_{1.13}$$

for all $T > 0$ P^x-a.s.

The paths just fail to be Hölder continuous of order $\gamma (< \tfrac{1}{2})$, so the following result is not surprising.

Corollary 1.8 (see Theorem 4.5 and Remark 4.6). *The paths of X are of infinite quadratic variation. In particular X is not a semimartingale.*

Further results on the local modulus of continuity of X and the d_w-variation of the paths are obtained in Sect. 4.

There are two main ingredients in the proof of Theorem 1.5. The first is the study of the law (under P^0) of

$$W = \inf\{s : |X(s)| = 1\} = T_1^0(X).$$

Let

$$Z_n = \min\{i : |X(T_i^n)| = 1\}.$$

Then $\{Z_n\}$ is a simple branching process with $Z_0 = 1$ and $EZ_1 = 5$ (in fact Z_1 is equal in law to $T_1^{-1}(Y)$ – see the remarks after Lemma 1.2). Therefore $Z_n 5^{-n}$ converges a.s. as $n \to \infty$ and it is easy to see that the limit is W. In Sect. 3, techniques from the theory of branching processes are used to derive estimates on the (smooth) distribution and density functions of W at zero. These estimates suffice for the derivation of (1.5) and the path properties of X in Sect. 4, including Corollaries 1.6, 1.7 and 1.8. The Laplace transform of W exhibits some surprising oscillatory behaviour at infinity (see Sect. 3 for details). This oscillatory behaviour is the main (but not the only!) reason for our inability to obtain exact constants in (1.4), (1.5) and (1.6).

The second ingredient in the derivation of Theorem 1.5 are the potential theoretic calculations on $X^{(n)}$ and X in Sect. 5 which lead to the existence and regularity properties of the resolvent densities of X. Of course most of these results follow from Theorem 1.5 but they are logically prior.

Theorem 1.9. *There is a function $u_\lambda(x, y)$, $(\lambda, x, y) \in (0, \infty) \times G \times G$ such that*

(a) *u_λ is the resolvent density of X*

$$U_\lambda f(x) = \int f(y)\, u_\lambda(x, y)\, \mu(dy) \qquad \text{for all } x \in G,\ \lambda > 0,\ f \in C_b(G).$$

(b) *$u_\lambda(x, y) = u_\lambda(y, x)$ for all $(x, y) \in G \times G$, $\lambda > 0$.*

(c) *$(\lambda, x, y) \to u_\lambda(x, y)$ is jointly continuous on $(0, \infty) \times G \times G$. In fact*

$$(1.7) \qquad |u_\lambda(x, y) - u_\lambda(x', y')| \leqq c_{1.14} |(x, y) - (x', y')|^{d_w - d_f}$$

for all (x, y), (x', y') in $G \times G$.

(d) *There are constants $c_{1.15}, c_{1.16}, c_{1.17}, c_{1.18}$ such that*

$$(1.8) \quad c_{1.15}\, \lambda^{d_s/2 - 1} \exp\{-c_{1.16}\, \lambda^\gamma |x - y|\} \leqq u_\lambda(x, y)$$
$$\leqq c_{1.17}\, \lambda^{d_s/2 - 1} \exp\{-c_{1.18}\, \lambda^\gamma |x - y|\} \qquad \text{for all} \quad \lambda > 0,\ (x, y) \in G \times G,$$

and in particular

$$(1.9) \qquad c_{1.15}\, \lambda^{d_s/2 - 1} \leqq u_\lambda(x, x) \leqq c_{1.17}\, \lambda^{d_s/2 - 1} \qquad \text{for all } \lambda > 0,\ x \in G.$$

Remark. $d_s/2 - 1 = -0.31739\ldots$.

Let $T_x^+ = \inf\{t > 0 : X_t = x\}$ be the hitting time of x. The calculations in Sect. 5 also give

Corollary 1.10. (a) *For each x in G, $P^x(T_x^+ = 0) = 1$.*

(b) *For each x, y in G, $P^x(T_y^+ < \infty) = 1$, and $\{t : X_t = y\}$ is P^1-a.s. perfect and unbounded. Thus X is point recurrent.*

Remark. It is clear that X must hit the points in G_∞: otherwise, since $G - G_\infty$ is totally disconnected it would be unable to move. The content of the corollary is that X also hits the points in $G - G_\infty$.

The estimates (1.7) and (1.9), together with a version of Garsia's lemma for a fractal (Lemma 6.1), lead easily to the existence and continuity of the local time of X.

Theorem 1.11. *There exists a jointly continuous version, L_t^x, $x \in G$, $t \geqq 0$, of the local time of X. L satisfies the density of occupation formula:*

$$\int_0^t g(X_s)\, ds = \int_G g(x)\, L_t^x\, \mu(dx),$$

and has modulus of continuity given by: for each $N > 0$ there exists $\delta_N(\omega) > 0$ such that

$$|L_t^x - L_t^y| < c_{1.18} |x - y|^{\frac{1}{2}(d_w - d_f)} \log \frac{1}{|x - y|} \qquad \text{if} \quad t \leqq N \quad \text{and} \quad |x - y| < \delta_N(\omega).$$

The proof is given in Sect. 6.

The results of Sect. 3 and 5 are combined to prove Theorem 1.5 in Sect. 7. The essential idea is to use the first entry decomposition

$$(1.10) \qquad p_t(x, y) = \int_0^t g_{x,y}(s)\, p_{t-s}(y, y)\, ds,$$

where $g_{x,y}(s)$ is the density of T_y^+ under P^x. If $x \neq y$, then the path must cross some G_n-triangle to reach y from x. Thus $g_{x,y}$ is the convolution of the C^∞ density of $5^{-n}W$ with another distribution. The results of Sect. 3 provide us with good estimates on the (necessarily) smooth function $g_{x,y}$. From (1.10) we have

$$p_t(x, y) \leqq \sup_{s \leqq t} g_{x,y}(s) \int_0^t p_s(y, y)\, ds. \tag{1.11}$$

To control, and in fact rigorously define, the second term on the right side of (1.11) we use the estimates on u_λ in Sect. 5.

This estimate on $p_t(x, y)$ is good if $|x-y| \geqq t^\gamma$, but for $|x-y| < t^\gamma$ (and in particular for $p_t(x, x)$) more work is required. Our basic idea is to let X go away from x until it hits a moving boundary and then use (1.11) on the return journey to y.

In Sect. 9 we take a brief look at the infinitesimal generator, $\mathscr{A}$, of X, acting on its domain $\mathscr{D}(\mathscr{A})$ in the Banach space $C_0(G)$ of continuous functions on G vanishing at ∞. Theorem 1.5 gives a large class of functions in $\mathscr{D}(\mathscr{A})$: $u_\lambda(\cdot, y)$ and $p_t(\cdot, y)$ are in $\mathscr{D}(\mathscr{A})$ for each $y \in G$, t, $\lambda > 0$. In fact p_t solves the heat equation

$$\frac{\partial p_t}{\partial t}(x, y) = \mathscr{A}(p_t(\cdot, y))(x) \tag{1.12}$$

(see Theorem 7.10). Nevertheless, it seems difficult to obtain an explicit description of a non-constant function in $\mathscr{D}(\mathscr{A})$. (1.7) can be used to show each function in $\mathscr{D}(\mathscr{A})$ is Hölder continuous of order $d_w - d_f$ (Theorem 5.22). We also show that functions in $\mathscr{D}(\mathscr{A})$ are almost everywhere Hölder continuous of order $d_w/2 - \varepsilon = 1.16096\ldots - \varepsilon$ for any $\varepsilon > 0$ (see Theorem 9.1 for a precise statement). This will imply that if $f \in \mathscr{D}(\mathscr{A})$ is the restriction of a C^1 function on $\mathbb{R}^2$ to G, then f is constant. The functions in $\mathscr{D}(\mathscr{A})$ appear to be Cantor-like functions on G. (The standard Cantor function on $[0, 1]$ is, in the sense of Theorem 9.1, almost everywhere Hölder continuous of index α for any $\alpha > 0$).

We expect that the methods and results in this paper extend to a large class of finitely ramified fractals. For example there is no problem in extending these results to the higher dimensional analogues of the Sierpinski gasket. These extensions are discussed in Sect. 10. It is interesting to note that even though the Hausdorff dimension of the gasket embedded in $\mathbb{R}^d$ is $\log(d+1)/\log 2$ which approaches ∞ as $d \to \infty$, the process still hits points and has a jointly continuous local time.

On the other hand infinitely ramified fractals such as the Sierpinski carpet appear to be much more difficult. Barlow and Bass (1987) have studied Brownian motion on the Sierpinski carpet but their results are much less complete.

We close this section by introducing some notation to allow us to work in G without introducing coordinates.

For each m, G is the union of G_m-triangles and hence each point, x, in G lies in a G_m-triangle, $\Delta_m(x)$. If $x \in G - G_m$, $\Delta_m(x)$ is unique, if $x \in G_m - \{0\}$ let $\Delta_m(x)$ denote the G_m-triangle containing x whose projection onto the x-axis

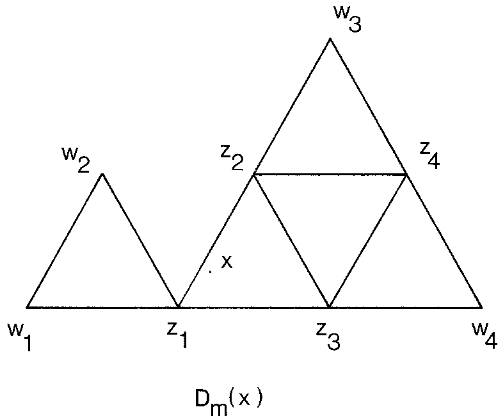

Fig. 3

is closer to zero and, finally, let $\Delta_m(0)=2^{-m}J_0\cap G$. Each vertex of a given G_m-triangle, Δ_m, has 4 neighbours in G_m, and thus Δ_m intersects exactly three other G_m-triangles, one at each vertex point. The exact configuration of these three triangles depends on the choice of Δ_m, but Δ_m and two of these neighbouring triangles will form a G_{m-1}-triangle. Let $D_m(x)$ be the union of $\Delta_m(x)$ and its three neighbouring triangles in $\mathcal{T}_m$. See Fig. 3.

If $A\subset G$, ∂A denotes the topological boundary of A, where A is considered as a topological subspace of G. For example in Fig. 3 $\partial D_m(X)=\{w_1, w_2, w_3, w_4\}$.

If $x, y\in G$, let $d(x, y)$ denote the length of the shortest path in G from x to y. d is a natural metric on G which we call the gasket metric. If $x, y\in G_m$ then $d(x, y)$ is clearly the length of the shortest path in $G^{(m)}$ from x to y. An elementary argument shows that for such x, y,

$$|x-y|\leqq d(x,y)\leqq c_{1.19}|x-y|. \tag{1.13}$$

These inequalities now extend easily to $x, y\in G$. The gasket metric is therefore equivalent to the restriction of the Euclidean distance to $G\times G$. Nonetheless it will sometimes be convenient to work with d. If $B_d(x, r)=\{y\in G; d(x, y)\leqq r\}$, then note that for $x\in G_m$, $B_d(x, 2^{-m})$ is the union of the two G_m-triangles which intersect at x.

Finally $\rho_G(\cdot,\cdot)$ is a metric on $C((0, \infty), G)$ which induces the compact-open topology on this function space.

2. Construction of a Brownian Motion on the Sierpinski Gasket

Notation. If X is a continuous time or discrete time G-valued process, let

$$\mathcal{F}_t^0(X)=\sigma(X_s\colon s\leqq t),$$

and for $A\subset G$, let

$$T(A, X)=\inf\{t\geqq 0\colon X(t)\in A\}.$$

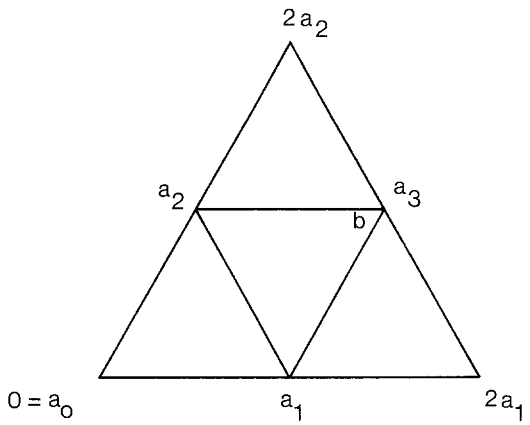

Fig. 4

We write $T_y(X)$ or T_y (if there is no ambiquity) for $T(\{y\}, X)$.

Let $S_m = S_m(X) = T(\partial D_m(X(0)), X)$.

In the next two results $\{Y_r: r=0, 1, 2, \ldots\}$ denotes a simple random walk on G_0, starting at zero. We begin with the simple but crucial property of decimation invariance mentioned in the Introduction.

Proposition 2.1. *For each* $m \in \mathbb{N}$,

$$Y^m(i) = Y(T_i^{-m}(Y)), \qquad i \in \mathbb{Z}_+,$$

is a simple random walk on G_{-m}.

Proof. An elementary argument using the Markov property, shows that $T_1^{-m}(y)$ has an exponentially small tail at infinity and in particular is finite a.s. The strong Markov property shows $T_i^{-m}(Y) < \infty$ a.s. for any $i \in \mathbb{N}$ and hence $Y^m(i)$ is well-defined.

$G_0 \cap B_d(0, 2^m)$ is symmetric in the y-axis and $G_0 \cap (\pm 2^m J_0)$ is symmetric in the perpendicular bisector of $\pm 2^m J_0$ through the origin (recall J_0 from the beginning of Sect. 1). These symmetries and a simple counting argument show that $Y(T_i^{-m}(Y))$ is uniformly distributed over the four points in $\partial B_d(0, 2^m)$. Of course the same is true if 0 is replaced by any starting point in G_{-m}. The strong Markov property of Y now completes the proof. $\square$

Notation. If $m \in \mathbb{N}$, let

$$N_m = T_1^{-m}(Y), \qquad H_m = \sum_{0 \leq r \leq N_m} 1(Y_r = 0).$$

N and H will be used to denote random variables equal in law to N_1 and H_1, respectively. Let

$$f(u) = E(u^N), \qquad h(u) = E(u^H) \qquad \text{for } u \in [0, 1].$$

Lemma 2.2. (a) $f(u)=u^2(4-3u)^{-1}$, $h(u)=3u(5-2u)^{-1}$. *Therefore* $N-1$ *is a geometric random variable with mean* 4 *and* H *is a geometric random variable with mean* 5/3.

(b) $E(N_m)=5^m$, $E(H_m)=(5/3)^m$.

Proof. (a) Let $\{a_i\colon i=0,1,2\}$ and F_1 be as in Sect. 1, and let $a_3=a_1+a_2$. By reflection in the y-axis, N is the first time a random walk on F_1, starting at zero, hits $\{2a_1, 2a_2\}$. See Fig 4. Let $f_i(u)=E^{a_i}(u^N)$, where the superscript denotes the starting point. Condition on the first step of the random walk and note that $f_1=f_2$ by symmetry to get

$$f_0(u)=uf_1(u), \quad f_1(u)=(u/4)(f_0(u)+f_1(u)+f_3(u)+1),$$
$$f_3(u)=(u/2)(f_1(u)+1).$$

Solving for $f_0(u)$, one gets $f_0(u)=u^2(4-3u)^{-1}$.

A similar argument works for h. If $h_i(u)=E^{a_i}(u^H)$, where H is the number of times a random walk on F_1 hits zero before hitting $\{2a_1, 2a_2\}$, the corresponding equations are

$$h_0(u)=uh_1(u), \quad h_1(u)=(\tfrac{1}{4})(h_0(u)+h_1(u)+h_3(u)+1),$$
$$h_3(u)=(\tfrac{1}{2})(h_1(u)+1).$$

(b) Consider $E(H_m)$. For $m\in\mathbb{N}$, inductively define stopping times by

$$V_0^m=0, \quad U_i^m=\min\{r>V_{i-1}^m\colon Y_r\in G_{-m}-\{0\}\},$$
$$V_i^m=\min\{r>U_i^m\colon Y_r=0\}\wedge N_{m+1}.$$

Let

$$H_{m,i}=\sum_r 1(V_i^m\leqq r<U_{i+1}^m, Y_r=0), \quad i\in\mathbb{Z}_+.$$

Then

$$E(H_{m+1})=E\left(\sum_{i=0}^{\infty} 1(Y(V_i^m)=0)H_{m,i}\right) \tag{2.1}$$
$$=E\left(\sum_{i=0}^{\infty} 1(Y(V_i^m)=0)\right)E(H_m),$$

the last by the strong Markov property at V_i^m. If $Y^m(i)=Y(T_i^{-m}(Y))$, then

$$\sum_{i=0}^{\infty} 1(Y(V_i^m)=0)=\sum_{i=0}^{\infty} 1(Y^m(i)=0, i\leqq N_{m+1}). \tag{2.2}$$

Decimation invariance (Lemma 2.1) implies $2^{-m}Y^m(\cdot)$ is a simple random walk on G_0. The left side of (2.2) is therefore equal in law to H. (2.1) and (a) imply $E(H_{m+1})=(\tfrac{5}{3})E(H_m)$.

A similar argument works for $E(N_m)$. □

Proposition 2.1 and the Kolmogorov Extension Theorem imply there is a family of discrete time processes, $\{\tilde{X}(n, x)\colon x \in G_n,\ n \in \mathbb{Z}\}$, defined on some (Ω_0, F_0, P) such that

(2.3) $\{\tilde{X}(n, x)(i),\ i \in \mathbb{Z}_+)$ is a simple random walk on G_n, starting at x.

(2.4) If $m \leqq n$ and $x \in G_m$, then

$$\tilde{X}(m, x)(i) = \tilde{X}(n, x)(T_i^m(\tilde{X}(n, x))), \qquad i \geqq 0.$$

(2.5) If $x, x' \in G_n$ and $x \neq x'$, then $\tilde{X}(n, x)$ and $\tilde{X}(n, x')$ are independent.

To obtain regularity in the initial point x, we couple the random walks so that if they enter G_m at the same point, they then trace out the same set of points thereafter. If $x \in G_m$ and $n \geqq m$, let

$$X(n, x)(i) = \tilde{X}(n, x)(i) \quad \text{if } i \leqq T^{m-1}(\tilde{X}(n, x)). \tag{2.6}$$

This last time therefore equals $T^{m-1}(X(n, x))$. Also let

$$X(n, 0)(i) = \tilde{X}(n, 0)(i) \quad \text{for all } i \in \mathbb{Z}_+. \tag{2.7}$$

To complete the definition of $X(n, x)$, if $x \in G_n$ set

$$\begin{aligned} &X(n, x)(i) = X(n, X(n, x)(T^j(X(n, x))))(i - T^j(X(n, x))) \\ &\text{if } T^j(X(n, x)) < i \leqq T^j(X(n, X)) + T^{j-1}(X(n, X(n, x)(T^j(X(n, x))))) \\ &\qquad = T^{j-1}(X(n, x)) - \infty < j \leqq n, \end{aligned}$$

and

$$\begin{aligned} &X(n, x)(i) = X(n, 0)(i - T_0(X(n, x))) \\ &\qquad \text{if } i \geqq T_0(X(n, x)) = \lim_{j \to \infty} T^j(X(n, x)). \end{aligned} \tag{2.9}$$

(2.8) is well-defined by (2.6) because $X(n, n)(T^j(X(n, x))) \in G_j$ if $T^j(X(n, x)) < \infty$. Note that (2.8) and (2.9) together define $X(n, x)(i)$ for all $i \in \mathbb{Z}_+$.

Some easy consequences of this definition are as follows:

Lemma 2.3. (a) *$X(n, x)$ is a simple random walk on G_n, starting at x.*

(b) *We have*

(2.10) $X(n, x)(i) = X(n, X(n, x)(T^j(X(n, x))))(i - T^j(X(n, x)))$ *for* $i \geqq T^j(X(n, x))$, $-\infty \leqq j \leqq n$, $x \in G_n$.

(c) *If* $n, j \in \mathbb{Z}$ *and* $n \geqq j$, *then* $\sigma(X(n, y), y \in G_j)$ *and* $\sigma(X(n, x)\ (\cdot \wedge T^j(X(n, x))), x \in G_n)$ *are independent σ-fields.*

Proof. (a) Use (2.5) and the strong Markov property of a simple random walk on G_n.

(b) This follows from (2.8), (2.9) and an easy induction argument.

(c) If $-\infty \leqq k \leqq m \leqq n$ and $x \in G_m$, then the above definition implies that $X(n, x)(\cdot \wedge T^k(X(n, x)))$ is $\sigma(\tilde{X}(n, y)\colon y \in G_m - G_k)$-measurable and $X(n, x)(\cdot + T^{-\infty}(X(n, x)))$ is $\sigma(\tilde{X}(n, 0))$-measurable. Take $m = j$ and $k = -\infty$ to see that

$$X(n, y) \in \sigma(\tilde{X}(n, y')\colon y' \in G_j) \quad \text{for } y \in G_j. \tag{2.11}$$

Take $k=j$ and $m=n$ to see that if $x\in G_n$,

$$X(n,x)(\cdot \wedge T^j(X(n,x)))\in\sigma(\tilde{X}(n,y'): y'\in G_n-G_j) \quad \text{for } x\in G_n.$$

This together with (2.11) and (2.5), implies the result. □

It is not hard to see that (2.4) (the nesting property) remains valid if $\tilde{X}$ is replaced by X. We provide a proof for the skeptics.

Lemma 2.4. *Let p, m, $n\in\mathbb{Z}$, $p\leqq m\leqq n$ and $x\in G_m$. Then*

$$X(m,x)(i)=X(n,x)(T_i^m(X(n,x))) \quad \textit{for all } i\in\mathbb{Z}_+ \tag{2.12}$$

$$X(m,x)(T^p(X(m,x))) \quad \textit{is independent of the choice of } m(\geqq p) \tag{2.13}$$

$$T^{p-1}(X(m,x))-T^p(X(m,x))=T^{p-1}(X(m,X(m,x)(T^p(X(m,x))))) \tag{2.14}$$

$$T_i^p(X(n,x))=T^m_{T_i^p(X(m,x))}(X(n,x)). \tag{2.15}$$

Proof. If $j\leqq m$, $x\in G_j$ and $i\leqq T^{j-1}(X(m,x))$, then

$$X(m,x)(i)=\tilde{X}(m,x)(i)=\tilde{X}(n,x)(T_i^m(\tilde{X}(n,x))) \quad \text{(by (2.6) and (2.4))}.$$

This shows $T_i^m(\tilde{X}(n,x))\leqq T^{j-1}(\tilde{X}(n,x))$ and so

$$X(m,x)(i)=X(n,x)(T_i^m(\tilde{X}(n,x))) \quad \text{(by (2.6))},$$

$$X(m,x)(i)=X(n,x)(T_i^m)X(n,x))) \quad \text{if } i\leqq T^{j-1}(X(m,x)),\ x\in G_j,\ j\leqq m. \tag{2.16}$$

The last equality follows from the former because $\tilde{X}(n,x)=X(n,x)$ on $[0, T^{j-1}(\tilde{X}(n,x))]$ by (2.6).

Let m, n, x be fixed as in the statement of the Lemma. We will prove that for $p\leqq m$,

$$X(m,x)(i)=X(n,x)(T_i^m(X(n,x))) \quad \text{for} \quad i\leqq T^p(X(m,x)), \tag{2.17}$$

by backwards induction on p. (2.17) holds with $p=m-1$ by setting $j=m$ in (2.16). Assume (2.17) holds for some $p<m$, let $k=T^p(X(m,x))$ and $y=X(m,x)(k)$. (2.17) implies

$$T_k^m(X(n,x))=T^p(X(n,x)) \tag{2.18}$$

and hence

$$y=X(m,x)(k)=X(n,x)(T^p(X(n,x)))\in G_p. \tag{2.19}$$

(2.10), (2.18) and (2.19) give us

$$T_{k+i}^m(X(n,x))-T^p(X(n,x))=T_i^m(X(n,y)), \quad i\in\mathbb{Z}_+, \tag{2.20}$$

and

$$T^{p-1}(X(m,x))-T^p(X(m,x))=T^{p-1}(X(m,y)). \tag{2.21}$$

If $i \leqq T^{p-1}(X(m,x)) - T^p(X(m,x))$, then

$$\begin{aligned} X(m,x)(k+i) &= X(m,y)(i) \quad \text{(by (2.10))} \\ &= X(n,y)(T_i^m(X(n,y))) \\ &\quad \text{(by (2.16) with } j=p \text{ because (2.21) shows } i \leqq T^{p-1}(X(m,y))) \\ &= X(n,y)(T_{k+i}^m(X(n,x)) - T^p(X(n,x))) \quad \text{(by (2.20))} \\ &= X(n,x)(T_{k+i}^m(X(n,x))) \quad \text{(by (2.10) and (2.19)).} \end{aligned}$$

(2.17) now holds for $i \leqq T^{p-1}(X(m,x))$ and hence for $i < T^{-\infty}(X(m,x))$ by induction on p.

To prove (2.12) for $i \geqq T^{-\infty}(X(m,x))$, argue as above with $y=0$, $k = T^{-\infty}(X(m,x))$ and use $X(m,0)=\tilde{X}(m,0)$. Here are the details:

$$\begin{aligned} X(m,x)(k+i) &= X(m,0)(i) \quad \text{((2.10) with } j=-\infty) \\ &= X(n,0)(T_i^m(X(n,0))) \quad \text{(by (2.4), (2.7))} \\ &= X(n,0)(T_{k+i}^m(X(n,x)) - T^{-\infty}(X(n,x))) \\ &\quad \text{(by the analogue of (2.20) with } p=-\infty) \\ &= X(n,x)(T_{k+i}^m(X(n,x))) \quad \text{(by (2.10)).} \end{aligned}$$

This completes the proof of (2.12). (2.19) implies (2.13), (2.21) is (2.14) and (2.15) is immediate from (2.12). □

Notation. In view of (2.13) we may let $Y^p(x) = X(m,x)(T^p(X(m,x)))$, $m \geqq p$, $x \in G_m$.

Lemma 2.5. (a) *If* $x \in G_\infty$, $m, n \in \mathbb{Z}$ *and* $n \geqq m$, *then* $\{W_i^m(X(n,x)) : i \in \mathbb{N}\}$ *are i.i.d. random variables whose common distribution does not depend on* x. *If* $x \in G_m$, *they are jointly independent of* $X(m,x)$.

(b) *If* $x \in G_m$ *and* $i \in \mathbb{N}$, *then* $r \to W_i^m(X(m+r,x))$ *is a supercritical branching process starting at* 1 (*when* $r=0$) *and with offspring distribution equal to the law of* N.

Proof. (a) The strong Markov property of $X(n,x)$ and the (local) translation and rotational invariance of $X(n,x)$ imply that $\{W_i^m(X(n,x)) : i \in \mathbb{N}\}$ are i.i.d. and have a common law independent of x. If $x \in G_m$, the joint independence with $X(m,x)$ follows by a symmetry argument. For example, by reflecting in an appropriate axis, one sees that given $W_1^m(X(n,x)) = k$, $X(n,x)(T_1^m(X(n,x)))$ equals any given vertex in $N_m(x)$ with probability $\frac{1}{4}$. Now proceed in general using the strong Markov property.

(b) Fix $x \in G_m$ and $i \in \mathbb{N}$. It follows from (2.15) that

$$T_i^m(X(m+r+1,x)) = T_{T_i^m(X(m+r,x))}^{m+r}(X(m+r+1,x)), \qquad r \in \mathbb{Z}_+,$$

and therefore

(2.22) $$W_i^m(X(m+r+1,x)) = \sum_{j=1}^{\infty} 1(j \leqq W_i^m(X(m+r,x)))\, W^{m+r}_{T^m_{i-1}(X(m+r,x))+j}(X(m+r+1,x)).$$

By (a), conditional on $X(m+r,x)$, the summands in (2.22) (excluding the indicator functions) are i.i.d. and equal in law to $W_1^{m+r}(X(m+r+1),x)$, which in turn is equal in law to N. (b) follows. □

Remark 2.6. The above proof shows that (b) holds for any family of simple random walks $\{X(m_0+r,x): r\in\mathbb{Z}_+\}$, providing $X(m_0+r,x)$ is a random walk on G_{m_0+r} starting at $x\in G_{m_0}$, and the nesting property (2.12) holds for $m_0\leqq m\leqq n$.

Notation. If $n\in\mathbb{Z}$ and $x\in G_n$, let $X_n(x)(j5^{-n})=X(n,x)(j)$ and extend $X_n(x)$ to $[0,\infty)$ by linear interpolation. Hence $X_n(x)(\cdot)\in C([0,\infty),G)$.

Proposition 2.7. *Let $m\in\mathbb{Z}$ and $x\in G_m$.*

(a) *For each $i\in\mathbb{N}$, $W_i^m(X_n(x))$ converges a.s. and in L^2 as $n\to\infty$ to a random variable $W_i^m(x)$ which is strictly positive a.s.*

(b) *$\{W_i^m(x): i\in\mathbb{N}\}$ are i.i.d. random variables and are jointly independent of $X_m(x)$.*

(c) *$W_i^m(x)$ is equal in law to $W_1^0(0)5^{-m}$. If*

(2.23) $$\phi(s)=E(e^{-sW_1^0(0)}), \qquad \mathrm{Re}(s)\geqq 0,$$

then ϕ is the unique characteristic function satisfying

(2.24) $$\phi(5s)=f(\phi(s)) \quad \textit{for } \mathrm{Re}(s)\geqq 0,\ \phi'(0)=-1.$$

Proof. Fix $m\in\mathbb{Z}$, $x\in G_m$ and $i\in\mathbb{N}$. If $r\in\mathbb{N}$ then

(2.25) $$W_i^m(X_{m+r}(x))=5^{-m}(5^{-r}W_i^m(X(m+r,x))).$$

Use Lemma 2.5(b), $E(N)=5$, and the convergence theorem for supercritical branching processes (Harris (1963), p. 13, Theorem 8.1) to conclude that

(2.26) $$5^{-r}W_i^m(X(m+r,x))\to \tilde{W}_i^m(x) \text{ a.s. and in } L^2 \quad \text{as} \quad r\to\infty,$$

$\tilde{W}_i^m(x)>0$ a.s. (Harris (1963, p. 14, Remark 1)), and

$$\phi(s)=E(e^{-s\tilde{W}_i^m(x)}), \qquad \mathrm{Re}(s)\geqq 0,$$

is the unique characteristic function satisfying (2.24) (Harris (1963, p. 15, Theorem 8.2)). (a) and (c) are now immediate from (2.25) and (2.26). (b) follows from Lemma 2.5(a) and the above convergence. □

Notation. $T_j^m(x)=\sum_{i=1}^{j} W_i^m(x)$ for $x\in G_m$.

W will be used to denote a random variable equal in law to $W_1^0(x)$.

The nesting property (2.12) shows that the set of points traced out by $X_n(x)(\cdot)$ converges a.s. as $n\to\infty$. Proposition 2.7 shows that the time it takes $X_n(x)$ to traverse this set of points also converges a.s. as $n\to\infty$. We now show that these results imply the a.s. convergence of $X_n(x)$ in $C([0,\infty), G)$ as $n\to\infty$. The argument is "soft" and may be used to simplify Knight's construction of Brownian motion (Knight (1981, p. 10)), which has motivated our approach.

Theorem 2.8. *For each $x\in G_\infty$, $X_n(x)$ converges a.s. in $C([0,\infty), G)$ as $n\to\infty$ to a process, $X(x)$. Moreover for all $m\in\mathbb{Z}$, $x\in G_m$ and $j\in\mathbb{Z}_+$,*

$$X(x)(T_j^m(x))=X(m,x)(j). \tag{2.27}$$

Proof. Fix $n_0\in\mathbb{N}$ and $x\in G_{n_0}$. By Proposition 2.7 we may choose ω outside a null set such that

$$\lim_{n\to\infty} W_i^m(X_n(x))=W_i^m(x)>0 \quad \text{for all } m\in\mathbb{Z},\ i\in\mathbb{N}$$

and

$$\lim_{j\to\infty} T_j^m(x)=\infty \quad \text{for all } m\in\mathbb{Z}.$$

Fix $m\geqq n_0$ and then choose $k=k(\omega)$ such that $T_k^m(x)>m$, and $n_1=n_1(\omega)$ such that

$$\max_{i\leqq k}\{T_i^m(X_n(n))-T_i^m(x)\}<\min\{W_i^m(x): i\leqq k\} \quad \text{if } n\geqq n_1. \tag{2.28}$$

Let $n, n'\geqq n_1$, $t\in[0,m]$ and choose $j=j(t)\in\{1,\dots,k\}$ such that $T_{j-1}^m(x)\leqq t<T_j^m(x)$. (2.28) implies that

$$T_{j-2}^m(X_n(x))<t<T_{j+1}^m(X_n(x))$$

and similarly with n' in place of n. Therefore we have

$$\begin{aligned} &|X_n(x)(t)-X_{n'}(x)(t)| \\ &\leqq 2^{-m+2}+|X_n(x)(T_j^m(X_n(x)))-X_{n'}(x)(T_j^m(X_n'(x)))| \\ &=2^{-m+2}. \end{aligned} \tag{2.29}$$

Since $t\in[0,m]$ and $m\in\mathbb{N}$ were arbitrary, this proves the a.s. convergence of $X_n(x)$ in $G([0,\infty), G)$ as $n\to\infty$. To derive (2.27) we used the nesting property (2.12) to see that

$$X_n(x)(T_j^m(X_n(x)))=X(m,x)(j) \quad \text{for } n\geqq m,\ x\in G_m. \tag{2.30}$$

Let $n\to\infty$ in the above and use Proposition 2.7 to obtain (2.27). □

Proposition 2.9. (a) *For each $x\in G_\infty$ and $j\in\mathbb{Z}$, $\lim_{n\to\infty} T^j(X_n(X))\equiv T^j(x)$ exists and is finite a.s., and*

$$X(x)(T^j(x))=Y^j(x)\in G_j. \tag{2.31}$$

(b) *If* $m\in\mathbb{Z}$ *and* $x\in G_m-G_{m-1}$, *then*

$$T^{m-1}(x)=\sum_{i=1}^{K_{m-1}(x)} W_i^m(x) \text{ a.s.,} \tag{2.32}$$

where $K_{m-1}(x)=T^{m-1}(X(m,x))$ *is a geometric random variable with mean two and is independent of* $\{W_i^m(x): i\in\mathbb{N}\}$.

$$E(T^{m-1}(x))=2(5^{-m}), \tag{2.33}$$

and

$$E(\exp(-\lambda T^{m-1}(x))=\phi(\lambda 5^{-m})(2-\phi(\lambda 5^{-m}))^{-1} \quad \text{for } \lambda\geqq 0, \tag{2.34}$$

where ϕ *is as in* (2.23), (2.24).

(c) *If* $j, n\in\mathbb{Z}, j\leqq n$, *and* $x\in G_n$, *then*

$$E(T^j(X_n(x)))\leqq 5^{-j}/2 \tag{2.35}$$

$$P(T^j(X_n(x))\geqq t))\leqq 2^{\frac{1}{2}}\exp\{-(\log 2/2)\,5^j t\} \quad \text{for all } t\geqq 0. \tag{2.36}$$

Therefore if $j\in\mathbb{Z}$ *and* $x\in G_\infty$, *then*

$$E(T^j(x))\leqq 5^{-j}/2 \tag{2.37}$$

$$P(T^j(x)\geqq t)\leqq 2^{\frac{1}{2}}\exp\{-(\log 2/2)\,5^j t\} \quad \text{for all } t\geqq 0. \tag{2.38}$$

Proof. (b) Let $x\in G_m-G_{m-1}(m\in\mathbb{Z})$. (2.15) (with $p=m-1$) implies that

$$T^{m-1}(X(n,x))=\sum_{i=1}^{\infty} 1(i\leqq T^{m-1}(X(m,x)))\,W_i^m(X(n,x)). \tag{2.39}$$

Replace $X(n,x)$ with $X_n(x)$ and let $n\to\infty$ in the above to derive (2.32) from Proposition 2.7(a). It is easy to see that $K_{m-1}(x)$ has a geometric distribution with mean 2. The independence of $K_{m-1}(x)$ and the summands in (2.32) is immediate from Proposition 2.7(b). Take expected values in (2.32) to see that

$$E(T^{m-1}(x))=2\,E(W_1^m(x))=2(5^{-m}).$$

If $\lambda\geqq 0$, then (2.32) and Proposition 2.7(c) imply

$$E(\exp(-\lambda T^{m-1}(x)))=\sum_{k=1}^{\infty} 2^{-k}E(\exp(-\lambda 5^{-m}W))^k$$
$$=\phi(\lambda 5^{-m})(2-\phi(\lambda 5^{-m}))^{-1}.$$

(a), (c) If $j<m\leqq n$ and $x\in G_m$, then (2.14) implies

$$T^j(X_n(x))=\sum_{i=j+1}^{m} T^{i-1}(X_n(Y^i(x))). \tag{2.40}$$

Let $n \to \infty$ and use the a.s. convergence obtained in (b) to get

$$\lim_{n\to\infty} T^j(X_n(x)) \equiv T^j(x) = \sum_{i=j+1}^{m} T^{i-1}(X(Y^i(x))) \text{ a.s.}$$

This establishes the a.s. convergence in (a), and (2.31) follows by letting $n \to \infty$ in

$$X_n(x)(T^j(X_n(x))) = Y^j(x).$$

The strong Markov property of $X_n(x)$ at $T^i(X_n(x))$ shows that if $j+1 \leqq i \leqq n$, and $x \in G_n$,

$$\begin{aligned} &E(T^{i-1}(X_n(Y^i(x))) \mid F^0_{T^i(X_n(x))}(X_n(x)))(\omega) \\ &= \int T^{i-1}(X_n(Y^i(x)(\omega))(\omega')\, dP(\omega') \\ &= 2(5^{-i})\, 1(Y^i(x)(\omega) \in G_{i-1}) \leqq 2(5^{-i}). \end{aligned}$$

In the last line we have taken expected values in (2.39) and used Lemma 2.5(a) and Proposition 2.7(c). (2.35) follows easily from this and (2.40) (with $m=n$).

Let $j, n \in \mathbb{Z}$, $j \leqq n$ and $x \in G_n$. Set $t_0 = 5^{-j}$. Note that

$$\begin{aligned} &P(T^j(X_n(x)) \geqq 2t_0) \\ &\quad \leqq E(1(T^j(X_n(x)(\omega)) \geqq t_0) \int 1(T^j(X_n(X_n(x)(t_0,\omega))(t_0,\omega')) \geqq t_0)\, dP(\omega')) \\ &\qquad \text{(by the Markov property of } X_n(x)) \\ &\quad \leqq E(1(T^j(X_n(x)(\omega)) \geqq t_0)(5^{-j}/2t_0)) \quad \text{(by (2.35) and Chebychev)} \\ &\quad \leqq 2^{-2} \quad \text{(by (2.35) and Chebychev again).} \end{aligned}$$

By induction one gets

$$P(T^j(X_n(x)) \geqq n t_0) \leqq 2^{-n} \qquad \text{for all } n \in \mathbb{N}.$$

For (2.36) we may assume $t \geqq t_0$ (or else the upper bound exceeds one). Choose $n \in \mathbb{N}$ such that $n t_0 \leqq t < (n+1)t_0$. Then

$$\begin{aligned} P(T^j(X_n(x)) \geqq t) &\leqq P(T^j(X_n(x)) \geqq n t_0) \\ &\leqq \exp\{-n \log 2\} \\ &= \exp\{-n t_0 (\log 2)\, 5^j\} \\ &\leqq \exp\left\{-\frac{\log 2}{2}\, 5^j t\right\}. \end{aligned}$$

(2.37) and (2.38) follow from (2.35) and (2.36), respectively, by letting $n \to \infty$ and using Fatou's lemma for (2.37). □

Remark. $X(x)$ is the process we will study. Note that it has not yet been shown that $T^j(x) = T^j(X(x))(x \in G_\infty)$, nor that $T_i^m(x) = T_i^m(X(x))(x \in G_m)$. These results will be easily obtained once the strong Markov property of $X(x)$ is established.

We call the times $\{T^j(X(x)), j\in\mathbb{Z}\}$ break-out times for $X(x)$. The above results will eventually give us some probability estimates on these times (Lemma 4.2 below) which will play an important role in the study of X.

Corollary 2.10. *$X(x)(t)=X(Y^j(x))(t-T^j(x))$ for all $t\geqq T^j(x)$,*

$$j\in\mathbb{Z}\cup\{-\infty\} \quad \text{and} \quad x\in G_\infty \text{ a.s.}$$

Proof. (2.10) implies that

$$X_n(x)(t)=X_n(Y^j(x))(t-T^j(X_n(x))) \qquad \text{for all } t\geqq T^j(X_n(x)),\ x\in G_n,\ -\infty\leqq j\leqq n \text{ a.s.}$$

Let $n\to\infty$ and use Proposition 2.9(a) and Theorem 2.8. □

To define $X(x)$ for all $x\in G$ we want to show it is uniformly continuous in probability on G_∞.

Lemma 2.11. *There is a $\rho<1$ such that if $j\in\mathbb{Z}$ and y_1, y_2 are neighbouring points in G_{j+1} then*

$$P(Y^j(y_1)\neq Y^j(y_2))\leqq\rho.$$

Proof. By scaling we may take $j=0$. One only need check that for any y_1, y_2 as above there is a strictly positive probability that $X(1, y_1)$ and $X(1, y_2)$ enter G_0 at the same point. If $y_1, y_2\in G_1-G_0$ these random walks, stopped at T^0, are independent by (2.6) and this is obvious. If, say, $y_1\in G_0$ then $y_2\in G_1-G_0$ and clearly there is positive probability that $X(1, y_2)$ first hits G_0 at y_1. □

(In fact one can take $\rho=\frac{17}{25}$, as the reader can easily check).

Lemma 2.12. *Let $m\in\mathbb{Z}$, $\Delta\in T_m$ and $x_0\in\partial\Delta$. Then for $j\leqq m$,*

$$P(X(x)(t+T^j(X(x)))\neq X(x_0)(t+T^j(X(x_0)))$$

for some $t\geqq 0$ and some $x\in\Delta\cap G_\infty)\leqq 2\rho^{m-j}$.

Proof. Let m, Δ, j be as above and $\partial\Delta=\{x_0, x_1, x_2\}$. By Corollary 2.10 it suffices to show

$$P(Y^j(x)\neq Y^j(x_0) \text{ for some } x \text{ in } \Delta\cap G_\infty)\leqq 2\rho^{m-j}.$$

An easy consequence of (2.10) is that

$$Y^j(Y^m(x))=Y^j(x) \quad \text{for } j\leqq m. \tag{2.41}$$

Since $Y^m(x)\in\partial\Delta$ it suffices to show

$$P(Y^j(x_1)\neq Y^j(x_0))\leqq\rho^{m-j}. \tag{2.42}$$

Note that

$$\begin{aligned}
&P(Y^j(x_1) \neq Y^j(x_0) \mid Y^{j+1}(x_1), Y^{j+1}(x_0))(\omega) \\
&\quad = 1(Y^{j+1}(x_1)(\omega) \neq Y^{j+1}(x_0)(\omega)) \int 1(Y^j(Y^{j+1}(x_1)(\omega))(\omega') \\
&\quad \neq Y^j(Y^{j+1}(x_0))(\omega))(\omega'))\, dP(\omega') \\
&\quad \leqq \rho\, 1(Y^{j+1}(x_1)(\omega) \neq Y^{j+1}(x_0)(\omega)) \quad \text{(Lemma 2.11).}
\end{aligned}$$

The next to last line uses Lemma 2.3(c) (with $j+1$ and m in place of j and n, respectively) to show that $\sigma(Y^j(y)\colon y \in G_{j+1})$ and $\sigma(Y^{j+1}(x)\colon x \in G_m)$ are independent σ-fields. (2.42) follows by iterating the above inequality. □

Notation. $L^0(C([0, \infty), G))$ denotes the complete metric space of $C([0, \infty), G)$-valued random vectors with the topology of convergence in probability.

Proposition 2.13. *The mapping*

$$X\colon G_\infty \to L^0(C([0, \infty), G))$$

is uniformly continuous on bounded subsets of G_∞ and hence has a unique continuous extension to G, which we also denote by X.

Proof. Fix $\varepsilon > 0$ and $M \in \mathbb{N}$. Let $m \in \mathbb{Z}$, $\Delta \in T_m$ and $x, y \in \Delta$. We claim that

(2.43) if $j \leqq m$ and $Y^j(x) = Y^j(y)$, then

$$\sup_{t \leqq M} |X(x)(t) - X(y)(t)| \leqq 2^{-j}$$
$$+ \sup\{|X(Y^j(x))(t'+u) - X(Y^j(x))(t')| \colon 0 \leqq t' \leqq M, 0 \leqq u \leqq |T^j(x) - T^j(y)|\}.$$

Assume $j \leqq m$, $Y^j(x) = Y^j(y)$, let $t \in [0, M]$ and consider four separate cases.

(i) $t \leqq T^j(x) \wedge T^j(y)$.
Then $|X(x)(t) - X(y)(t)| \leqq 2^{-j}$.

(ii) $T^j(x) < t \leqq T^j(y)$.
Then

$$\begin{aligned}
|X(x)(t) - X(y)(t)| &\leqq |X(Y^j(x))(t - T^j(x)) - Y^j(x)| + |Y^j(x) - X(y)(t)| \\
&\qquad \text{(Corollary 2.10)} \\
&\leqq \sup_{0 \leqq u \leqq T^j(y) - T^j(x)} |X(Y^j(x))(u) - X(Y^j(x))(0)| + 2^{-j}.
\end{aligned}$$

(iii) $T^j(y) < t \leqq T^j(x)$. As above.

(iv) $T^j(x) \vee T^j(y) < t$.

Then Corollary 2.10 implies

$$|X(x)(t) - X(y)(t)| = |X(Y^j(x))(t - T^j(x)) - X(Y^j(x))(t - T^j(y))|,$$

which is bounded by the supremum appearing in (2.43). (2.43) follows.

Assume $j \leqq k \leqq m$. Corollary 2.10 shows that if $Y^k(x) = Y^k(y)$, then $T^j(x) - T^j(y) = T^k(x) - T^k(y)$ and therefore

$$(2.44)\quad \begin{aligned} P(|T^j(x)-T^j(y)|>\delta) &\leqq P(|T^k(x)-T^k(y)|>\delta)+P(Y^k(x)\neq Y^k(y)) \\ &\leqq \delta^{-1} E(T^k(x)+T^k(y))+4\rho^{m-k} \quad \text{(by Lemma 2.12)} \\ &\leqq \delta^{-1} 5^{-k}+4\rho^{m-k} \quad \text{(by (2.37))}. \end{aligned}$$

Choose $j \in \mathbb{N}$ large enough so that $2^{-j} < \varepsilon/2$. Then (2.43) implies that for $m \geqq k \geqq j$, $\Delta \in \mathcal{T}_m$ and $x, y \in \Delta$ such that $x, y \in B(0, M)$

$$\begin{aligned} (2.45)\ & P(\sup_{t \leqq M} |X(x)(t)-X(y)(t)|>\varepsilon) \\ &\leqq P(Y^j(x) \neq Y^j(y)) + P(|T^j(x)-T^j(y)|>\delta) \\ &\quad + P(\sup\{|X(z)(t'+u)-X(z)(t')|: 0 \leqq t' \leqq M, 0 \leqq u \leqq \delta, z \in \partial\Delta_j(x)\} > \varepsilon/2) \\ &\leqq 4\rho^{m-j} + \delta^{-1} 5^{-k} + 4\rho^{m-k} \\ &\quad + 3 \sup_{z \in G_j, |z| \leqq M+1} P(\sup\{|X(z)(t'+u)-X(z)(t')|: \\ &\quad 0 \leqq t' \leqq M, 0 \leqq u \leqq \delta\} > \varepsilon/2). \end{aligned}$$

In the last line we have used Lemma 2.12 and (2.44). Choose $\delta = \delta(j, M, \varepsilon) > 0$ so that the last term is less than $\varepsilon/3$ (note that the supremum is over a finite number of z's). Then choose k so that $\delta^{-1} 5^{-k} < \varepsilon/3$. Finally we may choose m large enough (depending only on (ε, M)) so that the right side of (2.45) is less than ε. By considering adjacent triangles in $\mathcal{T}_m$ it follows that $X(\cdot)$ is uniformly continuous on bounded sets in G_∞, as required. □

Theorem 2.14. *If $x_n \in G_n$ and $\{x_n\}$ converges to $x \in G$, then*

$$X_n(x_n) \xrightarrow{P} X(x) \quad \text{in} \quad C([0, \infty), G),$$

where $\xrightarrow{P}$ denotes convergence in probability.

Proof. Let $\{x_n\}$ and x be as above and fix $M \in \mathbb{N}$. (2.27) shows that $X_n(x_n)(i5^{-n}) = X(x_n)(T_i^n(x_n))$ and therefore

$$\begin{aligned} \sup_{t \leqq M} |X_n(x_n)(t) - X(x)(t)| \leqq & \max_{i \leqq 5^n M} |X(x_n)(T_i^n(x_n)) - X(x_n)(i5^{-n})| + 2^{-n} \\ & + \max_{i \leqq 5^n M} |X(x_n)(i5^{-n}) - X(x)(i5^{-n})| \\ & + \sup_{u, v \in [0, M], |u-v| \leqq 5^{-n}} |X(x)(u) - X(x)(v)|. \end{aligned}$$

Proposition 2.13 and the continuity of $X(x)$ show that the last two terms converge to zero in probability as $n \to \infty$. In view of the convergence in probability of $X(x_n)$ to $X(x)$, the first term will converge to zero in probability, and hence the result will follow if

$$\max_{i \leqq 5^n M} |T_i^n(X_n) - i\,5^{-n}| \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2.46}$$

For n fixed, $T_i^n - i\,5^{-n}$ is a martingale in i by Proposition 2.7. Doob's maximal inequality implies

$$E(\max_{i \leqq 5^n M} T_i^n(x_n) - i\,5^{-n})^2) \leqq c_1\, E((T_{5^n M}^n(x_n) - M)^2)$$

$$\leqq c_1\, 5^n M\, 5^{-2n} E((W-1)^2)$$

(Proposition 2.7 (c)).

This proves (2.46) and hence the theorem. □

We now use $\{X(x) \colon x \in G\}$ to construct a Feller process on the canonical space of paths.

$\Omega = C([0, \infty), G)$, $P^x =$ law of $X(x)$ on Ω,

$\mathscr{F}$ is the Borel σ-field on Ω augmented in the usual manner (Blumenthal and Getoor (1968, p. 27)),

$X(t, \omega) = \omega(t)$ if $\omega \in \Omega$,

$\{\mathscr{F}_t, t \geqq 0\}$ is the usual augmentation of $\{\mathscr{F}_t^0(X), t \geqq 0\}$,

$\{\theta_t, t \geqq 0\}$ are the canonical shift operators on Ω.

Recall that P_t denotes the transition semigroup of X.

Theorem 2.15. *$(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ is a Feller process, that is, it is a Hunt process such that $P_t \colon C_b(G) \to C_b(G)$.*

Proof. Assume $0 \leqq s < t$, $x_n \in G_n$, $\{x_n\}$ converges to $x \in G$, and let $s_n = [5^n s]\,5^{-n}$, $t_n = [5^n t]\,5^{-n}$. Let $\phi \in C_b(G)$ and let ψ be a bounded continuous function on Ω. Then

$$\begin{aligned} (2.47) \qquad & E^x(\phi(X(t))\,\psi(X(s \wedge \cdot))) \\ &= \lim_{n \to \infty} E(\phi(X_n(x_n)(t_n))\,\psi(X_n(x_n)(s_v \wedge \cdot))) \quad \text{(by Theorem 2.14)} \\ &= \lim_{n \to \infty} \int_{\Omega_0} \psi(X_n(x_n)(s_n \wedge \cdot)(\omega))\, \mathrm{I}_n(\omega)\, \mathrm{d}P(\omega). \end{aligned}$$

where

$$I_n(\omega) = \int_{\Omega_0} \phi(X_n(X_n(x_n)(s_n, \omega))(t_n - s_n, \omega'))\, dP(\omega')$$

and we have used the Markov property of $X_n(x_n)$. Use Theorem 2.14 and pass to a subsequence, if necessary, so that we may fix ω outside a P-null set so that

$$\lim_{n \to \infty} X_n(x_n)(s_n, \omega) = X(x)(s, \omega).$$

Theorem 2.14 now shows that $I_n(\omega)$ converges to $E^{X(x)(s,\omega)}(\phi(X(t-s)))$ as $n\to\infty$. Use Theorem 2.14 once again to see that (2.47) equals

$$E^x(\psi((X(s\wedge\cdot))E^{X(s)}(\phi(X(t-s)))).$$

Therefore $X(t)$ is a Markov process with respect to $\{\mathscr{F}_t^0(X), t\geqq 0\}$. Proposition 2.13 implies the weak continuity of $\{P^x: x\in G\}$ and hence the Feller property. The rest of the theorem now follows from general Markov process theory (see Blumenthal and Getoor (1968, p. 27 and Theorem 8.11 on p. 41)). □

We want to show that $T_i^m(x)=T_i^m(X(x))$ so that we can carry over our estimates on $T_i^m(x)$ to the space of paths, Ω. To do this we require some continuity properties of the hitting times $T(A, X)$ which we may write as $T(A,\omega)$ on path space. Similarly $T_i^m(\omega)$ and $T^m(\omega)$ are just $T_i^m(X(\omega))$ and $T^m(X(\omega))$, respectively.

Lemma 2.16. *Let A be an open subset of G such that ∂A is finite subset of G_∞.*

(a) $\lim_{\substack{\omega'\to\omega\\ \omega'\in\Omega}} T(A^c,\omega')=T(A^c,\omega)$ *for P^x-a.a. $\omega\in\Omega$ and all $x\in G$.*

(b) *If A is also bounded, then*

$$\lim_{n\to\infty} E(|T(A^c, X_n(x))-T(A^c, X(x))|)=0 \qquad \text{for all } x\in G_\infty.$$

Proof. (a) If $x\notin cl(A)$, then both sides are zero. Assume therefore that $x\in cl(A)$ and let $\partial A=\{a_1,\ldots,a_k\}\subset G_m$. Then

(2.48) $$T(A^c,\omega)=T(\partial A,\omega)=\min\{T_{a_i}(\omega): i\leqq k\} \qquad P^x\text{-a.s.}$$

An elementary topological argument (use the openness of A) shows that for any $\omega\in\Omega$

$$T(A^c,\omega)\leqq \liminf_{\omega'\to\omega} T(A^c,\omega').$$

In view of (2.48) and the obvious inequality,

$$\limsup_{\omega'\to\omega} T(A^c,\omega')\leqq \limsup_{\omega'\to\omega} T_{a_i}(\omega'),$$

the result would follow from

(2.49) $$\limsup_{\omega'\to\omega} T_{a_i}(\omega')\leqq T_{a_i}(\omega) \qquad P^x \text{ a.a. } \omega \qquad \text{for each } i\leqq k.$$

Fix $i\leqq k$ and let A_1 and A_2 denote the interiors of the two adjacent triangles in $\mathscr{T}_m$ which intersect at a_i. A simple topological argument shows that (2.49) holds for each $\omega\in\Omega$ which satisfies

(2.50) if $T_{a_i}(\omega)<\infty$, then $T(A_j, \theta_{T_{a_i}}\omega)=0$ for $j=1,2$.

The strong Markov property and the zero-one law show that (2.50) would hold for P^x-a.a. ω if

$$(2.51)\qquad P^{a_i}(T(A_j,\omega)=0)>0 \qquad j=1,2.$$

For $n\geqq m$, we have

$$P(X(a_i)(W_1^n(a_i))\in A_1)=P(X(n,a_i)(1)\in A_1)=\tfrac{1}{2}\quad (\text{by } (2.27))$$
$$E(W_1^n(a_i))=5^{-n}\quad (\text{Proposition } 2.7\,(\text{c})).$$

These imply

$$P(X(a_i)(W_1^n(a_i))\in A_1 \text{ for infinitely many } n \text{ and } \lim_{n\to\infty} W_1^n(a_i)=0)\geqq\tfrac{1}{2}.$$

This proves (2.51) and hence gives (a).

(b) Assume A is bounded and $x\in A\cap G_\infty$ (if $x\in A^c$ the result is trivial). Choose m so that $x\in G_m$ and choose j such that $A\subset \mathrm{Int}(B_d(0,2^j))$. By (a) and Theorem 2.8 we only need show $\{T(A^c,X_n(x)): n\geqq m\}$ are uniformly integrable. Note that

$$(2.52)\qquad T(A^c,X_n(x))\leqq T(\mathrm{Int}(B_d(0,2^j))^c,X_n(x))\leqq T^{-j}(X_n(x))+W_1^{-j}(X_n(x)).$$

(2.36) shows that $\{T^{-j}(X_n(x)): n\geqq m\}$ is L^2-bounded and the L^2-convergence in Proposition 2.7 (a) shows the same is true of $\{W_1^{-j}(X_n(x)): n\geqq m\}$. The required uniform integrability is now immediate from (2.52). □

Corollary 2.17. *For each* $m\in\mathbb{Z}$, $i\in\mathbb{Z}_+$ *and* $x\in G$,

$$\lim_{\substack{\omega'\to\omega\\ \omega'\in\Omega}} T_i^m(\omega')=T_i^m(\omega)\qquad \textit{for } P^x\text{-a.a. } \omega \quad \textit{in } \Omega.$$

Proof. If $i=0$ the result is immediate from Lemma 2.16 (a) with $A=\mathrm{Int}(\Delta_m(x))$. Assume the result for T_i^m and consider T_{i+1}^m. If $T_i^m(\omega)=\infty$, then the results is clear because both sides are infinite for $T_{i+1}^m(\omega)$. On $\{T_i^m(\omega)<\infty\}$, we have

$$T_{i+1}^m(\omega)=T_i^m(\omega)+T(\mathrm{Int}(B_d(\omega(T_i^m),2^{-m}))^c,\theta_{T_i^m}\omega),$$

and hence we must show

$$(2.53)\qquad \lim_{\omega'\to\omega} T(\mathrm{Int}(B_d(\omega'(T_i^m),2^{-m}))^c,\theta_{T_i^m}\omega')$$
$$=T(\mathrm{Int}(B_d(\omega(T_i^m),2^{-m}))^c,\theta_{T_i^m}\omega)\qquad \text{for } P^x\text{-a.a. } \omega \quad \text{in } \{T_i^m<\infty\}.$$

The induction hypothesis implies

$$\lim_{\omega'\to\omega}\theta_{T_i^m}\omega'=\theta_{T_i^m}\omega\qquad \text{for}\quad P^x\text{-a.a. } \omega \quad \text{in} \quad \{T_i^m<\infty\},$$

and there is an $\varepsilon(\omega)$ such that

$$\omega'(T_i^m)=\omega(T_i^m) \quad \text{for } \rho_G(\omega',\omega)<\varepsilon(\omega) \quad \text{and}$$
$$\varepsilon(\omega)>0 \quad \text{for } P^x\text{-a.a. } \omega \text{ in } \{T_i^m<\infty\}.$$

These results show (2.53) would follow from

$$\lim_{\omega'\to\theta_{T_i^m}\omega} T(\operatorname{Int}(B_d(\omega(T_i^m),2^{-m}))^c,\omega')$$
$$=T(\operatorname{Int}(B_d(\omega(T_i^m),2^{-m}),2^{-m}))^c,\theta_{T_i^m}\omega) \quad \text{for } P^x\text{-a.a. } \omega \quad \text{in } \{T_i^m<\infty\},$$

which by the strong Markov property would in turn follow from

(2.54) $$\lim_{\omega''\to\omega'} T(\operatorname{Int}(B_d(\omega(T_i^m),2^{-m}))^c,\omega'')$$
$$=T(\operatorname{Int}(B_d(\omega(T_i^m),2^{-m}))^c,\omega') \quad \text{for} \quad P^{X(T_i^m,\omega)}\text{-a.a. } \omega',$$
$$\text{for } P^x\text{-a.a. } \omega \quad \text{in } \{T_i^m<\infty\}.$$

For each fixed ω such that $T_i^m(\omega)<\infty$, (2.54) holds by Lemma 2.16(a) with $A=\operatorname{Int}(B_d(\omega(T_i^m),2^{-m}))$ and $x=X_{T_i^m}(\omega)$. □

Lemma 2.18. (a) $T^m(x)=T^m(X(x))$ *for all* $m\in\mathbb{Z}$ *and* $x\in G_\infty$ a.s.

(b) $T_i^m(x)=T_i^m(X(x))$ *for all* $m\in\mathbb{Z}$, $i\in\mathbb{N}$ *and* $x\in G_m$ a.s.

Proof. (a) Let $x\in G\infty$ and $m\in\mathbb{Z}$.

$$T^m(x)=\lim_{n\to\infty} T^m(X_n(x)) \text{ a.s.}$$
$$=T^m(X(x)) \text{ a.s.,}$$

where in the last line we used Theorem 2.8 and Corollary 2.17,

(b) is similar. □

We now may carry over Propositions 2.7 and 2.9 over to the canonical space of paths and extend them to arbitrary starting points in G.

Notation. For each $m\in\mathbb{Z}$, $Y_m:\mathbb{Z}^+\times\Omega\to G_m$ is defined by

$$Y_m(i,\omega)=\begin{cases} X(T_i^m(\omega)) & \text{if } T_i^m(\omega)<\infty, \\ 0 & \text{if } T_i^m(\omega)=\infty. \end{cases}$$

Theorem 2.19. *For any* $x\in G$ *and* $m\in\mathbb{Z}$ *we have the following under* P^x:

(a) *Conditional on* $\mathscr{F}_{T^m}$, $Y_m(\cdot)$ *is a simple random walk on* G_m *starting at* $X(T^m)$.

(b) $\{W_i^m: i\in\mathbb{N}\}$ *are i.i.d. and are jointly independent of* $\mathscr{F}_{T^m}\vee\sigma(Y_m)$. $E(W_i^m)=5^{-m}$ *and* $5^m W_i^m$ *is equal in law to* W, *where* $\phi(s)=E(e^{-sW})$ $(\operatorname{Re}(s)\geq 0)$ *is the unique characteristic function satisfying* (2.24).

(c) *If* $x \in G_m - G_{m-1}$, *then*

$$E^x(T^{m-1}) = 2(5^{-m}) \tag{2.55}$$

$$E^x(\exp(-\lambda T^{m-1})) = \phi(\lambda 5^{-m})(2-\phi(\lambda 5^{-m}))^{-1} \in [\phi(\lambda 5^{1-m}), \phi(\lambda 5^{-m})], \quad \lambda \geqq 0. \tag{2.56}$$

(d)

$$E^x(T^m) \leqq 5^{-m}/2 \tag{2.57}$$

$$E^x(\exp(-\lambda T^m)) \geqq \prod_{i=m}^{\infty} \phi(\lambda 5^{-i}) \tag{2.58}$$

$$P^x(T^m \geqq t) \leqq 2^{\frac{1}{2}} \exp\left\{-\frac{\log 2}{2} 5^m t\right\} \quad \textit{for all } t \geqq 0 \tag{2.59}$$

$$P^x(S_m \geqq t) \leqq c_{2.1} \exp\{-c_{2.2} 5^m t\}. \tag{2.60}$$

Proof. (a) If A is a Borel subset of $G^{\mathbb{N}}$, then

$$P^x(Y_m \in A \mid \mathscr{F}_{T^m})(\omega) = P^{X(T^m)(\omega)}(Y_m \in A).$$

(a) follows now by (2.27) and Lemma 2.18.

(b) Let A be as above and $B_1, \ldots, B_n$ be Borel subsets of $[0, \infty)$. Then

$$\begin{aligned} &P^x(Y_m \in A, W_i^m \in B_i \text{ for } i \leqq m \mid \mathscr{F}_{T^m}) \\ &= P^{X(T^m)}(Y_m \in A, W_i^m \in B_i \text{ for } i \leqq m) \\ &= P^{X(T^m)}(Y_m \in A) \prod_{i=1}^{n} P^{X(T^m)}(W_i^m \in B_i) \\ &\qquad \text{(Proposition 2.7 (b), (2.27) and Lemma 2.18)} \\ &= P(Y_m \in A \mid \mathscr{F}_{T^m}) \prod_{i=1}^{n} P(5^{-m} W \in B_i) \\ &\qquad \text{(Proposition 2.7 (c) and Lemma 2.18).} \end{aligned}$$

(c) all but the bounds on $\phi(\lambda 5^{-m})(2-\phi(\lambda 5^{-m}))^{-1}$ are clear from Proposition 2.9 (b) and Lemma 2.18. $\phi(\lambda 5^{-m}) \leqq 1$ implies the upper bound in (2.56) and also

$$(2-\phi(\lambda 5^{-m}))^{-1} \geqq \phi(\lambda 5^{-m})(4-3\phi(\lambda 5^{-m}))^{-1}.$$

Multiply by $\Phi(\lambda 5^{-m})$ and use (2.24) to get the lower bound in (2.56).

(d) Let $\{x_n\}$ be a sequence in G_∞ converging to $x \in G$ such that $\lim_{n\to\infty} X(x_n) = X(x)$ P-a.s. By Corollary 2.17 we therefore have $\lim_{n\to\infty} T^m(X(x_n)) = T^m(X(x))$ P-a.s. Take $x = x_n$ in (2.37) and (2.38), let $n \to \infty$, and use Lemma 2.18 and the above to derive (2.57) and (2.59), respectively.

If $x \in G_n$ and $n \geqq m$, then $T^m = \sum_{i=m+1}^{n} T^{i-1} \circ \theta_{T^i}$. If $\lambda \geqq 0$, then

$$E^x(\exp(-\lambda T^m)) = E^x\left(\exp\left(-\lambda \sum_{i=m+1}^{n} T^{i-1} \circ \theta_{T^i}\right)\right)$$

$$\geqq \prod_{i=m+1}^{n} \phi(\lambda 5^{1-i})$$

(by (2.56) and the strong Markov property)

$$\geqq \prod_{i=m}^{\infty} \phi(\lambda 5^{-i}).$$

We may extend this inequality from x in G_∞ to all x in G as in the previous argument.

For (2.60) note that $S_m \leqq T_1^{m-2}$ and therefore

$$P^x(S_m \geqq t) \leqq P^x(T^{m-2} \geqq t/2) + P^x(W_1^{m-2} \geqq t/2)$$
$$\leqq P^x(T^{m-2} \geqq t/2) + P^y(T^{m-3} \geqq t/2),$$

where $y \in G_{m-2} - G_{m-3}$. (2.60) is now immediate from (2.59). □

Remark 2.20. If A is a bounded Borel subset of G and $m \in \mathbb{Z}$ satisfies $A \subset B_d(0, 2^{-m})$, then $T(A^c, X(x)) \leqq S_m(X(x))$ for all $x \in G$. (2.60) therefore shows that $\{T(A^c, X(x)): x \in G\}$ is uniformly integrable on $(\Omega_0, \mathscr{F}_0, P)$. This observation will prove useful in Sect. 5.

Theorem 2.21. *X is μ-symmetric, i.e.,*

$$\int P_t f(x)\, g(x)\, d\mu(x) = \int f(x)\, P_t g(x)\, d\mu(x) \qquad \text{for any} \quad f, g \in C_K(G).$$

Proof. Let $f, g \in C_K(G)$, and let $\phi_n: G \to G_n$ map x onto the southwest corner of $\Delta_n(x)$. Theorem 2.14 shows that if $t_n = [5^n t] 5^{-n}$, then

$$P_t f(x)\, g(x) = \lim_{n \to \infty} E(f(X_n(\phi_n(x))(t_n)))\, g(\phi_n(x))$$

for $x \in G$, and therefore

$$\int P_t f(x)\, g(x)\, d\mu(x) = \lim_{n \to \infty} \sum_{x \in G_n} E(f(X_n(x)(t_n)))\, g(x)\, 3^{-n}$$
$$= \lim_{n \to \infty} \sum_{x \in G_n} E(f(X(n, x)([5^n t])))\, g(x)\, 3^{-n}$$
$$= \lim_{n \to \infty} \sum_{x \in G_n} f(x)\, E(g\,(X(n, x)([5^n t])))\, 3^{-n}.$$

Here we have used the obvious symmetry of the G_n random walk $X(n, x)$ with respect to counting measure on G_n. Retracing the above steps gives the result. □

Remark 2.22. It is easy to see from its construction as a limit of random walks that X satisfies the following scaling property.

(2.61) $P^x(2X(\cdot/5)\in A)=P^{2x}(X\in A)$ for all Borel subsets of $C([0,\infty), G)$, A, and $x\in G$.

3. Branching Processes and the Distribution of Hitting Times

Let

$$f(z)=\frac{z^2}{4-3z}, \qquad z\in\mathbb{C}, \tag{3.1}$$

and let $Z_0=1, Z_1, \ldots$ be a simple branching process whose offspring distribution has probability generating function f. Then $f(z)=\sum_{r=0}^{\infty} p_r z^r$, where $p_0=p_1=0$, and $p_r=\frac{1}{4}(\frac{3}{4})^{r-2}$ for $r\geqq 2$. As $p_0=0$, the probability of extinction for Z is zero. From (3.1) we have $E(\exp(\lambda Z_1))<\infty$ for $\lambda<\log(\frac{4}{3})$, and $EZ_1^2=37<\infty$.

Standard results from branching process theory (Harris (1948)) show that there exists a random variable W, with $W>0$ a.s., such that $5^{-n}Z_n\to W$ a.s. and in L^2 as $n\to\infty$. Let

$$\phi(z)=Ee^{-zW} \qquad z\in\mathbb{C},\ \operatorname{Re}(z)\geqq 0.$$

Then there exists $\delta>0$ such that $\phi(z)$ is analytic in $D=\{z\colon \operatorname{Re}(z)>-\delta\}$, and ϕ is the unique solution of the Poincaré equation

$$\phi(5z)=f(\phi(z))=\frac{\phi(z)^2}{4-3\phi(z)}, \qquad z\in D \tag{3.2}$$

$$\phi(0)=1, \qquad \phi'(0)=-1.$$

Let

$$G(x)=P(W\leqq x),$$

be the distribution function of W, and let $g=G'$ be the probability density function of W. (This exists by Harris (1963, Theorem I.8.3)).

From Theorem 2.19 one sees that $5^{-m}W$ is equal in law to W_1^m, the "traversal time" of a triangle in $\mathcal{T}_m$ by X. In this section we use (3.2) to derive upper and lower bounds on G and g: these bounds provide the key to many properties of X.

Given the substantial literature on branching processes it is perhaps surprising that more is not known about the distribution of W. However, only a small proportion of the literature deals with the properties of the limiting distribution, and of the papers which do, many examine only the case $p_0+p_1>0$. It is known that if $p_0+p_1=0$ then the limiting distribution has quite different behaviour at 0. For references see Dubuc (1971, 1982), Bingham (1987).

We use z, w to denote elements of $\mathbb{C}$, and u, s, t to denote reals. Write $\mathbb{C}_+=\{z\in\mathbb{C}\colon \operatorname{Re}(z)\geqq 0\}$, and let

$$K=\{1\leqq|z|\leqq 5\}\cap\mathbb{C}_+.$$

From (3.2) we easily obtain (see also Asmussen and Hering (1981, p. 89)).

$$\sup_{z\in K}|\phi(z)|<1 \tag{3.3}$$

Clearly $|\phi(z)|\leqq 1$ for $z\in\mathbb{C}_+$, and it is easily seen from (3.2) that $|\phi(z)|>0$, $z\in\mathbb{C}_+$. Let

$$h(z)=-|z|^{-\gamma}\log|\phi(z)| \qquad z\in\mathbb{C}_+-\{0\}. \tag{3.4}$$

Note that $h\geqq 0$. Substituting in (3.2) we have

$$\begin{aligned} h(5z)&=h(z)+\tfrac{1}{2}|z|^{-\gamma}\log|4-3\phi(z)| \\ &=h(z)+|z|^{-\gamma}\log 2+\tfrac{1}{2}|z|^{-\gamma}\log|1-\tfrac{3}{4}\phi(z)|. \end{aligned} \tag{3.5}$$

Proposition 3.1. (a) *For each* $z\in\mathbb{C}_+-\{0\}$ *the limit*

$$k(z)=\lim_{n\to\infty} h(5^n z) \text{ exists.}$$

(b) *There exist constants* $c_{3.1}-c_{3.3}$ *such that*

$$e^{-c_{3.1}|z|^\gamma}\leqq|\phi(z)|\leqq c_{3.2}\,e^{-c_{3.3}|z|^\gamma}, \qquad z\in\mathbb{C}_+. \tag{3.6}$$

Proof. Since $0<|\phi(z)|\leqq 1$, $1<|4-3\phi(z)|\leqq 7$, so that, by (3.5),

$$h(z)<h(5z)\leqq h(z)+\tfrac{1}{2}|z|^{-\gamma}\log 7. \tag{3.7}$$

Thus $n\to h(5^n z)$ is increasing, $h(5^n z)\leqq h(z)+|z|^{-\gamma}\log 7$ for each n, and (a) follows.

As h is continuous on K, $\sup_{z\in K} h(z)\leqq c<\infty$, and thus $\sup_{z\in\mathbb{C}_+} h(z)\leqq c+2\log 7 =c_{3.1}$, say. This proves the left hand side of (3.6).

By (3.7) $h(w)\geqq\inf_{z\in K} h(z)$ for all $w\in\{|z|\geqq 5\}\cap\mathbb{C}_+$, while $\inf_{z\in K} h(z)>c_{3.3}>0$ by (3.3) so

$$|\phi(z)|\leqq e^{-c_{3.3}|z|^\gamma} \qquad \text{for } z\in\mathbb{C}_+,\ |z|\geqq 1. \tag{3.8}$$

Since $|\phi(z)|\leqq 1$ for $z\in\mathbb{C}_+$ the right hand side of (3.6) follows, with $c_{3.2} =\exp(c_{3.3})$. $\square$

Remark. (3.6) is proved in Dubuc (1971) in the case $\operatorname{Im}(z)=0$.

It is immediate from (3.6) that

$$c_{3.1}\leqq k(z)\leqq c_{3.3} \qquad \text{for all } z\in\mathbb{C}_+-\{0\}.$$

Given this bound it is natural to ask whether $k(z)$ is in fact constant, either on the real axis, or for all $z \in \mathbb{C}_+ - \{0\}$. We now describe some computer calculations, which prove that this is not the case.

We now restrict to the real axis. From (3.2) and (3.5) we have

$$h(5s) = h(s) + \tfrac{1}{2} s^{-\gamma} \log(4 - 3e^{-sh(s)}), \qquad s > 0 \tag{3.9}$$

$$s^{\gamma-1} h(s) \to 1 \qquad \text{as } s \downarrow 0.$$

Set $\psi(y) = \log(4 - 3e^{-y})$: for $0 \leqq y \leqq 1$ we have $3y - 6y^2 \leqq \psi(y) \leqq 3y$. Thus, by (3.9), $h(s) \leqq \frac{5}{2} h(5^{-1}s)$, and hence, for any $n \geqq 1$, $h(s) \leqq (5/2)^n h(5^{-n}s)$.

Let $\varepsilon > 0$: for sufficiently large n we have, by (3.9), $h(5^{-n}s) \leqq (1+\varepsilon)(5^{-n}s)^{1-\gamma}$, and putting these two estimates together we obtain

$$h(s) \leqq s^{1-\gamma} \qquad \text{for } s \geqq 0. \tag{3.10}$$

Similarly, $h(s) \geqq \frac{5}{2} h(5^{-1}s) - \frac{3}{2} s^{\gamma} h(5^{-1}s)^2$. By (3.10) therefore

$$h(s) \geqq \tfrac{5}{2} h(5^{-1}s) - 3(5^{-1}s)^{2-\gamma},$$

and so, for any $n \geqq 1$,

$$h(s) \geqq (\tfrac{5}{2})^n h(5^{-n}s) - 3s^{2-\gamma} \sum_{r=1}^{n-1} 5^{(\gamma-2)r}.$$

It follows that

$$h(s) \geqq s^{1-\gamma} - \tfrac{3}{10} s^{2-\gamma}, \qquad s > 0. \tag{3.11}$$

If $s_0 > 0$, $s_n = 5^n s_0$, $h(s_0) \geqq u_0$, and a sequence $(u_n, n \geqq 0)$ is defined by $u_{n+1} = u_n + \frac{1}{2} s_n^{-\gamma} \psi(s_n^{\gamma} u_n)$ then, by (3.9), $h(s_n) \geqq u_n$ for all n. Similarly, if $u_0 \geqq h(s_0)$, then $u_n \geqq h(s_n)$ for all n.

Combined with (3.10) and (3.11) this gives a method of computing bounds for $h(s)$, for $s > 0$, to any desired degree of accuracy. One first chooses s_0 (we took $s_0 \simeq 10^{-13}$), and uses (3.10) and (3.11) to derive bounds on $h(s_0)$. Iterating, one obtains upper and lower bounds on $h(s_n)$. For large y, $-e^{-y} \leqq \psi(y) - \log 4 \leqq 0$, and it is easy to derive upper and lower bounds for k from those for $h(5^n s)$.

Using a FORTRAN compiler with extended precision arithmetic, we calculated $k(r \times 10^{-14})$ for $r = 10, 11, \ldots, 50$ correct to 13 decimal places. In particular, we found

$$k(12 \times 10^{-14}) \simeq 1.959108167$$

$$k(27 \times 10^{-14}) \simeq 1.959102196.$$

Dubuc (1982) gives results which explain similar tiny oscillations found in the case $p_0 + p_1 > 0$.

Remark. Let $k_0 = \inf_{1 \leqq s \leqq 5} k(s)$, $k_1 = \sup_{1 \leqq s \leqq 5} k(s)$. The calculations above prove that

$$k_0 < 1.959103 < 1.959108 < k_1.$$

However, we have no rigorous lower bound on k_0, or upper bound on k_1. It appears from the table of values of $k(r \times 10^{-14})$ for $10 \leqq r \leqq 50$ that $k(s)$ is fairly smooth, and we suspect that

$$1.959102 < k_0, \qquad k_1 < 1.959109$$

But, without some theoretical bound on either the total oscillation of k (as in Dubuc (1982)), or on $|k'|$, these last bounds remain a guess.

We now use (3.6) to obtain bounds on G and g. The asymptotics of $G(x)$, as $x \downarrow 0$, are studied in Bingham (1987), where they are derived from (3.6) by a Tauberian theorem of exponential type for oscillating functions.

Proposition 3.2 (Theorem 1, Bingham (1987)). *There exist constants* $c_{3.4}$, $c_{3.5}$ *such that*

$$-c_{3.4} < \liminf_{x \downarrow 0} x^{\frac{\gamma}{1-\gamma}} \log G(x) \leq \limsup_{x \downarrow 0} x^{\frac{\gamma}{1-\gamma}} G(x) < -c_{3.5}. \tag{3.12}$$

For details of the constants see Bingham (1987): if $c_{3.1}$ and $c_{3.3}$ differ by ε, then $c_{3.4}$ and $c_{3.5}$ differ by $O(\varepsilon^{\frac{1}{2}})$.

Corollary 3.3.

$$c_{3.6}\, e^{-c_{3.4} x^{\frac{-\gamma}{1-\gamma}}} \leqq G(x) \leqq c_{3.7}\, e^{-c_{3.5} x^{\frac{-\gamma}{1-\gamma}}} \quad \text{for } 0 \leqq x < \infty. \tag{3.13}$$

Proof. This is immediate from (3.12). □

More work is needed to bound g. While the Laplace transform of G was sufficient to determine G, we will now need to use the Fourier transform.

Now write $a^\gamma = c_{3.3}$.

Lemma 3.4. *G has a density g, and g is C^∞. For each $k \geqq 0$*

$$\|g^{(k)}\|_\infty \leqq c_{3.8}\, a^{-(k+1)} \Gamma\left(\frac{k+1}{\gamma}\right). \tag{3.14}$$

Proof. From (3.8) we have

$$|\phi(it)| \leqq c_{3.2}\, e^{-a^\gamma t^\gamma}, \qquad t \in \mathbb{R}.$$

Hence $t^k |\phi(it)| \in L^1(\mathbb{R}, dt)$ for each $k \geqq 0$, and so g exists and is C^∞. By the Fourier inversion theorem

$$\|g^{(k)}\|_\infty \leqq \frac{1}{2\pi} \int_0^\infty |t|^k |\phi(it)|\, dt$$

$$\leqq c \int_0^\infty t^k e^{-(at)^\gamma}\, dt = c_{3.8}\, a^{-(k+1)} \Gamma\left(\frac{k+1}{\gamma}\right). \quad \square$$

Since $W \geqq 0$ a.s., $g(x) \equiv 0$ for $x<0$, and so $g^{(k)}(0)=0$ for $k \geqq 0$. Thus, writing $b_m = \|g^{(m)}\|_\infty$, we have, integrating k times

$$(3.15) \qquad |g^{(n)}(x)| \leqq b_{n+k} x^k/k! \quad \text{for } k \geqq 0,\ n \geqq 0,\ x \geqq 0.$$

Theorem 3.5. *For each $n \geqq 0$ there exist constants $c_{3.9}(n)$ and $c_{3.10}$ such that*

$$(3.16) \qquad |g^{(n)}(x)| \leqq c_{3.9}(n) \exp(-c_{3.10}\, x^{\overline{d_w-1}^{-1}}), \qquad x \geqq 0.$$

Proof. Let $n \geqq 0$ be fixed. It is enough to prove (3.16) for $0 \leqq x \leqq c(n)$, as adjusting the constant $c_{3.9}(n)$, and using (3.14) will then give (3.16).

Let $k \geqq 0$, and set $r=(k+1)/\gamma$, $s=n/\gamma$. From (3.15) and (3.14) we have

$$|g^{(n)}(x)| \leqq c_{3.8}\, a^{-s\gamma} x^{-1} (x/a)^{r\gamma} \Gamma(r+s)/\Gamma(r\gamma), \qquad x \geqq 0.$$

By Stirling's formula, as $r \to \infty$,

$$\frac{\Gamma(r+s)}{\Gamma(r\gamma)} \sim \left(\frac{\gamma r}{r+s}\right)^{\frac{1}{2}} e^{-r(1-\gamma)} \gamma^{-\gamma r} e^{-s} (r+s)^s \left(1+\frac{s}{r}\right)^r r^{r(1-\gamma)}.$$

Since $e^{-(1-\gamma)}\gamma^{-\gamma} < 0.82 < 1$, it follows that, for $r \geqq r_0(s)$,

$$\frac{\Gamma(r+s)}{\Gamma(r\gamma)} \leqq \exp(r(1-\gamma)\log r).$$

Thus

$$|g^{(n)}(x)| \leqq c(n)\, x^{-1} \exp(r(1-\gamma)\log r - r\gamma \log(a/x)).$$

Set $v(r) = r\log r - r\log b$, where $b=(a/x)^{\gamma/(1-\gamma)}$. Then the minimum of v is at $r_1 = be^{-1}$, and $v(r_1) = -be^{-1}$. Let x be fixed, and be small enough so that $r_1 \geqq r_0(s)$. Setting $r_2 = [r_1]$, and writing $\delta = r_1 - r_2$ we have

$$|v(r_2) - v(r_1)| \leqq \delta \sup_{0 \leqq y \leqq \delta} |v'(r_1 - y)| \leqq \left|\log\left(1 - \frac{e\delta}{b}\right)\right|.$$

So, provided x is sufficiently small, $v(r_2) \leqq -b/3$, and therefore

$$|g^{(n)}(x)| \leqq c(n)\, x^{-1} \exp(-\tfrac{1}{3}(1-\gamma)\, a^{\gamma/(1-\gamma)} x^{-\gamma/(1-\gamma)}) \qquad \text{for } 0 \leqq x \leqq c(n).$$

Absorb x^{-1} into the exponential and note $\gamma/(1-\gamma) = 1/d_w - 1)$, to prove (3.16). □

We now derive lower bounds for g. Here we use directly the probabilistic content of (3.2):

$$(3.17) \qquad W \overset{(d)}{=} \tfrac{1}{5} \sum_{i=1}^{N} W_i, \quad \text{where } W_1, \ldots \text{ are iid copies of } W,$$

N is independent of W, and $P(N=r) = \frac{1}{4}(\frac{3}{4})^{r-2}$ for $r \geqq 2$.

From (3.17) we obtain, writing $g^{*(r)}$ for the r-fold convolution of g with itself,

$$\frac{1}{5}g(u/5)=\sum_{r=2}^{\infty}\frac{1}{4}\left(\frac{3}{4}\right)^{r-2}g^{*(r)}(u), \tag{3.18}$$

and

$$g(u/5)\geqq g*g(u). \tag{3.19}$$

Lemma 3.6. $g(u)>0$ *for all* $u>0$.

Proof. Suppose that $g(u_0)=0$. Then, by (3.19) $g^{*(r)}(5u_0)=0$ for each $r\geqq 2$. Now

$$g^{*(r)}(5u_0)=\int\dots\int dt_1\dots dt_r\, g(5u_0-t_r)\, g(t_r-t_{r-1})\dots g(t_2-t_1),$$

and so, looking at the integral in a neighbourhood of $t_i=5u_0(i-1)/r$, $1\leqq i\leqq r$, and using the continuity of g, we have $g(5u_0/r)=0$. Taking $r=10$ or 2 and iterating it follows that $g(5^n 2^{-n-m}u_0)=0$ for each n, $m\geqq 1$. Thus $g\equiv 0$, and as $\int g=1$ this is a contradiction. □

Theorem 3.7. *For each* $N>0$ *there exist constants* $c_{3.11}(N)$, $c_{3.12}$ *such that*

$$g(x)\geqq c_{3.11}(N)\exp(-c_{3.12}x^{-\frac{1}{d_w-1}}), \qquad 0\leqq x\leqq N. \tag{3.20}$$

Proof. As g is continuous, and strictly positive in $(0,\infty)$, it is bounded away from 0 in $[\delta, N]$ for each $\delta>0$: thus, by adjusting the constants $c_{3.11}(N)$ it is enough to prove (3.20) for $x\in[0,1]$.

Let $\tilde{g}(x)=\inf_{1\wedge x\leqq y\leqq 1} g(y)$. The previous lemma shows $\tilde{g}(x)>0$ if $x>0$. (3.13) implies that $G(u)\geqq e^{-c_1u^{-\beta}}$ for all $u\in(0,1]$ and some $c_1>0$, where $\beta=\gamma(1-\gamma)^{-1}$. By enlarging c_1 we may also assume (use Lemma 3.6 again)

$$g(1)e^{-c_1}\leqq\inf\{g(v): \tfrac{1}{5}\leqq v\leqq 1\}. \tag{3.21}$$

If $x\in(0,\frac{1}{5})$, then (3.19) implies

$$\begin{aligned} g(x) &\geqq \int_0^{5x} g(5x-v)\,g(v)\,dv \\ &\geqq \int_0^{3x} g(5x-v)\,g(v)\,dv \\ &\geqq \tilde{g}(2x)\,G(3x), \end{aligned}$$

$$g(x)\geqq\tilde{g}(2x)\exp\{-c_1x^{-\beta}\}. \tag{3.22}$$

(3.21) shows that (3.22) also holds for $x\in[\frac{1}{5},1]$. Therefore we have

$$\tilde{g}(x)\geqq\tilde{g}(2x)\exp\{-c_1x^{-\beta}\} \qquad \text{for all } x\in(0,1]. \tag{3.23}$$

If $x \in (0, 1]$ choose $n \in \mathbb{Z}_+$ such that $2^n x \in (\frac{1}{2}, 1]$. Iterate (3.23) to get

$$\begin{aligned}\tilde{g}(x) &\geqq \tilde{g}(\tfrac{1}{2}) \exp\left\{-c_1 \sum_{r=0}^{n-1} (2^r x)^{-\beta}\right\} \\ &\geqq \tilde{g}(\tfrac{1}{2}) \exp\{-c_1 (1-2^{-\beta})^{-1} x^{-\beta}\},\end{aligned}$$

as required. □

4. Sample Path Properties

A proof of the following simple Tauberian theorem may be found in Fristedt-Pruitt (1971, Lemma 1).

Lemma 4.1. *If Y is a non-negative random variable and $\psi(\lambda) = E(e^{-\lambda Y})$ then*

$$(\psi(\lambda) - e^{-\lambda t})(1 - e^{-\lambda t})^{-1} \leqq P(Y \leqq t)$$

for all $\lambda, t > 0$. □

Using this and the estimates on ϕ from the previous section, we can easily convert the estimates on the Laplace transform of the break-out times into estimates on their distribution functions. Recall the notation $T^m(\omega)$ from Sect. 2 (preceding Lemma 2.16).

Lemma 4.2. *There are constants $c_{4.1}$ and $c_{4.2}$ such that*

$$P^x(T^m \leqq t) \geqq c_{4.1} \exp\{-c_{4.2} (t 5^m)^{-\gamma/(\gamma-1)}\} \quad \text{for all } t > 0,\ x \in G, \text{ and } m \in \mathbb{Z}. \tag{4.1}$$

Proof. If $t > 0$, $x \in G$ and $m \in \mathbb{Z}$ are fixed, then (2.58) and (3.6) imply

$$E^x(e^{-\lambda T^m}) \geqq \exp\{-c_{3.1} \lambda^{\gamma} 2^{1-m}\}. \tag{4.2}$$

The desired estimate is now a consequence of Lemma 4.1 with $\lambda = (c_{3.1} t^{-1} 2^{2-n})^{1/1-\gamma}$. □

Theorem 4.3. *There are constants $c_{4.3}$, $c_{4.4}$, $c_{4.5}$ and $c_{4.6}$ such that*

$$\begin{aligned} c_{4.5} \exp\{-c_{4.6} (\delta t^{-\gamma})^{1/1-\gamma}\} &\leqq P^x |X_t - X_0| \geqq \delta) \\ \leqq P^x(\sup_{s \leqq t} |X_s - X_0| \geqq \delta) &\leqq c_{4.3} \exp(-c_{4.4} (\delta t^{-\gamma})^{1/1-\gamma}\} \\ \text{for all } x \in G \quad \text{and} &\quad t,\ \delta \in (0, \infty).\end{aligned} \tag{4.3}$$

Proof. Let $x \in G$ and $t, \delta \in (0, \infty)$. Choose $n \in \mathbb{Z}$ such that $2^{-n+1} \leqq \delta < 2^{-n+2}$.

$$\begin{aligned} P^x(\sup_{s \leqq t} |X_s - X_0| \geqq \delta) &\leqq P^x(T_1^n \leqq t) \\ &\leqq P^x(W 5^{-n} \leqq t) \\ &\leqq c_{4.3} \exp\{-c_{4.4} (\delta t^{-\gamma})^{1/1-\gamma}\},\end{aligned}$$

by (3.13) and the choice of n.

For the lower bound choose $n\in\mathbb{Z}$ such that $2^{-n-2}<\delta\leqq 2^{-n-1}$.

$$\begin{aligned}
&P^x(|X_t-X_0|\geqq\delta)\\
&\quad\geqq E^x(1(T_1^n\leqq t,|X(T_1^n)-x|\geqq 2^{-n})\,P^{X(T_1^n)(\omega)}\sup_{s\leqq t-T_1^n(\omega)}|X_s-X_0|\leqq\delta))\\
&\quad\geqq E^x(1(T^n\leqq t/2)\,P^{X(T^n)}(W_1^n\leqq t/2,|X(W_1^n)-x|\geqq 2^{-n}))\\
&\qquad\times(1-c_{4.3}\exp\{-c_{4.4}(\delta t^{-\gamma})^{1/1-\gamma}\})\quad\text{(by the above).}
\end{aligned}$$

Choose K large enough so that the last factor exceeds $\frac{1}{2}$ if $\delta t^{-\gamma}\geqq K$. The independence of W_1^n and $X(W_1^n)$ (Theorem 2.19(b)) shows that if $\delta t^{-\gamma}\geqq K$, the above expression is bounded below by

$$\begin{aligned}
&4^{-1}P^x(T^n\leqq t/2)\,P(W\,5^{-n}\leqq t/2)\\
&\quad\geqq 4^{-1}c_{4.1}\exp\{-c_{4.2}(t5^n/2)^{-\gamma/\gamma-1}\}\,c_{3.6}\exp\{-c_{3.4}(t5^n/2)^{-\gamma/(\gamma-1)}\},
\end{aligned}$$

by (3.13) and (4.1). The required estimate now follows for $\delta t^{-\gamma}\geqq K$ by the choice of n, and hence for $\delta t^{-\gamma}<K$ as well by simply adjusting $c_{4.5}$. □

As an immediate corollary we obtain the sub-diffusive behaviour of X, mentioned in the Introduction.

Corollary 4.4. *There are constants* $\{c_{4.7}(p),\,c_{4.8}(p):p>0\}$ *such that*

$$c_{4.7}(p)\,t^{p\gamma}\leqq E^x(|X_t-X_0|^p)\leqq c_{4.8}(p)\,t^{p\gamma},\quad \textit{for all } t,p>0 \quad\textit{and}\quad x\in G.\quad\square$$

The scaling properties of X suggest that it has positive and finite d_w-variation (recall $d_w=\gamma^{-1}=\log 5/\log 2$). We first prove this for d_w-variation interpreted in the Lebesgue sense.

Notation. $N_n(t)=\max\{i\in\mathbb{Z}_+: T_i^n\leqq t\}$ $(\max\emptyset=0)$.

Theorem 4.5. $5^{-n}N_n(t)$ *converges uniformly in* $t\in[0,T]$ *to* t *as* $n\to\infty$ P^x-a.s., *for all* $T>0$ *and* $x\in G$.

Proof. By conditioning on $\mathscr{F}_{T^n}$ and using (2.57) to see that $T^n\downarrow 0$ a.s., we need only consider P^x where $x\in G_{n_0}$ for some n_0. Fix $t,\varepsilon>0$. If $n\geqq n_0$, then

$$\begin{aligned}
P^x(|5^{-n}N_n(t)-t|>\varepsilon)&\leqq P^x\left(\sum_{i=1}^{[5^n(t+\varepsilon)]}W_i^n<t\right)+P^x\left(\sum_{i=1}^{[5^n(t-\varepsilon)]+1}W_i^n>t\right)\\
&\leqq P^x\left(\sum_{i=1}^{[5^n(t+\varepsilon)]}W_i^n-E(W_i^n)<-\varepsilon+5^{-n}\right)\\
&\quad+P^x\left(\sum_{i=1}^{[5^n(t-\varepsilon)]}W_i^n-E(W_i^n)>\varepsilon-5^{-n}\right)\\
&\leqq(\varepsilon-5^{-n})^{-2}\,5^n(t+\varepsilon)\,5^{-2n}\operatorname{Var}(W)\\
&\quad+(\varepsilon-5^{-n})^{-2}(5^nt+1)\,5^{-2n}\operatorname{Var}(W).
\end{aligned}$$

The Borel-Cantelli lemma shows $\lim_{n\to\infty} 5^{-n} N_n(t) = t\ P^x$-a.s. A standard argument using the monotonicity of $N_n(\cdot)$ shows the convergence is uniform on compacts a.s. □

Remark 4.6. 1. Theorem 4.6 states that

$$\sum_{i=1}^{\infty} 1(T_i^n \leqq t)\,|X(T_i^n) - X(T_{i-1}^n)|^{d_w} \to t$$

uniformly on compacts as $n \to \infty\ P^x$-a.s.

Thus in describing the d_w-variation of X we have followed Chacon et al. (1981) and partitioned space rather than time.

If $t_i^n = i5^{-n}$, then w.p.1 for any $t > 0$

$$(4.4)\qquad c_{4.9}\, t \leqq \liminf_{n\to\infty} \sum_{i=1}^{[t5^n]} |X(t_i^n) - X(t_{i-1}^n)|^{d_w} \leqq \limsup_{n\to\infty} \sum_{i=1}^{[t5^n]} |X(t_i^n) - X(t_{i-1}^n)|^{d_w} \leqq c_{4.10}\, t.$$

To prove this, first use standard martingale arguments to show that, if

$$M_n(j) = \sum_{i>1}^{j} |X(t_i^n) - X(t_{i-1}^n)|^{d_w} - E^{X(t_{i-1}^n)}(|X(5^{-n}) - X(0)|^{d_w}),$$

then

$$\sup_{j \leqq [5^n t]} |M_n(j)| \to 0 \text{ a.s. and in } L^2 \quad \text{for any } t > 0.$$

(4.4) therefore follows from Corollary 4.4 with $p = d_w$. We conjecture that (4.4) may be strengthened to

$$(4.5)\qquad \lim_{n\to\infty} \sum_{i=1}^{[5^n t]} |X(t_i^n) - X(t_{i-1}^n)|^{d_w} = t \quad \text{for all } t \geqq 0\ P^x\text{-a.s. for all } x \in G.$$

2. It follows immediately from (4.4) that X has infinite quadratic variation and is therefore not a semimartingale.

Notation. $\psi_0(t) = t^\gamma (\log\log 1/t)^{1-\gamma},\ t \in (0, e^{-1})$

$$\psi_1(t) = t^\gamma (\log 1/t)^{1-\gamma}, \qquad t \in (0, 1).$$

We close this section by showing that ψ_0 and ψ_1 give the local and global moduli of continuity for X, respectively.

Theorem 4.7. *There are constants* $\{c_{4.11}(x)\colon x\in G\}$ *such that for all* $x\in G$,

(4.6) $$\limsup_{t\downarrow 0} |X_t - X_o|\,\psi_0(t)^{-1} = c_{4.11}(x) \qquad P^x\text{-a.s.},$$

where $c_{4.6}^{\gamma^{-1}} \leqq c_{4.11}(x) \leqq c_{4.4}^{\gamma^{-1}}$.

Proof. The usual Borel-Cantelli argument shows that (4.3) implies the lim sup in (4.6) is bounded above by $c_{4.4}^{\gamma^{-1}}$. In view of the Blumenthal zero-one law, it suffices to show that for each $x\in G$,

(4.7) $$\limsup_{t\downarrow 0} |X_t - X_0|\,\psi_0(t)^{-1} \geqq c_{4.6}^{\gamma^{-1}} \qquad P^x\text{-a.s.}$$

Let $\theta\in(0,1)$, $c=c(\theta)=c_{4.6}^{\gamma^{-1}}(1-\theta)^\gamma$, and

$$A_n = \{|X(\theta^n) - X(\theta^{n+1})| \geqq c\,\psi(\theta^n)) \in \mathscr{F}_{\theta^n}.$$

The Markov property and (4.3) imply

$$P^x(A_n \mid \mathscr{F}_{\theta^{n+1}}) \geqq c_{4.5}(n\log\theta^{-1})^{-1}.$$

The obvious conditioning argument now shows $P^x(A_n \text{ i.o.})=1$ and hence, for P^x-a.a. ω

$$\begin{aligned}
&\limsup_{n\to\infty} |X(\theta^n) - X(0)|\,\psi_0(\theta^n)^{-1}\\
&\quad \geqq \limsup_{n\to\infty} \{|X(\theta^n) - X(\theta^{n+1})|\,\psi_0(\theta^n)^{-1}\\
&\qquad - |X(\theta^{n+1}) - X(0)|\,\psi_0(\theta^n)^{-1}\}\\
&\quad \geqq c(\theta) - c_{4.4}^{\gamma^{-1}}\,\theta^\gamma,
\end{aligned}$$

by the above and the upper bound result. Let $\theta\downarrow 0$ to get (4.7). □

The local spatial homogeneity of X and G shows that $c_{4.11}(\cdot)$ is constant on each G_n and hence on G_∞. We conjecture that it is in fact constant on G.

Lévy's proof of the exact modulus of continuity of Brownian motion (see, for example Itô and McKean (1965, p. 36)) may be used together with (4.3) to show that ψ_1 gives the global modulus of continuity for X. More precisely Corollary 1.7 holds with $c_{1.12}=c_{4.6}^{\gamma^{-1}}$ and $c_{1.13}=c_{4.4}^{\gamma^{-1}}$.

5. Green's Functions and Resolvent Densities

In this section we write $T_i^n = T_i^n(X)$ for the stopping times introduced in Sect. 1 where $X_0 = x\in G_\infty$. For each n, $(X(T_i^n))_{i=0}^\infty$ is a simple random walk on $G^{(n)}$. Let

$$X^{(n)}(t) = X_t^{(n)} = X(T_{[5^n t]}^n).$$

By Theorem 2.8, $X^{(n)} \to X$ uniformly on compact subsets of $[0, \infty)$. For $x \in G_n$ we define the (normalized) occupation times for $X^{(n)}$ by

(5.1) $$K_t^n(x) = (\tfrac{3}{2})\, 3^n \int_0^t 1_{\{x\}}(X_s^{(n)})\, ds$$

Let A satisfy

(5.2) $A \subset G$, A is open in G, A is connected and bounded, and $\partial A \subset G_m$ for some m.

For A satisfying (5.2) let

$$R^n(A) = T(A^c, X^{(n)}) = \inf\{t \geqq 0 : X_t^{(n)} \in A^c\},$$
$$R(A) = T(A^c, X).$$

Now let

$$g_A^n(x, y) = E^x K_{R^n(A)}^n(y) \qquad \text{for } x, y \in G_n:$$

g_A^n is the (normalized) Green's function for $X^{(n)}$ in the region A. Let $\delta(x, y) = 1_{\{x\}}(y)$.

Lemma 5.1. *Let $m \leqq n$, $x \in G_m$, and suppose that A satisfies* (5.2). *Then, if $y \in G_m$*

(5.3) $$g_A^n(x, y) \leqq (\tfrac{3}{2})(\tfrac{3}{5})^m \delta(x, y) + \sum_{z \in N_m(x)} \tfrac{1}{4} g_A^n(z, y),$$

with equality if $\operatorname{int}(B_d(x, 2^{-m})) \subset A$.

Proof. Let $U_1 = T_1^m(X^{(n)})$, and $U_2 = \inf\{t \geqq U_1 : X_t^{(n)} \in A_c\}$. Then $R^n(A) \leqq U_2$, with equality if $\operatorname{int}(B_d(x, 2^{-m})) \subset A$. Thus

$$\begin{aligned} g_A^n(x, y) &= E^x K_{R^n(A)}^n(y) \leqq E^x K_{U_2}^n(y) \\ &= E^x K_{U_1}^n(y) + E^x(K_{U_2}^n(y) - K_{U_1}^n(y)). \end{aligned}$$

If $y \neq x$ the first term is zero, while if $y = x$ it equals $(\tfrac{3}{2})(\tfrac{3}{5})^m$ by Lemma 2.2(b). $X^{(n)}$ satisfies the strong Markov property at the time U_1, and thus the second term is $E^x g_A^n(X_{U_1}^{(n)}, y)$. Since $P(X_{U_1}^{(n)} = z) = \tfrac{1}{4}$ *for each* $z \in N_m(x)$, this proves (5.3). □

Proposition 5.2. *Let A satisfy* (5.2), *with $\partial A \subset G_m$. Let $n \geqq m$.*

(a) (i) $g_A^n(x, y) = g_A^n(y, x)$ *for all x, y in G_n*

(ii) $g_A^n(x, y) = 0$ *if $y \in A^c \cap G_n$, $x \in G_n$.*

(iii) $g_A^n(x, y) = (\tfrac{3}{2})(\tfrac{3}{5})^n \delta(x, y) + \sum_{u \in N_n(x)} \tfrac{1}{4} g_A^n(u, y)$, $x, y \in G_n \cap A$.

(b) *Let $m \leqq r \leqq n$. Then, for all $x, y \in G_r$, $g_A^r(x, y) = g_A^n(x, y)$.*

(c) $g_A^n(x, y) = P^x(T(y, X^{(n)}) < R^n(A))\, g_A^n(y, y)$, $x, y \in G_n$.

Proof. (a) (i) follows from the μ_n-symmetry of $X^{(n)}$. Let $h_k(x, y)$ be the number of paths in $G^{(n)} \cap A$ of length k from x to y. Then $h_k(x, y) = h_k(y, x)$, and so

$$P^x(X^{(n)}(k 5^{-n}) = y, \quad R^n(A) > k 5^{-n}) = 4^{-k} h_k(x, y)$$

is symmetric in x and y. As

$$g_A^n(x, y) = \sum_{k=0}^{\infty} P^x(X^{(n)}(k5^{-n}) = y, R^n(A) > k5^{-n})(\tfrac{3}{2})(\tfrac{3}{5})^n,$$

$g_A^n(x, y)$ is also symmetric in x and y. (ii) is evident, while (iii) comes from Lemma 5.1.

For (b) note that, by (a), both $g_A^n(x, y)$ and $g_A^r(x, y)$ are zero unless $x, y \in G_r \cap A$. Let $y \in G_r \cap A$ be fixed, and set $v(x) = g_A^n(x, y) - g_A^r(x, y)$. By (a) $v(x)$ is harmonic on $G_r \cap A$ for the random walk $X(T_i^r)$, $i = 0, 1, \ldots$, and $v(x) = 0$ if $x \in G_r \cap A^c$. Since A is bounded, $v(x) \equiv 0$.

(c) is immediate from the strong Markov property of $X(T^n)$. □

Let A satisfy (5.2). By Proposition 5.2, we can define, for $x, y \in G_\infty$,

$$g_A(x, y) = \lim_{n \to \infty} g_A^{(n)}(x, y), \tag{5.4}$$

where the limit in (5.4) is constant for all $n \geq n_0 = \min\{r: \partial A \cup \{x, y\} \subset G_r\}$. It is clear that g_A inherits the properties of the g_A^n.

Now write

$$p_A(x, y) = P^x(T_y < R(A)),$$

$$q_A(x, y) = 1 - p_A(x, y).$$

Note that p_A and q_A are not symmetric in x and y and that $p_A(x, x) = 1_A(x)$ by the definition of T_x. By the construction of $X^{(n)}$, if $x, y \in G_m$, $\partial A \subset G_m$, then $\{T(y, X^{(n)}) < R^n(A)\} = \{T_y < R(A)\}$ for all $n \geq m$. Thus, from Proposition 5.2(c) we deduce:

Lemma 5.3. *For all* $x, y \in G_\infty$

$$g_A(x, y) = p_A(x, y)\, g_A(y, y). \tag{5.5}$$

Lemma 5.4. *Let* $x \in G_\infty$, *and* A *satisfy* (5.2). *Then*

$$g_A(x, x) \leq (\tfrac{9}{2})(\tfrac{3}{5})^{m+1} + \max_{z \in \partial \Delta_m(x)} g_A(z, z). \tag{5.6}$$

Proof. The inequality is evident if $x \in G_m$. So suppose that $x \in G_n$, where $n > m$. Let $v_k = \max_{z \in \partial \Delta_k(x)} g_A(z, z)$. If $x \in G_{n-1}$, then $g_A(x, x) \leq v_{n-1}$ trivially. Otherwise let z_1, z_2, z_3 denote the vertices of $\Delta_{n-1}(x)$, and y_1, y_2 be the other two points in $\Delta_{n-1}(x) \cap G_n$. By (5.3) we have, since $g_A(z_i, x) \leq g_A(z_i, z_i) \leq v_{n-1}$,

$$g_A(x, x) \leq (\tfrac{3}{2})(\tfrac{3}{5})^n + \tfrac{1}{4}(2v_{n-1} + g_A(y_1, x) + g_A(y_2, x))$$

$$g_A(y_1, x) \leq \tfrac{1}{4}(2v_{n-1} + g_A(y_2, x) + g_A(x, x)),$$

with a similar inequality for y_2. Adding together the inequalities for $g_A(y_1, x)$ and $g_A(y_2, x)$, and substituting in the first inequality we deduce that

$$g_A(x, x) \leqq \tfrac{9}{5}(\tfrac{3}{5})^n + v_{n-1}.$$

Now let x_k be an element of $\partial \Delta_k(x)$ with $g_A(x_k, x_k) = v_k$. Repeating the above calculation for x_k, we have

$$g_A(x_k, x_k) \leqq (\tfrac{9}{5})(\tfrac{3}{5})^k + g_A(x_{k-1}, x_{k-1}).$$

Thus

$$\begin{aligned} g_A(x, x) &\leqq \sum_{k=m+1}^{n} (\tfrac{9}{5})(\tfrac{3}{5})^k + v_m \\ &\leqq (\tfrac{9}{2})(\tfrac{3}{5})^{m+1} + v_m, \end{aligned}$$

as required. □

Lemma 5.5. *Let $x \in G_\infty$, $A = \operatorname{int}(D_m(x))$, and let z_i, $1 \leqq i \leqq 4$ be the four points in $A \cap G_m$, labelled as in Fig. 3. Then*

$$g_A(z.\,, z.) = (\tfrac{3}{2})(\tfrac{3}{5})^m \begin{pmatrix} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{7}{5} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{5} & \frac{7}{5} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{5}{4} \end{pmatrix}. \tag{5.7}$$

Proof. We have, from Proposition 5.2,

$$g_A(z_1, z_1) = (\tfrac{3}{2})(\tfrac{3}{5})^m + \tfrac{1}{4}(g_A(z_2, z_1) + g_A(z_3, z_1)),$$

and 15 similar equations. Solving these (which is not hard, as there is a great deal of symmetry) we obtain (5.7). □

Corollary 5.6. *Let A satisfy* (5.2), *and $x \in G_\infty$ with* $\operatorname{int}(D_m(x)) \subset A$. *Then*

$$g_A(x, x) \geqq (\tfrac{3}{4})(\tfrac{3}{5})^m.$$

Proof. Set $B = \operatorname{int}(D_m(x))$: as $g_A \geqq g_B$ it is sufficient to prove the result for g_B. Let z_i, $1 \leqq i \leqq 4$ be as in Lemma 5.5, and $p_i = P^x(X_{T^m} = z_i)$ for $1 \leqq i \leqq 3$. Then

$$g_B(x, z_1) = \sum_{i=1}^{3} p_i\, g(z_i, z_1) \geqq \min_{1 \leqq i \leqq 3} g_B(z_i, z_1) = (\tfrac{3}{4})(\tfrac{3}{5})^m,$$

by Lemma 5.5, and thus $g_B(x, x) \geqq g_B(x, z_1) \geqq (\tfrac{3}{4})(\tfrac{3}{5})^m$. □

Now let $x \in G_m$, for some $m \in \mathbb{Z}$. By (5.3) we have

$$\sum_{z \in N_n(x)} (g_A(x, x) - g_A(z, x)) \leqq 6(\tfrac{3}{5})^m.$$

As $g_A(x,x)-g_A(z,x)\geqq 0$, we deduce that

$$(5.8)\qquad 0\leqq g_A(x,x)-g_A(z,x)\leqq 6(\tfrac{3}{5})^m \quad \text{for all } x\in G_m,\ z\in N_m(x),$$

and since $g_A(z,x)=g_A(x,z)\leqq g_A(z,z)$, we have

$$(5.9)\qquad g_A(x,x)\leqq 6(\tfrac{3}{5})^m+g_A(z,z) \quad \text{for all } x\in G_m,\ z\in N_m(x).$$

Lemma 5.7. *Let* $\Delta_m\in\mathscr{T}_m$, *and* A *satisfy* (5.2).

(a) (i) $g_A(y,y)-g_A(z,z)\leqq 9(\tfrac{3}{5})^m$ *for all* $y\in\Delta_m$, $z\in\partial\Delta_m$.

(ii) $g_A(y,z)-g_A(z,z)\leqq 0$ *for all* $y\in\Delta_m$, $z\in\partial\Delta_m$.

(b) *If* $\operatorname{int}(\Delta_m)\subset A$ *then*

$$-6(\tfrac{3}{5})^m\leqq g_A(y,z)-g_A(z,z)\leqq g_A(y,y)-g_A(z,z), \quad \textit{for } y\in\Delta_m,\ z\in\partial\Delta_m.$$

Proof. (a) (i) is immediate from Lemma 5.4 and (5.9), while (ii) is evident by (5.5).

(b) Since $g_A(\cdot,z)$ is harmonic in Δ_m, it attains its bounds on $\partial\Delta_m$. Thus $g_A(y,y)\geqq g_A(y,z)\geqq \min_{w\in\partial\Delta_m} g_A(w,z)\geqq g_A(z,z)-6(\tfrac{3}{5})^m$, by (5.8). □

Lemma 5.8. *Let* $\Delta_m\in\mathscr{T}_m$, *and* A *satisfy* (5.2). *Suppose there exists* $y\in\Delta_m\cap A^c\cap G_\infty$. *Then*

$$(5.10)\qquad g_A(z,z)\leqq 15(\tfrac{3}{5})^m \quad \textit{for } z\in\partial\Delta_m.$$

Proof. If $y\in G_m$, then (5.10) is immediate from (5.9). So suppose $y\notin G_m$, and let $n>m$ be such that $y\in G_n-G_{n-1}$. Let $B=A\cup(\operatorname{int}(\Delta_m)-\{y\})$. It is sufficient to prove (5.10) for g_B. Now define z_k, $m-1\leqq k\leqq n$ as follows. Let $z_n=y$. For $m\leqq k\leqq n-1$ let z_k be the unique point in $\partial\Delta_{k+1}(y)\cap\partial\Delta_k(y)$. Thus $z_m\in\partial\Delta_m$, and for $m+1\leqq k\leqq n$, z_k and z_{k+1} are vertices of $\Delta_k(y)$. (Note that it is possible to have $z_k=z_{k-1}$). Let $z_{m-1}\in\partial\Delta_m$. By (5.9) we have

$$g_B(z_{k-1},z_{k-1})\leqq 6(\tfrac{3}{5})^k+g_B(z_k,z_k) \quad \text{for } m\leqq k\leqq n.$$

Summing over k, we deduce that

$$g_B(z_{m-1},z_{m-1})\leqq 15(\tfrac{3}{5})^m+g_B(y,y).$$

As z_{m-1} is arbitrary, and $g_B(y,y)=0$, this implies (5.10). □

Proposition 5.9. *Let* A *satisfy* (5.2). *Then*

$$(5.11)\qquad |g_A(y,y)-g_A(x,y)|\leqq c_{5.1}|x-y|^{d_w-d_f}, \quad \textit{for all } x,y\in G_\infty$$

and

$$(5.12)\qquad q_A(x,y)\leqq\frac{c_{5.1}|x-y|^{d_w-d_f}}{g_A(y,y)} \quad \textit{for all } y\in A\cap G_\infty,\ x\in G_\infty.$$

Proof. Let $x, y \in G_\infty$. Choose m so that $2^{-m-2} \leqq |x-y| \leqq 2^{-m-1}$. Then $y \in \operatorname{int}(D_m(x))$, and so either $\Delta_m(x) = \Delta_m(y)$, or $\Delta_m(x)$ and $\Delta_m(y)$ are adjacent. In either case let $z \in \partial\Delta_m(x) \cap \partial\Delta_m(y)$. Then

$$\begin{aligned} & g_A(y, y) \times g_A(x, y) \\ & \quad = q_A(x, y)\, g_A(y, y) \\ & \quad \leqq q_A(x, z)\, g_A(y, y) + q_A(z, y)\, g_A(y, y) \\ & \quad = q_A(x, z)\, g_A(z, z) + q_A(x, z)(g_A(y, y) - g_A(z, z)) + g_A(y, y) - g_A(z, y) \\ & \quad = (q_A(x, z) + q_A(y, z))\, g_A(z, z) + (1 + q_A(x, z))(g_A(y, y) - g_A(z, z)). \end{aligned}$$

Now $q_A(x, z)\, g_A(z, z) = g_A(z, z) - g_A(x, z)$. If $\operatorname{int}(\Delta_m(x)) \subset A$, this is bounded by $6(\frac{3}{5})^m$, by Lemma 5.7(b). If $\operatorname{int}(\Delta_m(x)) \not\subset A$, then $g_A(z, z) \leqq 15(\frac{3}{5})^m$, by Lemma 5.8. Thus in either case $q_A(x, z)\; g_A(z, z) \leqq 15(\frac{3}{5})^m$, and the same bound holds for $q_A(y, z)\, g_A(z, z)$. Using Lemma 5.7(a)(i), we have

$$(1 + q_A(x, z))(g_A(y, y) - g_A(z, z)) \leqq 18(\tfrac{3}{5})^m.$$

Putting these estimates together, we have

$$g_A(y, y) - g_A(x, y) \leqq 48(\tfrac{3}{5})^m = 48(2^{-m})^{d_w - d_f}.$$

Since $2^{-m} \leqq 4|x-y|$, (5.11) now follows. (5.12) is immediate from (5.11). □

Remark. Taking $x \in \partial A$, so that $g_A(x, y) = 0$, we see that (5.11) implies that $g_A(y, y) \to 0$ as $y \to \partial A$.

Corollary 5.10. *Let A satisfy* (5.2). *Then*

(a) $|g_A(x, y) - g_A(x', y)| \leqq c_{5.1}|x-x'|^{d_w - d_f}\left(\dfrac{g_A(x, y)}{g_A(x, x)} \vee \dfrac{g_A(x', y)}{g_A(x', x')}\right)$
$\qquad \leqq c_{5.1}|x-x'|^{d_w - d_f}$, *for all* x, x', y *in* G_∞.

(b) $|g_A(x, y) - g_A(x', y')| \leqq c_{5.1}(|x-x'|^{d_w - d_f} + |y-y'|^{d_w - d_f})$, *for all* x, x', y, y' *in* G_∞.

Proof. Let $S = R(A - \{x'\})$. Then

$$g_A(x, y) \geqq E^x g_A(X_S, y) \geqq p_A(x, x')\, g_A(x', y) = g_A(x', y) - q_A(x, x')\, g_A(x', y).$$

Thus

$$\begin{aligned} g_A(x', y) - g_A(x, y) & \leqq q_A(x, x')\, g_A(x', y) \\ & \leqq c_{5.1}|x-x'|^{d_w - d_f}\, g_A(x', y)/g_A(x', x'), \end{aligned}$$

by (5.12). Reversing the roles of x, x' we obtain (a).

(b) is immediate from (a). □

The following result, on the random walks $X(n, x)$ will be needed in Sect. 8. (Recall from Sect. 2 that $X(n, x)$ is simple random walk on G_n starting at x).

Lemma 5.11. *Let $r<m\leqq n$, $x, y\in G_n$, and $d(x,y)<2^{-m}$. Then*

(5.13) $$P(S_r(X(n,x))\leqq T(y, X(n,x)))\leqq c_{5.2}(\tfrac{3}{5})^{m-r}.$$

Proof. Let $A=\operatorname{int}(D_r(x))$. By the construction of X the left hand side of (5.13) equals $P^x(S_r(X)\leqq T_y(X))=q_A(x,y)$. We have $D_{r+1}(y)\subset D_r(x)$, and so, by Corollary 5.6, $g_A(y,y)\geqq\frac{3}{4}(\frac{3}{5})^{r+1}$. So, by (5.12)

$$\begin{aligned} q_A(x,y) &\leqq (\tfrac{4}{3})\, c_{5.1}\, |x-y|^{d_w-d_f}(\tfrac{3}{5})^{-r-1} \\ &\leqq c_{5.2}(2^{-m})^{d_w-d_f}(\tfrac{3}{5})^{-r}=c_{5.2}(\tfrac{3}{5})^{m-r}. \quad \square \end{aligned}$$

Corollary 5.10(b) shows that $g_A(\cdot,\cdot)$ is uniformly continuous on $G_\infty\times G_\infty$.

Definition. For A satisfying (5.2) let $u_A(x,y)$, $(x,y)\in G\times G$, be the unique continuous extension to $G\times G$ of $g_A(x,y)$.

For A satisfying (5.2), and $f\in b\mathscr{B}(G)$ let

$$U_A f(x)=E^x\int_0^{R(A)} f(X_s)\,ds.$$

(As A is bounded, $E^x R(A)<\infty$ by Remark 2.20, and the expectation exists). U_A is the potential kernel for the process X killed on leaving the region A.

The estimates given in Corollary 5.10 transfer immediately to u_A, and in particular we have that u_A is bounded and continuous on $G\times G$.

Theorem 5.12. *Let A satisfy (5.2). u_A is the density of U_A, so that,*

(5.14) $$U_A f(x)=\int u_A(x,y)\, f(y)\,\mu(dy) \quad \textit{for all } x\in G,\ f\in b\mathscr{B}(G).$$

Proof. It is sufficient to prove the result for $f\in C_K(G)$. Let first $x\in G_\infty$, so that, for some $m\geqq 0$, $x\in G_m$. From the definition of g we have, for $n\geqq m$,

(5.15) $$E^x\int_0^{R^n(A)} f(X_s^{(n)})\,ds=\int g_A(x,y)\,f(y)\,d\mu_n(y).$$

Let $n\to\infty$ in (5.15). Since $u_A(x,\cdot)f(\cdot)\in C_K(G)$, the right hand side converges to the right hand side of (5.14) by Lemma 1.1(b). As $X_t^{(n)}\to X_t$ a.s., and the convergence is uniform on compacts, and $R^n(A)\to R(A)$ a.s. and in L^1 (by Lemma 2.16), the left hand side of (5.15) converges to $U_A f(x)$.

This proves (5.14) for $x\in G_\infty$. However, the right hand side is evidently continuous on a function of x, while the left hand side is continuous by Proposition 2.13, Lemma 2.16 and Remark 2.20. Thus (5.14) holds for all $x\in G$. $\square$

Proposition 5.13. (a) *Let $x\in G$, $x_n\in G_\infty$ for $n\geqq 1$, and $x_n\to x$. Then*

$$T_{x_n}\xrightarrow{P^x} 0 \quad \textit{as } n\to\infty.$$

(b) *Let A satisfy* (5.2). *Then*

$$u_A(x,y)=p_A(x,y)\,u_A(y,y) \quad \textit{for all } x,y\in G. \tag{5.16}$$

Proof. (The reason (5.16) is not evident is that we do not yet know that $p_A(\cdot,\cdot)$ is continuous).

Let A satisfy (5.2), $x\in G$, $y\in A\cap G_\infty$, and let $x_n\to x$, with $x_n\in G_\infty$. By Lemma 2.16 and Proposition 2.13 we have $p_A(x_n,y)\to p_A(x,y)$ as $n\to\infty$. Thus, since u_A is continuous,

$$u_A(x,y)=p_A(x,y)\,u_A(y,y) \quad \text{for } x\in G,\ y\in G_\infty. \tag{5.17}$$

Note that if $y\in A^c\cap G_\infty$ both sides of (5.17) are zero.

(a) Set $A_r=\mathrm{int}(D_r(x))$ for $r\geqq 1$. By (5.17) and (5.12)

$$q_{A_r}(x,x_n)\leqq c_{5.1}\,|x-x_n|^{d_w-d_f}/g_{A_r}(x_n,x_n).$$

Thus by Corollary 5.6, $P^x(T_{x_n}>R(A_r))\to 0$ as $n\to\infty$ for each r. By (2.60) $R(A_r)\to 0$ a.s. as $r\to\infty$, and hence $T_{x_n}\xrightarrow{P^x}0$.

(b) If $y\in A^c$, (5.16) is evident, so let $y\in G\cap A$, and $x\neq y$. Let $y_n\in G_\infty$, with $y_n\to y$. By the continuity of the paths of X we have $T_y\leqq \liminf_{n\to\infty} T_{y_n}$. On the other hand, by (a) we have $\lim_{n\to\infty} P(T_{y_n}>T_y+\varepsilon)=0$ for each $\varepsilon>0$. Thus $T_{y_n}\xrightarrow{P^x}T_y$, and so $p_A(x,y_n)\to p_A(x,y)$ as $n\to\infty$. This establishes (5.16) in the case $x\neq y$.

If $x=y\in A$ then $p_A(y,y)=1$, and (5.16) is immediate. □

Corollary 5.14. (a) *Let $x\in G$, $x_n\to x$, and $B=\{x_n, n\geqq 1\}$. Then $P^x(T(B)=0)=1$.*

(b) *$P^x(T_x^+=0)=1$ for all $x\in G$, where $T_x^+=\inf\{t>0\colon X_t=x\}$.*

(c) *The fine topology on G is the ordinary topology.*

Proof. (a) As $T_B=\inf_n T_{x_n}$, this is immediate from Proposition 5.13 (a).

(b) Let $A_r=\mathrm{int}(D_r(x))$, let $x_n\to x$, $x_n\neq x$, and let $U_n=\mathrm{int}\{t\geqq T_{x_n}\colon X_t=x\}$. Then

$$P^x(U_n<R(A_r))\geqq p_{A_r}(x,x_n)\,p_{A_r}(x_n,x),$$

and so, as $T_x^+\leqq \inf_n U_n$, $P^x(T_x^+<R(A_r))=1$ (use Proposition 5.13 and Corollary 5.6). Since $\lim_{r\to\infty} R(A_r)=0$, $T_x^+=0$ P^x-a.s.

(c) Let $A \subset G$, and suppose A is not open. Then there exists $x \in A$, and $x_n \in A^c$ with $x_n \to x$. Let $B = \{x_n, n \geqq 1\} \subset A^c$. By (a), $T(B) = 0$ P^x-a.s., and this implies that A is not finely open. As open sets are finely open, the two topologies agree. □

Remark. Note that, by the above, $T_x^+ = T_x$ P^y-a.s., for each $y \in G$.

For $\lambda > 0$ let R_λ be a negative exponential random variable with mean λ^{-1} which is independent of X.

Lemma 5.15. *Let* $A = \operatorname{int}(D_n(x))$. *Then*

(a) $P^x(R_\lambda \geqq R(A)) \leqq c_{3.2}\, e^{-c_{3.3}\lambda^\gamma 2^{-n}}$

(b) $P^x(R_\lambda \leqq R(A)) \leqq c_{5.3}\, \lambda 5^{-n}$.

Proof. (a) Since $R(A) \geqq W_1^n(X)$,

$$P^x(R_\lambda \geqq R(A)) \leqq P^x(R_\lambda \geqq W_1^n(x)) \leqq E^x e^{-\lambda W_1^n(X)} = \phi(\lambda 5^{-n}) \leqq c_{3.2}\, e^{-c_{3.3}\lambda^\gamma 2^{-n}},$$

by Proposition 3.1.

(b) By (2.60) we have $P^x(R(A) > t) \leqq c_{2.1}\, e^{-c_{2.2} t 5^n}$.
So,

$$\begin{aligned} P^x(R(A) \geqq R_\lambda) &= \int_0^\infty P^x(R(A) \geqq t)\lambda e^{-\lambda t}\, dt \\ &\leqq c_{2.1} \int_0^\infty \lambda e^{-t(\lambda + c_{2.2} 5^n)}\, dt \\ &= c_{2.1}\, \lambda(\lambda + c_{2.2}\, 5^n)^{-1} \leqq c_{5.3}\, \lambda 5^{-n}. \quad \square \end{aligned}$$

Let $\lambda > 0$, $x \in G$, $A = \operatorname{int}(D_n(x))$, and $f \in b\mathscr{B}(G)$, with $f \geqq 0$. Then

$$\begin{aligned} U_\lambda f(x) - U_A f(x) &= E^x\left(\int_0^{R_\lambda} f(X_s)\, ds - \int_0^{R(A)} f(X_s)\, ds\right) \\ &\leqq E^x\left(1_{(R_\lambda > R(A))} \int_{R(A)}^{R_\lambda} f(X_s)\, ds\right) \\ &= E^x(1_{(R_\lambda > R(A))}\, U_\lambda f(X_{R(A)})). \end{aligned} \tag{5.18}$$

Choosing A large enough so that $P^x(R_\lambda > R(A)) \leqq \frac{1}{2}$, we have from (5.18) that $U_\lambda f(x) \leqq U_A f(x) + \frac{1}{2} \sup_{z \in G} U_\lambda f(z)$. Hence

$$\sup_{x \in G} U_\lambda f(x) \leqq 2 \sup_{x \in G} U_A f(x) \leqq 2 \sup_{x \in G} u_A(x, x) \int_G f\, d\mu. \tag{5.19}$$

So, for each $\lambda > 0$, $x \in G$, the kernel $U_\lambda(\cdot, x)$ is absolutely continuous with respect to μ (the sup is bounded by $c(n)$ by Lemmas 5.4 and 5.8).

By Theorem 2.16 X is μ-symmetric, and thus U_λ is μ-symmetric. Applying Theorem VI.1.4 of Blumenthal-Getoor (1968) we deduce that there exist resolvent densities $u_\lambda(x, y)$, $\lambda>0$, $(x, y)\in G\times G$ satisfying

(5.20a) $U_\lambda f(x)=\int u_\lambda(x, y) f(y)\,\mu(dy), \qquad U_\lambda f(y)=\int u_\lambda(x, y) f(x)\,\mu(dx)$

for all $x, y\in G, f\in b\mathscr{B}(G)$

(5.20b) u_λ is λ-excessive in each variable.

The function u_λ is unique. For, if u'_λ also satisfies (5.20), by (5.20a) $u_\lambda(x,\cdot) =u'_\lambda(x,\cdot)$ μ-a.e. for each $x\in G$. As u_λ, u'_λ are both λ-excessive (so that $\lim_{\beta\to\infty} \beta U_{\lambda+\beta} u_\lambda(x,\cdot)=u_\lambda(x,\cdot)$ pointwise) $u_\lambda=u'_\lambda$ everywhere. Thus u_λ is symmetric in x, y: since $u'_\lambda(x, y)=u_\lambda(y, x)$ also satisfies (5.20). By Blumenthal and Getoor (1968, Theorem II.4.2) $u_\lambda(x,\cdot)$ and $u_\lambda(\cdot, y)$ are finely continuous, and so, by Corollary 5.14, u_λ is continuous in each variable.

Thus u_λ satisfies in addition

(5.20c) $u_\lambda(x, y)=u_\lambda(y, x)$ for all $x, y\in G\times G$.

(5.20d) For each x, y the functions $u_\lambda(x,\cdot)$, $u_\lambda(\cdot, y)$ are continuous.

Notation. For $\lambda>0$, $x, y\in G$ let

$$p_\lambda(x, y)=E^x e^{-\lambda T_y}=P^x(T_y<R_\lambda),$$

$$q_\lambda(x, y)=1-p_\lambda(x, y).$$

Lemma 5.16. *$u_\lambda(x, y)=p_\lambda(x, y)\, u_\lambda(y, y)$ for all $x, y\in G$.*

Proof. Let $n\geqq 1$, $T_n T(D_n(y), X)$, and let f be an approximate identity to δ_y, supported on $D_n(y)$. Then if $x\notin D_n(y)$,

$$U_\lambda f(x)=E^x 1_{(R_\lambda>T_n)} U_\lambda f(X_{T_n}),$$

and so

(5.21) $\int u_\lambda(x, z) f(z)\,\mu(dz)=E^x(1_{(R_\lambda>T_n)} u_\lambda(X_{T_n}, z)) f(z)\,\mu(dz).$

Since $X_{T_n}\in\partial D_n(y)$, which is finite, $z\to E^x(1_{(R_\lambda>T_n)} u_\lambda(X_{T_n}, z))$ is continuous. Thus, letting $f\to\delta_y$ in (5.21), we deduce

(5.22) $u_\lambda(x, y)=E^x 1_{(R_\lambda>T_n)} u_\lambda(X_{T_n}, y).$

Now let $n\to\infty$: since $\lim_n T_n=T_y$ and $u_\lambda(\cdot, y)$ is continuous, it follows that $u_\lambda(x, y)=p_\lambda(x, y)\, u_\lambda(y, y)$ for $x\neq y$. If $x=y$, the result is immediate. □

Proposition 5.17. *There exist constants $c_{5.4}, c_{5.5}$ such that*

(5.23) $c_{5.4}\,\lambda^{\frac{1}{2}d_s-1}\leqq u_\lambda(x, x)\leqq c_{5.5}\,\lambda^{\frac{1}{2}d_s-1}$ *for all $x\in G$ and $\lambda>0$.*

Proof. From (5.18) we have, if $\lambda>0$, $x\in G$, $A=\operatorname{int}(D_n(x))$ and $f\in b\mathscr{B}(G)$, $f\geqq 0$,

$$\int u_\lambda(x,z)f(z)\,\mu(dz)\leqq\int u_A(x,z)f(z)\,\mu(dz) + E^x\left(1_{(R_\lambda>R(A))}\int u_\lambda(X_{R(A)},z)f(z)\,\mu(dz)\right).$$

As in the previous lemma, we can let $f\to\delta_x$ to deduce

$$u_\lambda(x,x)\leqq u_A(x,x)+E^x(1_{(R_\lambda>R(A))}u_\lambda(X_{R(A)},x)).$$

Since $u_\lambda(z,x)\leqq u_\lambda(x,x)$, by Lemma 5.16, we have

$$u_\lambda(x,x)\leqq(1-P^x(R_\lambda>R(A)))^{-1}u_A(x,x).$$

By Lemmas 5.4 and 5.8, $u_A(x,x)\leqq c_1(\frac{3}{5})^n$, and so, using Lemma 5.15(a) and choosing n to be the largest m such that $c_{3.2}\exp(-c_{3.3}\lambda^\gamma 2^{-m})<\frac{1}{2}$, we obtain the right hand side of (5.23).

Exchanging the roles of $R(A)$ and R_λ we have

$$U_A f(x)\leqq U_\lambda f(x)+E^x(1_{(R(A)\geqq R_\lambda)}U_A f(X_{R_\lambda})),$$

and so

$$\begin{aligned}\int u_A(x,z)f(z)\,\mu(dz)&=\int u_\lambda(x,z)f(z)\,\mu(dz)\\&\quad+E^x\left(1_{(R(A)\geqq R_\lambda)}\int u_A(X_{R_\lambda},z)f(z)\,\mu(dz)\right)\\&\leqq\int u_\lambda(x,z)f(z)\,\mu(dz)\\&\quad+(\sup_z u_A(z,z))\,P^x(R(A)\geqq R_\lambda)\int f\,d\mu.\end{aligned}$$

Letting $f\to\delta_x$, we deduce that

$$u_A(x,x)-P^x(R(A)\geqq R_\lambda)\sup_z u_A(z,z)\leqq u_\lambda(x,x).$$

By Corollary 5.6, $u_A(x,x)\geqq\frac{3}{4}(\frac{3}{5})^n$, while $\sup_z u_A(z,z)\leqq c(\frac{3}{5})^n$ by Lemmas 5.4 and 5.8. Thus, using Lemma 5.15(b) to choose n so that $P^x(R(A)\geqq R_\lambda)$ is sufficiently small, we obtain the left hand side of (5.23). □

Remark. $\frac{1}{2}d_s-1=-0.31739\ldots$, so that $u_\lambda(x,x)\to\infty$ as $\lambda\to 0$.

Proposition 5.18. *There exists a constant $c_{5.6}$ such that*

(5.24) $$q_\lambda(x,y)\leqq c_{5.6}|x-y|^{d_w-d_f}\lambda^{1-\frac{1}{2}d_s}\qquad \textit{for}\ \ x,y\in G,\ \ \lambda>0.$$

Proof. Let n be such that $2^{-n-2}\leqq d(x,y)\leqq 2^{-n-1}$.

Let $U_m = \inf\{t \geqq 0 \colon X_t \in D_m(x)^c\}$. Then if $k \in \mathbb{Z}$, $k < n$

$$\begin{aligned} q_\lambda(x,y) &= P^x(T_y > R_\lambda) \\ &= P^x(T_y > R_\lambda, R_\lambda < U_n) + \sum_{m=k+1}^{n} P^X(T_y > R_\lambda, U_m < R_\lambda < U_{m-1}) \\ &\quad + P^x(T_y > R_\lambda, R_\lambda > U_k) \\ &\leqq P^x(R_\lambda < U_n) + P^x(T_y > U_k) + \sum_{m=k}^{n} P^x(T_y > R_\lambda, U_m < R_\lambda < U_{m-1}). \end{aligned}$$

For the m-th term in the sum we have

$$\begin{aligned} P^x(T_y > R_\lambda, U_m < R_\lambda < U_{m-1}) &\leqq P^x(R_\lambda < U_{m-1}, T_y > U_m, R_\lambda > U_m) \\ &= E^x 1_{(T_y > U_m, R_\lambda > U_m)} P^{X_{U_m}}(R_\lambda < U_{m-1}) \\ &\leqq P^x(T_y > U_m) \sup_{z \in D_m(x)} P^z(R_\lambda < U_{m-1}). \end{aligned}$$

Let $z \in D_m(x)$; then $D_{m-1}(x) \subset D_{m-2}(z)$. So, by Lemma 5.15, $P^z(R_\lambda < U_{m-1}) \leqq c_{5.3}\, \lambda 5^{-(m-2)}$. Also, by (5.12)

$$P^x(T_y > U_m) \leqq c_{5.1}\, |x-y|^{d_w - d_f} (u_{D_m(x)}(y,y))^{-1}.$$

As $d(x,y) \leqq 2^{-n-1}$, $D_{m+1}(y) \subset D_m(x)$ and by Corollary 5.6

$$P^x(T_y > U_m) \leqq c_1\, |x-y|^{d_w - d_f} (\tfrac{5}{3})^m.$$

Putting these estimates together we get

$$\begin{aligned} q_\lambda(x,y) &\leqq c_{5.3}\, \lambda 5^{-n} + c_1 |x-y|^{d_w-d_f} (\tfrac{5}{3})^k + \sum_{m=k+1}^{n} c_2 |x-y|^{d_w-d_f} (\tfrac{5}{3})^m \lambda 5^{-m} \\ &\leqq c_{5.3}\, \lambda 5^{-n} + c_1 |x-y|^{d_w-d_f} (\tfrac{5}{3})^k + c_2 \lambda |x-y|^{d_w-d_f} 3^{-k}. \end{aligned}$$

By the choice of n, $5^{-n} \leqq c_3 |x-y|^{d_w-d_f} 3^{-n}$, and so

$$q_\lambda(x,y) \leqq c_4 |x-y|^{d_w-d_f} (3^{-k}\lambda + (\tfrac{5}{3})^k).$$

If $\lambda < 5^n$ choose k such that $5^{k-1} \leqq \lambda \leqq 5^k$ to deduce (5.24). If $\lambda \geqq 5^n$ then $|x-y|^{d_w-d_f} \lambda^{1-\frac{1}{2}d_s} \geqq c_5 > 0$, and so, choosing $c_{5.6}$ appropriately we have (5.24). □

Corollary 5.19. (a) $P^x(T_y < \infty) = 1$ *for all* $x, y \in G$.

(b) $\{t \colon X_t = y\}$ *is* P^x-a.s. *unbounded, for each* $x, y \in G$.

Proof. This is essentially immediate from (5.24). □

Theorem 5.20. *For all* $\lambda > 0$, (x, y) *in* $G \times G$

(5.25) $\quad |u_\lambda(x,y) - u_\lambda(x',y)| \leqq c_{5.7}\, \lambda^{1-\frac{1}{2}d_s} |x-x'|^{d_w-d_f} (u_\lambda(x,y) \vee u_\lambda(x',y))$

(5.26) $\quad |u_\lambda(x,y) - u_\lambda(x',y')| \leqq c_{5.8}\, |(x,y)-(x',y')|^{d_w-d_f}.$

Proof. Lemma 5.16 shows that $u_\lambda(x, y) \geqq p_\lambda(x, x')\, u_\lambda(x', y)$, and so

$$u_\lambda(x', y) - u_\lambda(x, y) \leqq q_\lambda(x, x')\, u_\lambda(x', y).$$

(5.25) now follows, on using (5.24).

Since $u_\lambda(x, y) \leqq u_\lambda(y, y) \leqq c_{5.5}\, \lambda^{\frac{1}{2}d_s - 1}$ (by Proposition 5.17), we have from (5.25) that $|u_\lambda(x, y) - u_\lambda(x', y)| \leqq c_1 |x - x'|^{d_w - d_f}$. Using the symmetry of u_λ, (5.26) follows. □

To estimate $u_\lambda(x, y)$ for $|x-y|$ large we use the following:

Theorem 5.21. *For all* $\lambda > 0$ *and* $(x, y) \in G \times G$,

(a) $c_{5.9} \exp\{-c_{5.10}\, \lambda^\gamma |x-y|\} \leqq p_\lambda(x, y) \leqq c_{3.2} \exp\{-c_{5.11}\, \lambda^\gamma |x-y|\}$.

(b) $c_{5.12}\, \lambda^{d_s/2 - 1} \exp\{-c_{5.10}\, \lambda^\gamma |x-y|\}$
$$\leqq u_\lambda(x, y) \leqq c_{5.13}\, \lambda^{d_s/2 - 1} \exp\{-c_{5.11}\, \lambda^\gamma |x-y|\}.$$

Proof. Clearly we may assume $x \neq y$ (recall Proposition 5.17). Choose $n \in \mathbb{Z}$ so that $2^{1-n} < d(x, y) \leqq 2^{2-n}$. Then $y \in D_n(x)^c$ and so $T_y \geqq R(D_n(x))$. Then, by Lemma 5.15

$$p_\lambda(x, y) \leqq c_{3.2} \exp\{-c_{3.3}\, \lambda^\gamma 2^{-n}\} \leqq c_{3.2} \exp\{-c_{5.11}\, \lambda^\gamma |x-y|\},$$

proving the upper bound in (a).

Let $m = n-2$ so that $2^{-m-1} < d(x, y) \leqq 2^{-m}$. Either $\Delta_m(x) = \Delta_m(y)$ or they are neighbouring triangles in $\mathscr{T}_m$. In either case let $z \in \partial\Delta_m(x) \cap \partial\Delta_m(y)$. Note that

$$p_\lambda(x, y) \geqq E^x(e^{-\lambda T_z})\, E^z(e^{-\lambda T_y})$$

and

$$\begin{aligned} E^z(e^{-\lambda T_y}) &= u_\lambda(z, y)\, u_\lambda(y, y)^{-1} && \text{(Lemma 5.16)} \\ &\geqq c_1\, u_\lambda(y, z)\, u_\lambda(z, z)^{-1} && \text{(by (5.23))} \\ &= c_1\, E^y(e^{-\lambda T_z}). \end{aligned}$$

These results imply

$$p_\lambda(x, y) \geqq c_1\, E^x(e^{-\lambda T_z})\, E^y(e^{-\lambda T_z}). \tag{5.27}$$

Apply the strong Markov property at T^m to see that

$$\begin{aligned} E^x(e^{-\lambda T_z}) &= E^x(e^{-\lambda T_m} E^{X(T^m)}(e^{-\lambda T_z})) \\ &\geqq E^x(e^{-\lambda T^m})\, 4^{-1} E(e^{-\lambda 5^{-m} W}) && \text{(Theorem 2.19(b))} \\ &\geqq 4^{-1} \exp\{-c_{3.1}\, \lambda^\gamma 2^{1-m} - c_{3.1}\, \lambda^\gamma 2^{-m}\} && \text{(by (3.6) and (4.2))} \\ &\geqq c_2 \exp\{-c_3\, \lambda^\gamma |x-y|\}. \end{aligned}$$

The same estimate holds if on the left side we replace x with y. Substitute these estimates into (5.27) to derive the lower bound in (a).

(b) is immediate from (a) Lemma 5.16, and Proposition 5.17. □

Remark. The proof of Theorem 1.9 is now complete, as it consists of (5.20a), (5.20c), Theorem 5.20 and Theorem 5.21.

The distributional estimates in Theorem 4.3 show that P_t is a strongly continuous semigroup on $C_0(G)$ ($C_0(G)$ is the Banach space of continuous functions vanishing at ∞). Let $\mathscr{A}$ denote the infinitesimal generator of X, and let $\mathscr{D}(\mathscr{A})$ be the domain of $\mathscr{A}$. While it appears difficult to obtain a convenient characterization of $\mathscr{D}(\mathscr{A})$ (but see Sect. 9) our results on u_λ do give the following

Theorem 5.22. *Every function in $\mathscr{D}(\mathscr{A})$ is Holder continuous of order $d_w - d_f$. More precisely, if $f = U_\lambda g$ for some $g \in C_0(G)$, and $\lambda > 0$, then*

$$|f(x)-f(y)| \leqq 2 c_{5.7} \lambda^{-d_s/2} \|g\|_\infty |x-y|^{d_w-d_f} \qquad \text{for all } x, y \in G.$$

Proof. We have, using (5.25),

$$\begin{aligned}|U_\lambda g(x) - U_\lambda g(y)| &= |\textstyle\int (u_\lambda(x,z) - u_\lambda(y,z))\, g(z)\, \mu(dz)| \\ &\leqq \|g\|_\infty c_{5.7} \lambda^{1-\frac{1}{2}d_s} |x-y|^{d_w-d_f} \textstyle\int (u_\lambda(x,z) + u_\lambda(y,z))\, \mu(dz) \\ &\leqq \|g\|_\infty c_{5.7} |x-y|^{d_w-d_f} 2\lambda^{-d_s/2}. \quad \square\end{aligned}$$

6. Local Time

In this section we use the estimates on $u_\lambda(x, y)$ to prove Theorem 1.11. For each $x \in G$, $P^x(T_x^+ = 0) = 1$ (Corollary 5.14), and $y \to p_\lambda(y, x) = E^y e^{-\lambda T_x}$ is continuous. Thus, by Theorem 1 of Getoor and Kesten (1972), there exists a jointly measurable version $(\omega, t, x) \to L_t^x(\omega)$ of the local time, which satisfies the density of occupation formula. Further, for each x, $t \to L_t^x$ is a.s. continuous.

It remains to establish continuity in x. We begin with a version of Garsia's lemma for a fractal.

Lemma 6.1. *Let F be a closed subset of $\mathbb{R}^d$, and let μ be a measure on F such that there exist constants $c_1(F)$, $c_2(F)$, d_F so that if $B^F(x, r) = F \cap \{y \in \mathbb{R}^d : |x-y| \leqq r\}$, then*

$$c_1(F) r^{d_F} \leqq \mu(B^F(x,r)) \leqq c_2(F) r^{d_F} \qquad \text{for all } x \in F,\ r > 0. \tag{6.1}$$

Let p be an increasing function on $[0, \infty)$ with $p(0) = 0$, and $\psi: \mathbb{R} \to \mathbb{R}^+$ be a non-negative symmetric convex function, with $\lim_{u\to\infty} \psi(u) = \infty$. Let H be a compact set in F, and let $f: H \to \mathbb{R}$ be a measurable function. Suppose that

$$\Gamma = \int_{H \times H} \psi\left(\frac{|f(x)-f(y)|}{p(|x-y|)}\right) \mu(dx)\, \mu(dy) < \infty.$$

Then there exists a constant c_F (depending only on $c_1(F)$ and d_F) such that

$$|f(x)-f(y)| \leqq 8 \int_0^{|x-y|} \psi^{-1}\left(\frac{c_F \Gamma}{u^{2d_F}}\right) p(du), \tag{6.2}$$

for $\mu \times \mu$ almost all $x, y \in H \times H$. If f is continuous then (6.2) holds everywhere.

Proof. This is very much the same as the standard proof for $F=\mathbb{R}^d$ given in Garsia (1970): we shall just highlight a few differences. Let $(Q_n)_{n=0}^{\infty}$ be a sequence of spheres centred in F with $Q_n \subset Q_{n-1}$, $p(d(Q_n))=\frac{1}{2}p(d(Q_{n-1}))$, where $d(Q)$ is the diameter of Q. Set $x_n=d(Q_n)$, and note that, by (6.1), $\mu(Q_n)\geqq c_1(F)\,2^{-d_F}x_n^{d_F}$. As in Garsia (1970), using the convexity of ψ we have, writing $f_{Q_n}=\mu(Q_n)^{-1}\int_{Q_n} f\,d\mu$,

$$\psi\left(\frac{f_{Q_n}-f_{Q_{n-1}}}{p(x_{n-1})}\right)\leqq\frac{1}{\mu(Q_n)\,\mu(Q_{n-1})}\int_{Q_n\times Q_{n-1}}\psi\left(\frac{f(x)-f(y)}{p(x_{n-1})}\right)\mu(dx)\,\mu(dy)$$
$$\leqq\Gamma/\mu(Q_n)\,\mu(Q_{n-1})$$
$$\leqq\Gamma c_1(F)^{-2}\,2^{2d_F}x_n^{-2d_F}.$$

Thus, writing $c(F)=c_1(F)^{-2}\,2^{2d_F}$, we have, as in Garsia (1970),

$$|f_{Q_n}-f_{Q_{n-1}}|\leqq 4\int_{x_{n+1}}^{x_n}\psi^{-1}\left(\frac{c_F\Gamma}{u^{2d_F}}\right)p(du).$$

The remainder of the proof now proceeds as in Garsia (1970). Note that the upper bound in (6.1) allows the use of a Vitali covering lemma. □

Remark. (1) Some kind of uniform density condition, such as (6.1), seems to be necessary.

(2) If $F=G$ then (6.1) holds with $d_F=d_f=\dfrac{\log 3}{\log 2}$.

Proof of Theorem 1.11. From Proposition 5.18 we have, for $x, y\in G$

$$1-p_\lambda(x,y)\,p_\lambda(y,x)=1-(1-q_\lambda(x,y))(1-q_\lambda(y,x))$$
$$\leqq q_\lambda(x,y)+q_\lambda(y,x)$$
$$\leqq 2c_{5.6}|x-y|^{d_w-d_f}\lambda^{1-\frac{1}{2}d_s}.$$

Thus, if $p(u)=\sup_{\substack{x,y\in G\\|x-y|\leqq u}}(1-p_1(x,y)\,p_1(y,x))^{\frac{1}{2}}$ we have, writing $\beta=\frac{1}{2}(d_w-d_f)$

$$p(u)\leqq c_{6.1}\,u^\beta. \tag{6.3}$$

We now proceed as in Getoor and Kesten (1972). The estimate in Blumenthal and Getoor (1968, V.3.8) states that, for each x, y, z in G, $N>0$, $\delta>0$,

$$P^z\big(\sup_{0\leqq t\leqq N}|L_t^x-L_t^y|>2\delta\big)\leqq 2e^N e^{-\delta/p(|x-y|)}. \tag{6.4}$$

Let $\psi(x)=\exp(\frac{1}{2}|x|)$. Let $Y_N(x,y)=\sup_{0\leqq t\leqq N}|L_t^x-L_t^y|$: integrating (6.4) we obtain

$$E^z\,\psi\left(\frac{Y_N(x,y)}{2p(|x-y|)}\right)\leqq 2e^N. \tag{6.5}$$

Let $x_0 \in G$, $n \geqq 1$, and $A = D_{-n}(x_0)$: it is enough to prove the result in the region A. We have, for $0 \leqq t \leqq N$

$$(6.6)\quad \int_{A\times A} \psi\left(\frac{|L_t^x - L_t^y|}{p(|x-y|)}\right) \mu(dx)\,\mu(dy) \leqq \int_{A\times A} \psi\left(\frac{Y_N(x,y)}{p(|x-y|)}\right) \mu(dx)\,\mu(dy) \equiv \Gamma_N(A).$$

Applying (6.5) and Fubini we have, since $\mu(A) = 4(3^n)$,

$$(6.7)\qquad E\Gamma_N(A) \leqq 32\, e^N\, 9^n.$$

Thus

$$(6.8)\qquad P(\Gamma_N(A) > \lambda) \leqq c_{6.2}\, e^{N+3n} \lambda^{-1} \qquad \lambda > 0,$$

and in particular $\Gamma_N(A) < \infty$ P^{x_0}-a.s.

By Lemma 6.1, therefore, there exists a set $J(t,\omega) \subset A \times A$ with $\mu \times \mu(A \times A - J) = 0$ such that,

$$(6.9)\quad |L_t^x - L_t^y| \leqq 16 \int_0^{|x-y|} \log(c_F \Gamma_N(A) u^{-2d_f})\, dp(u) \qquad \text{for } (x,y) \in J,\ 0 \leqq t \leqq N.$$

Thus, for $(x, y) \in J$, by (6.3)

$$(6.10)\qquad |L_t^x - L_t^y| \leqq c_{6.3} |x-y|^\beta \log \Gamma_N(A) + c_{6.4} |x-y|^\beta \log |x-y|^{-1}.$$

Now let

$$\bar{L}_t^x = \limsup_{m\to\infty} \mu(D_m(x))^{-1} \int_{D_m(x)} L_t^y\, \mu(dy).$$

As in Getoor and Kesten (1972) we have that $\bar{L}_t^x$ satisfies (6.10) for all $x, y \in A$ and $0 \leqq t \leqq N$, that $\bar{L}_\cdot^x = L_\cdot^x$ a.s. for each $x \in A$, and that $(x,t) \to \bar{L}_t^x$ is continuous. □

Remarks. 1. Note that (6.10) and (6.8) give the inequalities

$$(6.11\text{a})\qquad |\bar{L}_t^x - \bar{L}_t^y| \leqq c_{6.5} |x-y|^\beta \log |x-y|^{-1}$$

$$\text{whenever } |x-y| \leqq \Gamma_N(A)^{-1},\ x, y \in A,\ 0 \leqq t \leqq N,$$

$$(6.11\text{b})\qquad P^z(\Gamma_N(A) > \lambda) \leqq c_{6.2}\, e^{N+3n} \lambda^{-1} (A = D_{-n}(x_0)).$$

2. We do not expect the modulus of continuity given above to be the best possible. It is likely that the estimate (6.4) can be improved to give a Gaussian tail, as was done for Lévy processes in Barlow (1985). This would give a modulus of continuity of the form

$$(6.12)\qquad \lim_{\delta \downarrow 0} \sup_{\substack{0 \leqq s \leqq t \\ |x-y| < \delta}} \frac{|L_s^x - L_s^y|}{\rho(|x-y|)} \leqq c_{6.6} (\sup_{z \in G} L_t^z)^{\frac{1}{2}},$$

where $\rho(u) = u^{\frac{1}{2}(d_w - d_f)} (\log 1/u)^{\frac{1}{2}}$.

In the next couple of results we use the local time to study the range of X. We now take L_t^x, $x \in G$, $t \geqq 0$ to be the jointly continuous version of the local time obtained in Theorem 1.11 and we set

$$R_t(X) = \{X_s, 0 \leqq s \leqq t\}.$$

Lemma 6.2. $\{x: L_t^x > 0\} \subset R_t(X)$, a.s., *for each* $t > 0$.

Proof. Fix $t > 0$. By Getoor and Kesten (1972, Theorem 1) for each x and a.a.ω, $L_\cdot^x$ has support $\{s: X_s = x\}$. Hence there exists a null set N such that on N^c

$$G_\infty \cap \{x: L_t^x > 0\} \subset G_\infty \cap R_t(X).$$

Now let $\omega \in N^c$, $x \in G$ with $L_t^x(\omega) > 0$. Let $x_n \to x$ with $x_n \in G_\infty$. By Theorem 1.11 $L_t^{x_n}(\omega) > 0$ for all large n, and so there exist $t_n = t_n(\omega) \in [0, t]$ such that $X_{t_n}(\omega) = x_n$. A compactness argument now shows that $x \in R_t$. □

Theorem 6.3. $R_\infty(X) = G$ P^y-a.s. *for all* $y \in G$.

Proof. For each $x \in G_\infty$, by Corollary 5.6,

$$E^x L_{S_0}^x = u_{\mathrm{int}(D_0(x))}(x, x) \geqq \tfrac{3}{4}.$$

As $L_{S_0}^x$ has a negative exponential distribution this implies that $P^x(L_{S_0}^x > 1) \geqq e^{-\frac{4}{3}}$. Hence, using Theorem 2.19 and (6.11 b), if $A = D_{-1}(x)$, we have

$$\begin{aligned} P^x(L_t^x > 1, \Gamma_t(A) \leqq \lambda) &\geqq P^x(L_{S_0}^x > 1, S_0 \leqq t, \Gamma_t(A) \leqq \lambda) \\ &\geqq e^{-\frac{4}{3}} - P^x(S_0 > t) - P^x(\Gamma_t(A) > \lambda) \\ &\geqq e^{-\frac{4}{3}} - c_{2.1} e^{-c_{2.2} t} - c_{6.2} \lambda^{-1} e^{3+t}. \end{aligned}$$

First choose t and then λ sufficiently large so that this final expression is greater than e^{-2}. By (6.11 a) there exists $n \geqq 1$ such that

$$P^x(L_t^x > 1, L_t^y > 0 \text{ for all } y \in D_n(x)) > e^{-2}, \qquad \text{for all } x \in G_\infty. \tag{6.12}$$

So, by Lemma 6.2,

$$P^x(D_n(x) \subset R_t(X)) > e^{-2} > 0 \qquad \text{for all } x \in G_\infty.$$

Fix $y \in G$. By Corollary 5.19 (b) $\{t: X_t = x\}$ is unbounded P^y-a.s., and an elementary argument using the strong Markov property shows that, for each $x \in G_n$, there exists U_n with $U_n < \infty$ P^y-a.s. such that $D_n(x) \subset R_{U_n}(x)$. Thus $D_n(x) \subset R_\infty(X)$ for each $x \in G_n$, P^y-a.s., and as $G = \bigcup_{x \in G_n} D_n(x)$ the result is immediate. □

7. Transition Densities

The existence of a transition density for X (with respect to μ) follows from the existence of the resolvent densities (Sect. 5) and a general result for symmetric Markov processes (see Fukushima (1980, p. 106)). We give a direct construction that will enable us to derive several properties of the transition density. The key ingredients are the estimates on $u_\lambda(x, y)$ and g, the C^∞ density of W, from Sect. 5 and 3, respectively. Note that the C^∞ density of W_1^n is (Theorem 2.19 (b))

$$g_n(t) = 5^n g(5^n t). \tag{7.1}$$

Notation. If $A \in \mathscr{B}(G)$ is non-empty and satisfies $d_0(x, A) = \inf\{|y-x| : y \in A\} > 2^{-n+1}$, then $T_1^n < T(A)$ ($<\infty$ by Corollary 5.19(a)) and we may let $H_{x,A,n}$ denote the law of the non-negative random variable $T(A) - W_1^n$ under P^x. If A is also finite and $z \in A$, let $H_{x,A,n}(ds|z)$ denote the conditional law (under P^x) *of* $T(A) - W_1^n$ given $X(T(A)) = z$. If $P^x(X(T(A) = z) = 0$, make an arbitrary choice for $H_{x,A,n}(ds|z)$ and similarly for the conditional law of $T(A)$ given $X(T(A)) = z$ in Lemma 7.1 below.

Lemma 7.1. *Let A be a non-empty Borel subset of G, let $x \notin cl(A)$ and choose $n \in \mathbb{Z}$ such that $2^{-n+1} < d_0(x, A)$.*

(a) *$T(A)$ has a C^∞ density under P^x given by*

$$g_{x,A}(t) = \int_0^\infty g_n(t-s) H_{x,A,n}(ds). \tag{7.2}$$

(b) *If A is finite and $z \in A$, then the C^∞ conditional density of $T(A)$ given $X(T(A)) = z$ (under P^x) is*

$$g_{x,A}(t|z) = \int_0^\infty g_n(t-s) H_{x,A,n}(ds|z). \tag{7.3}$$

More precisely if $B \subset A$ and $C \in \mathscr{B}([0, \infty))$, then

$$\sum_{z \in B} \int_C g_{x,A}(t|z)\, dt\, P^x(X(T(A)) = z) = P^x(T(A) \in C, X(T(A)) \in B). \tag{7.4}$$

Proof. Apply the strong Markov property at T_1^n and Theorem 2.19 (b) to see that W_1^n is independent of $\mathscr{F}_{T^n} \vee \sigma(X \circ \theta_{T_1^n})$. Therefore W_1^n is independent of

$$(T(A) - W_1^n, X(T(A))) = (T(A) \circ \theta_{T_1^n} + T^n, X(T(A)) \circ \theta_{T_1^n}).$$

(7.2) is immediate. The smoothness of $g_{x,A}$ follows by differentiating inside the integral in (7.2), which is justified by (3.16).

Assume now that A is also finite and $z \in A$ satisfies $P^x(X(T(A)) = z) > 0$. If $C \in \mathscr{B}([0, \infty))$, then, using the above independence, we obtain

$$\begin{aligned} P^x(T(A) \in C, X(T(A)) = z) &= P^x(W_1^n + (T(A) - W_1^n) \in C \mid X(T(A)) = z)\, P^x(X(T(A)) = z) \\ &= \int_C \int_0^\infty g_n(t-s) H_{x,A,n}(ds|z)\, dt\, P^x(X(T(A)) = z). \end{aligned}$$

This gives (7.3) and the smoothness of $g_{x,A}(\cdot|z)$ follows as for $g_{x,A}$. □

If $A = \{y\}$, we write $g_{x,y}$ and $H_{x,y,n}$ for $g_{x,A}$ and $H_{x,A,n}$, respectively.

Proposition 7.2. (a) *There are constants* $\{c_{7.1}(\delta)\colon \delta>0\}$ *and* $c_{7.2}$ *such that*

$$c_{7.1}(\delta)\,|x-y|^{-d_w}\exp\{-c_{7.2}(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}\leqq g_{x,y}(t), \tag{7.5}$$

for all (x, y) *in* $G\times G$ *and* $t>0$ *such that* $|x-y|\,t^{-\gamma}>\delta$.

(b) *There are constants* $\{c_{7.3}(k)\colon k\in\mathbb{Z}_+\}$ *and* $c_{7.4}$ *such that*

$$\left|\frac{\partial^k}{\partial t^k}\,g_{x,y}(t)\right|\leqq c_{7.3}(k)\,|x-y|^{-d_w(k+1)}\exp\{-c_{7.4}(|x-y|\,t^{-\gamma})^{1/1-\gamma}\} \tag{7.6}$$

for all (x, y) *in* $G\times G$ *such that* $x\neq y$ *and* $t>0$.

Proof. Let x, y be distinct elements of G and choose $n\in\mathbb{Z}$ such that $2^{1-n}\leqq|x-y|<2^{2-n}$.

(a) If $\delta>0$ and $t>0$ satisfies $|x-y|\,t^{-\gamma}\geqq\delta$, then (7.2) implies that

$$\begin{aligned} g_{x,y}(t) &\geqq \inf_{t/2\leqq s\leqq t} g_n(s)\,H_{x,y,n}([0,t/2])\\ &\geqq 5^n c_1(\delta)\exp\{-c_{3.12}(5^n t/2)^{-1/(d_w-1)}\}\,P^x(T_y\leqq t/2) \qquad \text{(Theorem 3.7)}\\ &\geqq c_2(\delta)\,|x-y|^{-d_w}\exp\{-c_3(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}\,P^x(T_y\leqq t/2), \end{aligned}$$

by the choice of n. Therefore (7.5) will follow from the above if we show there are universal constants c_4, c_5 such that

$$P^x(T_y\leqq t/2)\geqq c_4\exp\{-c_5(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\} \qquad \text{for all } t>0. \tag{7.7}$$

Apply Lemma 4.1 and Theorem 5.21 (a) to see that for all $\lambda>0$,

$$P^x(T_y\leqq t/2)\geqq(c_{5.9}\exp\{-c_{5.10}\,\lambda^\gamma\,|x-y|\}-e^{-\lambda t/2})(1-e^{-\lambda t/2})^{-1}.$$

If we set $\lambda=\alpha(|x-y|\,t^{-1})^{1/(1-\gamma)}$ and α is taken large enough (depending only on $(c_{5.9}, c_{5.10})$), then the above is bounded below by

$$c_4\exp\{-c_5(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}$$

providing t is small enough so that $|x-y|\,t^{-\gamma}\geqq 1$. This gives (7.7) for $t^\gamma\leqq|x-y|$, but it then follows trivially for larger t by simply adjusting c_4.

(b) Differentiate inside the integral in (7.2) to get

$$\begin{aligned} \left|\frac{\partial^k}{\partial t^k}\,g_{x,y}(t)\right| &= \left|\int_0^\infty 5^{n(k+1)}\,g^{(k)}(5^n(t-s))\,H_{x,y,n}(ds)\right|\\ &\leqq 5^{n(k+1)}\,c_{3.9}(k)\exp\{-c_{3.10}(5^n t)^{-1/(d_w-1)}) \qquad \text{(Theorem (3.5))}. \end{aligned}$$

The choice of n now gives (7.6). □

Lemma 7.3. *For each* $m\in\mathbb{N}$

$$\lim_{n\to\infty}\sup\{|g_{x,D_n(y)}(t)-g_{x,y}(t)|\colon t\geqq 0, (x, y)\in G\times G, |x-y|>2^{-m+2}\}=0.$$

Proof. Fix $m \in \mathbb{N}$. If $n > m$ the above supremum equals

$$\sup_{t \geqq 0, |x-y| > 2^{-m+2}} \left| \int_0^\infty g_m(t-s)(H_{x, D_n(y), m} - H_{x,y,m})(ds) \right|$$

$$\leqq \sup_{t \geqq 0, |x-y| > 2^{-m+2}} E^x(|g_m(t-(T(D_n(y)) - W_1^m)) - g_m(t-(T_y - W_1^m))|)$$

$$\leqq \sup_{|x-y| > 2^{-m+2}} E((25^m \|g'\|_\infty |T(D_n(y)) - T_y|) \wedge 5^m \|g\|_\infty) \quad \text{(Theorem 3.5)}$$

$$\leqq c(m) \sup_{y \in G} \sup_{z \in D_n(y)} E^z(T_y \wedge 1),$$

where in the last line we have used the strong Markov property at $T(D_n(y))$. The last expression approaches zero as $n \to \infty$ by Proposition 5.18. □

Notation. If $\{v_n\}$ and v are Radon measures on $\mathscr{B}([0, \infty))$, $v_n \xrightarrow{v} v$ denotes vague convergence of the sequence $\{v_n\}$ to v. That is, $v_n \xrightarrow{v} v$ if and only if $\lim_{n \to \infty} \int f dv_n = \int f dv$ for every continuous f with compact support in $[0, \infty)$.

If $\phi: G \times G \to [0, \infty)$ is a bounded measurable function and $(y, z) \in G \times G$, let $q(\phi, y, z)$ be the measure on $\mathscr{B}([0, \infty))$ defined by

$$q(\phi, y, z)([0, t]) = E^z\left(\int_0^t \phi(y, X_s)\, ds\right), \qquad t \geqq 0.$$

$\{\phi_n: n \in \mathbb{N}\}$ will always denote a sequence of bounded measurable functions from $G \times G$ to $[0, \infty)$ such that for all $y \in G$ and $n \in \mathbb{N}$,

$$\sup(\phi_n(y, \cdot)) \subset D_n(y) \tag{7.8}$$

$$\int \phi_n(y, z)\, \mu(dz) = 1. \tag{7.9}$$

Lemma 7.4. (a) *If* $\{y_n\}$ *and* $\{z_n\}$ *are G-valued sequences which converge to* y, *then* $q(\phi_n, y_n, z_n) \xrightarrow{v} q(y)$ *as* $n \to \infty$, *where* $q(y)$ *is the unique Radon measure on* $[0, \infty)$ *such that*

$$\int_0^\infty e^{-\lambda s} q(y)(ds) = u_\lambda(y, y).$$

(b) *If* $T > 0$, K_1 *is a relatively compact set in* $C([0, \infty), \mathbb{R})$, *whose elements are all supported on* $[0, T]$ *and* K_2 *is a compact subset of* G, *then*

$$\lim_{n \to \infty} \sup\{|\int h(s)\, q(\phi_n, y, z)(ds) - \int h(s)\, q(y)(ds)| : h \in K_1, y \in K_2, z \in D_n(y)\} = 0.$$

(c)

$$c_{7.5}\, t^{1-d_s/2} \leqq q(y)([0,t]) \leqq c_{7.6}\, t^{1-d_s/2} \tag{7.10}$$

for all $t>0$ *and* y *in* G.

Proof. (a)

$$\int_0^\infty e^{-\lambda s} q(\phi_n, y_n, z_n)(ds) = \int \phi_n(y_n, z)\, u_\lambda(z_n, z)\, \mu(dz).$$

The right side converges to $u_\lambda(y,y)$ as $n\to\infty$ by (7.8), (7.9) and the continuity of u_λ (Theorem 5.20). This proves (a) (see e.g., Feller (1971, p. 433)).

(b) If the limit in (b) is not zero, there are $h_n \in K_1$, $y_n \in K_2$, $z_n \in D_n(y_n)$ and $\varepsilon>0$ such that

$$\left|\int h_n(s)\, q(\phi_n, y_n, z_n)(ds) - \int h_n(s)\, q(y_n)(ds)\right| > \varepsilon \quad \text{for all } n\in\mathbb{N}. \tag{7.11}$$

By passing to a subsequence we may assume that $h_n \to h$ uniformly on $[0,T]$, $sup(h)\subset[0,T]$, and $\lim_{n\to\infty} y_n = y$. The continuity of $u_\lambda(\cdot,\cdot)$ and (a) implies $q(y)$ is vaguely continuous in y. This and the convergence obtained in (a) contradict (7.11) and complete the proof of (b).

(c) This follows from Proposition 5.17 and some standard Tauberian theorems. The upper bound is elementary and the required lower bound follows by making some elementary changes in the derivation of (5.2) in De Haan and Stadtmuller (1985, p. 350). □

Definition.

$$p_t(x,y) = \int_0^\infty g_{x,y}(t-s)\, q(y)(ds),\ t\geqq 0, (x,y) \text{ in } G\times G,\ x\neq y. \tag{7.12}$$

We will eventually prove that p has a jointly continuous extension to $\{(t,x,y): t>0, (x,y)\in G\times G\}$ but for now the reader may find it convenient to set $p_t(x,x)=0$.

Proposition 7.5. (a) $p_t(x,y)$ *is a transition density for* X *with respect to* μ.

(b) $p_t(x,y)$ *is jointly continuous on* $\{(t,x,y): t\geqq 0, (x,y)\in G\times G, x\neq y\}$

(c) *For each* $(x,y)\in G\times G$ *with* $x\neq y$, $p_t(x,y)$ *is* C^∞ *in* t *and there are constants* $\{c_{7.7}(k): k\in\mathbb{Z}_+\}$ *and* $\{c_{7.8}(\delta,k): \delta>0, k\in\mathbb{Z}_+\}$ *such that for all* $k\in\mathbb{Z}_+$,

$$\left|\frac{\partial^k}{\partial t^k} p_t(x,y)\right| \leqq c_{7.7}(k)\, |x-y|^{-d_w(k+1)}\, t^{1-d_s/2} \cdot \exp\{-c_{7.4}(|x-y|\, t^{-\gamma})^{1/(1-\gamma)}\} \tag{7.13}$$

for all $t>0$ *and* $x\neq y$ *in* G,

$$\left|\frac{\partial^k}{\partial t^x} p_t(x,y)\right| \leqq c_{7.8}(\delta,k)\, t^{-d_s/2-k} \exp\{-c_{7.4}(|x-y|\, t^{-\gamma})^{1/(1-\gamma)}\}, \tag{7.14}$$

for all $(t,x,y)\in(0,\infty)\times G\times G$ *such that* $|x-y|\, t^{-\gamma}\geqq\delta$.

(d) *There are constants* $\{c_{7.9}(\delta): \delta>0\}$ *such that*

$$p_t(x,y) \geqq c_{7.9}(\delta)\, t^{-d_s/2} \exp\{-2c_{7.2}(|x-y|\, t^{-\gamma})^{1/(1-\gamma)}\} \tag{7.15}$$

for all $(t,x,y)\in(0,\infty)\times G\times G$ *such that* $|x-y|\, t^{-\gamma}\geqq\delta$.

Proof. If $m \in \mathbb{N}$, $T>0$, $\{\phi_n\}$ satisfy (7.8) and (7.9), and K is a compact subset of G we claim that

$$\lim_{n\to\infty} \sup\{|E^x(\phi_n(y, X_t)) - p_t(x, y)|: x\in G, y\in K, |x-y|>2^{-m+2}, t\in[0, T]\}=0. \tag{7.16}$$

If $n>m$, $y\in K$, $x\in G$ and $|x-y|>2^{-m+2}$, then $d_0(x, D_n(y))>2^{-n+1}$ and $\phi_n(y, X_t)>0$ implies $T(\partial D_n(y))<t$ (P^x-a.s.) by (7.8). Apply the strong Markov property at $T(\partial D_n(y))$ and Lemma 7.1 (b) with $A=\partial D_n(y)$ to see that

$$\begin{aligned}
E^x(\phi_n(y, X_t)) &= \sum_{z\in\partial D_n(y)} E^x(1(T(\partial D_n(y))(\omega)<t, X(T(\partial D_n(y)))(\omega)=z) \\
&\quad \times E^z(\phi_n(y, X(t-T(\partial D_n(y))(\omega)))) \\
&= \sum_{z\in\partial D_n(y)} P^x(X(T(\partial D_n(y)))=z) \\
&\quad \times \int_0^t g_{x,\partial D_n(y)}(s|z)\, E^z(\phi_n(y, X(t-s)))\, ds \qquad \text{(by (7.4))} \\
&= \sum_{z\in\partial D_n(y)} P^x(X(T(\partial D_n(y)))=z) \int_0^\infty g_{x,\partial D_n(y)}(t-s|z)\, q(\phi_n, y, z)(ds).
\end{aligned}$$

Let

$$K_1=\{g_{x,\partial D_n(y)}(t-\cdot|z): t\in[0, T], (x, y)\in G\times G, |x-y|>2^{-m+2}, n>m, z\in\partial D_n(y)\} \subset C(([0,\infty), G).$$

Note that (7.3) holds with $A=\partial D_n(y)$ and m in place of n (because $d_0(x, \partial D_n(y)) >2^{-m+1}$). This, together with the smoothness of g_m and the Arzela-Ascoli theorem, show K_1 is a relatively compact subset of $C([0,\infty), G)$. Therefore

$$\begin{aligned}
&\sup\{|E^x(\phi_n(y, X_t)) - p_t(x, y)|: x\in G_\infty, y\in K, |x-y|>2^{-m+2}, t\in[0, T]\} \\
&\leqq \sup\Bigg\{\Bigg|\sum_{z\in\partial D_n(y)} P^x(X(T(\partial D_n(y)))=z) \\
&\quad \times\left(\int_0^\infty g_{x,\partial D_n(y)}(t-s|z)(q(\phi_n, y, z)-q(y)\right)(ds))\Bigg| \\
&\quad +\left|\int_0^\infty (g_{x,\partial D_n(y)}(t-s)-g_{x,y}(t-s))\, q(y)(ds)\right|: \\
&\quad x\in G, y\in K, |x-y|>2^{-m+2}, t\in[0, T]\Bigg\} \\
&\leqq \sup\{|\textstyle\int h(s)(q(\phi_n, y, z)-q(y))(ds)|: h\in K_1, y\in K, z\in D_n(y)\} \\
&\quad +(\sup_{y\in K} q(y)([0, T]))\sup\{|g_{x,\partial D_n(y)}(t)-g_{x,y}(t)|: \\
&\quad x, y\in G, |x-y|>2^{-m+2}, t\geqq 0\}.
\end{aligned}$$

The first term converges to zero as $n\to\infty$ by Lemma 7.4(b) and the second also approaches zero as $n\to\infty$ by Lemma 7.3 and (7.10). This establishes (7.16).

To prove (b), let $\psi(z)=(1-|z|)\vee 0$ and consider

$$\phi_n(y,z)=\psi(2^n|y-z|)(\int\psi(2^n|y-z'|)\,\mu(dz'))^{-1}.$$

(7.8) and (7.9) are obvious. It is also easy to see that ϕ_n is bounded and continuous on $G\times G$. Let us recall the processes $\{X_t(x): x\in G\}$ constructed in Proposition 2.13. That result and dominated convergence show that

$$(t,x,y)\to E^x(\phi_n(y,X_t))=E(\phi_n(y,X_t(x)))$$

is continuous on $[0,\infty]\times G\times G$. The uniform convergence in (7.16) implies the joint continuity of $p_t(x,y)$ on $[0,\infty)\times\{(x,y)\in G\times G: x\neq y\}$.

To prove (a), consider $\phi_n(y,z)=1(z\in\Delta_n(y))\,3^n$. Again (7.8) and (7.9) are obvious. (7.16) implies that for any $m\in\mathbb{N}$,

$$p_t(x,y)=\lim_{n\to\infty}P^x(X_t\in\Delta_n(y))\,\mu(\Delta_n(y))^{-1}$$

$$\text{uniformly in } \{(t,x,y): t\in[0,T],\ x\in G,\ y\in K,\ |x-y|>2^{-m+2}\}.$$

(a) will now follow easily providing we show $P^x(X_t=x)=0$ for all $t>0$, $x\in G$. If $P^x(X_t=x)>0$, the simple Markov property (at some $s<t$) and the fact that the $P^y(X_{t-s}\in\cdot)$ is absolutely continuous with respect to μ except perhaps for an atom at y shows that $P^x(X_s=x)>0$ for all $s<t$. This contradicts the existence of a resolvent density with respect to μ (Theorem 1.9).

(c) (7.6) allows us to differentiate inside the integral in (7.12) k times and hence prove the smoothness of $p_t(x,y)$ in t and obtain (for $x\neq y$, $t>0$, and $k\in\mathbb{Z}^+$)

$$\begin{aligned}\left|\frac{\partial^k}{\partial t^k}p_t(x,y)\right|&=\left|\int_0^\infty g^{(k)}_{x,y}(t-s)\,q(y)(ds)\right|\\&\leqq c_{7.3}(k)\,|x-y|^{-d_w(k+1)}\exp\{-c_{7.4}(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}\,c_{7.6}\,t^{1-d_s/2}\\&\quad\text{(by (7.6) and (7.10)).}\end{aligned}$$

This gives (7.13), and (7.14) follows trivially.

(d) (7.12), (7.10) and (7.5) show that if $\delta>0$ and $|x-y|\,t^{-\gamma}\geqq\delta$, then

$$\begin{aligned}p_t(x,y)&\geqq\inf_{t/2\leqq s\leqq t}g_{x,y}(s)\,q(y)([0,t/2])\\&\geqq c_{7.1}(\delta)\,|x-y|^{-d_w}\exp\{-c_{7.2}\,2^{\gamma/(1-\gamma)}(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}\,c_{7.5}(t/2)^{1-d_s/2}\\&\geqq c_1(\delta)\,t^{-d_s/2}(|x-y|\,t^{-\gamma})^{-d_w}\exp\{c_{7.2}(2-2^{\gamma/(1-\gamma)})(|x-y|\,t^{-\gamma})^{1/1-\gamma}\})\\&\quad\times\exp\{-2c_{7.2}(|x-y|\,t^{-\gamma})^{1/1-\gamma}\}\\&\geqq c_{7.9}(\delta)\,t^{-d_s/2}\exp\{-2c_{7.2}(|x-y|\,t^{-\gamma})^{1/(1-\gamma)}\}.\quad\square\end{aligned}$$

Although (7.13) gives a poor estimates for $|x-y|t^{-\gamma}$ small, it will be helpful in controlling $p_t(x, y)$ and its derivatives in t near the diagonal. In order to extend (7.14) and (7.15) to all (t, x, y) we introduce a class of stopping times at which (7.14) and (7.15) are valid.

Notation. If $\delta, t>0$ and $y \in G$, let

$$V(t, \delta, y)=\inf\{s>0: |X_s-y| \geqq \delta(t-s)^\gamma\} \in [0, t]$$

$$U(t, \delta, y)=(t-V(t, \delta, y))t^{-1} \in [0, 1].$$

Lemma 7.6. *There are constants $c_{7.10}$ and $c_{7.11}$ such that if $\delta \in (0, c_{7.10})$, then $k(\delta)=(\log 1/\delta)/\log 2-c_{7.11}>0$ and*

(7.17) $P^x(U(t, \delta, y) \leqq u) \leqq \delta^{-d_w} u^{k(\delta)}$ *for all u in $[0, 1]$, $t>0$ and (x, y) in $G \times G$.*

If $\delta \in (0, c_{7.10})$ and $k(\delta)>a>0$, then

$$E^x(U(t, \delta, y)^{-a}) \leqq \delta^{-d_w}(a(k(\delta)-a)^{-1}+1) \tag{7.18}$$

for all $t>0$ and (x, y) in $G \times G$.

Proof. Fix $\delta, t>0$ and x, y in G, and write U and V for $U(t, \delta, y)$ and $V(t, \delta, y)$, respectively. If $V>(1-5^{-(r-1)})t$ $(r \in \mathbb{N})$, then

$$\begin{aligned}(7.19)\quad P^x(V>(1-5^{-r})t \mid \mathscr{F}_{(1-5^{-(r-1)})t}) &\leqq P^{X((1-5^{-(r-1)})t)}(|X(s)-y|<\delta(5^{-(r-1)}t)^\gamma \\ &\qquad \text{for all } s \leqq 4(5^{-r}t)) \\ &\leqq \sup_{z \in G} P^z(|X(s)-z|<2\delta(5^{-(r-1)}t)^\gamma \\ &\qquad \text{for all } s \leqq 4(5^{-r}t)),\end{aligned}$$

where in the last line we have used our assumption on V. Choose $n_r \in \mathbb{N}$ such that

$$2^{-n_r-1}<2\delta(5^{-(r-1)}t)^\gamma=4\delta\, 2^{-r}t^\gamma \leqq 2^{-n_r}. \tag{7.20}$$

(7.19) is bounded by

$$\begin{aligned}&\sup_{z \in G} P^z(T_1^{n_r-1}>4(5^{-r}t)) \\ &\quad \leqq 4^{-1}t^{-1}5^r(5^{-n_r+1}/2+5^{-n_r+1}) \text{ (Theorem 2.19 and Markov's inequality)} \\ &\quad \leqq c_1\delta^{d_w},\end{aligned}$$

where $c_1=15(8^{d_w-1})$ and we have used (7.20). These bounds on the conditional probabilities lead easily to

$$P^x(V>(1-5^{-r})t) \leqq (c_1\delta^{d_w})^r, \qquad r \in \mathbb{Z}_+,$$

or equivalently,

$$P^x(U<5^{-r}) \leqq (c_1\delta^{d_w})^r.$$

If $u \in (0, 1]$, choose $r \in \mathbb{Z}_+$ such that $5^{-r-1} \leqq u < 5^{-r}$. If $\delta < c_1^{-\gamma}$, we have

$$P^x(U \leqq u) \leqq (c_1 \delta^{d_w})^{-1} u^{k(\delta)} \leqq \delta^{-d_w} u^{k(\delta)},$$

where

$$k(\delta) = (\log c_1^{-1}/\log 5) + (\log \delta^{-1}/\log 2) > 0.$$

This gives (7.17) and (7.18) follows easily. □

Lemma 7.7. *There is a $c_{7.12}$ and constants $\{c_{7.13}(\delta): \delta > 0\}$ such that if $\delta \in (0, c_{7.12}]$ then $k(\delta) > d_s/2$ and for all $t > 0$ and (x, y) in $G \times G$,*

$$\text{(7.21)} \qquad \lim_{n \to \infty} P^x(X_t \in \Delta_n(y)) 3^n = E^x(p_{t-V(t,\delta,y)}(X(V(t,\delta,y)), y)) \leqq c_{7.13}(\delta) t^{-d_s/2}.$$

Proof. Choose $c_{7.12}$ small enough so that $c_{7.12} \leqq c_{7.10}$ and $k(c_{7.12}) > d_s/2$. Let δ, t, x, y be as above and write U and V for $U(t, \delta, y)$ and $V(t, \delta, y)$, respectively. The strong Markov property at $V(< t$ a.s.) shows that

$$\text{(7.22)} \quad \begin{aligned} P^x(X_t \in \Delta_n(y)) 3^n &= E^x\Big(\int_{\Delta_n(y)} p_{t-V}(X(V), z)\, \mu(dz)\Big) 3^n \\ &= E^x\Big(1(U < t^{-1}(2^{1-n}\delta^{-1})^{d_w}) \int_{\Delta_n(y)} p_{t-V}(X(V), z)\, \mu(dz)\Big) 3^n \\ &\quad + E^x\Big(1(U \geqq t^{-1}(2^{1-n}\delta^{-1})^{d_w}) \int_{\Delta_n(y)} p_{t-V}(X(V), z)\, \mu(dz)\Big) 3^n. \end{aligned}$$

Use (7.17) to bound the first term by

$$\delta^{-d_w} t^{-k(\delta)} \delta^{-d_w k(\delta)} 2^{d_w k(\delta)} 2^{n((\log 3/\log 2) - d_w k(\delta))}$$

which approaches zero as $n \to \infty$ because $k(\delta) > d_s/2$. If $U \geqq t^{-1}(2^{1-n}\delta^{-1})^{d_w}$, then $2^{-n} \leqq (\delta/2)(t - V)^\gamma$ and hence for each $z \in \Delta_n(y)$,

$$|X(V) - z| \geqq |X(V) - y| - 2^{-n} \geqq (\delta/2)(t - V)^\gamma.$$

Therefore (7.14), with $k = 0$, shows that on $\{U \geqq t^{-1}(2^{1-n}\delta^{-1})^{d_w}\}$,

$$\text{(7.23)} \qquad \begin{aligned} \int_{\Delta_n(y)} p_{t-V}(X(V), z)\, \mu(dz) 3^n &\leqq c_{7.8}(\delta/2, 0)(t - V)^{-d_s/2} \\ &= c_{7.8}(\delta/2, 0)\, U^{-d_s/2} t^{-d_s/2}. \end{aligned}$$

This upper bound is P^x-integrable by the choice of δ and (7.18). Hence we may use dominated convergence in (7.22) to conclude that

$$\text{(7.24)} \qquad \begin{aligned} \lim_{n \to \infty} P^x(X_t \in \Delta_n(y)) 3^n &= E^x\Big(\lim_{n \to \infty} 1(U \geqq t^{-1}(2^{1-n}\delta^{-1})^{d_w}) \\ &\qquad \times \int_{\Delta_n(y)} p_{t-V}(X(V), z)\, \mu(dz) 3^n\Big) \\ &= E^x(P_{t-V}(X(V), y)), \end{aligned}$$

where in the last we have used $X(V) \neq y$ a.s., and the continuity of $p_t(\cdot,\cdot)$ away from the diagonal (Proposition 7.5(b)). Use (7.23) to bound (7.24) by

$$c_{7.8}(\delta/2, 0)\, E^x(U^{-d_s/2})\, t^{-d_s/2}.$$

The upper bound in (7.21) now follows from (7.18) and the choice of δ. □

It follows from (7.21) and the continuity of $p_t(x,\cdot)$ away from x that for a fixed δ satisfying the hypotheses of the above result,

$$p_t(x, y) \leqq c_{7.1}(\delta)\, t^{-d_s/2} \quad \text{for all } t>0,\ x \neq y. \tag{7.25}$$

Fix $s>0$ and consider the Chapman-Kolmogorov equations for $t>s$. For each $x \in G$,

$$p_t(x, y) = \int p_s(x, z)\, p_{t-s}(z, y)\, \mu(dz) \quad \text{for } \mu\text{-a.a. } y. \tag{7.26}$$

We claim that the right side is a jointly continuous extension of the left side to $\{(t, x, y): t>s, (x, y) \in G \times G\}$. If $t_n \to t > s$ $(t_n > s)$, $x_n \to x$, $y_n \to y$ (x_n, y_n in G), then

$$\lim_{n\to\infty} p_s(x_n, z)\, p_{t_n-s}(z, y_n) = p_s(x, z)\, p_{t-s}(z, y) \quad \text{for all } z \notin \{x, y\},$$

and

$$p_s(x_n, z)\, p_{t_n-s}(z, y_n) \leqq c_1\, p_s(x_n, z) \quad \text{for all } n$$

by (7.25). Note that $\{p_s(x_n,\cdot), n \in \mathbb{N}\}$ is a uniformly integrable family with respect to μ because they converge μ-a.s. to $p_s(x,\cdot)$ and all of these functions integrate to one. Dominated convergence therefore allows us to take the limit in n inside the integral on the right side of (7.26) and hence prove the claim. Now let $s \downarrow 0$ to see that there is a jointly continuous extension of $p_t(x, y)$ to $\{(t, x, y): t >0, (x, y) \in G \times G\}$ which we also denote by $p_t(x, y)$. It is clear from (7.21) that

$$p_t(x, y) = E^x(p_{t-(V(t,\delta,y)}(X\, V(t, \delta, y)) \leqq c_{7.13}(\delta)\, t^{-d_s/2} \tag{7.27}$$

for all $t>0$, (x, y) in $G \times G$ and δ in $(0, c_{7.12})$.

Theorem 7.8. *X has a transition density, $p_t(x, y)$, with respect to μ such that*

(a) *$p_t(x, y)$ is jointly continuous on $\{(t, x, y): t>0, (x, y)$ in $G \times G\}$,*
(b) *$p_t(x, y) = p_t(y, x)$ for all $t>0$, (x, y) in $G \times G$,*
(c) *$p_t(x, y) = p_{t/5}(x/2, y/2)/3$ for all $t>0$, (x, y) in $G \times G$,*
(d) *(1.4) holds.*

Proof. (a) was proved in the above discussion. (b) is obvious from the symmetry of the semigroup (Theorem 2.21) and (a). (c) is clear from the scaling property of X, (2.61).

The upper bound in (1.4) is immediate from the upper bound in (7.27) and (7.14) (with $k=0$). For the lower bound choose δ in $(0, c_{7.12})$. If $|x-y|\, t^{-\gamma} \leqq \delta$ then

$$|X(V(t, \delta, y)) - y|\, (t - V(t, \delta, y))^{-\gamma} = \delta,$$

and so (7.27) and (7.15) imply

$$p_t(x, y) \geqq c_{7.9}(\delta) \exp\{-2c_{7.2}\,\delta^{1/(1-\gamma)}) E^x((t-V(t, \delta, y))^{-d_s/2})$$
$$\geqq c_1(\delta) t^{-d_s/2}.$$

This and (7.15) give the lower bound in (1.4). □

Dan Stroock has shown us a quite general technique that allows one to directly convert the upper bound on $\|u_\lambda(\cdot,\cdot)\|_\infty$, obtained in Proposition 5.17, into the upper bound on $\|p_t(\cdot,\cdot)\|_\infty$ obtained in (7.25). Let $\mathscr{E}(\cdot,\cdot)$ be the Dirichlet form associated with a symmetric Markov process which has a resolvent density $u_\lambda(\cdot,\cdot)$ satisfying $\|u_\lambda(\cdot,\cdot)\|_\infty \leqq c_1 \lambda^{\nu/2-1}$. An elementary argument then established the following generalized Nash inequality:

$$\|\phi\|_2^{2+4/\nu} \leqq c_2 \mathscr{E}(\phi, \phi) \|\phi\|_1^{4/\nu} \quad \text{for all } \phi \text{ in } C_0(G).$$

Theorem (2.1) of Carlen, Kusuoka and Stroock (1987) shows that this implies $\|p_t(\cdot,\cdot)\|_\infty \leqq c_3 t^{-\nu/2}$.

This proof gives the upper bound (7.25) without the moving boundary arguments that led to (7.27). We kave kept our derivation of (7.27) because in addition to the lower bound on $p_t(x, y)$, it will also give us information about the derivatives of $p_t(x, y)$, and because it is more in the spirit of the highly probabilistic "bare-hands" approach of the rest of the paper.

The next result extends the smoothness in t of $p_t(x, y)$ and the earlier estimates on $p_t^{(k)}(x, y) = \partial^k/\partial t^k (p_t(x, y))$ to the diagonal $x = y$.

Theorem 7.9. *For every $k \in \mathbb{Z}_+$:*

(a) *$p_t^{(k)}(x, y)$ exists and is jointly continuous in (t, x, y) in $(0, \infty) \times G \times G$.*

(b)

$$\limsup_{h \to 0} \{|(p_{t+h}^{(k)}(x, y) - p_t^{(k)}(x, y)) h^{-1} - p_t^{(k+1)}(x, y)| : x, y \in G, \varepsilon \leqq t \leqq \varepsilon^{-1}\} = 0 \tag{7.28}$$

for every $\varepsilon > 0$.

(c) *There is a $c_{7.14}(k)$ such that*

$$|p_t^{(k)}(x, y)| \leqq c_{7.14}(k) t^{-k-d_s/2} \exp(-c_{7.4}(|x-y| t^{-\gamma})^{1/1-\gamma}\} \tag{7.29}$$

for all $t > 0$, (x, y) in $G \times G$.

Proof. If $t > u > 0$, $(x, y) \in G \times G$, $k \in \mathbb{Z}_+$ and $\delta \in (0, c_{7.12}]$ satisfies $k(\delta) > k+1$, then

$$\begin{aligned} &E^x(|p_{t-V(u,\delta,y)}^{(k)}(X(V(u, \delta, y)), y)|) \\ &\leqq c_{7.7}(k) t^{1-d_s/2} E^x(|X(V(u, \delta, y)) - y|^{-d_w(k+1)}) \quad \text{(by (7.13))} \\ &\leqq c_{7.7}(k) \delta^{-d_w(k+1)} t^{1-d_s/2} u^{-(k+1)} E^x(U(u, \delta, y)^{-(k+1)}) \\ &\leqq c_1(k, \delta) t^{1-d_s/2} u^{-(k+1)}, \end{aligned} \tag{7.30}$$

where in the last we have used (7.18) and the fact that $k(\delta) > k+1$.

We now induct on $k\in\mathbb{Z}_+$ to prove (a), (b) and

(7.31) $p_t^{(k)}(x,y)=E^x(p_{t-V(u,\delta,y)}^{(k)}(X(V(u,\delta,y)),y))$ for all $t>u>0$, (x,y) in $G\times G$ and δ in $(0,c_{7.12}]$ such that $k(\delta)>k+2$.

If $t>u$ are fixed then the Chapman-Kolmogorov equations show $s\to p_{t-s}(X_s,y)$ is an $\mathscr{F}_s$-martingale (with respect to P^x) for s in $[0,t]$ and is uniformly bounded on $[0,u]$ by Theorem 7.8 (d). (7.31) is therefore valid for $k=0$ for any $\delta>0$ by the optional stopping theorem. (a) has already been proved for $k=0$ in Theorem 7.8 (a).

Assume (a) and (7.31) hold for k and consider $k+1$. Choose δ in $(0,c_{7.12}]$ such that $k(\delta)>k+3$, let $t>u>0$ and fix (x,y) in $G\times G$. If $|h|<1\wedge(t-u)$, then

$$\begin{aligned}
&|(p_{t+h}^{(k)}(x,y)-p_t^{(k)}(x,y))h^{-1}-E^x(p_{t-V(u,\delta,y)}^{(k+1)}(X(V(u,\delta,y)),y))|\\
&\quad\leqq E^x(|p_{t+h-V(u,\delta,y)}^{(k)}(X(V(u,\delta,y)),y)\\
&\qquad -p_{t-V(u,\delta,y)}^{(k)}(X(V(u,\delta,y)),y))h^{-1}\\
&\qquad -p_{t-V(u,\delta,y)}^{(k+1)}(X(V(u,\delta,y)),y)|) \qquad \text{(induction hypothesis)}\\
&\quad\leqq hE^X(\sup\{|p_s^{(k+2)}(X(V(u,\delta,y)),y)|:\\
&\qquad s \text{ between } t-V(u,\delta,y) \text{ and } t+h-V(u,\delta,y)\})\\
&\quad\leqq hc_{7.7}(k)(t+|h|)^{1-d_s/2}E^x(|X(V(u,\delta,y))-y|^{-d_w(k+3)}) \qquad \text{(by (7.13))}\\
&\quad\leqq hc_{7.7}(k)(t+1)^{1-d_s/2}\delta^{-d_w(k+3)}u^{-(k+3)}E^x(U(u,\delta,y)^{-(k+3)})\\
&\quad\leqq hc_1(\delta,k)(t+1)^{1-d_s/2}u^{-(k+3)},
\end{aligned}$$

the last by (7.18). This shows that

(7.32) $$\lim_{h\to 0}(p_{t+h}^{(k)}(x,y)-p_t^{(k)}(x,y))h^{-1}=E^x(p_{t-V(u,\delta,y)}^{(k+1)}(X(V(u,\delta,y)),y))$$

and the convergence is uniform on $\{(t,x,y):t\in[u+\varepsilon,u+\varepsilon^{-1}],$ (x,y) in $G\times G\}$ for any $\varepsilon>0$.

(7.31) follows for $k+1$ in place of k. The uniform convergence in (7.32) and assumed continuity of $p_t^{(k)}(x,y)$ show that $p_t^{(k+1)}(x,y)$ is continuous on $(0,\infty)\times G\times G$ (let $\varepsilon, u\downarrow 0$). To complete the induction it remains only to prove (b) for k. This, however, is immediate from (7.32) since ε and u may be taken arbitrarily small.

To prove (c), let $u\uparrow t$ in (7.30) and use (7.31) to see that

$$|p_t^{(k)}(x,y)|\leqq c_1(k,\delta)t^{-k-d_s/2}$$

for all $t>0$, (x,y) in $G\times G$, $\delta\in(0,c_{7.12}]$ and $k(\delta)>k+2$. This, together with (7.14), implies (7.29). □

Consider now the regularity properties of $p_t(\cdot,\cdot)$ in the spatial variables. We will see in Sect. 9 (Corollary 9.3) that they do not have a C^1 extension to a neighbourhood of G. On the other hand they are regular in the sense of being in the domain of arbitrarily many iterations of $\mathscr{A}$.

Inductively define

$$\mathscr{A}^{(l)} = \mathscr{A} \circ \mathscr{A}^{(l-1)}$$

on

$$\mathscr{D}(\mathscr{A}^{(l)}) = \{f \in \mathscr{D}(\mathscr{A}^{(l-1)}) \colon \mathscr{A}^{(l-1)} f \in \mathscr{D}(\mathscr{A})\}.$$

Theorem 7.10. (a) *For all* $t>0$, $y \in G$ *and* $k, l \in \mathbb{Z}_+$, $x \to p_t^{(k)}(x, y)$ *belongs to* $\mathscr{D}(A^{(l)})$ *and*

$$\mathscr{A}^{(l)}(p_t^{(k)}(\cdot, y))(x) = p_t^{(k+l)}(x, y).$$

(b) $|p_t^{(k)}(x, y) - p_t^{(k)}(x', y')| \leqq c_{7.15}(k) t^{-1-k} |(x\text{-}x', y-y')|^{d_w - d_f}$ *for all* $t>0$, $(x, y) \in G \times G$, $k \in \mathbb{Z}_+$.

Proof. (a) Fix $t>0$, $y \in G$ and $k \in \mathbb{Z}_+$. Theorem 7.9 shows that $p_t^{(k)}(\cdot, y) \in C_0(G)$ and (7.29) allows one to differentiate inside the integral in the Chapman-Kolmogorov equations to obtain

$$p_t^{(k)}(x, y) = \int p_s(x, z) p_{t-s}^{(k)}(z, y)\, \mu(dz), \qquad 0<s<t.$$

This implies that if $h>0$, then $s \to p_{t+h-s}^{(k)}(X_s, y)$ is a P^x-martingale for $s \in [0, t+h)$. Therefore, if $h>0$, then

$$E^x(p_t^{(k)}(X_h, y) - p_t^{(k)}(x, y)) h^{-1} = (p_{t+h}^{(k)}(x, y) - p_t^{(k)}(x, y)) h^{-1}.$$

By (7.28) the latter expression converges to $p_t^{(k+1)}(x, y)$ uniformly in $(x, y) \in G \times G$. (a) follows for $l=1$ and in general by a trivial induction on l.

(b) If $\lambda>0$ then the above shows that

$$p_t^{(k)}(x, y) = U_\lambda h_{k,t,y,\lambda}(x),$$

where

$$h_{k,t,y,\lambda}(x) = \lambda p_t^{(k)}(x, y) - p_t^{(k+1)}(x, y).$$

(7.29) implies

$$\|h_{k,t,y,\lambda}\|_\infty \leqq \lambda c_{7.14}(k) t^{-k-d_s/2} + c_{7.14}(k+1) t^{-k-1-d_s/2}.$$

Theorem 5.22 therefore gives us

$$\|p_t^{(k)}(x, y) - p_t^{(k)}(x', y)| \leqq 2 c_{5.7} \lambda^{-d_s/2} (\lambda c_{7.14}(k) t^{-k-d_s/2} + c_{7.14}(k+1) t^{-k-1-d_s/2}) |x-x'|^{d_w - d_f}.$$

Take $\lambda = t^{-1}$ and use the symmetry of $p_t^{(k)}(x, y)$ to complete the proof. □

Remark. Theorems 7.8, 7.9 and 7.10 together imply Theorem 1.5.

Add ∂ to G as the point at ∞, and if $A = \operatorname{int}(B_d(z, r))$ for $z \in G$ and $r>0$, consider the killed process

$$X^A(t) = \begin{cases} X(t) & \text{if } t < R(A) \\ \partial & \text{if } t \geqq R(A). \end{cases}$$

It is clear that X^A has a transition density with respect to μ, $p_t^A(x, y)$, such that

$$(7.33) \qquad p_t^A(x, y) \leqq p_t(x, y).$$

Using the strong Markov property at $R(A)$ (as in Fabes-Stroock (1986)) one may take

$$(7.34) \qquad p_t^A(x, y) = p_t(x, y) - E^x(1(R(A) \leqq t)\, p_{t-R(A)}(X(R(A)), y)).$$

It is not hard to use Proposition 2.13, Lemma 7.16, Lemma 7.1 (a) and Theorem 7.8 to prove the continuity in (t, x, y) of the right side of (7.34), first on $(0, \infty) \times B_d(z, \delta r) \times B_d(z, \delta r)$ for $0 < \delta < 1$ and hence on all of $(0, \infty) \times A \times A$. The argument in Fabes-Stroock (1986, Lemma (5.1)) and (1.4) may be used to prove the lower bound in

Theorem 7.11. *Let $A = B_d(z, r)$ where $z \in G$, $r > 0$.*

(a) (7.34) *defines a transition density for X^A which is jointly continuous on $(0, \infty) \times A \times A$.*

(b) *For any $\delta \in (0, 1)$ there are constants $c_{7.16}(\delta)$, $c_{7.17}(\delta)$ (independent of A) such that*

$$(7.35) \qquad p_t^A(x, y) \geqq c_{7.16}(\delta)\, t^{-d_s/2} \exp\{-c_{7.17}(\delta)(|y - x|\, t^{-1/d_w})^{d_w/(d_w - 1)}\}$$

for $t \in (0, r^{d_w})$ and $x, y \in B(z, \delta r)$.

8. Uniqueness of the Brownian Motion

Among continuous, $\mathbb{R}^d$-valued strong Markov processes, Brownian motion is uniquely determined (up to a trivial rescaling of time of the form $t \to ct$) by the properties of rotation and translation invariance. To state a corresponding theorem for X_t we introduce a group of local symmetries on G.

Definition. Let Δ_1 and Δ_2 denote two adjacent triangles in $\mathcal{T}_n$ which intersect at $x \in G_n$. $\Pi(\Delta_1 \cup \Delta_2)$ denotes the group of symmetries of $\Delta_1 \cup \Delta_2$ which fix x and are generated by the two mappings

π_0 = reflection of $\Delta_1 \cup \Delta_2$ in the perpendicular bisector of the common base of Δ_1 and Δ_2

π_1 = reflection of Δ_1 in the perpendicular bisector of Δ_1 through x, and the identity on Δ_2.

Consider the following hypotheses on G-valued processes.

(H$_1$) $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, Y_t, \theta_t, \tilde{P}^x)$ is a G-valued diffusion defined on the canonical space of continuous paths, i.e., a normal, continuous strong Markov process with state space G (in the sense of Blumenthal and Getoor (1968, p. 20, 30, 37)).

(H_2) For any $n\in\mathbb{Z}$ and any pair of adjacent triangles in $\mathscr{T}_n$, Δ_1 *and* Δ_2, which intersect at $x\in G_n$, we have

$$\tilde{P}^x(\pi(Y(\cdot\wedge T_1^n(Y)))\in A)=\tilde{P}^x(Y(\cdot\wedge T_1^n(Y))\in A)$$

for all $A\in\mathscr{B}(C([0,\infty),G))$ and all $\pi\in\Pi(\Delta_1\cup\Delta_2)$.

(H_3) If Δ_1, Δ_2, n and x are as in (H_2) and $\Psi\colon\Delta_1\cup\Delta_2\to\Psi(\Delta_1)\cup\Psi(\Delta_2)$ is the composition of a translation and a rotation of $\pm 2\pi/3$ or 0 which maps $\Delta_1\cup\Delta_2$ onto another pair of adjacent triangles in $\mathscr{T}_n$, then

(8.1) $\tilde{P}^x(\Psi(Y(\cdot\wedge T(\partial(\Delta_1\cup\Delta_2),Y)))\in A)=\tilde{P}^{\Psi(x)}(Y(\cdot\wedge T(\partial(\Psi(\Delta_1\cup\Delta_2)),Y))\in A)$
for all $A\in\mathscr{B}(C([0,\infty),G))$.

It Y is a simple random walk on G_m for $m\geqq n$, then (H_2) and (H_3) are obvious (A is now a Borel subject of $G^{\mathbb{Z}_+}$) by the corresponding invariance of G_m. The construction of X as a limit of these random walks shows that (H_1), (H_2) and (H_3) hold if $(Y,\tilde{P}^x)$ is replaced by (X,P^x). Clearly these hypotheses are also satisfied by the diffusion $X(ct)$ for any $c\in[0,\infty)$.

Theorem 8.1. *$(\Omega,\tilde{\mathscr{F}},\tilde{\mathscr{F}}_t,Y_t,\theta_t,\tilde{P}^x)$ satisfies* (H_1), (H_2) *and* (H_3) *if and only if there is a $c\in[0,\infty)$ such that for every $x\in G$, $\tilde{P}^x$ is the P^x-law of $X(c\cdot)$.*

Proof. By the above remarks it suffices to prove the sufficiency of (H_1)–(H_3). Assume throughout the rest of this section that $(\Omega,\tilde{\mathscr{F}},\tilde{\mathscr{F}}_t,Y_t,\theta_t,\tilde{P}^x)$ satisfies (H_1)–(H_3). We require some preliminary lemmas.

In the next lemma, a geometric random variable with mean ∞ is understood to be identically $+\infty$.

Lemma 8.2. *Let $x\in G_n$ and work with respect to $\tilde{P}^x$.*

(a) *$\{W_i^n(Y)\colon i\in\mathbb{N}\}$ are i.i.d. $[0,\infty]$-valued random variables, whose common law does not depend on x.*

(b) *If*

$$\rho(n)=\min\{i\colon W_i^n(Y)=\infty\},$$

and

$$Y(n)(i)=\begin{cases} Y(T_i^n(Y)) & \text{if } 0\leqq i<\rho(n)\\ \partial & \text{if } i\geqq\rho(n),\end{cases}$$

then conditional on $\{W_i^n(Y)\colon i\in\mathbb{N}\}$, $Y(n)$ is a simple random walk on G_n starting at x and killed at $\rho(n)$. Therefore (unconditionally) $Y(n)$ is a simple random walk on G_n, killed at an independent geometric time with mean $\tilde{P}^0(W_1^n(Y)=\infty)^{-1}$.

Proof. Let $M\in\mathbb{N}$, $\{A_i\colon 1\leqq i\leqq M\}\subset\mathscr{B}([0,\infty))$ and $\{x_i\colon 0\leqq i\leqq M\}\subset G_n$ satisfy $x_0=x$ and $x_i\in N_n(x_{i-1})$. If

$$B_i=\{Y(n)(i)=x_i,\quad W_i^n(Y)\in A_i\},$$

then

$$\tilde{P}^x\left(\bigcap_{i=1}^M B_i\right)=\tilde{E}^x\left(1\left(\bigcap_{i=1}^{M-1} B_i\right)\tilde{P}^{x_{M-1}}(Y(n)(1)=x_M,\,W_1^n(Y)\in A_M)\right)$$

(strong Markov property)

$$=\tilde{P}^x\left(\bigcap_{i=1}^{M-1} B_i\right)4^{-1}\,\tilde{P}^{x_{M-1}}(W_1^n(Y)\in A_M).$$

In the last line we have used the symmetry given by (H_2). Moreover (H_3) shows that $\tilde{P}^{x_{M-1}}(W_1^n(Y)\in A_M)$ is independent of x_{M-1}. By induction we get

$$\tilde{P}^x\left(\bigcap_{i=1}^{M} B_i\right)=4^{-M}\prod_{i=1}^{M}\tilde{P}^0(W_1^n(Y)\in A_i).$$

The results follow easily. □

Lemma 8.3. *If* n, Δ_1, Δ_2 *and* Ψ *are as in* (H_3) *and* $y\in(\Delta_1\cup\Delta_2)\cap G_\infty$, *then* (8.1) *holds if* x *is replaced by* y.

Proof. Let $y\in(\Delta_1\cup\Delta_2)\cap G_{n'}$ where $n'\geqq n$. We may assume $\tilde{P}^y(W_1^m(Y)<\infty)>0$ for large enough m because otherwise the result holds trivially (*both* processes in question will be constant by (H_3)). Choose $m\geqq n'$ such that $\tilde{P}^y(W_1^m(Y)<\infty)>0$. If $Y(m)(\cdot)$ is defined as in Lemma 8.2 and $x\in\Delta_1\cap\Delta_2$, then Lemma 8.2 implies that

$$\tilde{P}^x(T_y(Y(m))\leqq T(\partial(\Delta_1\cup\Delta_2), Y(m)),\ T_y(Y(m))<\infty)>0,$$

and therefore (use (8.1) as well),

$$\begin{aligned}(8.2)\qquad 0<\tilde{P}^x(T_y(Y)&\leqq T(\partial(\Delta_1\cup\Delta_2), Y),\ T_y(Y)<\infty)\\ &=\tilde{P}^{\Psi(x)}(T_{\Psi(y)}(Y)\leqq T(\partial(\Psi(\Delta_1\cup\Delta_2)), Y),\ T_{\Psi(y)}(Y)<\infty).\end{aligned}$$

If $A\in\mathscr{B}(C([0,\infty), G))$, then (8.1) shows that

$$\begin{aligned}&\tilde{P}^x(\Psi(Y((T_y+\cdot)\wedge T(\partial(\Delta_1\cup\Delta_2))))\in A\,|\,T_y(Y)\leqq T(\partial(\Delta_1\cup\Delta_2), Y),\ T_y(Y)<\infty)\\ &\quad=\tilde{P}^{\Psi(x)}(Y((T_{\Psi(y)}+\cdot)\wedge T(\partial(\Psi(\Delta_1\cup\Delta_2))))\in A\,|\\ &T_{\Psi(y)}(Y)\leqq T(\partial(\Psi(\Delta_1\cup\Delta_2))\ Y),\ T_{\Psi(y)}(Y)<\infty),\end{aligned}$$

and hence

$$\tilde{P}^y(\Psi(Y(\cdot\wedge T(\partial(\Delta_1\cup\Delta_2))))\in A)=\tilde{P}^{\Psi(y)}(Y(\cdot\wedge T(\partial(\Psi(\Delta_1\cup\Delta_2))))\in A)$$

by (8.2) and the strong Markov property. □

Recall the notation $S_m(x)$ introduced at the beginning of Sect. 2.

Lemma 8.4. *If*

$$(8.3)\qquad \sup_{x\in G}\tilde{P}^x(S_m(Y)>m)=1\quad \textit{for all}\ m\in\mathbb{N},$$

then

$$(8.4)\qquad \tilde{P}^x(Y(t)=x\ \textit{for all}\ t\geqq 0)=1\quad \textit{for all}\ x\in G.$$

Proof. We claim that it suffices to show

$$(8.5)\qquad \tilde{P}^0(W_1^m(Y)=\infty)\geqq\varepsilon\quad \text{for all}\ m\in\mathbb{N}\quad \text{and some}\quad \varepsilon>0.$$

Assume (8.5) and let $x \in G$. Then

$$\begin{aligned}
&\tilde{P}^x(\sup_t |Y(t)-x|>(m+1)2^{-m})\\
&\leqq \tilde{E}^x(1(T^m(Y)<\infty)\tilde{P}^{Y(T^m)}(W_i^m(Y)<\infty \text{ for all } i\leqq m))\\
&\leqq (1-\varepsilon)^m \quad \text{(by Lemma 8.2 and (8.5))}.
\end{aligned}$$

The Borel-Cantelli lemma implies (8.4) and the claim is proved.

Assume (8.3). If $\varepsilon_m \downarrow 0$, there are $\{x_m\}$ in G such that

$$\tilde{P}^{x_m}(S_m(Y)>m)\geqq 1-\varepsilon_m. \tag{8.6}$$

Choose $y_m \in \partial\Delta_{2m}(x_m)$ such that $D_m(y_m)=D_m(x_m)$. If $n>2m$ and $\tilde{P}^{y_m}(S_m(Y)<\infty)>0$, Lemma 8.2 shows that under $\tilde{P}^{y_m}$ and conditional on $\{S_m(Y)<\infty\}$, $i \to Y(T_i^n \wedge S_m)$ is a simple random walk on G_n starting at y_m and stopped when it hits $\partial D_m(y_m)$. Let $z_n \in \partial\Delta_n(x_m)$. We may apply Lemma 5.11 to the above random walk to see that for $n>2m$,

$$\tilde{P}^{y_m}(T_{z_n}(Y)\leqq S_m(Y)\,|\,S_m(Y)<\infty)\geqq(1-c_{5.2}(\tfrac{3}{5})^m).$$

Continuity of $Y(\cdot)$ shows that

$$\tilde{P}^{y_m}(T_{x_m}(Y)\leqq \liminf_{n\to\infty} T_{z_n}(Y)\leqq S_m(Y)\,|\,S_m(Y)<\infty)\geqq(1-c_{5.2}(\tfrac{3}{5})^m),$$

and therefore

$$\begin{aligned}
\tilde{P}^{y_m}(S_m(Y)>m)&\geqq \tilde{E}^{y_m}(1(T_{x_m}(Y)\leqq S_m(Y))\tilde{P}^{x_m}(S_m(Y)>m))\\
&\geqq(1-c_{5.2}(\tfrac{3}{5})^m)(1-\varepsilon_m).
\end{aligned}$$

(The same inequality is trivial if $\tilde{P}^{y_m}(S_m(Y)<\infty)=0$). Hence by redefining $\{\varepsilon_m\}$ and replacing x_m with y_m we may assume without loss of generality that $\{x_m\}\subset G_\infty$.

Use the strong Markov property at $T^m(Y)$ to see that (8.6) implies either

(i) $\tilde{P}^{z_m}(T(\partial D_m(x_m), Y)>m/2)>\frac{7}{8}$ for some $z_m \in \partial\Delta_m(x_m)$ and arbitrarily large m

or (ii) $\tilde{P}^{x_m}(T^m(Y)>m/2)>\frac{1}{9}$ for arbitrarily large m.

Assume (i) holds and consider an m for which the inequality in (i) holds. Choose $y_m \in \partial D_m(x_m)$ such that $|y_m-z_m|=2^{-m}$ (z_m as in (i)). Then

$$\begin{aligned}
\tfrac{1}{8}&>\tilde{P}^{z_m}(T(\partial D_m(x_m), Y)\leqq m/2)\\
&\geqq \tilde{P}^{z_m}(W_1^m(Y)\leqq m/2,\ Y(W_1^m)=y_m)\\
&=(\tfrac{1}{4})\tilde{P}^{z_m}(W_1^m(Y)\leqq m/2) \quad \text{(Lemma 8.2)}.
\end{aligned}$$

Therefore for arbitrarily large m, one has

$$\begin{aligned}
\tilde{P}^0(W_1^m(Y)>m/2)&=\tilde{P}^{z_m}(W_1^m(Y)>m/2) \quad \text{(Lemma 8.2(a))}\\
&>\tfrac{1}{2}.
\end{aligned}$$

The monotonicity of $\{W_1^m\}$ now proves (8.5), completing the argument in this case.

Assume (ii) and consider an m for which the inequality in (ii) holds. Lemma 8.3 allows us to translate and rotate and therefore assume without loss of generality that $d(0, x_m) < 2^{-m}$ (recall $x_m \in G_\infty$). If $j < m-1$, then

$$\begin{aligned}\tilde{P}^0(W_1^j > m/2) &\geqq \tilde{P}^0(T_{x_m} \leqq W_1^j)\, \tilde{P}^{x_m}(T^m > m/2)\\ &\geqq \tilde{P}_0(T_{x_m} \leqq S_{j+1})/9 \qquad \text{(by (ii))}\\ &\geqq (1 - c_{5.2}(\tfrac{3}{5})^{m-j-1})/9.\end{aligned}$$

In the last line we have again used Lemmas 5.11 and 8.2 as before. Now let $m \to \infty$ through an appropriate subsequence to derive $\tilde{P}^0(W_1^j = \infty) \geqq \frac{1}{9}$ and hence obtain (8.5) and complete the proof. □

Lemma 8.5. *If*

$\tilde{P}^x(\exists t \geqq 0, \varepsilon > 0$ such that $Y(s) = Y(t)\ \forall s \in [t, t+\varepsilon)) > 0$ for some x in G, then $\tilde{P}^x(Y_t = Y_0\ \forall t \geqq 0) = 1$ for all x in G.

Proof. A standard argument, using only (H_1), shows that the hypothesis implies the set of traps,

$$S = \{x \in G : \tilde{P}^x(Y_t = x \text{ for all } t \geqq 0) = 1\},$$

is non-empty. Lemma 8.4 now implies $S = G$, as required. □

We are now ready to return to the

Proof of Theorem 8.1. We may assume, without loss of generality, that for every $x \in G$, $\tilde{P}^x(Y_t \neq Y_0$ for some $t > 0) > 0$ because if this fails for some x, Lemma 8.4 proves the theorem with $c = 0$.

The above assumption and Lemma 8.4 shows there is an $n_1 \in \mathbb{N}$ such that

$$\sup_{y \in G} \tilde{P}^y(T_1^{n_1}(Y) > n_1) \equiv p_0 < 1.$$

Use the Markov property at n_1 to conclude that

$$\sup_{y \in G} \tilde{P}^y(T_1^{n_1}(Y) > 2n_1) \leqq p_0^2.$$

Proceeding inductively we see that for some universal constants θ, $T > 0$,

$$\tilde{P}^y(T_1^{n_1}(Y) > t) \leqq e^{-\theta t} \quad \text{for } t \geqq T \text{ and all } y \in G. \tag{8.7}$$

Fix $x \in G_{n_0}$ where $n_0 \geqq n_1$ and work with respect to $\tilde{P}^x$. If $n \geqq n_0$, then $T_1^n(Y) < \infty$ $\tilde{P}^x$-a.s. and therefore $Y(n)(i) = Y(T_i^n(Y))$ is a random walk on G_n by Lemma 8.2. Define

$$\tau_n(t) = T^n_{[5^n t]}(Y),$$

$$Y_n(t) = Y(\tau_n(t)) = Y(n)([5^n t]). \tag{8.8}$$

The collection of random walks $\{Y(n): n \geqq n_0\}$ satisfies the nesting property (2.12). Theorem 2.8 (with $\{Y(n): n \geqq n_0\}$ in place of $\{X(n, x): n \in \mathbb{N}\}$) shows that

$$\lim_{n \to \infty} Y_n = X \quad \text{in } C([0, \infty), G) \ \tilde{P}^x\text{-a.s.},$$

where $\tilde{P}^x(X \in \cdot) = P^x(\cdot)$ on $\mathscr{B}(C([0, \infty), G))$. Lemma 2.5(b) and Remark 2.6 show that

$$N_n = W_1^{n_0}(Y(n_0 + n))$$

is a supercritical branching process starting at 1 (when $n = 0$) and with offspring distribution equal to the law of N (see Lemma 2.2). The definition of N_n gives

$$W_1^{n_0}(Y) = \sum_{i=1}^{N_n} W_i^{n+n_0}(Y), \qquad n \in \mathbb{N}. \tag{8.9}$$

The summands in (8.9) are i.i.d. and are independent of N_n by Lemma 8.2. Let $\mu_n = \tilde{E}^x(W_1^n(Y))$ and $\sigma_n^2 = \mathrm{Var}(W_1^n(Y))$ (independent of x). By (8.7) we may take expected values in (8.9) to get

$$\mu_{n+n_0} = 5^{-n} \mu_{n_0} \tag{8.10}$$

and therefore

$$\tilde{E}^x(\tau_{n+n_0}(t)) = [5^{n+n_0} t] \, 5^{-n} \mu_{n_0}. \tag{8.11}$$

(8.7) shows that $\sigma_n^2 < \infty$ for $n \geqq n_1$, and (8.9) and the above independence properties show that

$$\tilde{E}^x(W_1^{n_0}(Y)^2) = \sum_{j=1}^{\infty} \tilde{P}^x(N_n = j)(j \sigma_{n+n_0}^2 + j^2 \mu_{n+n_0}^2) \geqq \tilde{E}^x(N_n) \sigma_{n+n_0}^2 = 5^n \sigma_{n+n_0}^2.$$

Therefore

$$\sigma_{n+n_0}^2 \leqq \tilde{E}^x(W_1^{n_0}(Y)^2) \, 5^{-n},$$

and (8.11) now shows

$$\tilde{E}^x(\tau_{n+n_0}(K)^2) \leqq K \, 5^{n_0} \tilde{E}^x(W_1^{n_0}(Y)^2) + K^2 \, 25^{n_0} \mu_{n_0}^2 < \infty \qquad \text{for all } K \in \mathbb{N}. \tag{8.12}$$

We claim that

$$\max_{i \leqq 5^{n_0+n} K} W_i^{n+n_0}(Y) \xrightarrow{\tilde{P}^x} 0 \quad \text{as } n \to \infty \qquad \text{for all } K \in \mathbb{N}. \tag{8.13}$$

If ε, $M > 0$, then

$$\begin{aligned}\tilde{P}^x(\max_{i \leqq 5^{n_0+n}K} W_i^{n+n_0}(Y) > \varepsilon) \\ \leqq \tilde{P}^x(T^{n+n_0}_{5^{n_0+n}K}(Y) > M) + \tilde{P}^x(\inf_{t \leqq M} \sup_{u \in [t, t+\varepsilon]} |Y(u) - Y(t)| \leqq 2^{-(n+n_0)}).\end{aligned}$$

The last term approaches zero as $n \to \infty$ by Lemma 8.5 and our earlier assumption. The first term is bounded by $M^{-1} 5^{n_0} K \mu_{n_0}$ by (8.11) and hence can be made arbitrarily small, uniformly in n, by taking M large. This proves (8.13).

For $n \in \mathbb{N}$ and ε, $t > 0$, we have

$$\begin{aligned}(8.14) \qquad \tau_{n+n_0}(t) - \tilde{E}^x(\tau_{n+n_0}(t)) &= \sum_{i=1}^{[5^{n+n_0}t]} (W_i^{n+n_0}(Y) - W_i^{n+n_0}(Y) \wedge \varepsilon) \\ &+ \sum_{i=1}^{[5^{n+n_0}t]} W_i^{n+n_0}(Y) \wedge \varepsilon - \tilde{E}^x(W_i^{n+n_0}(Y) \wedge \varepsilon) \\ &- \sum_{i=1}^{[5^{n+n_0}t]} \tilde{E}^x(W_i^{n+n_0}(Y) - W_i^{n+n_0}(Y) \wedge \varepsilon).\end{aligned}$$

(8.13) implies the first term converges to zero in probability as $n \to \infty$, and (8.12) shows that the first term is L^2 bound in n. Therefore by dominated convergence, the last term also approaches zero as $n \to \infty$. The square of the L^2-norm of the second term is bounded by

$$\begin{aligned}5^{n+n_0} t E((W_1^{n+n_0}(Y) \wedge \varepsilon)^2) \leqq \varepsilon\, 5^{n+n_0} t\, 5^{-n} \mu_{n_0} \qquad \text{(by (8.10))} \\ \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly in } n.\end{aligned}$$

It therefore follows from (8.14) and (8.11) that

$$\tau_n(t) \xrightarrow{\tilde{P}^x} 5^{n_0} \mu_{n_0} t \equiv c^{-1} t \quad \text{as } n \to \infty, \qquad \text{for all } t \geqq 0.$$

Let $n \to \infty$ in (8.8) to get

$$(8.15) \qquad X(t) = Y(c^{-1} t) \quad \text{for all } t \geqq 0 \ \tilde{P}^x\text{-a.s.}$$

The choice of c is independent of $x \in G_{n_0}$ for each n_0 and hence cannot depend on n_0 (because $G_{n_0} \subset G_{n_0+1}$). Therefore (8.15) is valid for all $x \in G_\infty$.

Now consider $x \in G - G_\infty$ and choose a sequence $\{x_n\}$ in G_∞ which converges to x. If $0 < \varepsilon < t$ and $f \in C_b(G)$, then

$$\begin{aligned}(8.16) \quad \tilde{E}^x(f(Y(t))) = \lim_{n \to \infty} \tilde{E}^x(1(T^n(Y(\omega)) < \varepsilon)\, \tilde{E}^{Y(T^n)(\omega)}(f(Y(t - T^n(Y(\omega)))))) \\ + \tilde{E}^x(1(T^n(Y) \geqq \varepsilon)\, f(Y(t))).\end{aligned}$$

Lemma 8.5 and our initial assumption imply

$$\lim_{n\to\infty} T^n(Y)=0 \quad \tilde{P}^x\text{-a.s.,} \tag{8.17}$$

and therefore the last term in (8.16) approaches zero as $n\to\infty$. The first term on the right side of (8.16) equals

$$\lim_{n\to\infty} \tilde{E}^x(1(T^n(Y(\omega))<\varepsilon)\, E^{Y(T^n)(\omega)}(f(X(c(t-T^n(Y(\omega)))))),$$

by the above. Proposition 2.13 shows $E^x(f(X(ct)))$ is jointly continuous in (t, x) and hence (8.17) shows that the above limit, and therefore $\tilde{E}^x(f(Y(t)))$, equals $E^x(f(X(ct)))$. □

9. Some Remarks on the Infinitesimal Generator

We showed in Sect. 5 (Theorem 5.22) that every function in $\mathscr{D}(\mathscr{A})$ is Hölder continuous of index $d_w-d_f=0.736966\ldots$. Our conjecture is that no non-constant function in $\mathscr{D}(\mathscr{A})$ is Hölder continuous of index $d_w-d_f+\varepsilon$ for any $\varepsilon>0$. Although this suggests that functions in $\mathscr{D}(\mathscr{A})$ are fairly rough, they must also, in a certain sense, be exceptionally smooth. If $f\in\mathscr{D}(\mathscr{A})$, then $f(X_t)$ is a semimartingale by Dynkin's formula. Therefore f maps a path of positive d_w-variation ($d_w>2$) (Theorem 4.5, Remark 4.6) into a path of finite quadratic variation. We therefore expect $|f(y)-f(z)|\leqq|y-z|^{d_w/2}$ ($d_w/2=1.160964\ldots$) for "most points (y, z) sufficiently close in G". This argument and conclusion are made precise in

Theorem 9.1. *Let $f\in\mathscr{D}(\mathscr{A})$ and for $R>0$, let*

$$B_R=\{z\in G: |z|\leqq R\}.$$

There is a constant $c(f, R)$ such that for any $\varepsilon>0$ and $N\in\mathbb{N}$,

$$\frac{\mu\times\mu(\{(y,z)\in B_R\times B_R: |y-z|\leqq 2^{-N}, |f(y)-f(z)|\geqq|y-z|^{(d_w/2)-\varepsilon}\}}{\mu\times\mu(\{(y,z)\in B_R\times B_R: |y-z|\leqq 2^{-N}\})} \leqq c(f,R)\,2^{-2\varepsilon N}. \tag{9.1}$$

In particular, this ratio approaches zero as $N\to\infty$.

Proof. Let $f\in\mathscr{D}(\mathscr{A})$ and $R>0$. By Dynkin's formula,

$$f(X_t)=M_t^f+\int_0^t Af(X_s)\,ds,$$

where M^f is a martingale which is evidently uniformly bounded on compact time intervals. If $t_i^n = i\,5^{-n}$, then

$$\sum_{i=5^n+1}^{2(5^n)} (f(X(t_i^n)) - f(X(t_{i-1}^n)))^2 \xrightarrow{L^1} [f(X), f(X)]_2 - [f(X), f(X)]_1 \quad \text{as } n \to \infty$$

(see Meyer (1976, p. 355)). Here and throughout the proof, the underlying measure is P^0. Let $I_n = \{i \in \mathbb{N}: 5^n < i \leqq 2(5^n)\}$ and

$$A_n = \{(y, z) \in G \times G: 2^{-n-1} < |y-z| \leqq 2^{-n}\}.$$

Choose $c_1(f)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned}
c_1(f) &\geqq E^0\Big(\sum_{i\in I_n} (f(X(t_i^n)) - f(X(t_{i-1}^n)))^2\Big) \\
&= \sum_{i\in I_n} \int_G \int_G p_{t_{i-1}^n}(0, y)\, p_{5^{-n}}(y, z)(f(z) - f(y))^2\, d\mu(z)\, d\mu(y) \\
&\geqq \int_{B_R} \sum_{i\in I_n} p_{t_{i-1}^n}(0, y) \Big(\int 1(|y-z| \leqq 2^{-n}, |f(y) - f(z)| \geqq |y-z|^{(d_w/2)-\varepsilon}) \\
&\quad \times c_{1.2}\, 3^n e^{-c_{1.5}} |z-y|^{d_w - 2\varepsilon}\, d\mu(z)\Big)\, d\mu(y) \qquad \text{(by (1.4))} \\
&\geqq c_2(R) \int_{B_R} \int_G 5^n\, 1(|f(y) - f(z)| \geqq |y-z|^{(d_w/2)-\varepsilon}, (y, z) \in A_n)\, 5^{-n}\, 2^{2\varepsilon n} \\
&\qquad \times 3^n\, d\mu(z)\, d\mu(y) \qquad \text{(by (1.4) again).}
\end{aligned}$$

Rearrange the above inequality to get

$$\int_{B_R} \int_G 1((y, z) \in A_n, |f(y) - f(z)| \geqq |y-z|^{d_w/2-\varepsilon})\, d\mu(z)\, d\mu(y) \leqq c_1(f)\, c_2(R)^{-1}\, 2^{-2\varepsilon n}\, 3^{-n}.$$

An elementary calculation shows that

$$\mu \times \mu(A_n \cap (B_R \times B_R)) \geqq c_3(R)\, 3^{-n} \qquad \text{for all } n \in \mathbb{N}.$$

Combine the last two inequalities and sum over $n \geqq N$ ($N \in \mathbb{N}$) to conclude that

$$\begin{aligned}
&\int_{B_R} \int_G 1(|y-z| \leqq 2^{-N}, |f(y) - f(z)| \geqq |y-z|^{(d_w/2)-\varepsilon})\, d\mu(z)\, d\mu(y) \\
&\leqq c_1(f)\, c_2(R)^{-1}\, c_3(R)^{-1} \sum_{n=N}^{\infty} 2^{-2\varepsilon n}\, \mu \times \mu(A_n \cap (B_R \times B_R)) \\
&\leqq c_1(f)\, c_2(R)^{-1}\, c_3(R)^{-1}\, 2^{-2\varepsilon N}\, \mu \times \mu(\{(y, z) \in B_R \times B_R: |y-z| \leqq 2^{-N}\}). \qquad \square
\end{aligned}$$

Non-constant infinitely differentiable functions are much too rough to satisfy (9.1).

Corollary 9.2. *If U is an open set which intersects G and $f\in\mathcal{D}(\mathcal{A})$ has a C^1 extension to U, then f must be constant on $U\cap G$. In particular, the only functions in $\mathcal{D}(\mathcal{A})$ which have a C^1 extension to a neighbourhood of G are the constant functions.*

Proof. Assume U and f are as above and let f also denote the C^1 extension to U. Assume f is not constant on $U\cap G$. Then $\nabla f(x_0)\neq 0$ for some $x_0\in U\cap G_{n_0}$ and some $n_0\in\mathbb{N}$ (use the fact that f must be non-constant on some graph $G^{(n_0)}$). The mean value theorem shows there are $\delta, \eta>0$ such that if $y, z\in B(x_0,\delta)$, $y\neq z$ and $\left|\frac{(z-y)}{|z-y|}\cdot\frac{\nabla f(x_0)}{|\nabla f(x_0)|}\right|\geqq 2^{-\frac{1}{2}}$, then $|f(z)-f(y)|\geqq\eta|z-y|$. This implies

$$(9.2)\quad \mu\times\mu\{(y,z)\in B(x_0,\delta)^2: |z-y|\leqq 2^{-N}, |f(z)-f(y)|\geqq\eta|z-y|\}$$
$$\geqq \mu\times\mu\left\{(y,z)\in B(x_0,\delta)^2: 0<|z-y|\leqq 2^{-N}, \left|\frac{(z-y)}{|z-y|}\cdot\frac{\nabla f(x_0)}{|\nabla f(x_0)|}\right|\geqq 2^{-\frac{1}{2}}\right\}$$
$$\geqq c_1\, 3^{-N},$$

where $c_1>0$ does not depend on N. Choose R such that $B(x_0,\delta)\subset B_R$ (B_R as in Theorem 9.1). Note that

$$(9.3)\qquad \mu\times\mu\{(y,z)\in B_R^2: |y-z|\leqq 2^{-N}\}\leqq c_2\, 3^{-N}\qquad \text{for all } N\in\mathbb{N}.$$

(9.2) and (9.3) together contradict Theorem 9.1 with $\varepsilon>0$ chosen so that $(d_w/2)-\varepsilon>1$. □

An immediate consequence of the above and Theorem 7.10 is

Corollary 9.3. *For every $t>0$ and $y\in G$, $x\to p_t(x,y)$ does not have a C^1 extension to a neighbourhood of G.* □

Functions in $\mathcal{D}(\mathcal{A})$ appear to have Cantor-like properties reminiscent of functions in the domain of a one-dimensional diffusion with a singular increasing scale function. Unlike the latter setting, however, an explicit construction of a function in $\mathcal{D}(\mathcal{A})$ appears to be difficult. It is interesting to ponder the behaviour of $p_t(\cdot,\cdot)$ (the one class of functions we have found in $\mathcal{D}(\mathcal{A})$) in light of the above results.

10. Concluding Remarks

Density of States

Let $z\in G$, $r>0$, $D=\operatorname{int}(B_d(x,r))$ and $\mathcal{A}^D$ be the infintesimal generator of X^D (the process X killed on hitting ∂D). Let $0>-\lambda_0>-\lambda_1>\dots$ denote the eigenvalues of $\mathcal{A}^D$ and $\{\Psi_n\}$ the complete orthonormal system (in $L^2(D,\mu)$) of corresponding eigenfunctions. There is considerable interest in the physics literature in the asymptotic frequency ("density of states") of $\{\lambda_n\}$. Theorem 7.11 and (7.33) show that P_t^D, the semigroup of X^D at time t, is a Hilbert-Schmidt integral operator on $L^2(D,\mu)$ with a continuous, symmetric kernel $p_t^D(\cdot,\cdot)$ and eigenvalues

$\{e^{-\lambda_n t}: n \in \mathbb{Z}_+\}$. Therefore Mercer's theorem (see Riesz-Nagy (1952, p. 242)) shows that

$$\int_D p_t^D(x,x)\,d\mu(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} = \int_0^{\infty} e^{-st}\,Q(ds), \tag{10.1}$$

where

$$Q(B) = \mathrm{card}\{\lambda_n : \lambda_n \in B\}.$$

By (7.33), (1.4) and (7.35), the left side of (10.1) is bounded between $c_1(D)\,t^{-d_s/2}$ and $c_2(D)\,t^{-d_s/2}$ as $t \downarrow 0$. A Tauberian theorem (see De Haan and Stadtmüller (1985, Theorem 1)) shows that

$$c_{10.1}(D)\,x^{d_s/2} \leqq Q([0,x]) = \mathrm{card}\{\lambda_n : \lambda_n \leqq x\} \leqq c_{10.2}(D)\,x^{d_s/2}, \qquad x \geqq \lambda_0,$$

which agrees with the results in the physics literature.

d-Dimensional Gaskets

There is an obvious d-dimensional analogue of the 2-dimensional gasket. Let $0 = x_0, x_1, \ldots, x_d$ be points in $\mathbb{R}^d$, with $|x_i - x_j| = \delta_{ij}$, let $G_0 = \{x_0, \ldots, x_d\}$, and let $G^{(0)}$ be the graph with vertices G_0, and edges connecting x_i and x_j for $i \neq j$. Thus $G^{(0)}$ is a d-dimensional tetrahedron, and has $d+1$ vertices, and $\frac{1}{2}d(d+1)$ edges. Now let x_{ij} be the midpoint of the line joining x_i to x_j, write $x_{ii} = x_i$, let $G_1 = \{x_{ij},\ 0 \leqq i,\ j \leqq d+1\}$, and let $G^{(1)}$ be the graph with vertices G_1, and edges between x_{ij} and x_{ik}, x_{jl}, $0 \leqq k,\ l \leqq d$, $k \neq j$, $l \neq i$. $G^{(1)}$ consists of $d+1$ d-dimensional tetrahedrons, with each pair sharing exactly one vertex.

Repeating this procedure, one obtains $G_2, G_3, \ldots$, and setting $G' = cl(\cup G_n)$ gives a bounded set in $\mathbb{R}^d$ with dimension $d_f = \log(d+1)/\log 2$. As before, we can now build outwards to form an unbounded set G, with the same kind of local scale, rotation and translation self-similarity as in the 2-dimensional gasket.

When we wish to emphasize the dependency on the dimension d, we will refer to $G(d)$, $d_f(d)$, etc.

Lemma 10.1. *Let Y be a simple symmetric random walk on $G^{(1)}$, starting at x_0 let*

$$N = \min\{r \geqq 0 : Y_r \in \{x_{11}, \ldots, x_{dd}\}\};$$

$$H = \sum_{0 \leqq r \leqq N} 1_{(Y_r = x_0)}, \qquad f(u) = E^{x_0} u^N, \qquad h(u) = E^{x_0} u^H.$$

Then

(a)
$$f(u) = \frac{u^2}{2d - 3(d-1)u + (d-2)u^2}$$

$$h(u) = \frac{(d+1)u}{(d+3-2u)}$$

(b)
$$EN = d+3,\ EH = \frac{d+3}{d+1}.$$

Proof. Set $f_{ij}(u)=E^{x_{ij}}u^N$. Note that each vertex of $G^{(1)}$ has $2d$ neighbours, and that, by symmetry, we need to consider only 4 kinds of point: x_{00}, x_{0i}, $1\leqq i\leqq d$, x_{ij}, $1\leqq i\neq j\leqq d$, and x_{ii}, $1\leqq i\leqq d$. We have, conditioning on Y_1, and using the symmetry,

$$f_{00}(u)=uf_{01}(u),$$

$$f_{01}(u)=\frac{u}{2d}(f_{00}(u)+f_{11}(u)+(d-1)f_{01}(u)+(d-1)f_{12}(u)),$$

$$f_{12}(u)=\frac{u}{2d}(2f_{11}(u)+2f_{01}(u)+2(d-2)f_{12}(u)),$$

$$f_{11}(u)=1.$$

Solving these equations, one obtains (a) for $f_{00}=f$. The expression for h is obtained in the same way, and by evaluating $f'(1)$, $h'(1)$ one proves (b). □

Note that $N-1$ is only geometric in the case $d=2$.

We now set $d_w=d_w(d)=\log(d+3)/\log 2$, $d_s=d_s(d)=2\log(d+1)/\log(d+3)$.

The proofs and results of the paper extend without any difficulty to the higher dimensional gaskets, with, of course, appropriate changes of constants. In particular, the Brownian motion on $G(d)$ a set with dimension $\log(d+1)/\log 2$, has a continuous local time. This emphasizes that it is the spectral dimension, d_s, rather than the fractal dimension d_f, which governs the behaviour of the Brownian motion on G.

Remark. If $d=1$, the construction of the d-dimensional gasket just gives us $\mathbb{R}$, and the Brownian motion on $G(1)$ is just ordinary Brownian motion, run at the usual speed. We have $d_f(1)=1$, $d_w(1)=2$, $d_s(1)=1$, and Theorems 1.5 and 1.9 become (weak) restatements of well known facts.

We have $f(u)=\frac{u^2}{2-u^2}$, and in this case the functional equation (3.2) has the explicit solution

$$\phi(u)=(\cosh(2u)^{\frac{1}{2}})^{-1}.$$

Thus the hitting time of the set $\{1, -1\}$ by a 1-dimensional Brownian motion is the limiting distribution of a supercritical branching process.

We conclude the paper with a list of problems.

The Distribution of W

Although the analysis in Sect. 3 leads to a fairly detailed account of the behaviour of $g(x)$ and $G(x)$, a number of problems remain.

Problem *10.2*. Obtain other bounds on the oscillation of the function $k(z)$, defined in Sect. 3.

This has been done by Dubuc (1982) for branching processes with $p_0 + p_1 > 0$.

While $|c_{3.5} - c_{3.4}|$ is small if $|c_{3.3} - c_{3.1}|$ is small, our methods do not give good values for the constants $c_{3.10}, c_{3.12}$.

Problem 10.3. Obtain good estimates for $c_{3.10}, c_{3.12}$.

We do not know anything about the sign of $g'(x)$:

Problem 10.4. (i) Is $g(x)$ monotone in a neighbourhood of 0?

(ii) Is $g(x)$ unimodal?

Oscillation in $p_t(x, x)$

From Theorem 1.5 we have

$$c_{1.2} \leqq t^{d_s/2} p_t(x, x) \leqq c_{1.4}, \qquad \text{for all } x \in G, \; t \geqq 0.$$

Problem 10.5. Does the limit

$$\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x) \tag{10.2}$$

exist?

We suspect not: the tiny oscillations in $\phi(u)$ are likely to lead to similar small oscillations in $p_t(x, x)$. However, the methods of this paper are too crude to give us any information on this.

It is natural to approach $p_t(x, x)$ via $u_\lambda(x, x)$, and indeed standard Tauberian arguments show that (10.2) exists if and only if the limit

$$\lim_{\lambda \to \infty} \lambda^{1 - \frac{1}{2} d_s} u_\lambda(x, x) \tag{10.3}$$

exists.

Dimension of the Filtration of X

Let $\mathscr{F}_t$ be the usual augmentation of the filtration $\sigma(X_s, s \leqq t)$. Since X is a continuous strong Feller process, by Meyer's theorem every $\mathscr{F}_t$ martingale is continuous.

Problem 10.6. What is the dimension of $(\mathscr{F}_t)$, in the sense of Davis and Varaiya (1974)?

At present no effective technique exists for beginning to answer this problem. We are inclined to guess that the answer is 2 (and d for the d-dimensional case).

Fourier Analysis on G

It seems possible that there are, on G, natural analogues of the sin and cos functions on $\mathbb{R}$, that would enable one to express the action of the generator in terms of appropriate multipliers.

This seems to be the only hope of being able to do effective computations with $\mathscr{A}$ and $\mathscr{D}(\mathscr{A})$.

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