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Probability Theory and Related Fields

Random Non-Linear Wave Equations: Smoothness of the Solutions

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Summary. We show existence and uniqueness for the solution of a one-dimensional wave equation with non-linear random forcing. Then we give sufficient conditions for the solution at a given time and a given point, to have a density and for this density to be smooth. The proof uses the extension of the Malliavin calculus to the two parameters Wiener functionals.

Introduction

We consider the stochastic partial differential equation:

$$\frac{\partial^2 X}{\partial t^2}(t, x)-\frac{\partial^2 X}{\partial x^2}(t, x)=a(X(t, x))\,\xi(t, x)+b(X(t, x)) \tag{I.1}$$

where the "time variable" t varies in $[0, \infty)$ and the "space variable" x varies in an interval I which could be bounded or semibounded or even be the whole real line. When I has a finite endpoint, we will impose a Dirichlet boundary condition for the solutions. This equation is a wave equation (because of its left hand side) with a nonlinear forcing term (because of its right hand side). The functions a and b are assumed to be continuously differentiable on $\mathbf{R}$ with bounded derivatives. ξ is the source of the stochasticity. We will assume it is a white noise in time as well as in space.

The problem of a vibrating string forced by a space-time white noise has been considered in the linear case of $b \equiv 0$ and $a \equiv 1$ by Cabana and Orsingher. See for example [3, 4, 16, and 17]. Even if complications due to damped vibrations or random initial conditions are introduced, the situation is relatively simple in the sense that one does not go out of the Gaussian (or conditionally Gaussian) case. Cabana introduced the "planar Brownian motion" in order to solve the equation of the vibrating string and he obtained some probabilistic bounds on the energy in [3]. He also solved some particular barrier problems

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in [4]. Orsingher considered two different kinds of sources for the randomness in [16]. On one hand he considered a space-time white noise and on the other, a noise of the form $\xi(t, x) = z(t)\, w(x)$ where $z(t)$ is a white noise in time and $w(x)$ a stationary Gaussian process independent of z. He computed some upper bounds for the probability that the string exceeds some given level. He analyzed the damped case and the case of an initial white noise disturbance in [17]. The analysis of the Eq. (I.1) has been proposed in Walsh's lectures [21] on stochastic partial differential equations.

Equation (I.1) is obviously formal. We use ideas of [21] to rewrite it as an integral equation in order to give a rigorous meaning to the notion of solution. We prove existence and uniqueness in Sect. II below. The techniques involved are standard. See for example [21]. We also define line integrals in the same spirit as in [24] and we check a Markov property in the spirit of the work of Wong and Zakai in [23].

In the companion paper [6], we defined a singularity of a solution as an anomaly in the local modulus of continuity and we proved the existence of such singularities and their propagations along the characteristic curves. Also we showed how these singularities were reflected at the boundary of the interval I.

The main part of the present paper is Sect. III. There, we investigate the following problem: given a time $t > 0$ and a position $x \in I$, the displacement $X(t, x)$ of the string is a random variable, and we would like to know if its distribution has a density and if such a density is smooth. Since this distribution is not the solution of an equation of a diffusion type, we cannot use Hormander's theory of hypoelliptic operators. We will use instead the calculus developed in [10] by Malliavin to show the existence and the regularity of the density for the distribution of the solution of a stochastic differential equation. Various approaches to the Malliavin calculus have been proposed in [1, 8, 18–20, 22] and applications to multiparameter stochastic differential equations have already been given by Nualart and Sanz in [13] and [14].

Typically, the Malliavin calculus gives sufficient conditions for the absolute continuity of the law of a functional of a Gaussian process and for the smoothness of the density. We will use it in the following form. Let W be an L^2-Gaussian measure with orthogonal increments on the Borel subsets of our parameter space $T = [0, \infty) \times I$ and let F be a functional of the form $F = f(W(A_1), \ldots, W(A_N))$ where $A_1, \ldots, A_N$ are Borel subsets of T and f a bounded smooth function with bounded derivatives on $\mathbf{R}^N$. Then we set:

$$[DF](\mathbf{r}) = \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(W(A_1), \ldots, W(A_N))\, \mathbf{1}_{A_j}(\mathbf{r}).$$

The derivative DF so defined can be regarded as a random variable taking values in the Hilbert space $L^2(T)$. More generally, the n-th derivative of F, say $D^n F$, is the $L^2(T^n)$-valued random variable defined by:

$$[D^n F](\mathbf{r}_1, \ldots, \mathbf{r}_n) = \sum_{i_1, \ldots, i_n = 1}^{N} \frac{\partial^n f}{\partial x_{i_1} \ldots \partial x_{i_n}}(W(A_1), \ldots, W(A_N))\, 1_{A_{i_1}}(\mathbf{r}_1) \ldots 1_{A_{i_n}}(\mathbf{r}_n).$$

We will use the notation $D_{\mathbf{r}}F$ for $[DF](\mathbf{r})$ and, with this notation, $D^n F$ appears as the n-th iteration of the operator D. Let $\mathscr{L}$ be the space of smooth functionals considered above and, for each integer $n \geqq 0$ and for each real number $p \geqq 1$, let $\mathscr{D}_{p,n}$ be the completion of the space $\mathscr{L}$ with respect to the seminorm

$$\|F\|_{p,n} = \mathbf{E}\{|F|^p\}^{1/p} + \sum_{k=1}^{n} \mathbf{E}\{\|D^k F\|^p_{L^2(T^k)}\}^{1/p}$$

and let

$$\mathscr{D}_\infty = \bigcap_{n \geqq 0} \bigcap_{p \geqq 1} \mathscr{D}_{p,n}.$$

The operators D^n extend naturally form $\mathscr{L}$ to $\mathscr{D}_{p,n}$. Moreover:

a) The operator D can be interpreted as a directional derivative. Indeed, for each $F \in \mathscr{L}$ and $h \in L^2(T)$ the random variable $\langle DF, h\rangle = \int_T D_{\mathbf{r}} F \cdot h(\mathbf{r})\, d\mathbf{r}$ coincides with the derivative at $t=0$ of the function

$$t \to F(W(A_1) + t\langle 1_{A_1}, h\rangle, \ldots, W(A_N) + t\langle 1_{A_N}, h\rangle).$$

b) If we use the notation $\mathscr{F}_A$ for the σ-field generated by $\{W(B);\, B \in \mathscr{B}_T,\, B \subset A\}$, then $D_{\mathbf{r}}F = 0$ almost surely for almost every $\mathbf{r} \in A^c$ the complementary set of A, whenever the functional F is in $\mathscr{D}_{2,1}$ and is $\mathscr{F}_A$-measurable.

c) The operator D satisfies the chain rule in the sense that:

$$D g(F_1, \ldots, F_m) = \sum_{j=1}^{m} \frac{\partial g}{\partial x_j}(F_1, \ldots, F_m)\, D F_j$$

whenever g is a continuously differentiable function with bounded derivatives and $F_1, \ldots, F_m$ are in $\mathscr{D}_{2,1}$.

We can now recall the two fundamental results of the Malliavin calculus which we will use. If $F = (F_1, \ldots, F_m)$ is an m-dimensional random vector and if $Q = [\langle DF_i, DF_j\rangle_{L^2(T)}]_{i,j=1,\ldots,m}$ denotes the so-called Malliavin matrix when the components $F_1, \ldots, F_m$ are in $\mathscr{D}_{2,1}$, then we have:

(1) sufficient condition for the existence of a density for the law of F:
   (i) the components of F are in $\mathscr{D}_{2,1}$
   (ii) $\det Q \neq 0$ almost surely.

(2) sufficient condition for the existence of an infinitely differentiable density:
   (i) the components of F are in $\mathscr{D}_\infty$
   (ii) $[\det Q]^{-1}$ has finite moments of all orders.

The first result is proven in [18, 25], under the stronger hypothesis that the components of Q are in $\mathscr{D}_{2,1}$ and the components of F in a somewhat smaller space and in [15] under assumptions which are weaker than those of [18] and [25] and stronger than (1) above. In its present form, it was proven in [2]. For the second result, the basic references are [8, 10, 20 and 22].

Note that we will need only the one-dimensional version (i.e., $m=1$) of these results, so that the matrix Q will actually be a nonnegative real random variable and $\det Q = Q$. We will prove that the solution of our nonlinear wave equation at a point (t, x) is in $\mathscr{D}_{p,1}$ when the coefficients are continuously differentiable with bounded derivatives, or in $\mathscr{D}_\infty$ when the coefficients are infinitely differentiable with bounded derivatives, by using an approximation argument very much in the same spirit as in [8, 22, or 13]. Even though natural, our sufficient conditions are too technical to be reproduced in this introduction.

Section IV is devoted to the investigation of the half line $I = [0, \infty)$. The proofs that are similar to the case of the whole line are merely outlined and a counter example is given to illustrate the differences.

The case of a bounded interval I is discussed in Sect. V. The problems created by the boundary conditions at the endpoints of I are of a technical nature and their solutions are not as illuminating as in the case of the propagation of singularities studied in [6].

II. Preliminaries, Existence and Uniqueness

Let $\mathbf{W} = \{W(A); A \in \mathscr{B}_f(\mathbf{R}_+ \times \mathbf{R})\}$ be a mean zero Gaussian process with covariance given by:

$$E\{W(A)\, W(B)\} = |A \cap B| \tag{II.1}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbf{R}_+ \times \mathbf{R}$ and $\mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$ its Borel σ-field. (We use the notation $\mathscr{B}_f(\mathbf{R}_+ \times \mathbf{R})$ for the set of Borel subsets of $\mathbf{R}_+ \times \mathbf{R}$ with finite Lebesgue measure). $\mathbf{W}$ is often called a random (Gaussian) measure with orthogonal increments. On the complete probability space, say $(\Omega, \mathscr{F}, \mathbf{P})$, on which $\mathbf{W}$ is defined, we consider a right continuous nondecreasing family of sub-σ-fields of $\mathscr{F}$ say $\{\mathscr{F}_t; t \geqq 0\}$, which contain all the null sets of $\mathscr{F}$ (according to the terminology in use this is a "filtration satisfying the usual conditions") such that:

(i) for each $t \geqq 0$, $\{W(A); A \subset [t, \infty) \times R\}$ is independent of $\mathscr{F}_t$.
(ii) $W(A)$ is $\mathscr{F}_t$-measurable whenever $A \subset [0, t] \times \mathbf{R}$. (II.2)

For each $\mathbf{z} = (t, x) \in \mathbf{R}_+ \times \mathbf{R}$ we denote by $D(\mathbf{z})$ the triangle

$$D(\mathbf{z}) = \{(s, y) \in \mathbb{R}_+ \times \mathbf{R};\ 0 \leqq s \leqq t,\ x-(t-s) \leqq y \leqq x+(t-s)\},$$

and we write $\mathbf{W}(\mathbf{z})$ for $W(D(\mathbf{z}))$. Note that one can choose a continuous version of the two-parameter stochastic process $\{W(\mathbf{z}); \mathbf{z} \in \mathbf{R}_+ \times \mathbf{R}\}$ such that $W(0, x) = 0$ for all $x \in \mathbb{R}$.

Next we assume that a and b are real valued measurable functions on $\mathbf{R}$ and we consider the stochastic partial differential equation:

$$\frac{\partial^2 X}{\partial t^2}(t, x) - \frac{\partial^2 X}{\partial x^2}(t, x) = a(X(t, x))\, \dot{W}(t, x) + b(X(t, x)) \tag{II.3}$$

with some initial conditions $X(0, \cdot)$ and $\frac{\partial X}{\partial t}(0, \cdot)$. This equation is only formal and we try to give a meaning to a solution in the sense of distributions. Let us consider a C^∞-function $f(t,x)$ with compact support in $[0,\infty)\times\mathbf{R}$, multiply both sides of (II.3) by $f(t,x)$ and integrate over $[0,T]\times\mathbf{R}$ where T is chosen so that $f(T,x)=\frac{\partial f}{\partial t}(T,x)=0$ for all $x\in\mathbb{R}$. Integrating by parts twice the left hand side one gets:

$$\int_0^T\int_{\mathbf{R}}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right]X(t,x)\,dx\,dt+\int_R\frac{\partial f}{\partial t}(0,x)\,X(0,x)\,dx$$

$$-\int_{\mathbf{R}} f(0,x)\,\frac{\partial X}{\partial t}(0,x)\,dx$$

$$=\int_0^T\int_{\mathbf{R}}[a(X(t,x))\,\dot W(t,x)+b(X(t,x))]\,f(t,x)\,dt\,dx. \tag{II.4}$$

At this point we remark that the left hand side makes sense if $\{X(0,x);\,x\in\mathbb{R}\}$ is a stochastic process with continuous sample paths and if $\frac{\partial X}{\partial x}(0,x)$ – which will not exist as a function of x – is interpreted as a random measure on $\mathbf{R}$. Also, $\{W([0,t]\times A);\, t\geqq 0,\, A\in\mathscr{B}_f(\mathbf{R})\}$ is a martingale measure in the sense of Walsh (see Chap. II of [21]) and consequently, for each measurable process $\{Y(t,x);\,(t,x)\in\mathbb{R}_+\times\mathbf{R}\}$ such that $Y(t,x)$ is $\mathscr{F}_t$-measurable for each $x\in\mathbf{R}$ and

$$\int_0^T\int_{\mathbf{R}}\mathbf{E}\{|Y(t,x)|^2\}\,dx\,dt<+\infty \tag{II.5}$$

for each $T>0$, we can define the stochastic integral $\int_0^T\int_{\mathbf{R}} Y(t,x)\,dW(t,x)$. This will be used to give a meaning to the right hand side of (II.4) provided one can check (II.5). We can now state the following:

Definition II.1. Let $\{F(x);\,x\in\mathbb{R}\}$ be an $\mathscr{F}_0$-measurable stochastic process with continuous sample paths and let $\mu\colon\mathscr{B}_f(\mathbb{R})\to L^2(\Omega,\mathscr{F},P)$ be a σ-finite random L^2-measure with a continuous distribution function. Then a continuous process $X=\{X(\mathbf{z});\,\mathbf{z}\in\mathbf{R}_+\times\mathbf{R}\}$ such that $X(t,x)$ is $\mathscr{F}_t$-measurable for all $x\in\mathbf{R}$ and each $t\geqq 0$ is said to be a weak solution of (II.3) with initial condition (F,μ) if:

$$\int_0^T\int_{\mathbf{R}}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right]X(t,x)\,dx\,dt+\int_R\frac{\partial f}{\partial t}(0,x)\,F(x)\,dx-\int_{\mathbf{R}} f(0,x)\,\mu(dx)$$

$$=\iint_{[0,T]\times\mathbf{R}} a(X(t,x))\,f(t,x)\,dW(t,x)+\int_0^T\int_{\mathbf{R}} b(X(t,x))\,f(t,x)\,dx\,dt \tag{II.6}$$

$\mathbf{P}$ almost surely for each C^∞ function f with compact support contained in $[0,T)\times\mathbf{R}$ for some $T>0$.

The reader should be aware that the above definition may not make sense when the process $Y(t,x)=a(X(t,x))\,f(t,x)$ does not satisfy the condition (II.5) which guarantees the existence of the stochastic integral in the right hand side of (II.6) or when the function $b(y)$ is not locally bounded. Nevertheless, in the applications to follow, the function $a(y)$ and $b(y)$ will be continuous so that the process $Y(t, x)$ is continuous with compact support and the stochastic integral can be defined via a standard localization argument. For example, one could use the fact that, for each integer $N > 1$, the process $Y \mathbf{1}_{B_N}$ with

$$B_N = \{(t, x) \in \mathbb{R}_+ \times \mathbf{R};\ \sup_{(s,y) \in D(t,x)} |Y(s, y)| < \mathrm{N}\},$$

satisfies (II.5).

Proposition II.2. *Let us assume that the coefficients a and b are locally Lipschitz, in the sense that for each $C > 0$ one can find a constant $K_C > 0$ such that:*

$$|a(x) - a(y)| + |b(x) - b(y)| \leqq K_C |x - y| \tag{II.7}$$

for all x and y in $[-C, +C]$. Then there exists at most one weak solution.

Proof. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two weak solutions. For each C^∞-function with compact support contained in $(0, T) \times \mathbf{R}$, say g, we set:

$$f(t, x) = \int_0^T \int_{\mathbf{R}} g(s, y) \mathbf{1}_{D(s,y)}(t, x)\, dy\, ds. \tag{II.8}$$

f so defined is C^∞, has compact support contained in $[0, T) \times \mathbf{R}$ and satisfies:

$$\frac{\partial^2 f}{\partial t^2}(t, x) - \frac{\partial^2 f}{\partial x^2}(t, x) = g(t, x).$$

Consequently, using this function f in (II.6) with $\mathbf{X}_1$ and $\mathbf{X}_2$ we obtain the relation

$$\begin{aligned} \int_0^T \int_{\mathbf{R}} g(t, x)\, Y(t, x)\, dx\, dt = & \iint_{[0,T] \times \mathbf{R}} [a(X_1(t, x)) - a(X_2(t, x))] f(t, x)\, dW(t, x) \\ & + \int_0^T \int_{\mathbf{R}} [b(X_1(t, x)) - b(X_2(t, x))] f(t, x)\, dx\, dt \end{aligned} \tag{II.9}$$

for $\mathbf{Y} = \mathbf{X}_1 - \mathbf{X}_2$. For each integer N the random set:

$$B_N = \{(t, x) \in \mathbb{R}_+ \times \mathbf{R};\ \sup_{(s,y) \in D(t,x)} (|X_1(s, y)| + |X_2(s, y)|) < N\}$$

is such that $1_{B_N}(t, x)$ is $\mathscr{F}_t$-measurable for all $x \in \mathbf{R}$ and each $t \geqq 0$. We choose an approximate identity $\{j_\varepsilon; \varepsilon > 0\}$ in the plane. $\mathbf{Y}$ being continuous we have:

$$\lim_{n \to \infty} \int_0^T \int_{\mathbf{R}} Y(s, y)\, j_{1/n}(t - s, x - y)\, dy\, ds = Y(t, x)$$

for each $(t, x) \in (0, \infty) \times \mathbf{R}$. Consequently, if $T>0$, $C>0$ and $(t, x) \in D(T, C)$ are fixed, for each $\varepsilon>0$ we have:

$$\begin{aligned}
&\mathbf{E}\{\mathbf{1}_{B_N}(t+\varepsilon, x)\,|Y(t, x)|^2\}\\
&\quad\leqq \liminf_{n\to\infty} \mathbf{E}\left\{\mathbf{1}_{B_N}(t+\varepsilon, x)\left|\int_0^T \int_{\mathbf{R}} Y(s, y)\, j_{1/n}(t-s, x-y)\, dy\, ds\right|^2\right\}\\
&\qquad \text{(by Fatou's lemma)}\\
&\quad\leqq 2 \liminf_{n\to\infty} \mathbf{E}\{\mathbf{1}_{B_N}(t+\varepsilon, x)\,|\iint_{[0,T]\times R} [a(X_1(s, y))\\
&\qquad -a(X_2(s, y))]\, f_n(s, y)\, dW(s, y)|^2\}\\
&\qquad +2 \liminf_{n\to\infty} \mathbf{E}\left\{\mathbf{1}_{B_N}(t+\varepsilon, x)\left|\int_0^T \int_{\mathbf{R}} [b(X_1(s, y))-b(X_2(s, y))]\, f_n(s, y)\, dy\, ds\right|^2\right\}\\
&\qquad \text{(by using (II.9) with } g(\cdot,\cdot)=j_{1/n}(t-\cdot, x-\cdot))
\end{aligned}$$

for n large enough and with f_n given now by the formula

$$\begin{aligned}
f_n(s, y) &= \int_0^T \int_{\mathbf{R}} j_{1/n}(t-\sigma, x-\xi)\, \mathbf{1}_{D(\sigma,\xi)}(s, y)\, d\sigma\, d\xi\\
&\leqq 2 \liminf_{n\to\infty} \mathbf{E}\{|\iint_{[0,T]\times\mathbf{R}} [a(X_1(s, y))\\
&\quad -a(X_2(s, y))]\, \mathbf{1}_{B_N}(s, y)\, f_n(s, y)\, dW(s, y)|^2\}\\
&\quad +2 \liminf_{n\to\infty} \mathbf{E}\left\{\left|\int_0^T \int [b(X_1(s, y))-b(X_2(s, y))]\, \mathbf{1}_{B_N}(s, y)\, f_n(s, y)\, dy\, ds\right|^2\right\}\\
&\quad \text{(because of the local properties of the integrals)}\\
&\leqq C(N, T, C) \liminf_{n\to\infty} \int_0^T \int_{\mathbf{R}} \mathbf{E}\{\mathbf{1}_{B_N}(s, y)\,|Y(s, y)|^2\}\, f_n(s, y)|^2\, dy\, ds \qquad \text{(II.10)}\\
&\quad \text{(because of the local Lipschitz assumption).}
\end{aligned}$$

Note that $\lim_{n\to\infty} f_n(s, y) = 1_{D(t,x)}(s, y)$ for any point (s, y) away from the boundary of $D(t, x)$ which is of Lebesgue's measure zero. Moreover, $|f_n(s, y)| \leqq 1$ and $|Y(s, y)|^2\, \mathbf{1}_{B_N}(s, y) \leqq N$ for all integers n so that, by Lebesgue's dominated convergence theorem we have:

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_{\mathbf{R}} \mathbf{E}\{\mathbf{1}_{B_N}(s, y)\, Y(s, y)^2\}\, f_n(s, y)^2\, dy\, ds\\
&\quad = \iint_{D(t,x)} \mathbf{E}\{\mathbf{1}_{B_N}(s, y)\,|Y(s, y)|^2\}\, dy\, ds.
\end{aligned}$$

Plugging this in (II.10) gives:

$$E\{\mathbf{1}_{B_N}(t+\varepsilon, x)\,|Y(t,x)|^2\} \leqq C(N,T,C) \iint_{D(t,x)} \mathbf{E}\{\mathbf{1}_{B_N}(s,y)\,|Y(s,y)|^2\}\,dy\,ds$$

and

$$E\{\mathbf{1}_{B_N}(t, x)\,|Y(t,x)|^2\} \leqq C(N,T,C) \iint_{D(t,x)} \mathbf{E}\{\mathbf{1}_{B_N}(s,y)\,|Y(s,y)|^2\}\,dy\,ds$$

by letting $\varepsilon \searrow 0$ and using again Fatou's lemma. Using this estimate recursively gives $\mathbf{E}\{\mathbf{1}_{B_N}(t,x)\,|Y(t,x)|^2\}=0$ and hence $Y=0$ almost surely on B_N, but since N was arbitrary this implies $Y=0$ a.s. □

Our existence results require the use of notions like martingale, weak martingale and strong martingale with respect to a filtration obtained by a rotation of the natural filtration of $\mathbf{R}_+ \times \mathbf{R}$.

Let $\mathbf{u}=(2^{-1/2}, 2^{-1/2})$ and $\mathbf{v}=(2^{-1/2}, -2^{-1/2})$. If $\mathbf{z}=(t,x)\in\mathbb{R}_+\times\mathbf{R}$, we will call t and x its natural coordinates and if $z=\lambda\mathbf{u}+\mu\mathbf{v}$, we will call λ and μ its rotated coordinates.

Now, for each $\mathbf{z}\in\mathbf{R}_+\times\mathbf{R}$ we define $\mathscr{F}_z$ as the σ-field obtained by completing $\mathscr{F}_0 \vee \sigma\{W(A);\ A\in\mathscr{B}(\mathbb{R}_+\times\mathbf{R}),\ A\subset D(z)\}$ with the null sets of $\mathscr{F}$ say $\mathscr{N}$, and if $\mathbf{z}=(t,x)\in\mathbf{R}^2$ is such that $t<0$ we set $\mathscr{F}_\mathbf{z}=\mathscr{F}_0$. The filtration $\{\mathscr{F}_\mathbf{z};\ \mathbf{z}\in\mathbf{R}^2\}$ so obtained satisfies the conditions $(F1)$–$(F4)$ of Cairoli and Walsh (see [5]) if we use the rotated coordinates (λ,μ).

Proposition II.3. *Let us assume that a and b are Lipschitz (in the sense that* (II.7) *holds with a constant K independent of C). Then, for each $\mathscr{F}_0$-measurable continuous process $\{X_0(t,x);\ (t,x)\in\mathbb{R}_+\times\mathbf{R}\}$ satisfying:*

$$\iint_{D(t,x)} \mathbf{E}\{|X_0(s,y)|^2\}\,ds\,dy < +\infty \tag{II.12}$$

for every $(t,x)\in\mathbb{R}_+\times\mathbf{R}$, there exists a unique continuous solution, say $\{X(t,x);\ (t,x)\in\mathbb{R}_+\times\mathbf{R}\}$, of the following stochastic integral equation:

$$X(t,x)=X_0(t,x)+\iint_{D(t,x)} [a(X(s,y))\,dW(s,y)+b(X(s,y))\,ds\,dy]. \tag{II.13}$$

Proof. We use the Picard iterative scheme to construct a solution. For each integer $n\geqq 1$ we set:

$$X_n(t,x)=X_0(t,x)+\iint_{D(t,x)} a(X_{n-1}(s,y))\,dW(s,y)+b(X_{n-1}(s,y))\,dy\,ds.$$

Notice that, for each $n\geqq 1$ the process $\{X_n(\mathbf{z});\ \mathbf{z}\in\mathbf{R}_+\times\mathbf{R}\}$ is $\mathscr{F}_\mathbf{z}$-adapted. Besides, $\{\iint_{D(\mathbf{z})} a(X_{n-1}(s,y))\,dW(s,y);\ \mathbf{z}\in\mathbf{R}_+\times\mathbf{R}\}$ is a two-parameter martingale with respect to the rotated coordinates. We can use the two-parameter version of the maximal inequality and obtain:

$$\begin{aligned}
&\mathbf{E}\{\sup_{\substack{0\leqq s\leqq t\\ x-(t-s)\leqq y\leqq x+(t-s)}} |X_{n+1}(s,y)-X_n(s,y)|^2\}\\
&\leqq \mathbf{E}\{\sup_{(s,y)\in D(t,x)} |\iint_{D(s,y)} [a(X_n(s',y'))-a(X_{n-1}(s',y'))]\,dW(s',y')\\
&\quad+\iint_{D(s,y)} [b(X_n(s',y'))-b(X_{n-1}(s',y'))]\,dy'\,ds'|^2\}\\
&\leqq c(t,x)(\mathbf{E}\{|\iint_{D(t,x)} [a(X_n(s,y))-a(X_{n-1}(s,y))]\,dW(s,y)|^2\}\\
&\quad+\mathbf{E}\{|\iint_{D(t,x)} [b(X_n(s,y))-b(X_{n-1}(s,y))]\,dy\,ds|^2\})\\
&\quad\text{(for some constant } c(t,x)>0 \text{ depending only on the choice}\\
&\quad\text{of } (t,x) \text{ in } \mathbf{R}_+\times\mathbf{R})\\
&\leqq c(t,x)\iint_{D(t,x)} \mathbf{E}\{|X_n(s,y)-X_{n-1}(s,y)|^2\}\,dy\,ds\\
&\leqq c(t,x)\frac{t^{2n}}{n!}\iint_{D(t,x)} \mathbf{E}\{|X_0(s,y)|^2\}\,dy\,ds.
\end{aligned}$$

Consequently we have:

$$\sum_{n=0}^{\infty} \mathbf{E}\{\sup_{(s,y)\in D(t,x)} |X_{n+1}(s,y)-X_n(s,y)|^2\}<\infty$$

for each $(t,x)\in\mathbf{R}_+\times\mathbf{R}$ and this implies the local uniform convergence of

$$X_0(t,x)+\sum_{n=0}^{\infty}(X_{n+1}(t,x)-X_n(t,x))$$

to a process $X(t,x)$ which is continuous and satisfies (II.13). The proof of the uniqueness is standard and we omit it. □

Remarks. 1. The maximal inequality implies that the above solution satisfies:

$$\mathbf{E}\{\sup_{(s,y)\in D(t,x)} |X(s,y)|^p\}<+\infty$$

provided $\iint_{D(t,x)} |X_0(s,y)|^p\,dy\,ds<+\infty$.

2. In the particular case of:

$$X_0(t,x)=\tfrac{1}{2}([F(x+t)+F(x-t)]+\tfrac{1}{2}\mu([x-t,x+t])$$

with F and μ as in Proposition II.4 below, the above uniqueness result is also a consequence of Proposition II.2 and the following result.

Proposition II.4. *Let us assume that a and b are globally Lipschitz and that:*

(i) $F=\{F(x); x\in\mathbf{R}\}$ *is a continuous* $\mathscr{F}_0$*-measurable process for which* $\int_I \mathbf{E}\{|F(x)|^2\}\,dx<+\infty$ *for each bounded interval I,*

(ii) μ: $\mathscr{B}_f(\mathbf{R})\to L^2(\Omega,\mathscr{F}_0,P)$ *is an* L^2*-measure with a continuous distribution function G satisfying* $\int_I \mathbf{E}\{|G(x)|^2\}\,dx<+\infty$ *for each bounded interval I.*

Then, the unique solution of the integral equation:

$$X(t,x)=\tfrac{1}{2}[F(x+t)+F(x-t)]+\tfrac{1}{2}\mu([x-t,x+t]) +\tfrac{1}{2}\iint_{D(t,x)}[a(X(s,y))\,dW(s,y)+b(X(s,y))\,dy\,ds] \qquad \text{(II.14)}$$

is a weak solution of (II.3) *with initial condition* (F,μ) *in the sense of Definition* (II.2).

Proof. Let us denote by $X(t,x)$ the unique solution of (II.14) (recall Proposition (II.3) above). Notice that $X(t,x)$ is $\mathscr{F}_{(t,x)}$-measurable, so $\mathscr{F}_t$-measurable since $\mathscr{F}_{(t,x)}\subset\mathscr{F}_t$. If f is any C^∞-function with compact support contained in $[0,t)\times\mathbf{R}$ we have:

$$\begin{aligned}
&\int_0^T\int_{\mathbf{R}}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right]\Big(\iint_{D(t,x)}[a(X(s,y))\,dW(s,y)+b(X(s,y))\,dy\,ds]\,dx\,dt\\
&=\int_0^T\int_{\mathbf{R}}\left(\int_s^T\left(\int_{y-(t-s)}^{y+(t-s)}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right]dx\right)dt\right)\cdot[a(X(s,y))\,dW(s,y)+b(X(s,y))]\,dy\,ds\\
&=\int_0^T\int_{\mathbf{R}} f(s,y)[a(X(s,y))\,dW(s,y)+b(X(s,y))\,dy\,ds]\,dy\,ds
\end{aligned} \qquad \text{(II.15)}$$

because of a standard version of Fubini's Theorem for stochastic integrals. Moreover,

$$\tfrac{1}{2}\int_0^T\int_{\mathbf{R}}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right](F(x+t)+F(x-t))\,dx\,dt+\int_R\frac{\partial f}{\partial t}(0,x)\,F(x)\,dx=0 \qquad \text{(II.16)}$$

by simple integration by parts. Finally we similarly have:

$$\tfrac{1}{2}\int_0^T\int_{\mathbf{R}}\left[\frac{\partial^2 f}{\partial t^2}(t,x)-\frac{\partial^2 f}{\partial x^2}(t,x)\right]\nu([x-t,x+t])\,dx\,dt-\int_R f(0,x)\,\nu(dx)=0 \qquad \text{(II.17)}$$

for each deterministic continuous σ-finite measure ν. Using (II.17) with ν defined by $\nu(A)=\mathbf{E}\{1_B\,\mu(A)\}$ for $A\in\mathscr{B}_f(\mathbf{R})$ for each $B\in\mathscr{F}$ we can put (II.17), (II.16) and (II.15) together with (II.14) to check (II.6). □

Remark. In the deterministic case the Cauchy data are the solution at time $t=0$, say $X(0,\cdot)$, and the partial derivative of the solution with respect to t at time $t=0$, say $\frac{\partial X}{\partial t}(0,\cdot)$. Coming back to Definition II.1 one sees that $F(\cdot)$ has to be $X(0,\cdot)$ and that $\mu(\cdot)$ has to be $\int \frac{\partial X}{\partial t}(0,x)\,dx$. The solution we obtained in Proposition II.4 above obviously satisfies $F(x)=X(0,x)$ for all $x\in\mathbb{R}$. Unfortunately, our solution is almost surely nondifferentiable and the interpretation of the second half of the initial condition as the primitive of the partial derivative with respect to t at $t=0$ is less obvious. We will justify it by means of line integrals in the spirit of Wong and Zakai. See [24].

Line Integrals and the Markov Property

Let us assume for a while that $X=\{X(t,x);\ (t,x)\in\mathbf{R}_+\times\mathbf{R}\}$ is any continuous stochastic process of order 2. For each $t\geqq 0$ and $a<b$ in $\mathbf{R}$ we denote by $L(t;a,b)$ the horizontal segment:

$$L(t;a,b)=\{(s,y)\in\mathbb{R}_+\times\mathbf{R};\ s=t,\ a\leqq y\leqq b\}.$$

For each subdivision $\pi=\{a=x_0<x_1<\ldots<x_n=b\}$ of the interval $[a,b]$ we set:

$$\mathbf{z}_i=(t,x_i)\qquad i=0,1,\ldots,n$$

and

$$\mathbf{z}'_i=(t-(x_{i+1}-x_i)/2,\ (x_i+x_{i+1})/2)\qquad i=0,1,\ldots,n-1.$$

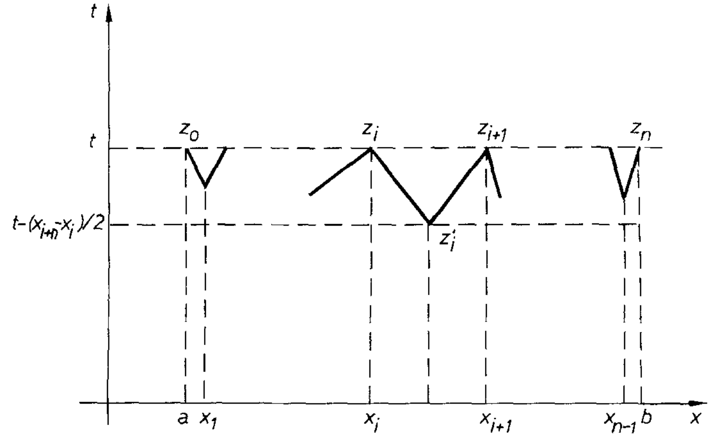

Fig. 1

Consider the sums:

$$S_{1,n}=\sum_{i=0}^{n-1}(X(\mathbf{z}_i)-X(\mathbf{z}'_i))\quad\text{and}\quad S_{2,n}=\sum_{i=0}^{n-1}(X(\mathbf{z}_{i+1})-X(\mathbf{z}'_i))$$

and define the line integrals

$$\int_{L(t;a,b)}\partial_1 X=\lim_{\text{mesh}(\pi)\to 0}S_{1,n}\quad\text{and}\quad\int_{L(t;a,b)}\partial_2 X=\lim_{\text{mesh}(\pi)\to 0}S_{2,n}$$

whenever the limits exist in probability. Recall that $\operatorname{mesh}(\pi)=\max_{i=0,\ldots,n-1}|x_{t+1}-x_i|$. We also set:

$$\int_{L(t;a,b)} \partial X = \int_{L(t;a,b)} \partial_1 X + \int_{L(t;a,b)} \partial_2 X$$

whenever the two line integrals in the right hand side make sense.

Examples. 1. If the sample paths of **X** are continuously differentiable functions of (t, x) one easily checks that:

$$\int_{L(t;a,b)} \partial_1 X = \frac{1}{\sqrt{2}} \int_a^b \mathbf{v} \cdot \nabla X(t, x)\, dx$$

$$\int_{L(t;a,b)} \partial_2 X = \frac{1}{\sqrt{2}} \int_a^b \mathbf{u} \cdot \nabla X(t, x)\, dx$$

and consequently:

$$\int_{L(t;a,b)} \partial X = \int_a^b \frac{\partial X}{\partial t}(t, x)\, dx$$

(recall the definitions of **u** and **v** given before Proposition II.3).

2. If X is the continuous process given in Proposition II.4 by (II.14), then the line integrals exist and are given by:

$$\int_{L(t;a,b)} \partial_1 X = \tfrac{1}{2}[F(a-t)-F(b-t)] + \tfrac{1}{2}\mu([a-t, b-t]) + \tfrac{1}{2} \iint_{D'(t;a,b)} [a(X(s, y))\, dW(s, y) + b(X(s, y))\, dy\, ds]$$

and

$$\int_{L(t;a,b)} \partial_2 X = \tfrac{1}{2}[F(b+t)-F(a+t)] + \tfrac{1}{2}\mu([a+t, b+t]) + \tfrac{1}{2} \iint_{D''(t;a,b)} [a(X(s, y))\, dW(s, y) + b(X(s, y))\, dy\, ds]$$

where $D'(t; a, b)$ and $D''(t; a, b)$ are the parallelograms described in Fig. 2 below.

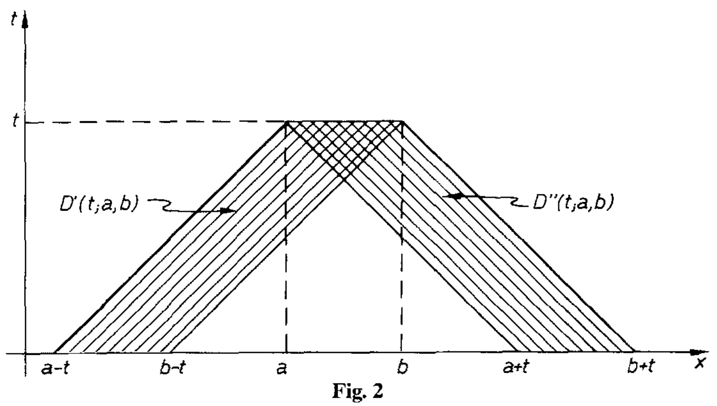

Fig. 2

Finally:

$$\int_{L(t;a,b)} \partial X = \tfrac{1}{2}[F(b+t)-F(a+t)-F(b-t)+F(a-t)]$$
$$+\tfrac{1}{2}[\mu([a+t,b+t])+\mu([a-t,b-t])]$$
$$+\tfrac{1}{2} \iint_{D'(t;a,b)} + \iint_{D''(t;a,b)} [a(X(s,y))\,dW(s,y)+b(X(s,y))\,dy\,ds]. \quad \text{(II.18)}$$

From now on the process **X** will be the one given in Proposition II.4.

Remarks. 1. In the limiting case $t \downarrow 0$ formula (II.18) gives

$$\int_{L(0;a,b)} \partial X = \mu([a,b]) \quad \text{(II.19)}$$

and this is the interpretation of μ as the primitive of $\frac{\partial X}{\partial t}$ which we were looking for (recall the remark following Proposition II.4).

2. When F happens to be the distribution function of an L^2-measure, for each fixed $t>0$, the mapping $[a,b] \to \int_{L(t;a,b)} \partial X$ defines an L^2-measure having the same properties as μ.

3. When F and μ are identically zero, $\int_{L(t;a,b)} \partial X$ defines a martingale measure in the sense of Walsh. See [21].

In trying to understand the real meaning of the Cauchy data (F, μ) we were motivated by a possible Markov property of our process. In the case of stochastic ordinary differential equations the strong uniqueness result obtained in the Lipschitz case essentially guarantees the Markov property of the process. In the case of stochastic partial differential equation, a uniquenes result for each given Cauchy data $\left(X(0,\cdot), \frac{\partial X}{\partial t}(0,\cdot)\right)$ should also imply the Markov property for the process $\left\{X(t,\cdot), \frac{\partial X}{\partial t}(t,\cdot); t \geqq 0\right\}$ once interpreted as a process taking values in an appropriate infinite dimensional space. This is the purpose of the present discussion.

If $t_0 \geqq 0$ is fixed one has, for each $(t,x) \in [t_0, \infty) \times \mathbf{R}$

$$X(t,x) = \tfrac{1}{2}[X(t_0, x-(t-t_0)) + X(t_0, (t-t_0))]$$
$$+\tfrac{1}{2} \int_{L(t_0; x-(t-t_0), x+(t-t_0))} \partial X$$
$$+\tfrac{1}{2} \iint_{D(t,x) \cap [t_0,\infty) \times \mathbf{R}} [a(X(s,y))\,dW(s,y)+b(X(s,y))\,dy\,ds]$$

and this shows that the continuous process $\mathbf{X}^{t_0} = \{X(t,x); (t,x) \in [t_0,\infty) \times \mathbf{R}\}$ is the unique weak solution (in the sense of Definition II.1) of the equation (II.3) on $[t_0,\infty) \times \mathbf{R}$ with initial condition (F^{t_0}, μ^{t_0}) given by:

$$F^{t_0}(x) = X(t_0, x),$$
$$\mu^{t_0}([a,b]) = \int_{L(t_0;a,b)} \partial X.$$

Consequently, the process $\{(X(t,\cdot), \int_{L(t,\cdot,\cdot)} \partial X); t \geq 0\}$ is Markovian in the state space product of the space of continuous functions on $\mathbf{R}$ and of additive continuous functions of the bounded intervals of $\mathbb{R}$.
Consequently:

$$\int_{L(t;a,b)} \partial X = \tfrac{1}{2}[X(t_0, b+(t-t_0)) - X(t_0, a+(t-t_0))$$
$$- X(t_0, b-(t-t_0)) + X(t_0, a-(t-t_0))]$$
$$+ \tfrac{1}{2}\Big[\int_{L(t_0; a+(t-t_0), b+(t-t_0))} \partial X + \int_{L(t_0; a-(t-t_0), b-(t-t_0))} \partial X\Big]$$
$$+ \tfrac{1}{2} \iint_{D'(t_0,t;a,b) D''(t_0,t;a,b)} [a(X(s,y))\, dW(s,y) + b(X(s,y))\, dy\, ds].$$

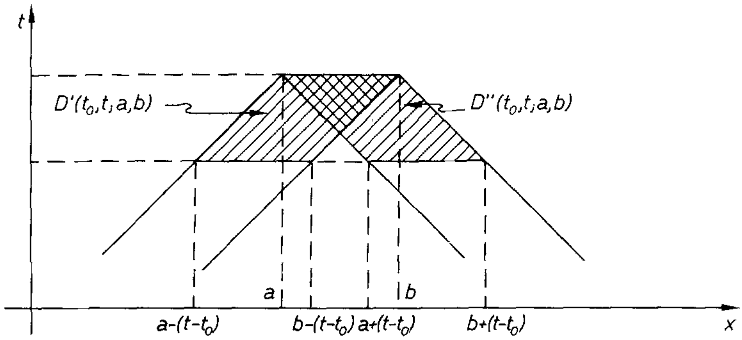

Fig. 3

III. Smoothness of the Solutions

This section is devoted to the investigation of the stochastic process $\mathbf{X} = \{X(t,x); (t,x) \in \mathbf{R}_+ \times \mathbf{R}\}$ which solves the integral equation:

$$X(t,x) = \tfrac{1}{2}[f(x+t) + f(x-t)] + \tfrac{1}{2}[g(x+t) - g(x-t)]$$
$$+ \iint_{D(t,x)} [a(X(s,y))\, dW(s,y) + b(X(s,y))\, dy\, ds] \qquad \text{(III.1)}$$

where f and g are given deterministic continuous functions on $\mathbf{R}$, the coefficients a and b are deterministic C^1-functions with bounded derivatives on $\mathbf{R}$ and W is a mean zero Gaussian measure with orthogonal increments on $\mathbf{R}_+ \times \mathbf{R}$ and intensity the Lebesgue measure (recall (II.1)). According to Proposition II.4, $\mathbf{X}$ is the unique weak solution of the nonlinear wave equation (II.3) with deterministic initial condition (F, μ) with $F = f$ and $\mu([a,b]) = \int_a^b g(x)\, dx$.

We look for conditions on the coefficients a and b and on the Cauchy data (f, g) at $t=0$ which will insure that for positive time $t>0$, the random variable $X(t, x)$ has a density and then, we will try to see when this density is smooth. As explained in the introduction we will use the Malliavin calculus for which we recalled the basic notations in the introduction.

Proposition III.1. *For each $(t, x)\in(0, \infty)\times\mathbf{R}$, the random variable $X(t, x)$ belongs to the space $\mathscr{D}_{p,1}$ for all $p\geqq 2$ and the Malliavin derivative $D_{\mathbf{r}} X(\mathbf{z})$ is uniquely determined by:*

$$D_{\mathbf{r}} X(\mathbf{z})=a(X(\mathbf{r}))+\iint_{[\mathbf{r},\mathbf{z}]}[a'(X(\mathbf{z}'))\, D_{\mathbf{r}} X(\mathbf{z}')\, dW(\mathbf{z}') + b'(X(\mathbf{z}'))\, D_{\mathbf{r}}(X(\mathbf{z}'))\, d\mathbf{z}'] \tag{III.2}$$

whenever $\mathbf{r}\in D(\mathbf{z})$.

Proof. By the interval $[\mathbf{r}, \mathbf{z}]$ we mean the set $\{\mathbf{z}'; \mathbf{r}\leqq\mathbf{z}'\leqq\mathbf{z}\}$ where the order is the usual component by component order for the rotated coordinates. The idea of the proof is to rotate the coordinate axes and then to use the polygonal approximation method of Ikeda and Watanabe [8] (see also [22]). More precisely, we fix a point $(T, 0)$ in $(0, \infty)\times\mathbf{R}$ and for each integer $n\geqq 1$ we consider the grid $\mathscr{S}_n$ made out of the points of the form

$$i T 2^{-n}\mathbf{u}+j T 2^{-n}\mathbf{v},$$

where i and j are integers such that $i+j\geqq 0$. Then, for each $\mathbf{z}\in D(T, 0)$ we set

$$f_n(\mathbf{z})=\sup\{\mathbf{z}'\in\mathscr{S}_n;\ \mathbf{z}'\leqq\mathbf{z}\}$$

and

$$g_n(\mathbf{z})=\inf\{\mathbf{z}'\in\mathscr{S}_n;\ \mathbf{z}'\geqq\mathbf{z}\}$$

where the order is the usual component by component order for the rotated coordinates. Next we define the process $X_n=\{X_n(\mathbf{z});\ \mathbf{z}\in D(T, 0)\}$ by

$$X_n(\mathbf{z})=\tfrac{1}{2}[f(x+t)+f(x-t)]+\tfrac{1}{2}[g(x+t)-g(x-t)] + \iint_{D(\mathbf{z})}[a(X_n(f_n(\mathbf{z}')))\, dW(\mathbf{z}')+b(X_n(f_n(\mathbf{z}')))\, d\mathbf{z}'] \tag{III.3}$$

where $\mathbf{z}=(t, x)$. By convention, we will set $X_n(\mathbf{z}')=0$ whenever $\mathbf{z}'=(t', x')$ is such that $t'<0$. Note that, because of the definition of f_n, (III.3) is more of a recursive definition than an integral equation. Then, for each $p\geqq 2$ one can easily prove that:

$$\lim_{n\to\infty}\mathbf{E}\{\sup_{\mathbf{z}\in D(T,0)}|X(\mathbf{z})-X_n(\mathbf{z})|^p\}=0. \tag{III.4}$$

Actually, $\{X_n(\mathbf{z});\ n\geqq 1\}$ is an approximating sequence for $X(\mathbf{z})$ in the sense of the $\mathscr{D}_{p,1}$-norm for all $p\geqq 2$. In fact, if $f_n(\mathbf{z})=\mathbf{z}_n$ and if the notations $\mathbf{z}_n\odot\mathbf{z}$ and $\mathbf{z}\odot\mathbf{z}_n$ are defined by Fig. 4,

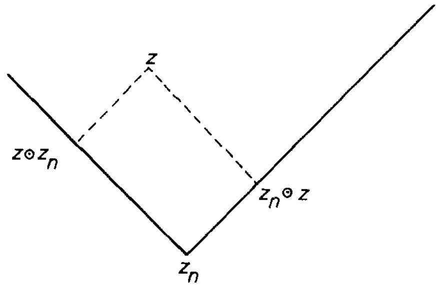

Fig. 4

we can write:

$$X_n(\mathbf{z}) = X_n(\mathbf{z} \odot \mathbf{z}_n) + X_n(\mathbf{z}_n \odot \mathbf{z}) - X_n(\mathbf{z}_n) + a(X_n(\mathbf{z}_n))\, W((\mathbf{z}_n, \mathbf{z}]) + b(X_n(\mathbf{z}_n))\, |(\mathbf{z}_n, \mathbf{z}]|$$

and consequently, we have

$$D_{\mathbf{r}} X_n(\mathbf{z}) = D_{\mathbf{r}} X_n(\mathbf{z} \odot \mathbf{z}_n) + D_{\mathbf{r}} X_n(\mathbf{z}_n \odot \mathbf{z}) - D_{\mathbf{r}} X_n(\mathbf{z}_n) + a'(X_n(\mathbf{z}_n))\, D_{\mathbf{r}} X_n(\mathbf{z}_n)\, W((\mathbf{z}_n, \mathbf{z}]) + a(X_n(\mathbf{z}_n))\, \mathbf{1}_{(\mathbf{z}_n, \mathbf{z}]}(\mathbf{r}) + b'(X_n(\mathbf{z}_n))\, D_{\mathbf{r}} X_n(\mathbf{z}_n)\, |(\mathbf{z}_n, \mathbf{z}]|.$$

This shows that the process $\{D_{\mathbf{r}}(X_n(\mathbf{z}')); \mathbf{r} \leqq \mathbf{z}'\}$ is the unique solution of the following stochastic integral equation:

$$D_{\mathbf{r}} X_n(\mathbf{z}') = a(X_n(f_n(\mathbf{r}))) + \iint_{[g_n(r) \wedge \mathbf{z}, \mathbf{z}]} [a'(X_n(f_n(\mathbf{z}')))\, D_{\mathbf{r}} X_n(f_n(\mathbf{z}'))\, dW(\mathbf{z}') + b'(X_n(f_n(\mathbf{z}')))\, D_{\mathbf{r}} X_n(f_n(\mathbf{z}'))\, d\mathbf{z}'].$$

Notice that $D_{\mathbf{r}} X_n(\mathbf{z}') = 0$ unless $\mathbf{z}' \geqq \mathbf{r}$. Now, we fix $\mathbf{r}$ and we consider the continuous process $\{D_{\mathbf{r}}(\mathbf{z}); \mathbf{z} \geqq \mathbf{r}\}$ solution of the stochastic integral Eq. (III.2) (where we assume that $\{X(\mathbf{z}); \mathbf{z} \geqq \mathbf{r}\}$ is known). Using as before a technical lemma on the convergence of polygonal approximations for stochastic differential equations on the plane (see [13]), we obtain:

$$\lim_{n \to \infty} \sup_{\mathbf{r} \in D(T,0)} \mathbf{E}\{\sup_{\mathbf{r} \leqq \mathbf{z} \leqq (T,0)} |D_{\mathbf{r}} X(\mathbf{z}) - D_{\mathbf{r}} X_n(\mathbf{z})|^p\} = 0 \tag{III.5}$$

for all $p \geqq 2$. Putting together (III.4) and (III.5) we obtain $X(\mathbf{z}) \in \bigcap_{p \geqq 2} \mathscr{D}_{p,1}$. □

Let us assume that $\mathbf{r} \in [0, \infty) \times \mathbf{R}$ is fixed and (assuming again that the process $\{X(\mathbf{z}); \mathbf{z} \in [0, \infty) \times \mathbf{R}\}$ is known) let $Y = \{Y(\mathbf{z}, \mathbf{r}); \mathbf{z} \geqq \mathbf{r}\}$ be the unique solution of the following linear stochastic integral equation:

$$Y(\mathbf{z}, \mathbf{r}) = 1 + \iint_{[\mathbf{r}, \mathbf{z}]} [a'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, dW(\mathbf{z}') + b'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, d\mathbf{z}']. \tag{III.6}$$

One easily checks that for each $T>0$ there exists a constant $C_T>0$ such that:

$$\mathbf{E}\{|Y(\mathbf{z},\mathbf{r})-Y(\mathbf{z}',\mathbf{r}')|^p\}\leqq C_T(|\mathbf{r}-\mathbf{r}'|^{p/2}+|\mathbf{z}-\mathbf{z}'|^{p/2})$$

for $\mathbf{z}$, $\mathbf{z}'$, $\mathbf{r}$ and $\mathbf{r}'$ in $D(T,0)$. Using Kolmogorov's criterion and varying $T>0$ one concludes that Y possesses a continuous version. In what follows, we will always use this continuous version.

In particular, one can easily check that the process $\{a(X(\mathbf{r}))\,Y(\mathbf{z},\mathbf{r});\ \mathbf{z}\geqq\mathbf{r}\}$ satisfies

$$a(X(\mathbf{r}))+\iint_{[\mathbf{r},\mathbf{z}]}[a'(X(\mathbf{z}'))\,a(X(\mathbf{r}))\,Y(\mathbf{z}',\mathbf{r})\,dW(\mathbf{z}')$$
$$+b(X(\mathbf{z}'))\,a(X(\mathbf{r}))\,Y(\mathbf{z}',\mathbf{r})\,d\mathbf{z}']=a(X(\mathbf{r}))\,Y(\mathbf{z},\mathbf{r})$$

and because of the uniqueness of the solution of Eq. (III.2) we can conclude that:

$$D_{\mathbf{r}}X(\mathbf{z})=a(X(\mathbf{r}))\,Y(\mathbf{z},\mathbf{r}). \tag{III.7}$$

This equation will play a crucial role in the sequel. We are now ready to state and prove our first important result on smoothness of X.

Theorem III.2. *Let us assume that $t>0$ and $x\in\mathbf{R}$ are fixed and that one of the following two conditions holds:*

(i) *$a(y)\neq 0$ for some y in the closed interval with end points $f(x-t)$ and $f(x+t)$*

(ii) *$a(y)=0$ for all y in this interval and letting J be the maximal closed interval containing $f(x-t)$ and $f(x+t)$ on which a vanishes one of the following condition holds*

(ii)$_1$ *$J=\{y_0\}$ and $a(f(\xi_0))\neq 0$ for some $\xi_0\in(x-t,x+t)$*

(ii)$_2$ *$J=\{y_0\}$, $a(f(\xi))=0$ for all $\xi\in[x-t,x+t]$ and either $g'_+(x-t)\neq 0$, or $g'_-(x+t)\neq 0$, or $g''_+(x-t)\neq -b(y_0)$, or $g''_-(x+t)\neq -b(y_0)$, or one of these derivatives does not exist*

(ii)$_3$ *J does not reduce to a singleton, $a(f(\xi_0))\neq 0$ for some $\xi_0\in(x-t,x+t)$ and $b'\geqq 0$ on J.*

Then the distribution of $X(t,x)$ is absolutely continuous.

Proof. $X(\mathbf{z})$ with $\mathbf{z}=(t,x)$ is in $\mathscr{D}_{2,1}$ because of Proposition III.1. Also, because $D_{\mathbf{r}}X(\mathbf{z})$ is continuous in $\mathbf{r}$, we need only to show that

$$\iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}>0 \tag{III.8}$$

almost surely (recall the introduction). Let us call G the subset of Ω where the left hand side of (III.8) vanishes and let us assume $\mathbb{P}(G)>0$ and let us try to contradict our assumptions. Using the continuity in $\mathbf{r}$ we can conclude that

$$a(X(\mathbf{r}))\,Y(\mathbf{z},\mathbf{r})=D_{\mathbf{r}}X(\mathbf{z})=0 \tag{III.9}$$

for all $\mathbf{r} \leqq \mathbf{z}$ almost surely in G. Let us consider the line segments:

$$L_0 = \{\mathbf{z}_0 + \lambda\,\mathbf{u};\ 0 \leqq \lambda \leqq \sqrt{2}\} \quad \text{and} \quad L_1 = \{\mathbf{z}_1 + \lambda\,\mathbf{v};\ 0 \leqq \lambda \leqq \sqrt{2}\,t\} \tag{III.10}$$

where $\mathbf{z}_0 = (0, x-t)$ and $\mathbf{z}_1 = (0, x+t)$.

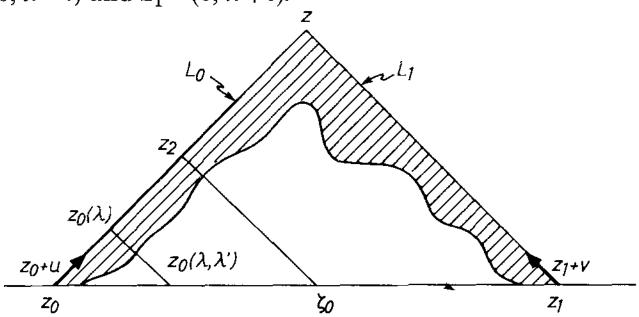

Fig. 5

We know that $Y(\mathbf{z}, \mathbf{r}) = 1$ for $\mathbf{r} \in L_0 \cup L_1$, so that, there exists an open neighborhood of $L_0 \cup L_1$, say U, such that $Y(\mathbf{z}, \mathbf{r}) \neq 0$ for all $\mathbf{r} \in U$. Consequently, (III.9) implies that $a(X(\mathbf{r})) = 0$ for all $\mathbf{r}$ in U. In particular $a(X(\mathbf{r})) = 0$ for all $\mathbf{r} \in L_0 \cup L_1$ and the function a must vanish on the closed interval with end points $f(x-t)$ and $f(x+t)$. This contradicts assumption i).

Let us now assume that $J = \{y_0\}$. If $a(f(\xi_0)) \neq 0$ for some $\xi_0 \in (x-t, x+t)$, then $\mathbf{z}_2 = \mathbf{z}_0 + 2^{-1/2}(\xi_0 - (x-t))\,\mathbf{v}$ is on L_0 and by the above argument, this implies that

$$\iint_{D(\mathbf{z}_2)} (D_{\mathbf{r}} X(\mathbf{z}_2))^2\, d\mathbf{r} \neq 0$$

almost surely and, as a consequence the random variable $X(\mathbf{z}_2)$ has a density. On the other hand $\{X(\mathbf{z}_2) = y_0\} \supset G$ because $J = \{y_0\}$ and this implies $\mathbf{P}\{X(\mathbf{z}_2) = y_0\} > 0$ which is impossible. Consequently we contradicted ii)$_1$.

Now, for each $\lambda \geqq 0$ we set $\mathbf{z}_0(\lambda) = \mathbf{z}_0 + \lambda\,\mathbf{u}$ and $\mathbf{z}_0(\lambda, \lambda') = (0, x-t+\sqrt{2}\lambda) + \lambda' v$ for $0 \leqq \lambda' \leqq \lambda$. (See Fig. 5).

$$\rho = \inf\{\lambda > 0;\ a(X(\mathbf{z}_0(\lambda, \lambda'))) \neq 0 \text{ for some } \lambda' \in [0, \lambda]\}$$

is a stopping time for the filtration $\{\mathscr{B}_{0,\lambda};\ \lambda \geqq 0\}$ where

$$\mathscr{B}_{0,\lambda} = \sigma\{W(B);\ B \subset D(\mathbf{z}_0(\lambda))\} \vee \mathscr{N} \tag{III.11}$$

As we already noticed, $\{\rho > 0\} \supset G$, so that $\mathbf{P}\{\rho > 0\} > 0$ and $\mathbf{P}\{\rho > 0\} = 1$ by the zero-one law. Finally, $a(X(\mathbf{z}_0(\lambda, \lambda'))) = 0$ for all $\lambda < \rho$ and $\lambda' \in [0, \lambda]$. Note that $J = \{y_0\}$ and the continuity of the solution imply that $X(\mathbf{z}_0(\lambda, \lambda')) = y_0$ for $\lambda < \rho$ and $\lambda' \in [0, \lambda]$. Consequently:

$$0 = \tfrac{1}{2}[g(x-t+\sqrt{2}\lambda) - g(x-t)] + \tfrac{1}{2}\lambda^2 b(y_0)$$

for $\lambda < \rho$. This implies that $g'_+(x-t)$ and $g''_+(x-t)$ exist, $g'_+(x-t) = 0$ and $g''_+(x-t) = -b(y_0)$. Similarly, one shows that $g'_-(x+t)$ and $g''_-(x+t)$ exist and that

$g'_-(x+t)$ and $g''_-(x+t)+b(y_0)$ are equal to zero, which contradicts our assumption $(\text{ii})_2$.

Finally we assume that J does not reduce to a single point, that $b'\geqq 0$ on J and that $a(f(\xi_0))\neq 0$ for some $\xi_0\in(x-t, x+t)$. Consider the random time:

$$\tau=\sup\{t'\geqq 0;\ a(X(t',\xi))\neq 0 \text{ for some } \xi\in[x-(t-t'), x+(t-t')]\}.$$

Note that $G\subset\{\tau<t\}$ so that $\mathbf{P}\{0<\tau<t\}>0$. Let us also consider on G the random points $\tilde{\mathbf{z}}_0=(\tau, x-(t-\tau))$ and $\tilde{z}_1=(\tau, x+(t-\tau))$ and the random triangle Δ determined by the three points $\mathbf{z}$, $\tilde{\mathbf{z}}_0$, and $\tilde{z}_1$. Δ is non degenerated and $a(X(\mathbf{r}))=a'(X(r))=0$ for all $\mathbf{r}\in\Delta$ because $X(\mathbf{r})\in J$ if $r\in\Delta$. We claim that

$$1_\Delta(\mathbf{r}) \iint_{[\mathbf{r},\mathbf{z}]} a'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, dW(\mathbf{z}')=0$$

on G for all $\mathbf{r}$. Indeed, since this stochastic integral is almost surely continuous in $\mathbf{r}$, it is enough to consider $\mathbf{r}$'s with rational coordinates. For such $\mathbf{r}$'s, we approximate the stochastic integral by an L^2-convergent sequence of "Riemann sums" S_n. Then:

$$1_\Delta S_n \xrightarrow{L^2} 1_\Delta \iint_{[\mathbf{r},\mathbf{z}]} a'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, dW(\mathbf{z}')$$

as $n\to\infty$ as claimed. Consequently, equation (III.6) defining $\mathbf{Y}$ gives:

$$Y(\mathbf{z}, \mathbf{r})=1+\iint_{[\mathbf{r},\mathbf{z}]} b'(X(\mathbf{z}'))\, Y(\mathbf{z}'\, \mathbf{r})\, dz'.$$

This implies that $Y(\mathbf{z}, \mathbf{r})\geqq 1$ for all $\mathbf{r}\in\Delta$ on G and by continuity, $Y(\mathbf{z}, \mathbf{r})>0$ for $\mathbf{r}\in\Delta^\varepsilon$ for some $\varepsilon\in(0, \tau)$ where Δ^ε is the triangle determined by $\mathbf{z}$, $(\tau-\varepsilon, x-(t-\tau-\varepsilon))$ and $(\tau-\varepsilon, x+(t-\tau+\varepsilon))$. This implies that $a(X(\mathbf{r}))=0$ for $\mathbf{r}\in\Delta^\varepsilon$ (recall that we are on G) which is in contradiction with the definition of τ. □

Next we tackle the problem of the smoothness of the density whose existence we just established. We will need some additional regularity properties on the coefficients of our wave equation. We will assume that:

a and b are C^∞-functions with bounded derivatives of all orders larger than or equal to one

and

f and g are locally Holder continuous of order α for some $\alpha>0$.

We first prove

Proposition III.3. *For each fixed $t>0$ and $x\in\mathbb{R}$, the random variable $X(t, x)$ belongs to the space $\mathscr{D}_\infty$.*

Proof. It suffices to check that the sequence $\{X_n(\mathbf{z}); n\geqq 1\}$ defined by (III.3) in the proof of Proposition III.1 with $\mathbf{z}=(t, x)$ converges to $X(t, x)$ in the topology of $\mathscr{D}_\infty$. But first, we need to compute the successive derivatives of $X_n(\mathbf{z}')$. A simple induction argument gives:

$$\begin{aligned} D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X_n(\mathbf{z}) = & \sum_{j=1}^{N} D_{\mathbf{r}_1} \dots D_{\mathbf{r}_{j-1}} D_{\mathbf{r}_{j+1}} \dots D_{\mathbf{r}_N} X_n(\mathbf{f}_n(\mathbf{r}_j)) \\ & + \iint_{[g_n(\mathbf{r}_1 \vee \dots \vee \mathbf{r}_N) \wedge \mathbf{z}, \mathbf{z}]} [D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} a(X_n(f_n(\mathbf{z}'))) \, dW(\mathbf{z}') \\ & + D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} b(X_n(f_n(\mathbf{z}'))) \, d\mathbf{z}']. \end{aligned} \tag{III.12}$$

Using the chain rule for the Malliavin derivatives we obtain

$$D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} a(X_n(\mathbf{z}')) = \sum a^{(\nu)}(X_n(\mathbf{z}')) \, D_{\mathbf{r}(I_1)} X_n(\mathbf{z}') \dots D_{\mathbf{r}(I_\nu)} X_n(\mathbf{z}')$$

and

$$D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} b(X_n(\mathbf{z}')) = \sum b^{(\nu)}(X_n(\mathbf{z}')) \, D_{\mathbf{r}(I_1)} X_n(\mathbf{z}') \dots D_{\mathbf{r}(I_\nu)} X_n(\mathbf{z}')$$

where the sums are over all the partitions $\{I_1, \dots, I_\nu\}$ of $\{1, \dots, N\}$ and where, for each subset $I = \{j_1 < \dots < j_k\}$ of $\{1, \dots, N\}$ we used the notation $D_{\mathbf{r}(I)} = D_{\mathbf{r}_{j_1}} \dots D_{\mathbf{r}_{j_k}}$. We now consider, for $\mathbf{r}_1, \dots, \mathbf{r}_N$ fixed in $D(T, 0)$, the processes $\{D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X(\mathbf{z});\ \mathbf{z} \in D(T, 0),\ \mathbf{z} \geqq \mathbf{r}_1 \vee \dots \vee \mathbf{r}_N\}$ defined inductively by the integral equations:

$$\begin{aligned} D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X(\mathbf{z}) = & \sum_{j=1}^{N} D_{\mathbf{r}_1} \dots D_{\mathbf{r}_{j-1}} D_{\mathbf{r}_{j+1}} \dots D_{\mathbf{r}_N} X(\mathbf{r}_j) \\ & + \iint_{[(\mathbf{r}_1 \vee \dots \vee \mathbf{r}_N) \wedge \mathbf{z}, \mathbf{z}]} D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} a(X(\mathbf{z}')) \, dW(\mathbf{z}') \\ & + D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} b(X(\mathbf{z}')) \, d\mathbf{z}' \end{aligned} \tag{III.13}$$

where, as above:

$$D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} a(X(\mathbf{z}')) = \sum a^{(\nu)}(X(\mathbf{z}')) \, D_{\mathbf{r}(I_1)} X(\mathbf{z}') \dots D_{\mathbf{r}(I_\nu)} X(\mathbf{z}')$$

and

$$D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} b(X(\mathbf{z}')) = \sum b^{(\nu)}(X(\mathbf{z}')) \, D_{\mathbf{r}(I_1)} X(\mathbf{z}') \dots D_{\mathbf{r}(I_\nu)} X(\mathbf{z}').$$

Up to a rotation by $\pi/4$ of the coordinate axes, equation (III.12) is a particular case of the type of equations discussed in Lemma 3.1 of [13]. So it has a unique continuous solution. Moreover, we claim that:

$$\lim_{n \to \infty} \sup_{\mathbf{r}_1, \dots, \mathbf{r}_N \in D(T, 0)} \mathbf{E}\{ \sup_{\substack{\mathbf{z} \in D(T, 0) \\ \mathbf{z} \geqq \mathbf{r}_1 \vee \dots \vee \mathbf{r}_N}} |D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X(\mathbf{z}) - D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X(\mathbf{z})|^p \} = 0$$

for all $p \geqq 2$. This can be shown by induction on N. Indeed, we already proved the result for $N = 1$ in Proposition III.1. If we assume that it is true up to $N - 1$, we observe that $D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} a(X(\mathbf{z}))$ is equal to $a'(X(\mathbf{z})) \, D_{\mathbf{r}_N}(X(\mathbf{z}))$ plus a sum of higher order derivatives of a, i.e., $a^{(\nu)}(X(\mathbf{z}))$ with $\nu \geqq 2$ times products of quantities of the form $D_{\mathbf{r}(I)} X(\mathbf{z})$ with $\#(I) \leqq N - 1$. Then, the convergence in (III.14) follows from the induction hypothesis and Lemma 3.2 of [13] applied to $\lambda = (\mathbf{r}_1, \dots, \mathbf{r}_N) \in D(T, 0)^N$, $\mathbf{r}(\lambda) = \mathbf{r}_1 \dots \mathbf{r}_N$, $Y(\tau, \lambda) = D_{\mathbf{r}_1} \dots D_{\mathbf{r}_N} X(\tau)$ and $V(\tau, \lambda)$

being the (2^N-1)-dimensional process whose components are $X(\tau)$ and the $D_{\mathbf{r}(I)}X(\tau)$ with $I\{1, \ldots, N\}$, $0<\#(I)<N$ (the processes $\alpha(\lambda)$, $\alpha_n(\lambda)$ and $V_n(\tau, \lambda)$ also appearing in this lemma are defined in an obvious way). □

Theorem III.4. *Let us assume that $t>0$ and $x\in\mathbf{R}$ are fixed and that one of the following two conditions holds:*

(i) *$a(y)\neq 0$ for some y in the closed interval with endpoints $f(x-t)$ and $f(x+t)$*

(ii) *$f(x-t)=f(x+t)=y_0$, $a(y_0)=0$, $a^{(n)}(y_0)\neq 0$ for some $n\geqq 1$ and either*

ii)$_1$ *$a(f(\xi_0))\neq 0$ for some $\xi_0\in(x-t, x+t)$ and the Holder exponent is equal to 1 or*

ii)$_2$ *$a(f(\xi))=0$ for all $\xi\in[x-t, x+t]$ and either g is C^2 and $g'(x-t)\neq 0$ or $g'(x+t)\neq 0$, or g is C^3 $g''(x-t)+b(y_0)$ or $g''(x+t)+b(y_0)\neq 0$.*

Then $X(t, x)$ possesses an infinitely differentiable density.

Proof. We prove that, for each $p\geqq 2$ there exists $\varepsilon_0(p)>0$ such that

$$P\{\iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}\leqq\varepsilon^p \tag{III.15}$$

for all $\varepsilon<\varepsilon_0(p)$. This implies that, for each $p\geqq 2$

$$E\{|\iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}|^{-p}\}<+\infty$$

which, together with Proposition III.3 gives the desired result according to the Malliavin calculus as recalled in the introduction. As before, we use the notation $\mathbf{z}=(t, x)$. The proof of (III.15) is rather long, so we divide it in several steps.

1. Let us assume that $a(f(x-t))\neq 0$, let $0<\varepsilon<1$ and let $A_0^\varepsilon(\mathbf{z})$ be the strip of width ε^σ contained in $D(\mathbf{z})$ and limited by the line L_0 defined in (III.10) and the line segment:

$$L_0^\varepsilon=\{\mathbf{z}_0+(0, \sqrt{2}\,\varepsilon^\sigma)+\lambda\,\mathbf{u};\ 0\leqq\lambda\leqq\sqrt{2}\,t-\varepsilon^\sigma\}$$

(See Fig. 6) where σ is chosen so that $[1+(1\wedge 2\alpha)]^{-1}<\sigma<1$. We have:

$$\begin{aligned}
&\mathbf{P}\{\iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}\leqq\mathbf{P}\{\iint_{A_0^\varepsilon(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}\\
&\leqq\mathbf{P}\left\{\int_{\varepsilon^\sigma}^{\sqrt{2}t}a(X(\mathbf{z}_0(\lambda)))^2\,d\lambda\leqq 4\,\varepsilon^{1-\sigma}\right\}\\
&+\mathbf{P}\left\{\int_{\varepsilon^\sigma}^{\sqrt{2}t}a(X(\mathbf{z}_0(\lambda)))^2\,d\lambda>4\,\varepsilon^{1-\sigma},\ \iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\right\}\\
&=\mathrm{i})+\mathrm{ii}).
\end{aligned} \tag{III.16}$$

Now:

$$\mathrm{ii})\leqq\mathbf{P}\{\iint_{A_0^\varepsilon(\mathbf{z})}a(X(\mathbf{r}_0))^2\,d\mathbf{r}>4\,\varepsilon,\ \iint_{A_0^\varepsilon(\mathbf{z})}(D_{\mathbf{z}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}$$

where we use the notation $\mathbf{r}_0$ for the orthogonal projection of $\mathbf{r}$ onto L_0.

$$\leqq \mathbf{P}\{ \iint_{A_0^\delta(\mathbf{z})} [D_{\mathbf{r}} X(\mathbf{z}) - a(X(\mathbf{r}_0))]^2 \, d\mathbf{r} > \varepsilon\}$$

$$\leqq \varepsilon^{-q} \mathbf{E}\{| \iint_{A_0^\delta(\mathbf{z})} [D_{\mathbf{r}} X(\mathbf{z}) - a(X(\mathbf{r}_0))]^2 \, d\mathbf{r}|^q\}$$

for any real number $q \geqq 1$,

$$\leqq \varepsilon^{-q/3} (\sqrt{2}\, t)^q \sup_{r \in A_0^\delta(\mathbf{z})} \mathbf{E}\{|a(X(\mathbf{r}))\, Y(\mathbf{z}, \mathbf{r}) - a(X(\mathbf{r}_0))|^{2q}\}.$$

Furthermore:

$$\begin{aligned} &|a(X(\mathbf{r}))\, Y(\mathbf{z}, \mathbf{r}) - a(X(\mathbf{r}_0))| \\ &\leqq |a(X(\mathbf{r}))|\, |Y(\mathbf{z}, \mathbf{r}) - 1| + |a(X(\mathbf{r})) - a(X(\mathbf{r}_0))| \\ &\leqq |a(X(\mathbf{r}))|\, |Y(\mathbf{z}, \mathbf{r}) - 1| + cst\, |X(\mathbf{r}) - X(\mathbf{r}_0)|. \end{aligned}$$

Using the local Holder continuity of f and g one can show that:

$$\mathbf{E}\{|X(\mathbf{r}) - X(\mathbf{r}_0)|^{2q}\} \leqq cst(\mathbf{z})\, |\mathbf{r} - \mathbf{r}_0|^{q \wedge 2\alpha q} \leqq c\,s\,t(\mathbf{z})\, \varepsilon^{\sigma q(1 \wedge 2\alpha)}.$$

In much the same way one shows that:

$$\mathbf{E}\{|a(X(\mathbf{r}))|^{2q}\, |Y(\mathbf{z}, \mathbf{r}) - 1|^{2q}\} \leqq c\,s\,t(\mathbf{z})\, \varepsilon^{\sigma q}$$

and this gives

$$\text{(ii)} \leqq c\,s\,t(\mathbf{z})\, \varepsilon^{q[\sigma(1 + (1 \wedge 2\alpha)) - 1]}. \tag{III.17}$$

To estimate the first term we define:

$$S = \sqrt{2}\, t \wedge \inf\{\lambda > 0;\, |a(X(\mathbf{z}_0(\lambda))) - a(X(\mathbf{z}_0))| \geqq c/2\}$$

where $c = |a(f(x-t))| \neq 0$. S is a stopping time with respect to the filtration $\{\mathscr{B}_{0,\lambda};\, 0 \leqq \lambda \leqq \sqrt{2}\, t\}$ defined in (III.11). If $\lambda \leqq S$ we have:

$$|a(X(\mathbf{z}_0(\lambda)))| \geqq |a(X(\mathbf{z}_0))| - |a(X(\mathbf{z}_0(\lambda))) - a(X(\mathbf{z}_0))| \geqq c/2$$

so, if $S > \varepsilon^\beta$ with $\beta < \min(\sigma, 1-\sigma)$ we have:

$$\int_{\varepsilon^\sigma}^{\sqrt{2}} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda \geqq \int_{\varepsilon^\sigma}^{S} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda \geqq \frac{c^2}{4} (\varepsilon^\beta - \varepsilon^\sigma)$$

and consequently:

$$\{S > \varepsilon^\beta\} \cap \left\{ \int_{\varepsilon^\sigma}^{\sqrt{2}t} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda \leqq 4\varepsilon^{1-\sigma} \right\} = \phi$$

for ε small enough, and in such a case:

$$\begin{aligned}
(\mathrm{i}) &\leqq P\{S \leqq \varepsilon^{\beta}\} \\
&\leqq \mathbf{P}\{\sup_{0 \leqq \lambda \leqq \varepsilon^{\beta}} |a(X(\mathbf{z}_0(\lambda))) - a(X(\mathbf{z}_0))| > c/2\} \\
&\leqq (2/c)^q \, \mathbf{E}\{\sup_{0 \leqq \lambda \leqq \varepsilon^{\beta}} |a(X(\mathbf{z}_0(\lambda))) - a(X(\mathbf{z}_0))|^q\} \\
&\leqq c\,s\,t\;\varepsilon^{2q\beta(1 \wedge 2\alpha)}.
\end{aligned} \tag{III.18}$$

(III.16), (III.17) and (III.18) give (III.15). The proof of (III.15) in the case $a(f(x+t)) \neq 0$ is similar and we omit it.

2. We now assume that $a(f(x-t)) = a(f(x+t)) = 0$ but $a(y) \neq 0$ for some y in the closed interval with endpoints $f(x-t)$ and $f(x+t)$. We used the same notations as before for $\mathbf{z}_0$, $\mathbf{z}_1$, $\mathbf{z}_0(\lambda)$ and we add $\mathbf{z}_1(\lambda) = \mathbf{z}_1 + \lambda \mathbf{v}$. Moreover:

$$S_0 = \sqrt{2}\,t \wedge \inf\{\lambda > 0;\, |a(X(\mathbf{z}_0(\lambda)))| > |a(y)|/2\}$$

and

$$T_0 = \sqrt{2}\,t \wedge \inf\{\lambda > S_0;\, |a(X(\mathbf{z}_0(\lambda))) - a(X(\mathbf{z}_0(S_0)))| > |a(y)|/4\}$$

are stopping times for the filtration $\{\mathscr{B}_{0,\lambda};\, \lambda \geqq 0\}$ defined in (III.11) while S_1 and T_1 defined by replacing $\mathbf{z}_0(\lambda)$ by $\mathbf{z}_1(\lambda)$ are stopping times for the filtration $\{\mathscr{B}_{1,\lambda};\, \lambda \geqq 0\}$ obtained by replacing $\mathbf{z}_0(\lambda)$ by $\mathbf{z}_1(\lambda)$ in (III.11).

The notation $A_0^{\varepsilon}(\mathbf{z})$ still have the same meaning as before and we introduce the strip $A_1^{\varepsilon}(\mathbf{z})$ of width ε^{σ} in the triangle $D(\mathbf{z} - \varepsilon^{\sigma} v)$ limited by the line L_1 and the segment

$$L_1^{\varepsilon} = \{(0,\, x + t - \sqrt{2}\,\varepsilon^{\sigma}) + \lambda \mathbf{v};\, 0 \leqq \lambda \leqq \sqrt{2}\,t - 2\,\varepsilon^{\sigma}\}$$

(see Fig. 6 below).

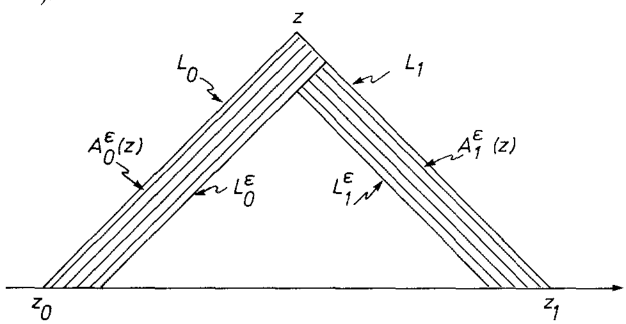

Fig. 6

With these notations we have:

$$\begin{aligned}
\mathbf{P}\{\iint_{D(\mathbf{z})} (D_{\mathbf{r}} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon\} &\leqq \mathbf{P}\{\iint_{A_0^{\varepsilon}(\mathbf{z}) \cup A_1^{\varepsilon}(\mathbf{z})} (D_{\mathbf{r}} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon\} \\
&\leqq \mathbf{P}\{\iint_{A_0^{\varepsilon}(\mathbf{z})} (D_{\mathbf{z}} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon;\, 0 \leqq S_0 < T_0 < \sqrt{2}\,t\} \\
&\quad + \mathbf{P}\{\iint_{A_0^{\varepsilon}(\mathbf{z})} (D_{\mathbf{r}} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon;\, 0 \leqq S_1 < T_1 < \sqrt{2}\,t\} \\
&= \mathrm{i}) + \mathrm{ii}),
\end{aligned}$$

because our assumption implies that $\Omega=\{0<S_0<T_0<\sqrt{2}t\}\cup\{0<S_1<T_1<\sqrt{2}t\}$. Now:

$$\text{(i)}\leqq \mathbf{P}\left\{\int_{\varepsilon^\sigma}^{\sqrt{2}t} a(X(\mathbf{z}_0(\lambda)))^2\,d\lambda<4\varepsilon^{1-\sigma},\ 0<S_0<T_0<\sqrt{2}t\right\}$$

$$+\mathbf{P}\left\{\int_{\varepsilon^\sigma}^{\sqrt{2}} a(X(\mathbf{z}_0(\lambda)))^2\,d\lambda>4\varepsilon^{1-\sigma},\ \iint_{A_0^\varepsilon(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\right\}$$

$$=\text{iii})+\text{iv}).$$

iv) can be controlled in the same way as in part 1 of the proof. Moreover we notice that $|a(X(\mathbf{z}_0(\lambda)))|$ is bounded below by $|a(y)|/4$ in the interval (S_0, T_0) and consequently

$$\left\{\int_{\varepsilon^{2/3}}^{\sqrt{2}t} a(X(\mathbf{z}_0(\lambda)))^2\,d\lambda\leqq 4\varepsilon^{1-\sigma},\ 0<S_0<T_0<\sqrt{2}t,\ T_0-S_0>\varepsilon^\beta\right\}=\phi$$

if $\beta<\min(\sigma, 1-\sigma)$ and ε is small enough. In this case we have:

$$\begin{aligned}\text{iii})&\leqq \mathbf{P}\{0<S_0<T_0<\sqrt{2}t,\ T_0-S_0\leqq\varepsilon^\beta\}\\ &\leqq \mathbf{P}\{\sup_{S_0\leqq\lambda\leqq S_0+\varepsilon^\beta}|a(X(\mathbf{z}_0(\lambda)))-a(X(\mathbf{z}_0(S_0)))|\geqq|a(y)|/4\}\\ &\leqq (4/|a(y)|)^q\,\mathbf{E}\{\sup_{S_0\leqq\lambda\leqq S_0+\varepsilon^\beta}|a(X(\mathbf{z}_0(\lambda)))-a(X(\mathbf{z}_0(S_0)))|^q\}\\ &\leqq cst(\mathbf{z})\,\varepsilon^{2\beta q(1\wedge 2\alpha)}.\end{aligned}$$

Since the quantity ii) can be controlled similarly (essentially by replacing all the subscripts 0's by 1's) we obtain (III.15) again.

For the proof of (ii), namely for the steps 3. and 4. below, one can take $\sigma=\frac{2}{3}$.

3. We now assume that $y_0=f(x-t)=f(x-t)$, $a(y_0)=0$ and let $n\geqq 1$ the smallest integer such that $a^{(n)}(y_0)\neq 0$. In this case, we can find $C>0$ and $\delta>0$ such that $|a(y)|\geqq C\,|y-y_0|^n$ for $|y-y_0|<\delta$.

We also assume that $a(f(\xi_0))\neq 0$ for some $\xi_0\in(x-t, x+t)$ and we let z_3 be the orthogonal projection of $(0, \xi_0)$ on the line L_1 and we let B^ε be the strip of width $\varepsilon^{2/3}$ contained in the triangle $D(z_3)$ and limited by the lines L_1 and L_1^ε.

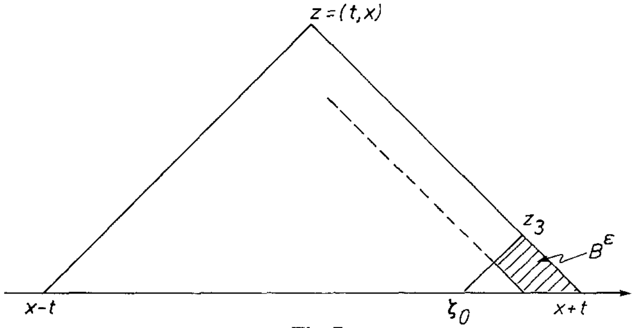

Fig. 7

Then:

$$\mathbf{P}\{\iint_{D(\mathbf{z})}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}\leqq\mathbf{P}\{\iint_{B^\varepsilon}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\}$$

$$\leqq\mathbf{P}\left\{\int_{\varepsilon^{2/3}}^{\lambda_0}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda\leqq 4\,\varepsilon^{1/3}\right\}$$

$$+\mathbf{P}\left\{\int_{\varepsilon^{2/3}}^{\lambda_0}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda>4\,\varepsilon^{1/3},\ \iint_{B^\varepsilon}(D_{\mathbf{r}}X(\mathbf{z}))^2\,d\mathbf{r}\leqq\varepsilon\right\}=\text{i)}+\text{ii)}$$

where $\lambda_0\in[0,\sqrt{2}\,t]$ is such that $\mathbf{z}_3=\mathbf{z}_1(\lambda_0)$. ii) can be controlled as before.

We define the $\mathscr{B}_{1,\lambda}$-stopping times:

$$S_1=\inf\{\lambda>0;\,|X(\mathbf{z}_1(\lambda))-y_0|>\varepsilon^\gamma/2\}$$

and

$$S_2=\inf\{\lambda>S_1;\,|X(\mathbf{z}_1(\lambda))-X(\mathbf{z}_1(S_1))|>\varepsilon^\gamma/4\}.$$

Let $\beta>0$ and $\gamma>0$ be such that $2\gamma<\beta<\frac{2}{3}$ and $\beta+2n\gamma<\frac{1}{3}$. If $\varepsilon>0$ is small enough, $\lambda<S_2$ implies $|X(\mathbf{z}_1(\lambda))-y_0|<\delta$ so that

$$|a(X(\mathbf{z}_1(\lambda)))|\geqq C\,|X(\mathbf{z}_1(\lambda))-y_0|^n\geqq C\,2^{-2n}\,\varepsilon^{n\gamma}$$

if $\lambda>S_1$. Consequently:

$$\int_{\varepsilon^{2/3}}^{\lambda_0}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda\geqq\int_{\varepsilon^{2/3}\vee S_1}^{S_2}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda\geqq cst\,\varepsilon^{2n\gamma+\beta}$$

if $S_2-S_1>\varepsilon^\beta$, $S_2<\lambda_0$ and ε is small enough. Therefore,

$$\left\{\int_{\varepsilon^{2/3}}^{\lambda_0}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda\leqq 4\,\varepsilon^{1/3},\ S_2<\lambda_0\right\}\cap\{S_2-S_1>\varepsilon^\beta\}=\phi$$

and hence

$$\text{i)}\leqq\mathbf{P}\{S_2\geqq\lambda_0\}+\mathbf{P}\left\{\int_{\varepsilon^{2/3}}^{\lambda_0}a(X(\mathbf{z}_1(\lambda)))^2\,d\lambda\leqq 4\,\varepsilon^{1/3},\ S_2<\lambda_0\right\}$$

$$\leqq\mathbf{P}\{S_2\geqq\lambda_0\}+P\{S_2-S_1\leqq\varepsilon^\beta,\ S_2<\lambda_0\}. \tag{III.19}$$

The second term of the above expression can be bounded as follows

$$\mathbf{P}\{S_2-S_1\leqq\varepsilon^\beta,\ S_2<\lambda_0\}$$

$$\leqq\mathbf{P}\{\sup_{S_1\leqq\lambda\leqq S_1+\varepsilon^\beta}|X(\mathbf{z}_1(\lambda))-X(\mathbf{z}_1(s_1))|>\varepsilon^\gamma/4\}$$

$$\leqq 4q\,\varepsilon^{-q\gamma}\,\mathbf{E}\{\sup_{s_1\leqq\lambda\leqq s_1+\varepsilon^\beta}|X(\mathbf{z}_1(\lambda))-X(\mathbf{z}_1(s_1))^q\}$$

$$\leqq cst\,\varepsilon^{-q\gamma+q\beta/2},$$

which gives the desired estimate.

The functions a and f are continuous, so $a(f(\xi)) \neq 0$ for any ξ in some interval $[\xi_0, \xi_1]$, $\xi_0 < \xi_1 < x+t$. Let $\mathbf{z}_4$ be the orthogonal projection of $(0, \xi_1)$ on the line L_1, and set $\mathbf{z}_4 = \mathbf{z}_1(\lambda_1)$ for some $\lambda_1 \in (0, \lambda_0)$.

In order to estimate the first term of (III.19) we write

$$\mathbf{P}\{S_2 \geqq \lambda_0\} \leqq \mathbf{P}\left\{\int_{\lambda_1}^{\lambda_0} |X(\mathbf{z}_1(\lambda)) - X(\mathbf{z}_1(\lambda_1))|^2 \, d\lambda \leqq 4(\lambda_0 - \lambda_1)\, \varepsilon^{2\gamma}\right\}.$$

Consider the process defined by

$$\begin{aligned} Y(\lambda) &= X(\mathbf{z}_1(\lambda)) - X(\mathbf{z}_1(\lambda_1)) \\ &= \tfrac{1}{2}\{f(\xi_1 + \lambda\sqrt{2}) - f(\xi_1) + g(\xi_1) - g(\xi_1 + \lambda\sqrt{2})\} \\ &\quad + \int_{D(\mathbf{z}_1(\lambda)) - D(\mathbf{z}_1(\lambda_1))} [a(X(\mathbf{z}'))\, dW(\mathbf{z}') + b(X(\mathbf{z}'))\, d\mathbf{z}'], \end{aligned}$$

for $\lambda_1 < \lambda < \lambda_0$.

$Y(\lambda)$ is a continuous semimartingale with respect to the filtration $\{\mathscr{B}_{1,\lambda};\ \lambda_1 \leqq \lambda \leqq \lambda_0\}$. Its quadratic variation is given by:

$$\langle Y\rangle(\lambda) = \int_{D(\mathbf{z}_1(\lambda)) - D(\mathbf{z}_1(\lambda_1))} a(X(\mathbf{z}'))^2 \, d\mathbf{z}'.$$

Fix $\varepsilon > 0$ and define the $\mathscr{B}_{1,\lambda}$-stopping time

$$T = \inf\{\lambda > \lambda_1 : \sup_{\mathbf{z}' \in D(\mathbf{z}_1(\lambda)) - D(\mathbf{z}_1(\lambda_1))} |X(\mathbf{z}')| > \varepsilon\}.$$

We have

$$\begin{aligned} \mathbf{P}\{T \leqq \lambda_0\} &\leqq \mathbf{P}\{\sup_{\mathbf{z}' \in D(\mathbf{z}_1(\lambda_0)) - D(\mathbf{z}_1(\lambda_1))} |X(\mathbf{z}')| \geqq \varepsilon\} \\ &\leqq \varepsilon^{-q} \mathbf{E}\{\sup_{\mathbf{z}' \in D(\mathbf{z}_1(\lambda_0)) - D(\mathbf{z}_1(\lambda_1))} |X(\mathbf{z}')|\} \leq cst\, \varepsilon^q. \end{aligned}$$

Consequently it suffices to estimate the following probability

$$\begin{aligned} &P\left\{\int_{\lambda_1}^{\lambda_0} |Y(\lambda)|^2 \, d\lambda \leqq 4(\lambda_0 - \lambda_1)\, \varepsilon^{2\gamma},\ T > \lambda_0\right\} \\ \leqq\ & P\left\{\int^{\lambda_0 \wedge T} |Y(\lambda)|^2 \, d\lambda \leqq 4(\lambda_0 - \lambda_1)\, \varepsilon^{2\gamma},\ \langle Y\rangle(\lambda_0 \wedge T) \geqq \varepsilon^\eta\right\} \\ &+ P\{\langle Y\rangle(\lambda_0) < \varepsilon^\eta\}. \end{aligned} \tag{III.20}$$

Assume that $2\eta < \gamma$ and $\xi < \frac{1}{4}(\gamma - 2\eta)$. Then the first summand of (III.20) can be bounded using a slight modification of Lemma 4.2 of [13] (this lemma is actually a version of Theorem 8.26 of Stroock [20] for continuous semimartingales). In fact, let M be a bound for both the derivative of the bounded variation part of $Y(\lambda)$ and the square root of derivative of the quadratic variation $\langle Y\rangle(\lambda)$, on the interval $[\lambda_1, \lambda_0 \wedge T]$. The definition of T and the Lipschitz hypothesis on f and g imply that M is less or equal than a multiple of $\varepsilon^{-\xi}$. Consequently we can apply Lemma 4.2 of [13] with a constant M depending on ε and obtain the desired estimate for the first term of (III.20).

The second term of (III.20) can be estimated by the arguments used in step 1 because $a(f(\xi)) \neq 0$ for any $\xi \in [\xi_0, \xi_1]$. This gives the desired estimate.

4. We assume that $y_0 = f(x-t) = f(x+t)$, $a(y_0) = 0$, and $a^{(n)}(y_0) \neq 0$ as before.

$$\mathbf{P}\{\iint_{D(\mathbf{z})} (D_\mathbf{r} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon\}$$

$$\leqq \mathbf{P}\left\{\iint_{A_0^\xi(\mathbf{z})} (D_\mathbf{r} X(\mathbf{z}))^2 \, d\mathbf{r} \leqq \varepsilon, \int_{\varepsilon^{2/3}}^{\sqrt{2}t} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda > 4\varepsilon^{1/3}\right\}$$

$$+\mathbf{P}\left\{\int_0^{\varepsilon^{2/3}} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda > 4\varepsilon^{1/3}\right\} + \mathbf{P}\left\{\int_0^{\sqrt{2}t} a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda > 8\varepsilon^{1/3}\right\}$$

$$= \text{i}) + \text{ii}) + \text{iii}).$$

The quantity (i) is estimated in the same way as in part 1 of the proof while ii) is controlled by Chebichev's inequality. Now:

$$S = \sqrt{2}\,t \wedge \inf\{\lambda > 0;\ \sup_{0 \leqq \lambda' \leqq \lambda} |X(\mathbf{z}_0(\lambda, \lambda')) - y_0| > \delta\}$$

is an $\mathscr{B}_{0,\lambda}$-stopping time and

$$\text{iii}) \leqq \mathbf{P}\left\{\int_0^S a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda \leqq 8\varepsilon^{1/3}\right\}$$

$$\leqq \mathbf{P}\left\{\int_0^S a(X(\mathbf{z}_0(\lambda)))^2 \, d\lambda \leqq 8\varepsilon^{1/3}, S > \varepsilon^\gamma\right\} + \mathbf{P}\{S \leqq \varepsilon^\gamma\} \tag{III.21}$$

for some number $\gamma > 0$ to be chosen later. The second term is easily estimated:

$$\mathbf{P}\{S \leqq \varepsilon^\gamma\} \leqq \mathbf{P}\{\sup_{\substack{0 \leqq \lambda \leqq \varepsilon^\gamma \\ 0 < \lambda' < \lambda}} |X(\mathbf{z}_0(\lambda, \lambda')) - y_0| \geqq \delta\}$$

$$\leqq \delta^{-q} \mathbf{E}\{\sup_{\substack{0 \leqq \lambda \leqq \varepsilon^\gamma \\ 0 \leqq \lambda' \leqq \lambda}} |X(\mathbf{z}_0(\lambda, \lambda')) - y_0|^q\} \leqq cst\, \varepsilon^{\gamma q/2}.$$

The first term in (III.21) is bounded from above by:

$$\mathbf{P}\left\{\int_0^S |X(\mathbf{z}_0(\lambda)) - y_0|^{2n} \, d\lambda \leqq 8C^{-2}\varepsilon^{1/3}, S > \varepsilon^\gamma\right\}$$

(because $|X(\mathbf{z}_0(\lambda)) - y_0| < \delta$)

$$\leqq \mathbf{P}\left\{\int_0^S |X(\mathbf{z}_0(\lambda)) - y_0|^2 \, d\lambda \leqq c'\varepsilon^\eta, S > \varepsilon^\gamma\right\}$$

provided we set $c' = 2^{3/n} C^{-2/n} (\sqrt{2}\, t)^{1-1/n}$ and $\eta = 1/(3n)$.

$$\leq \mathbf{P}\left\{\int_0^S |X(\mathbf{z}_0(\lambda)) - y_0|^2 \, d\lambda \leq c' \varepsilon^{\eta},\right.$$

$$\left.\int_0^S \left[\frac{1}{\sqrt{2}} g'(x - t + \sqrt{2}\lambda) + \int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'\right]^2 d\lambda \geq \varepsilon^{\beta},\ S > \varepsilon^{\gamma}\right\}$$

$$+ \mathbf{P}\left\{\int_0^S \left[\frac{1}{\sqrt{2}} g'(x - t + \sqrt{2}\lambda) + \int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'\right]^2 d\lambda < \varepsilon^{\beta},\ S > \varepsilon^{\gamma}\right\}. \tag{III.22}$$

Our assumptions imply that $f(\xi) = y_0$ for all ξ in $[x-t, x+t]$ and consequently that:

$$X(\mathbf{z}_0(\lambda)) = y_0 + \tfrac{1}{2}[g(x - t + \sqrt{2}\lambda) - g(x-t)]$$
$$= \iint_{D(\mathbf{z}_0(\lambda))} [a(X(\mathbf{z}')) \, dW(\mathbf{z}') + b(X(\mathbf{z}')) \, d\mathbf{z}'].$$

The first term in (III.22) can be estimated using Lemma 4.1 of [11] provided $\eta > 8\beta$ which is true if $\beta > 0$ is chosen small enough. In fact, $\frac{1}{\sqrt{2}} g'(x - t + \sqrt{2}\lambda) + \int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'$ is a semimartingale in λ satisfying the desired properties. We use here our assumption $g \in C^2$ but note that g' with bounded variation is in fact all we need. The second term in (III.22) is always bounded above by:

$$\mathbf{P}\left\{\int_0^{\varepsilon^{\gamma}} \left[\frac{1}{\sqrt{2}} g'(x - t + \sqrt{2}\lambda) + \int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'\right]^2 d\lambda < \varepsilon^{\beta},\right.$$

$$\left.\sup_{0 \leq \lambda \leq \varepsilon^{\lambda}} \left|\int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'\right| < c''/2\right\}$$

$$+ \mathbf{P}\left\{\sup_{0 \leq \lambda \leq \varepsilon^{\gamma}} \left|\int_0^\lambda b(X(\mathbf{z}_0(\lambda, \lambda'))) \, d\lambda'\right| \geq c''/2\right\}$$

but if $c'' > 0$ is chosen so that $|g'(x - t + \sqrt{2}\lambda)| > \sqrt{2}\, c''$ for all $\lambda \in [0, \varepsilon^{\gamma}]$, then if $\varepsilon > 0$ is small enough and if $\beta > \gamma$, the first probability is zero and we estimate the second one as usual.

The proof for the case $g'(x+t) \neq 0$ is similar.

Finally we assume that g is three times continuously differentiable or at least that g'' exists and is of bounded variation, that $g'(x-t) = 0$ and that $g''(x-t) + b(y_0) \neq 0$. In order to conclude, we need only to control the second term in (III.22) since the other terms can be estimated in the same way. Using the expression:

$$X(\mathbf{z}_0(\lambda)-\mu\mathbf{v})=y_0+\tfrac{1}{2}[g(x-t+\sqrt{2}t)-g(x-t+\sqrt{2}\mu)] + \iint_{D(\mathbf{z}_0(\lambda)-\mu\mathbf{v})} [a(X(\mathbf{z}')\,dW(\mathbf{z}')+b(X(\mathbf{z}'))\,dz']$$

and applying Ito's formula one obtains:

$$\frac{1}{\sqrt{2}}g'(x-t+\sqrt{2}\lambda)+\int_0^\lambda b(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\,d\mu$$
$$=\int_0^\lambda\Bigg[\iint_{D(\mathbf{z}_0(\lambda)-\mu\mathbf{v})} b'(X(\mathbf{z}_0(\mu+\lambda')-\mu\mathbf{v}))$$
$$\cdot\, a(X(\mathbf{z}_0(\lambda'+\mu)-(\mu'+\mu)\mathbf{v}))\,dW(\lambda',\mu')\Bigg]\,d\mu$$
$$+\int_0^\lambda\left(\int_0^{\lambda-\mu}\int_0^{\lambda'} b'(X(\mathbf{z}_0(\mu+\lambda')-\mu\mathbf{v}))\right.$$
$$\left.\cdot\, b(X(\mathbf{z}_0(\lambda'+\mu)-(\mu'+\mu)\mathbf{v}))\,d\mu'\,d\lambda'\right)d\mu+\lambda\, b(y_0)$$
$$+\int_0^\lambda g''(x-t+\sqrt{2}\lambda')\,d\lambda'+\frac{\sqrt{2}}{2}\int_0^\lambda\int_0^{\lambda-\mu} b'(X(\mathbf{z}_0(\mu+\lambda')-\mu\mathbf{v}))$$
$$\cdot\, g'(x-t+\sqrt{2}\lambda')\,d\lambda'\,d\mu$$
$$+\tfrac{1}{2}\int_0^\lambda\left(\int_0^{\lambda-\mu}\int_0^{\lambda'} b''(X(\mathbf{z}_0(\lambda'+\mu)-\mu\mathbf{v}))\,a(X(\mathbf{z}_0(\lambda'+\mu)\right.$$
$$\left.-(\mu'+\mu)\mathbf{v}))^2\,d\mu'\,d\lambda'\right)d\mu.$$

Consequently, the second term in (III.22) is bounded above by the sum of:

$$\mathbf{P}\left\{\int_0^S\left[\frac{1}{\sqrt{2}}g'(x-t+\sqrt{2}\lambda)+\int_0^\lambda b(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\,d\mu\right]^2 d\lambda\right.$$
$$\leqq\varepsilon^\beta,\int_0^{\varepsilon^\gamma}\Bigg[\int_0^\lambda\int_0^{\lambda-\mu} b'(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\,b(X(\mathbf{z}_0(\lambda)-(\mu+\mu')\mathbf{v}))\,d\mu'\,d\mu$$
$$+\tfrac{1}{2}\int_0^\lambda\int_0^{\lambda-\mu} b''(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\,a(X(\mathbf{z}_0(\lambda)-(\mu+\mu')\mathbf{v}))^2\,d\mu'\,d\mu$$
$$+\frac{1}{\sqrt{2}}\int_0^\lambda b'(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\,g'(x-t+\sqrt{2}(\lambda-\mu))\,d\mu+b(y_0)$$
$$\left.+g''(x-t+\sqrt{2}\lambda)\Bigg]\,d\lambda>\varepsilon^\alpha,\ S>\varepsilon^\gamma\right\}$$

which can be bounded from above by using Lemma 4.1 of [11] provided $\beta > 8\alpha$, and

$$\mathbf{P}\left\{\int_0^{\varepsilon^\gamma}\left[\int_0^{\lambda}\int_0^{\lambda-\mu} b'(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\, b(X(\mathbf{z}_0(\lambda)-(\mu+\mu')\mathbf{v}))\, d\mu'\, d\mu\right.\right.$$
$$+\tfrac{1}{2}\int_0^{\lambda}\int_0^{\lambda-\mu} b''(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\, a(X(\mathbf{z}_0(\lambda)-(\mu+\mu')\mathbf{v}))^2\, d\mu'\, d\mu$$
$$+\frac{1}{\sqrt{2}}\int_0^{\lambda} b'(X(\mathbf{z}_0(\lambda)-\mu\mathbf{v}))\, g'(x-t+\sqrt{2}(\lambda-\mu))\, d\mu + b(y_0)$$
$$\left.\left. + g''(x-t+\sqrt{2}\lambda)\right]^2 d\lambda \leqq \varepsilon^\alpha\right\}$$

which can be controlled using the same technique as before, as long as $\gamma < \alpha$ and $b(y_0)+g''(x-t) \neq 0$. □

IV. The Case of the Half Line

Throughout this section, $W=\{W(A);\ A\in\mathscr{B}_f(\mathbf{R}_+\times\mathbf{R}_+)\}$ is a mean zero Gaussian process with covariance $\mathbf{E}\{W(A)\,W(B)\}=|A\cap B|$, on a complete probability space $(\Omega, \mathscr{F}, P)$. For each $\mathbf{z}=(t, x)\in\mathbf{R}_+\times\mathbf{R}_+$ we set

$$E(\mathbf{z})=\{(s, y)\in\mathbf{R}_+\times\mathbf{R}_+;\ 0\leqq s\leqq t,\ |x-(t-s)|\leqq y\leqq x+(t-s)\}. \qquad \text{(IV.1)}$$

Notice that $E(\mathbf{z})=D(\mathbf{z})$ when $x\geqq t$. In the case $x<t$, $E(\mathbf{z})$ is the shaded region in the Fig. 8 below:

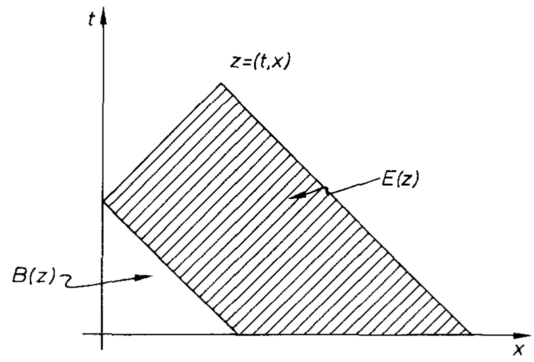

Fig. 8

For $x<t$ we also set:

$$C(t, x)=\{(s, y)\in\mathbf{R}_+\times\mathbf{R}_+;\ s+y\leqq t-x\}$$

and $B(t, x)=C(t, x)\cup E(t, x)$. Proposition II.3 has the following analogue in the present situation:

Proposition IV.1. *Let us assume that a and b are Lipschitz functions on* $\mathbf{R}$ *and that* $\mathbf{X}_0 = \{X_0(t,x);\ (t,x) \in \mathbb{R}_+ \times \mathbf{R}_+\}$ *is a continuous process independent of* W *satisfying*

$$\int_0^t \int_0^x \mathbf{E}\{|X_0(s,y)|^2\}\, dy\, ds < +\infty$$

for all (t,x) *in* $\mathbf{R}_+ \times \mathbf{R}_+$. *Then, there exists a unique continuous solution, say* $\mathbf{X} = \{X(t,x);\ (t,x) \in \mathbf{R}_+ \times \mathbf{R}_+\}$, *of the following stochastic integral equation:*

$$X(t,x) = X_0(t,x) + \iint_{E(t,x)} [a(X(s,y))\, dW(s,y) + b(X(s,y))\, ds\, dy]. \qquad \text{(IV.2)}$$

Proof. We define inductively the sequence $\{\mathbf{X}_n;\ n \geqq 0\}$ of processes by:

$$X_n(t,x) = X_0(t,x) + \iint_{E(t,x)} [a(X_{n-1}(s,y))\, dW(s,y) + b(X_{n-1}(s,y))\, ds\, dy].$$

For any fixed triangle $T = \{(t,x) \in \mathbb{R}_+ \times \mathbf{R}_+;\ x + t \leqq \mathbf{K}\}$ we have:

$$\begin{aligned}
&\mathbf{E}\{\sup_{(t,x)\in T} |X_{n+1}(t,x) - X_n(t,x)|^2\} \\
&\quad \leqq 2\,\mathbf{E}\{\sup_{(t,x)\in T} |\iint_{E(t,x)} [a(X_n(s,y)) - a(X_{n-1}(s,y))]\, dW(s,y)|^2\} \\
&\qquad + 2\,\mathbf{E}\{\sup_{(t,x)\in T} |\iint_{E(t,x)} [b(X_n(s,y)) - b(X_{n-1}(s,y))]\, ds\, dy|^2\}.
\end{aligned}$$

If $x \geqq t$, the stochastic integral is a two-parameter martingale with respect to the rotated coordinates. If $x < t$, this stochastic integral is written as the stochastic integral over $B(t,x)$ which is a two-parameter martingale for the rotated coordinates, minus the stochastic integral over $C(t,x)$ which is a martingale with respect to the parameter $y + s$. In any case one can apply maximal martingale inequalities and obtain:

$$\begin{aligned}
&\leqq cst \iint_T \mathbf{E}\{|X_n(t,x) - X_{n-1}(t,x)|^2\}\, dt\, dx \\
&\leqq cst\, \frac{K^{2n}}{n!} \iint_T \mathbf{E}\{|X_0(t,x)|^2\}\, dt\, dx.
\end{aligned}$$

Therefore,

$$\sum_{n\geqq 0} \mathbf{E}\{\sup_{(t,x)\in T} |X_{n+1}(t,x) - X_n(t,x)|^2\} < +\infty$$

which implies the uniform convergence of the series

$$X_0(t,x) + \sum_{n\geqq 0} [X_{n+1}(t,x) - X_n(t,x)]$$

over T. As usual, one shows that the resulting process is the unique continuous solution of (IV.2). □

As in the case of the whole real line (recall Proposition II.4), the solution process constructed above can be shown to be a weak solution of the nonlinear random wave Eq. (II.3) on the interval $I=[0,\infty)$. In fact, if:

(i) $\mathbf{F}=\{F(x);\ x\geqq 0\}$ is a continuous stochastic process satisfying $F(0)=0$ and $\int_0^x \mathbf{E}\{F(y)^2\}\,dy<+\infty$ for all $x>0$, and

(ii) $\mu\colon \mathscr{B}_f(\mathbf{R}_+)\to L^2(\Omega,\mathscr{F},P)$ is an L^2-measure with a continuous distribution function G satisfying $\int_0^x \mathbf{E}\{G(y)^2\}\,dy<+\infty$ for all $x>0$ which are both independent of $\mathbf{W}$, then, the unique solution of

$$X(t,x)=X_0(t,x)+\tfrac{1}{2}\iint_{E(t,x)}[a(X(s,y))\,dW(s,y)+b(X(s,y))\,ds\,dy]$$

with

$$X_0(t,x)=\begin{cases}\tfrac{1}{2}[F(x+t)+F(x-t)]+\tfrac{1}{2}\mu([x-t,x+t]) & \text{if } x\geqq t\\ \tfrac{1}{2}[F(x+t)-F(t-x)]+\tfrac{1}{2}\mu([t-x,x+t]) & \text{if } x<t\end{cases}\tag{IV.3}$$

is a weak solution of the random wave equation (II.3) with initial condition (F,μ) in the sense that: equation (II.6) with $\mathbf{R}_+$ instead of $\mathbf{R}$ is satisfied for all C^∞ function $f\colon \mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}$ with compact support in $[0,\infty)\times(0,\infty)$.

The proof is identical to the one of the case of the whole real line and we omit it.

As before, we address the problem of the absolute continuity of the solution at a fixed time $t>0$ and location $x\in(0,\infty)$, and of the smoothness of the possible density.

We will first assume that

the functions a and b are C^1 with bounded derivatives

and that

$$X_0(t,x)=\begin{cases}\tfrac{1}{2}[f(x+t)+f(x-t)]+\tfrac{1}{2}[g(x+t)-g(x-t)] & \text{if } t\leqq x\\ \tfrac{1}{2}[f(x+t)-f(t-x)]+\tfrac{1}{2}[g(x+t)-g(t-x)] & \text{if } t>x\end{cases}\tag{IV.4}$$

where f and g are continuous functions on $I=[0,\infty)$ such that $f(0)=0$.

Notice that:

$$X(t,0)=X_0(t,0)=0\quad t\geqq 0$$

and that

$$X(0,x)=X_0(0,x)=f(x)\quad x\geqq 0.$$

We fix $\mathbf{z}=(t,x)$ with $t>0$ and $x>0$. As in the case of the whole real line, the random variable $X(\mathbf{z})$ belongs to $\mathscr{D}_{p,1}$ for all $p\geqq 2$ and the Malliavin derivative $D_{\mathbf{r}}X(\mathbf{z})$ satisfies:

$$D_{\mathbf{r}}X(\mathbf{z})=a(X(\mathbf{r}))\,1_{E(\mathbf{z})}(\mathbf{r})+\iint_{E(\mathbf{z})}[a'(X(\mathbf{z}'))\,D_{\mathbf{r}}X(\mathbf{z}')\,dW(\mathbf{z}')+b'(X(\mathbf{z}'))\,D_{\mathbf{r}}X(\mathbf{z}')\,dz'].\tag{IV.5}$$

The following remarks are in order:

a) When $t \leqq x$, $E(\mathbf{z}) = D(\mathbf{z})$ and the Malliavin derivative $D_{\mathbf{r}} X(\mathbf{z})$ has the same properties as in the case of the whole real line.

b) When $x < t$, for $\mathbf{r} \in E(\mathbf{z})$ we write $D_{\mathbf{r}} X(\mathbf{z}) = a(X(\mathbf{r}))\, Y(\mathbf{z}, \mathbf{r})$ where the process $\mathbf{Y} = \{Y(\mathbf{z}, \mathbf{r});\, r \in E(\mathbf{z})\}$ satisfies:

$$Y(\mathbf{z}, \mathbf{r}) = 1 + \iint_{E(\mathbf{z})} a'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, dW(\mathbf{z}') + b'(X(\mathbf{z}'))\, Y(\mathbf{z}', \mathbf{r})\, d\mathbf{z}'. \tag{IV.6}$$

As before, one can use Kolmogorov's criterion to show that the process $\mathbf{Y}$ possesses a version which is continuous in $\mathbf{r}$.

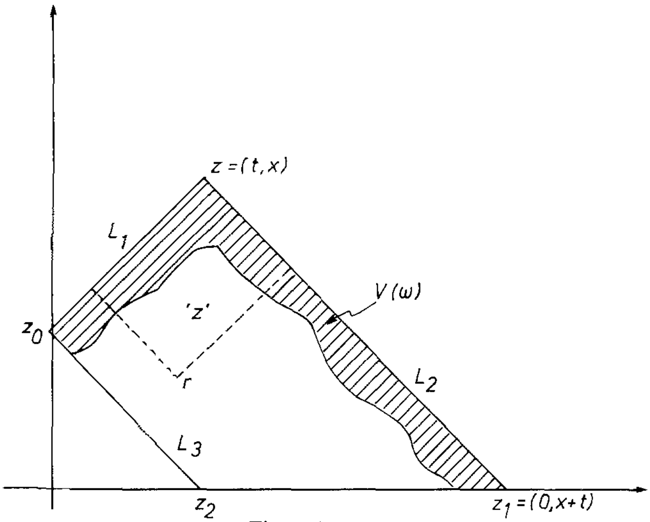

Fig. 9. Case $x < t$

Notice also that $D_{\mathbf{r}} X(\mathbf{z}')$; is zero unless $\mathbf{r} \leqq \mathbf{z}'$ so that we can replace $E(\mathbf{z})$ by the rectangle $[\mathbf{r}, \mathbf{z}]$ in the double integrals appearing in (IV.5) and (IV.6). Let L_1 be the line segment with endpoints $\mathbf{z}_0 = (t-x, 0)$ and $\mathbf{z}$ and L_2 the line segment between $\mathbf{z}$ and $\mathbf{z}_1 = (0, x+t)$. Because of (IV.6) we have $Y(\mathbf{z}, \mathbf{r}) = 1$ a.s. for any $\mathbf{r}$ in $L_1 \cup L_2$, and in this case we must have $D_{\mathbf{r}} X(\mathbf{z}) = a(X(\mathbf{r}))$. Moreover, by continuity of $\mathbf{Y}$, for $\mathbf{P}$-almost every $\omega \in \Omega$ there exists an open neighborhood of $L_1 \cup L_2$, say $V(\omega)$, such that $Y(\mathbf{z}, \mathbf{r}) > 0$ for $r \in V(\omega)\, E(\mathbf{z})$.

c) If $\mathbf{r} \in B(\mathbf{z})$, i.e., if $r = (s, y)$ with $y + s \leqq t - x$, then

$$D_{\mathbf{r}} X(\mathbf{z}) = \iint_{E(\mathbf{z})} [a'(X(\mathbf{z}'))\, D_{\mathbf{r}} X(\mathbf{z}')\, dW(\mathbf{z}') + b'(X(\mathbf{z}'))\, D_{\mathbf{r}} X(\mathbf{z}')\, dz']$$

because of (IV.5). In particular, if $r = (s, 0)$ with $0 \leqq s < t - x$, or $\mathbf{r} = (0, y)$ with $0 \leqq y < t - x$, an induction argument shows that $D_{\mathbf{r}} X(\mathbf{z}) = 0$. Consequently, the process $\{D_{\mathbf{r}} X(\mathbf{z});\, r \leqq \mathbf{z}\}$ may be discontinuous below the line segment L_3 joining z_0 and $z_2 = (0, t-x)$.

The analogue of Theorem III.2 in the present situation is:

Theorem IV.2. *Let us assume that $t > x > 0$ and that one of the following two conditions hold:*

(i) *$a(y) \neq 0$ for some y in the closed interval with endpoints 0 and $f(x+t)$*

(ii) *$a(y)=0$ for all y between 0 and $f(x+t)$ and letting J be the maximal closed interval containing 0 and $f(x+t)$ on which a vanishes, one of the following conditions holds:*

(ii)$_1$ *$J=\{0\}$ and $a(f(\xi)) \neq 0$ for some $\xi \in (0, x+t)$.*

(ii)$_2$ *$J=\{0\}$ and $a(f(\xi))=0$ for all $\xi \in [0, x+t]$ and either $g'_-(x+t) \neq 0$ or $g''_-(x+t)+b(0) \neq 0$ or $g'_+(t-x)+2(t-x)\,b(0) \neq 0$ or $g''_+(t-x)+b(0) \neq 0$ or one of these derivatives does not exist.*

(ii)$_3$ *J does not reduce to a singleton, $a(f(\xi)) \neq 0$ for some $\xi \in [t-x, x+t]$ and $b' \geqq 0$ on J.*

Then, the distribution of $X(t,x)$ is absolutely continuous.

Proof. It suffices to show that

$$\iint_{B(\mathbf{z})} (D_{\mathbf{z}} X(\mathbf{z}))^2 \, d\mathbf{r} > 0 \tag{IV.7}$$

almost surely. As before we denote by G the subset of Ω where the left hand side of (IV.7) vanishes, we assume $\mathbf{P}(G)>0$ and we try to find a contradiction with our assumptions.

First we notice that we always have $X(\mathbf{z}_1)=f(x+t)$ and $X(\mathbf{z}_0)=0$. Moreover, since $D_{\mathbf{r}} X(\mathbf{z})=a(X(\mathbf{r}))\,Y(\mathbf{z},\mathbf{r})$, we must have $a(X(\mathbf{r}))=0$ on $L_1 \cup L_2$ on G and $\mathbf{P}(G)>0$ implies that a must vanish on the closed interval with endpoints 0 and $f(x+t)$ which contradicts i).

Let us now assume $J=\{0\}$; then if $a(f(\xi)) \neq 0$ for some $\xi \in [0, x+t]$, then there exists a point $\mathbf{z}_3$ on L_2 such that $X(\mathbf{z}_3)$ has a density. But this would contradict $\mathbf{P}(G)>0$ because $\{X(\mathbf{z}_3)=0\} \supset G$.

Now, if $J=\{0\}$ and $a(f(\xi))=0$ for all $\xi \in [0, x+t]$, the argument used in Theorem III.2 shows that $\mathbf{P}(G)>0$ implies that $g'_-(x+t)=0$ and $g''_-(x+t)+b(0)=0$. For any $\lambda \geqq 0$ we set $\mathbf{z}_0(\lambda)=\mathbf{z}_0+\lambda\,\mathbf{u}$ and

$$S=\inf\{\lambda>0;\ a(X(\mathbf{z}_0(\lambda))) \neq 0\}.$$

S is a stopping time with respect to the filtration $\{\mathscr{B}_\lambda;\ \lambda \geqq 0\}$ defined by:

$$\mathscr{B}_\lambda=\sigma\{W(A);\ A \subset B(\mathbf{z}_0(\lambda))\} \vee \mathcal{N}.$$

$S>0$ on G so that $\mathbf{P}\{S>0\}>0$; and $\mathbf{P}\{S>0\}=1$ by the zero-one law. We have:

$$X(\mathbf{z}_0(\lambda))=\tfrac{1}{2}[g(t-x+\sqrt{2}\,\lambda)-g(t-x)] + \iint_{E(\mathbf{z}_0(\lambda))} [a(X(\mathbf{z}'))\,dW(\mathbf{z}')+b(X(\mathbf{z}'))\,d\mathbf{z}']. \tag{IV.8}$$

By definition of S we have $a(X(\mathbf{z}_0(\lambda)))=0$ for $\lambda<S$, and hence $X(\mathbf{z}_0(\lambda))=0$ for $\lambda<S$ because $J=\{0\}$. Consequently, from (IV.8) one gets:

$$\iint_{E(\mathbf{z}_0(\lambda))} a(X(\mathbf{z}'))^2\,d\mathbf{z}'=0$$

and

$$\tfrac{1}{2}[g(t-x+\sqrt{2}\,\lambda)-g(t-x)]+\iint_{E(\mathbf{z}_0(\lambda))} b(X(\mathbf{z}'))\,d\mathbf{z}'=0.$$

The first equality implies that $a(X(\mathbf{z}'))=0$ for all $\mathbf{z}'\in E(\mathbf{z}_0(\lambda))$ and $X(\mathbf{z}')=0$ for all $\mathbf{z}'\in E(\mathbf{z}_0(\lambda))$ follows by continuity. Using this fact in the second equality gives:

$$\tfrac{1}{2}[g(t-x+\sqrt{2}\lambda)-g(t-x)]+b(0)[\sqrt{2}\lambda(t-x)+\tfrac{1}{2}\lambda^2]=0$$

which implies that $g'_+(t-x)+2b(0)(t-x)=0$ and $g''_+(t-x)+b(0)=0$ which in turns contradicts assumption ii)$_2$.

Finally let us assume that J is not a singleton and that $b'\geqq 0$ on J. Let us set:

$$S_1=\sup\{\lambda\in[0,x+t];\ a(X(\mathbf{r}))\neq 0 \text{ for some } \mathbf{r}\in I(\lambda)\}$$

with $I(\lambda)=B(\mathbf{z})\cap\{\lambda\mathbf{u}+\mu\mathbf{u};\ \mu\in\mathbf{R}\}$. Obviously $G\subset\{S_1<(x+t)/\sqrt{2}\}$. Moreover, $\mathbf{P}\{G\cap\{S_1>(t-x)/\sqrt{2}\}\}>0$ leads to a contradiction in the same way as in the proof of Theorem III.2. Consequently $S_1\leqq(t-x)/\sqrt{2}$ on G which implies that $a(f(\xi))=0$ for all $\xi\in[t-x,x+t]$ and this contradicts (ii)$_3$. □

The following example shows that condition (ii)$_3$ in Theorem IV.2 cannot be weakened into assuming that $a(f(\xi))\neq 0$ for some $\xi\in[0,x+t]$.

Example. The present example has been inspired by the results of [12].

Let us assume that g and b vanish identically and that a is a C^∞-function such that $0\leqq a\leqq 1$, $a(x)=0$ for $x\leqq\alpha$ and $a(x)=1$ for $x\geqq 2\alpha$ for some $\alpha>0$. Furthermore, we assume that $f(y)=2\alpha(1-|y-\xi_0|/\delta)_+$ and $\mathbf{z}=(t,x)$ with $x<t$ and $0<\xi_0-\delta<\xi_0<\xi_0+\delta<t-x$. For each $\varepsilon>0$, let $\mathbf{X}_\varepsilon=\{X_\varepsilon(\mathbf{z});\ \mathbf{z}\in\mathbf{R}_+\times\mathbf{R}_+\}$ be the continuous solution of the stochastic integral equation:

$$X_\varepsilon(\mathbf{z}')=X_0(\mathbf{z}')+\iint_{E(\mathbf{z}')}\varepsilon\, a(X_\varepsilon(\mathbf{z}''))\,dW(\mathbf{z}'')$$

where $\mathbf{X}_0$ is defined by (IV.4) as before. We claim that:

$$\mathbf{P}\{X_\varepsilon(\mathbf{z}')\leqq\alpha \text{ for all } z'\in E(\mathbf{z})\}>0 \tag{IV.9}$$

for ε small enough. (IV.9) implies that $\mathbf{P}\{X_\varepsilon(\mathbf{z})=0\}>0$ and that the distribution of $X_\varepsilon(\mathbf{z})$ has an atom at 0, so it cannot be absolutely continuous even though $a(f(\xi_0))=a(2\alpha)=1$.

In order to show (IV.9) we consider the process Y_ε which solves:

$$Y_\varepsilon(\mathbf{z}')=X_0(\mathbf{z}')+\iint_{E(\mathbf{z})}\varepsilon\, a(Y_\varepsilon(\mathbf{z}''))\,dW(\mathbf{z}'')-\iint_{E(\mathbf{z}')}a(Y_\varepsilon(\mathbf{z}''))\,d\mathbf{z}'' \tag{IV.10}$$

and the probability measure $\mathbf{P}_\varepsilon$ defined by its restriction to $\sigma\{W(A);\ A\subset[0,t']\times[0,x']\}$ by

$$\frac{d\mathbf{P}_\varepsilon}{d\mathbf{P}}=\exp\left[\frac{1}{\varepsilon}W([0,t']\times[0,x'])-t'x'/2\varepsilon^2\right]$$

for each $(t',x')\in(0,\infty)\times(0,\infty)$. Then, the multiparameter analog of Girsanov theorem tells us that the process $\mathbf{W}_\varepsilon$ defined by $W_\varepsilon(A)=W(A)-|A|/\varepsilon$ for $A\in\mathscr{B}_f(\mathbf{R}_+\times\mathbf{R}_+)$ has the same distribution for $\mathbf{P}_\varepsilon$ as $\mathbf{W}$ for $\mathbf{P}$ and that the $\mathbf{Y}_\varepsilon$ has for $\mathbf{P}_\varepsilon$ the same distribution as $\mathbf{X}_\varepsilon$ for $\mathbf{P}$. Using these facts, one can easily check that:

$$\lim_{\varepsilon\searrow 0}\mathbf{E}\{\sup_{\mathbf{z}'\in[0,t']\times[0,x']}|Y_\varepsilon(\mathbf{z}')-F(\mathbf{z}')|^2\}=0 \tag{IV.11}$$

where F is the solution of the deterministic equation:

$$F(\mathbf{z}') = X_0(\mathbf{z}') - \iint_{E(\mathbf{z}')} a(F(\mathbf{z}''))\, d\mathbf{z}''.$$

Notice that $X_0(\mathbf{z}')=0$ if $\mathbf{z}' \in E(\mathbf{z})$ so that

$$F(\mathbf{z}') = -\iint_{E(\mathbf{z}')} a(F(\mathbf{z}''))\, d\mathbf{z}'' \leqq 0$$

and

$$\mathbf{P}\{\sup_{\mathbf{z}' \in E(\mathbf{z})} Y_\varepsilon(\mathbf{z}') \leqq \alpha\} \geqq \mathbf{P}\{\sup_{\mathbf{z}' \in E(\mathbf{z})} |Y_\varepsilon(\mathbf{z}') - F(\mathbf{z}')| \leqq \alpha\}$$
$$\geqq \mathbf{P}\{\sup_{\mathbf{z}' \in [0,t] \times [0, x+t]} |Y_\varepsilon(\mathbf{z}') - F(\mathbf{z}')| \leqq \alpha\}$$

which converges to 1 as $\varepsilon \searrow 0$ because of (IV.11). Consequently

$$\mathbf{P}\{\sup_{\mathbf{z}' \in E(\mathbf{z})} Y_\varepsilon(\mathbf{z}') \leqq \alpha\} > 0$$

for $\varepsilon > 0$ small enough. We can replace the measure $\mathbf{P}$ by $\mathbf{P}_\varepsilon$ and this gives (IV.9). □

The smoothness of the density exhibited in Theorem IV.2 can be investigated along the lines of the proof of Theorem III.4. The reader will easily convinced himself that the following result holds.

Theorem IV.3. *Let us assume that a and b are C^∞ functions with bounded derivatives of all orders larger than or equal to one and that f and g are locally Hölder continuous functions. Then, for each $z=(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$, the random variable $X(t,x)$ belongs to $\mathscr{D}_\infty$. If $x \geqq t > 0$, it has a C^∞ density under the conditions of Theorem III.4 and if $t > x > 0$, it has a C^∞ density provided one of the following two conditions holds:*

(i) *$a(y) \neq 0$ for some y in the closed interval with endpoints 0 and $f(x+t)$.*

(ii) *$f(x+t)=0$, $a(0)=0$ and $a^{(n)}(0) \neq 0$ for some $n \geqq 1$ and either*

  (ii)$_1$ *$a(f(\xi)) \neq 0$ for some $\xi \in (0, x+t)$ or*

  (ii)$_2$ *$f(\xi)=0$ for all $\xi \in [0, x+t]$ and either g is C^2 and $g'(x+t) \neq 0$ or g is C^3 and $g''(x+t) + b(0) \neq 0$.*

V. The case of a bounded interval

Let us assume that $I=[0,L]$ for some $L \in (0,\infty)$, and let us consider the random wave equation (II.3) on $[0,\infty) \times I$. We will assume that $f\colon I \to \mathbf{R}$ is a continuous function satisfying $f(0)=f(L)=0$ because we want to deal with the operator of second derivation with respect to the space variable $x \in I$ with Dirichlet boundary condition at the endpoints of I. For simplicity we will assume that g vanishes identically.

We obtain a weak solution by finding the continuous solution, say $\mathbf{X} = \{X(\mathbf{z}); z \in \mathbf{R}_+ \times [0,L]\}$ of the following stochastic integral equation:

$$X(t,x) = X_0(t,x) + \iint_{[0,\infty) \times [0,L]} [\psi_{(t,x)}(s,y)\, a(X(s,y))\, dW(s,y)$$
$$+ \psi_{(t,x)}(s,y)\, b(X(s,y))\, ds\, dy]$$

where

$$\psi_{(t,x)}(s, y)=\frac{1}{2}\sum_{k=-\infty}^{+\infty}(1_{[0,\infty)\times[0,L]\cap\{|y-x-2kL|\leqq t-s\}}(s, y)$$
$$-1_{[0,\infty)\times[0,L]\cap\{|y+x-2kL|\leqq t-s\}}(s, y))$$

($\psi_{(t,x)}$ vanishes except on the shaded region of Fig. 10 where it takes successively the values $+1$ and -1) and where

$$X_0(t, x)=\frac{1}{2}[\varepsilon_1 f(\alpha)+\varepsilon_2 f(\beta)]$$

where ε_1, ε_2, α and β are the functions of (t, x) defined for $k\in\mathbf{Z}$ by:

$$\alpha=2kL-(t, x), \quad \beta=(2k+2)L-(t+x),$$
$$\varepsilon_1=+1 \quad \text{and} \quad \varepsilon_2=-1 \quad \text{if } t-x-2kL\leqq 0\leqq t+x-(2k+1)L$$
$$\alpha=t-x-2kL, \quad \beta=(2k+2)L-(t+x),$$
$$\varepsilon_1=\varepsilon_2=-1 \quad \text{if } t+x-(2k+2)L\leqq 0\leqq t-x-2kL$$
$$\alpha=t+x-(2k+2)L, \quad \beta=t-x-2kL,$$
$$\varepsilon_1=1 \quad \text{and} \quad \varepsilon_2=-1 \quad \text{if } t-x-(2k+1)L\leqq 0\leqq t+x-(2k+2)L$$

and

$$\alpha=t+x-(2k+2)L, \quad \beta=2kL-(t-x),$$

and

$$\varepsilon_1=\varepsilon_2=1 \quad \text{if } t+x-(2k+3)L\leqq 0\leqq t-x-(2k+1)L$$

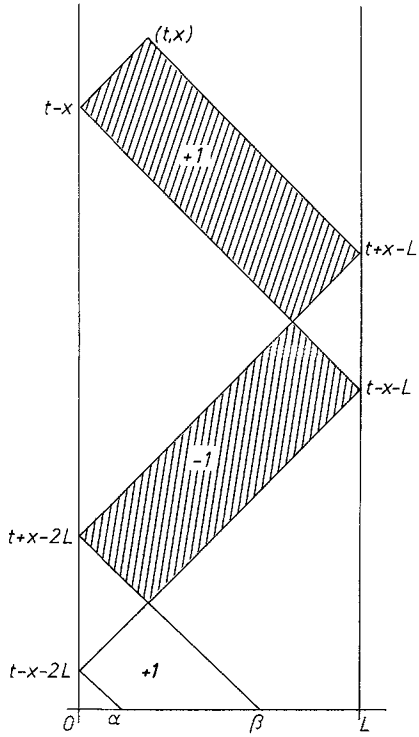

Fig. 10

Using the ideas presented in Sects. III and IV one can show that the distribution of the random variable $X(t, x)$ is absolutely continuous if

(i) $a(0) \neq 0$ and either $t > x$ and $0 < x \leq L/2$ or $t > L - x$ and $L/2 < x < L$ (i.e., (t, x) is in the shaded region of Fig. 11.a).

(ii) $a(0) = 0$ and there exist a sequence $\{x_n; n \geq 1\}$ decreasing to 0 and a sequence $\{x'_n; n \geq 1\}$ increasing to zero such that $a(x_n) \neq 0$ and $a(x'_n) \neq 0$ for all n and $a(f(\xi_0)) \neq 0$ for some $\xi_0 \in (0, L) \cap (x - t, x + t)$. Note that if we define:

$$A = \sup\{\xi \in [0, L]; a(f(\xi)) = 0\}$$

and:

$$B = \inf\{\xi \in [0, L]; a(f(\xi)) = 0\}$$

condition ii) means that $A < B$ and (t, x) is in the shaded region of Fig. 11.b.

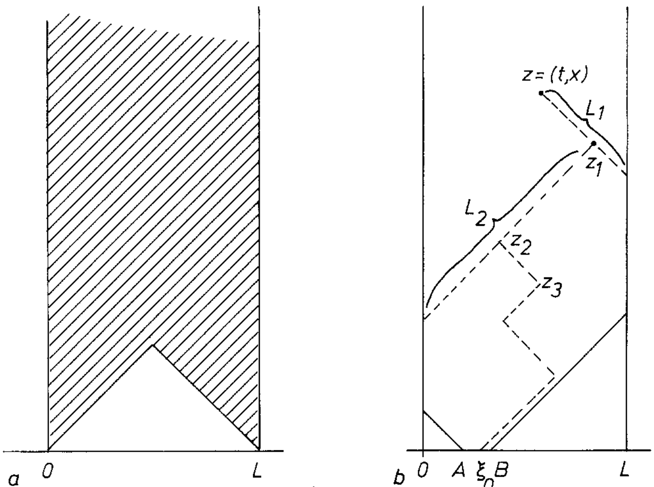

Fig. 11

The proof under condition (i) is immediate. Now, if we assume condition (ii) the proof is as follows. If one assumes that:

$$\mathbf{P}\{\iint_{[0,\infty)\times[0,L]} |D_\mathbf{r} X(\mathbf{z})|^2 \, d\mathbf{r} = 0\} > 0,$$

then $\mathbf{P}\{X(\mathbf{z}_1) = 0\} > 0$ for any point $\mathbf{z}_1$ in the open segment L_1 with endpoints $\mathbf{z} = (t, x)$ and $(t + x - L, L)$. This implies:

$$\mathbf{P}\{\iint_{[0,\infty)\times[0,L]} |D_\mathbf{r} X(\mathbf{z})|^2 \, d\mathbf{r} = 0\} > 0$$

and therefore, that $\mathbf{P}\{X(\mathbf{z}_2) = 0\} > 0$ for any point $\mathbf{z}_2$ in the open segment L_2 with endpoints $\mathbf{z}_1 = (t_1, x_1)$ and $(0, t_1 - x_1)$. Iterating this argument along a broken line joining $\mathbf{z}$ and $(0, \xi_0)$ as shown in Fig. 11.b, we obtained $a(f(\xi_0)) \neq 0$ which contradicts ii).

Finally, the reader will easily be convinced that the density so-obtained is actually smooth when the coefficients a and b are smooth, f is α-Hölder continuous for some $\alpha>0$ and one of the above conditions hold. The proof is in the same spirit as the previous ones.

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