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Probability Theory and Related Fields

Couplings of Markov Chains by Randomized Stopping Times

Part II: Short Couplings for 0-Recurrent Chains and Harmonic Functions

Andreas Greven

Universität Heidelberg, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, D-6900 Heidelberg 1, Federal Republic of Germany

Summary. We consider a 0-recurrent ergodic Markov chain on $(E,\mathscr{B})$, generated by a kernel P. Again we consider couplings of two chains $({}_\nu X_n)$, $({}_\mu X_n)$ starting with the initial distributions ν respectively μ and evolving with P. The coupling consists of two randomized stopping times: T, S, with $\mathscr{L}({}_\nu X_T)=\mathscr{L}({}_\mu X_S)$.

Under additional regularity assumptions we characterize the existence of "short" couplings for two chains $({}_\nu X_n)$, $({}_\mu X_n)$ by the property: $\langle \nu-\mu, h\rangle = 0$ for all harmonic functions h fulfilling certain growth conditions. By "short" we mean that the probability to hit $\complement \tilde{A}_m$ before T respectively S decays faster than the analogue quantity for the recurrence times of ν and μ. Here the $\tilde{A}_m$ are constructed in terms of recurrence times for a certain class of measures $(\tilde{A}_m \uparrow E)$.

We will also show that the couplings of the chains $({}_\nu X_k^m)_{k\in\mathbb{N}}$, $({}_\mu X_k^m)_{k\in\mathbb{N}}$ obtained by stopping the original chains, when first hitting $\complement \tilde{A}_m$, converge to a coupling of the original chains, which is also distinguished from other exact couplings by space-time properties. We use these results to characterize the recurrent potential kernel.

0. Introduction and Main Theorems

a) The Problem

In this paper our interest is focussed on a Markov kernel P on $(E,\mathscr{B})$, which is m-Harris recurrent, ergodic and has a non-finite (but σ-finite) invariant measure π. The Markov chain with initial distribution ν and transition kernel P is denoted by $({}_\nu X_n)_{n\in\mathbb{N}}$. $\mathscr{A}$ denotes all "special sets" introduced in [4].

Our main goal in this paper is to sharpen and extend results on the longterm behaviour of the chain obtained in Part I. Specializing the results of Part I gives the following:

For any two probability measures ν, μ on $(E,\mathscr{B})$ we can find memoryless randomized stopping times T for $({}_\nu X_n)$ and S for $({}_\mu X_n)$ such that

$$\text{(i)}\quad \mathscr{L}({}_\nu X_T)=\mathscr{L}({}_\mu X_S) \quad (\mathscr{L}(X)=\text{Law of } X) \tag{1}$$

$$\text{(ii)}\quad T,S<\infty \text{ a.s.} \tag{2}$$

$$\text{(iii)}\quad \eta\wedge\xi=0 \quad \left(\eta(A)=\sum_0^T 1_A({}_\nu X_k) \quad \xi(A)=\sum_0^S 1_A({}_\mu X_k)\right) \tag{3}$$

$\eta+\xi$ is σ-finite.

Or in our terminology (compare Part I): T, S form a successful coupling for ν, μ with disjoint effects η, ξ. The effects are the negative respectively positive part of a solution of the Poisson equation $(I-P)(\cdot)=\nu-\mu$.

For 0-recurrent Markov kernels one can go further and ask whether a *short coupling* exists for a given pair of measures ν, μ. What do we mean by "short" here? There are two possible approaches to this. Let us look at examples.

1. Consider a simple random walk on $\mathbb{Z}$. Let ν be concentrated on the origin and let μ be a probability measure with mean 0 and finite variance. Then we can conclude from well known results (Skorohod's lemma) that there exists not only a coupling S, T with S, $T<\infty$, but one which is "very short", namely

$$E(S)<\infty, \qquad T\equiv 0 \quad \text{a.s.} \tag{4}$$

2. Another way to look at "short" couplings can be derived from Theorem 1 (Part I) and results of Ornstein [7]: Take for example a symmetric random walk on $\mathbb{R}^1$ with finite variance and absolutely continuous transition density. For two probability measures ν, μ on $\mathbb{R}^1$ with equal means and compact support there exists a successful coupling for ν and μ such that the effects η, ξ have the property

$$\lim_{x\to-\infty}\eta(x)=\lim_{x\to\infty}\xi(x)=0; \qquad \eta\wedge\xi=0. \tag{5}$$

So this coupling is not only successful but is short in the sense, that sets "far out" are visited seldom by the chains $({}_\nu X_n)$, $({}_\mu X_n)$ before the stopping times T respectively S.

Phenomena as described in our first example have been studied in the transient case by H. Rost [5]. We will focus here on 0-recurrent kernels and ask when two measures have a "short" coupling, where "short" refers here to a sharpened and generalized version of Eq. (5). Again we want to show that the question whether ν, μ have a short coupling or not can be resolved by looking at $\langle\nu-\mu,h\rangle$ for certain (unbounded) harmonic functions h. This will be in Theorem 3. These considerations will also shed some new light on the interpretation of the recurrent potential kernel as introduced in [6]. We will show that this potential kernel defines a solution to the Poisson equation $(I-P)(\cdot)=\nu-\mu$ which is related to the effects of a coupling with very curious spatial and space-time properties. This will be formulated in our Theorem 4.

b) The Concept of a Short Coupling

Our concept of a *short coupling is based on the comparison of the stopping times T, S constituting the coupling with the recurrence times of* ν *and* μ. So let us first have a look at recurrence times.

Definition 1. Suppose that P generates a m-Harris recurrent chain.

a) We call a randomized stopping time R a *recurrence time of* ν if:

(i) $1 \leqq R < \infty$ a.s. (6)

(ii) $\mathscr{L}({}_\nu X_R)=\nu$. (7)

b) We call a recurrence time for ν *minimal*, if the expected number of visits of ${}_\nu X_n$ to any set $A \in \mathscr{B}$ before R is smaller or equal than for any other recurrence time of ν.

Proposition 0 ([8]). *If the measure* ν *is absolutely continuous with respect to* m *then a recurrence time of* ν *exists. If we denote by* π *an invariant measure of* P *and assume that* ν *is bounded by a multiple of* π *then a minimal recurrence time* R *of* ν *has the property*

$$\sum_{n=0}^{\infty} \mathscr{L}({}_\nu X_n, n<R)=a\pi \qquad a=\inf(a \mid a\pi-\nu \geqq 0) \tag{8}$$

In order to compare stopping times forming a coupling of (ν, μ), with the minimal recurrence times for ν, μ, we need a class of sets characteristic for recurrence times of measures in a class $\mathscr{E}_1$, to be introduced precisely in Sect. 1a). This class contains for example measures with finite mean in the random walk situation.

Definition 2. We call a sequence of sets $A_m \uparrow E$ *a sequence of characteristic sets for* P if for every measure $\nu \in \mathscr{E}_1$, a minimal recurrence time R for ν has the property:

$$\operatorname{Prob}({}_\nu X_k \text{ hits } \complement A_m \text{ before } R) \underset{m\to\infty}{\sim} m^{-1}. \tag{9}$$

These sets describe the diffusive behaviour of our chain. To construct these sets we need a solution $U^*(g)$ of the Poisson equation $(P^*-I)(\cdot)=g$ $(g \in \mathscr{F}_0^+)$ with certain additional properties, which will characterize the solution uniquely. We will construct the map U^* in Sect. 1b) (45)-(47). Here $\mathscr{F}_0^+$ denotes bounded functions supported by sets in $\tilde{\mathscr{A}}$ (compare 29). We generate characteristic sequences $\tilde{A}_m$ now as follows: (compare Sect. 1, Proposition 3)

$$\tilde{A}_m=\{U^*(g) \leqq m\} \tag{10}$$

We will see later that the asymptotic shape ($m \to \infty$) of the $\tilde{A}_m$ does not depend on the choice of g. This special choice of $(\tilde{A}_m)$ will be justified through Theorem 4 and via (45)-(47).

Furthermore, we are able to define the for the following very important cone of harmonic functions:

Definition 3. A function f, measurable on $(E, \mathscr{B})$, belongs to $\mathscr{F}$ iff

(i) $P^* f = f$ (11)

(ii) $\underset{g \in \mathscr{F}_+^0}{\exists} \ \underset{a, b \in \mathbb{R}^+}{\exists}: \ |f| \leqq a + b\, U^*(g).$ (12)

For example in the case of a simple random walk on $\mathbb{Z}^1$ we have:

$$\mathscr{F} = \{\alpha x + \beta \mid \alpha, \beta \in \mathbb{R}^+\} \tag{13}$$

and characteristic sequences have (asymptotically) the form $[-m, m]$.

Sequences with (8) and U^* exist if we require that our kernel P fulfills certain regularity conditions which will be summarized in a condition (N^*) in Sect. 1. Now we are able to define the crucial notion of this paper.

Definition 4. We say that two randomized stopping times S, T of $({}_\nu X_k)$ respectively $({}_\mu X_k)$ form a *short coupling of* ν *and* μ if for every characteristic sequence $(\tilde{A}_m)$ generated as in (10) we have

$$\begin{aligned} &\operatorname{Prob}({}_\nu X_k \text{ hits } \complement \tilde{A}_m \text{ before } S) + \operatorname{Prob}({}_\mu X_k \text{ hits } \complement \tilde{A}_m \text{ before } T) \\ &\quad = o(m^{-1}). \end{aligned} \tag{14}$$

The following notation will be used frequently:

Definition 5. Fix a characteristic sequence $(\tilde{A}_m)$ for the kernel P. Then we denote by $({}_\nu X_k^m)_{k \in \mathbb{N}}$ the Markov chain which is obtained by freezing ${}_\nu X_k$ when it first reaches the set $\complement \tilde{A}_m$.

c) The Results on Short Couplings

For our main result we need a regularity condition (N^*) for P which we will introduce in Sect. 1 in detail.

Theorem 3a. *Let P be a Markov transition kernel with property (N^*). Consider two measures $\nu, \mu \in \mathscr{E}_1(P)$.*

The following statements are equivalent (compare Definitions 3–5):

(i) *There exists a short coupling for ν, μ.*

(ii) $\langle \nu - \mu, h \rangle = 0 \quad \forall h \in \mathscr{F}.$ (15)

(iii) *Let $(\tilde{A}_m)$ be a characteristic sequence of P generated by some $g \in \mathscr{F}_+^0$. For any sequence of couplings (S_m, T_m) with disjoint effects of the chains $({}_\nu X_k^m)_{k \in \mathbb{N}}, ({}_\mu X_k^m)_{k \in \mathbb{N}}$ we have*

$$\operatorname{Prob}(S_m = +\infty) + \operatorname{Prob}(T_m = +\infty) = o(m^{-1}). \tag{16}$$

Corollary 3.1 *(Application to hitting probabilities). Let $(\tilde{A}_n)$ be a characteristic sequence and (B_n) any sequence with $B_n \subseteq \complement \tilde{A}_n$ $\forall n \in \mathbb{N}$. Denote by H_n^1, H_n^2 the hitting times of ${}_\nu X_k$ respectively ${}_\mu X_k$ for the sets B_n. If ν and μ have a short coupling then*

$$\|\mathscr{L}({}_\nu X_{H_n^1}) - \mathscr{L}({}_\mu X_{H_n^2})\| = o(n^{-1}). \tag{17}$$

If the above relation holds for all sequences B_n with $B_n \subseteq \complement A_n$ then ν and μ have a short coupling.

An example:

Let $P(x, y)$ by the transition kernel of a symmetric random walk on $\mathbb{Z}^1$ with finite variance σ. Then P fulfills the regularity conditions (N) and (N^*). We can show:

1. The characteristic sequences generated by a function $g \in \mathscr{F}_0^+$ have asymptotically the form:

$$\tilde{A}_m = \sigma^2 [-m, m]. \tag{18}$$

2. Probability measures ν, μ on $\mathbb{Z}^1$ with finite means have a short coupling if and only if

$$\sum_{-\infty}^{+\infty} k\, \nu(k) = \sum_{-\infty}^{+\infty} k\, \mu(k). \tag{19}$$

Theorem 3b. *The effects η, ξ and the final distribution ρ of a short coupling for ν and μ can be calculated as follows:*

$$\begin{aligned} \eta(A) &= \lim_{n\to\infty} \left(\sum_{k=0}^{n} (\nu - \mu) P^k \right)^+ (A) \\ \xi(A) &= \lim_{n\to\infty} \left(\sum_{k=0}^{m} (\nu - \mu) P^k \right)^- (A) \qquad \forall A \in \mathscr{A} \end{aligned} \tag{20}$$

(Here $\mathscr{A}$ denotes all special sets [4]).

$$\rho = -\eta + \eta P + \nu = -\xi + \xi P + \mu. \tag{21}$$

In the case of a random walk on $\mathbb{Z}^1$ for example this means that η, ξ can be calculated by the recurrent potential kernel $a(i,j)$ [6] through

$$\eta(j) = \left(\sum_{-\infty}^{+\infty} (\nu(k) - \mu(k))\, a(k,j) \right)^+, \qquad \xi(j) = \left(\sum_{-\infty}^{+\infty} (\nu(k) - \mu(k)\, a(k,j)) \right)^-,$$

whenever the measures ν, μ have finite mean and fulfill (19).

If we compare Theorem 1 (Part I) and Theorem 3 we notice a formal resemblance. In both cases the existence of the desired coupling is equivalent to the requirement $\langle \nu - \mu, h \rangle = 0$ for an appropriate set of harmonic functions. In the case of successful coupling we consider the set of all bounded harmonic functions, while in the case of a short coupling (for recurrent P) we consider all harmonic functions which are bounded by a slowly growing subharmonic function. Furthermore, the effects of successful or short couplings (with disjoint effects) are obtained via the potential kernel for the kernel P generating the chains.

d) Space-Time Structure of a Coupling; Recurrent Potential Kernel

In this section we will show that effects η, ξ given by (20) and the corresponding final distribution (given by (21)) are distinguished by two remarkable properties stated in Theorem 4. We need some preparations:

1. Consider a characteristic sequence $\tilde{A}_m$ generated by some $g' \in \mathscr{F}_+^0$. For two measures ν, $\mu \in \mathscr{E}_1$, look at the chains $({}_\nu X_k^m)$, $({}_\mu X_k^m)$ obtained by freezing the original chains when they first hit $\complement \tilde{A}_m$.

By Theorem 1 of Part I, there exist couplings (T_m, S_m) of these chains which have disjoint effects. These effects denoted by $\eta^{(m)}$, $\xi^{(m)}$ and the final distribution $\rho^{(m)}$ are uniquely determined. In order to understand the spatial structure of the original-coupling we study the behaviour of $(\eta^{(m)}, \xi^{(m)}, \rho^{(m)})$ for $m \to \infty$.

2. Now we want to include the space-time structure into our considerations.

Definition 6. Consider the time-space chains $({}_\nu X_k, k)$, $({}_\mu X_k, k)$. We call a pair S', T' of randomized stopping times for these chains a *space-time coupling* of ν and μ, if

$$\mathscr{L}({}_\nu X_{S'}, S') = \mathscr{L}({}_\mu X_{T'}, T'). \tag{22}$$

Now given any successful coupling T, S of ν and μ with disjoint effects, we can find a space time-coupling (not necessarily successful) such that

$$S \leqq S' \quad \text{a.s.,} \qquad T \leqq T' \quad \text{a.s.} \tag{23}$$

(Apply Theorem 1 to the space-time kernel and the measures $\mathscr{L}({}_\nu X_S, S)$, $\mathscr{L}({}_\mu X_T, T)$!).

Consider now special couplings S, T of ν and μ, which single out the purely spatial aspect in the different position of ν and μ with respect to P; or more precisely:

Definition 7. We call a *coupling* S, T of ν and μ *regular*, if there exists a space-time coupling S', T' of ν, μ with the property (23) and for which exists a sequence $A_j \uparrow E$ such that for every $j \in \mathbb{N}$:

$$\lim_{n \to \infty} \left(E \sum_{k=0}^{n} 1_{\{{}_\nu X_k \in A_j, k \in [T, T')\}} - 1_{\{{}_\mu X_k \in A_j, k \in [S, S')\}} \right) = 0.$$

Theorem 4. *Let P be an ergodic m-Harris recurrent Markov kernel on $(E, \mathscr{B})$.*

(a) *Assume that P has the property (N^*). Let $(\tilde{A}_m)$ be a characteristic sequence for P. Then we have: (with the notation as introduced in* 1 *above)*

$$\lim_{m \to \infty} \eta^{(m)}(A) = \eta(A); \quad \lim_{m \to \infty} \xi^{(m)}(A) = \xi(A) \quad \forall\colon A \text{ with } \pi(A) < \infty. \tag{24}$$

$$\rho = \|\cdot\| - \lim_{m \to \infty} (\rho^{(m)}). \tag{25}$$

With

$$\left.\begin{aligned}\eta(A)&=\left(\lim_{n\to\infty}\sum_0^n(\nu-\mu)P^k\right)^+(A)\\ \xi(A)&=\left(\lim_{n\to\infty}\sum_0^n(\nu-\mu)P^k\right)^-(A)\end{aligned}\right\}\quad\forall A\in\mathscr{A}(P)\qquad\begin{aligned}(26)\\(27)\end{aligned}$$

$$\rho=-\eta+\eta P+\nu=-\xi+\xi P+\mu. \tag{28}$$

Statements (24) *and* (25) *hold also m-a.e. (i.e., if we consider the m-densities of the involved measures).*

(b) *Assume that P is normal. Then for every pair ν, μ of probability measures in $\mathscr{E}_1(P)$, there exists a regular successful coupling with disjoint effects.*

The effects η, ξ of any regular successful coupling for $\nu, \mu\in\mathscr{E}_1(P)$ with disjoint effects are given by (26), (27) *and the final distribution by* (28).

Remark. This result justifies especially our choice of sequences among the characteristic sequences for P which we performed in (10).

1. Recurrent Chains from the Point of View of Stopping Sequences

In this chapter we will introduce the regularity conditions (N), (N^*) and their most important consequences. (Proposition 2, 3). We also study the class of measures $\mathscr{E}_1$ appearing in Theorems 3 and 4, and prove a crucial lemma on subharmonic functions.

a) Some Classes of Measures

Our ultimate goal will be to study couplings of measures in a certain class $\mathscr{E}_1$. In order to obtain a better understanding for this class $\mathscr{E}_1$, we study also related classes $\mathscr{E}_0$, $\mathscr{E}_2$, $\mathscr{E}_3$, which are easier to identify. The basic class is $\mathscr{E}_0$. In the case of a discrete state space E the basic class $\mathscr{E}_0$ would contain all measures with finite support. In the case where the dynamical system (P, E) has a suitable topological structure $\mathscr{E}_0$ would contain all measures with compact support which are dominated by an invariant measure. We need again the following class of sets (which is a subclass of the special sets in the sense of Neveu):

$$A\in\tilde{\mathscr{A}}\Leftrightarrow\underset{b>0}{\exists}\ \underset{K<\infty}{\exists}:\ \left(\sum_{k=0}^{K}P^k(x,B)\geqq b\,\pi(B)\ \ \forall B\subseteq A, x\in A\right). \tag{29}$$

In the general case we define the classes $\mathscr{E}_i$ as follows:

Definition 8. Let P be an ergodic m-Harris-recurrent Markov kernel who admits a positive σ-finite invariant measure π which is not finite. For a probability measure ν on $(E,\mathscr{B})$ we define:

(a) $\nu\in\mathscr{E}_0\Leftrightarrow$ There exists a set $A\in\tilde{\mathscr{A}}$ which supports ν and ν is dominated by a multiple of π.

(b) $\nu \in \mathscr{E}_2 \Leftrightarrow$ There exists a $\mu \in \mathscr{E}_0$ and two stopping sequences $(\nu, (\nu_n))$, $(\mu, (\mu_n))$ with final distribution μ respectively ν and σ-finite effects.

(c) $\nu \in \mathscr{E}_1 \Leftrightarrow$ There exists a decomposition $E = \sum A_k$ with $A_k \in \tilde{\mathscr{A}}$ and a measure $\lambda \in \mathscr{E}_0$ such that we can find for every $\nu 1_{A_k}$ a recurrence time of the form $(\nu_n^k) \oplus (\mu_n^k)$ where $(\nu_n^k)_{n \in \mathbb{N}}$ starts in $\nu 1_{A_k}$ and has final distribution $\|\nu 1_{A_k}\| \lambda$, while $(\mu_n^k)_{n \in \mathbb{N}}$ starts in $\|\nu 1_{A_k}\| \lambda$ and has final distribution $\nu 1_{A_k}$ and the effects of these recurrence times have the property

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \nu_n^k + \mu_n^k \quad \text{is a } \sigma\text{-finite measure.} \tag{30}$$

(d) Assume that P has the property that for some $A \in \tilde{\mathscr{A}}$ there exists a positive and finite solution f of the equation

$$\begin{aligned} &(P^* - I)(f) = 1_A && \text{with the additional property:} \\ &\langle \nu - \mu, P^{*n}(f) \rangle && \text{converges to 0 for all } \nu, \mu \in \mathscr{E}_0. \end{aligned} \tag{31}$$

Let f be such a subharmonic function. Then we define for a probability measure ν on $(E, \mathscr{B})$

$$\nu \in \mathscr{E}_3(f) \Leftrightarrow \langle \nu, f \rangle < \infty \text{ and } \nu \text{ is dominated by a multiple of } \pi. \tag{32}$$

One easily shows that: $\mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \mathscr{E}_3$. (33)

Example 1. For a symmetric random walk on $\mathbb{Z}^1$ with finite mean the classes $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$ coincide. In fact in this case a probability measure on $\mathbb{Z}^1$ is in $\mathscr{E}_i$ ($i = 1, 2, 3$) if it has finite mean.

Example 2. For a left continuous random walk on $\mathbb{Z}^1$ with mean 0 and infinite variance the classes $\mathscr{E}_i$ do not coincide.

We come now to our first sufficient criterion for a measure ν to be in $\mathscr{E}_1$ (besides the trivial one $\nu \in \mathscr{E}_0$).

Proposition 1. *Assume that P is ergodic and m-Harris recurrent with σ-finite invariant measure π, satisfying $\pi(E) = +\infty$. Let f be a subharmonic function with property* (31) *above. Let λ be a measure in $\mathscr{E}_0$. Denote by $F_\lambda(\nu)$ the effect of the (ν, λ)-filling scheme.* (34)

If there exists a positive function $a(\cdot)$ such that:

$$F_\lambda(\nu) \leqq \left(\int a \, d\nu\right) \pi \tag{35}$$

$$P^*(a) - a \in L_1^+(E, \pi) \tag{36}$$

then a measure $\nu \in \mathscr{E}_2$ belongs to $\mathscr{E}_1$, if there exists a partition $E = \sum A_k$, $A_k \in \tilde{\mathscr{A}}$ such that:

$$\sum_{k=0}^{\infty} \|\nu 1_{A_k}\|^\infty < \infty. \tag{37}$$

Here $\|\tau\|^\infty$ denotes the essential supremum of the π-density of the measure τ.

Remark. In Sect. 2b) there will be a proposition guaranteeing $\mathscr{E}_2 = \mathscr{E}_3$ in certain cases.

b) The Property (N) and main Consequences

For our further considerations the following notion will be very useful: Let $\mathscr{C}$ be a collection of measurable subsets of E and let m be a positive σ-finite measure on $(E,\mathscr{B})$. We say a sequence (f_n) of functions in $L^1(E,m)$ converges locally with respect to $(\mathscr{C},m)$ to a function $f\in L^1(E,m)$, if

$$f=\lim_{n\to\infty} f_n \quad \text{a.e.,} \qquad \lim_{n\to\infty} \|(f_n-f)\,1_A\|^1=0 \qquad \forall,\ A\in\mathscr{C}. \tag{38}$$

A sequence of measures (v_n) with $v_n \ll m$ is said to converge locally with respect to $(\mathscr{C},m)$ to a measure $v(v\ll m)$, if the m-densities of the v_n converge locally to the m-density of v. We write shortly

$$v=(\mathscr{C},m)-\lim_{n\to\infty} v_n. \tag{39}$$

Now we formulate a regularity condition on transition kernels which is central for our theory. The main and most important consequences of (N) are contained in Proposition 2.

Definition 10. The property (N).

Let P be a Markov transition kernel on the state space $(E,\mathscr{B})$. We say P has the property (N), with respect to some σ-finite measure m on $(E,\mathscr{B})$, if:

(i) P is null recurrent and ergodic. We denote by π a positive and m-continuous σ-finite invariant measure of P. (40)

(ii) P is normal in the following sense:
$\sum_{k=0}^{n}(v-\mu)P^k$ converges for $n\to\infty$ locally with respect to $(\tilde{\mathscr{A}},m)$; for all measures $(v,\mu)\in\mathscr{E}_0\times\mathscr{E}_0$ (41)

(iii) $P(x,\cdot)\in\mathscr{E}_1$, $\forall\ x\in E$. (42)

(iv) There exists a partition $E=\sum A_k$ with $(A_k)\subseteq\tilde{\mathscr{A}}$ such that we have the following inequality for the π-densities $p(x,\cdot)$ of the measures $P(x,\cdot)$

$$\sum_{k=0}^{\infty}\|p(x,\cdot)\,1_{A_k}\|^\infty<\infty \qquad \forall\ \ x\in E. \tag{43}$$

Let P be a Markov transition kernel with the property (N). For all pairs $v,\mu\in\mathscr{E}_0$ we define an operator U by

$$(v-\mu)\,U=(\tilde{\mathscr{A}},m)-\lim_{n\to\infty}\left(\sum_0^n(v-\mu)P^k\right) \qquad \forall\, v,\mu\in\mathscr{E}_0. \tag{44}$$

An important property of the class $\mathscr{E}_1$ is the possibility to extend the domain of the operator U to the set $\{v-\mu\,|\,v,\mu\in\mathscr{E}_1\}$. The most important consequence of property (N) is (for proof see Chap. 4):

Proposition 2. *Let P be a Markov transition kernel with property (N).*

(a) $\sum_0^n(v-\mu)P^k$ converges locally with respect to $(\tilde{\mathscr{A}},m)$ for all $v,\mu\in\mathscr{E}_1$.

(b) For any function $g \in \mathscr{F}_0^+$ there exists a positive and finitely valued function k with the properties

(i) $\langle \nu, k\rangle < \infty \ \forall \nu \in \mathscr{E}_0$ (45)

(ii) $(P^* - I)(k) = g$ m-a.e. (46)

(iii) $\langle (\nu-\mu) U, g\rangle = -\langle \nu-\mu, k\rangle \quad \forall \nu, \mu \in \mathscr{E}_1$. (47)

In fact this function is uniquely determined if we require that it is the minimal function with (i)–(iii). We denote it henceforth by $k = U^*(g)$. Note that:

$$(P^* - I)\, U^* = I, \qquad \langle (\nu-\mu) U, g\rangle = -\langle \nu-\mu, U^*(g)\rangle \tag{47'}$$

Furthermore, if g_1, $g_2 \in \mathscr{F}_0^+$ then there exists a real number c such that with setting $a = \langle \pi, g_2\rangle$, $b = \langle \pi, g_1\rangle$

$$|a\, U^*(g_1) - b\, U^*(g_2)| \leqq c\, 1 \qquad m\text{-a.e.}, \tag{48}$$

Remark. The last estimate shows that the class $\mathscr{E}_3(f)$ as defined in (32) does not depend on f, if P has the property (N) and furthermore that $\tilde{A}_m = \{U^*(g) \leqq m\}$ does not depend on g very much if $m \to \infty$.

c) The Property (N^*)

First, we need some preparations.

Definition 11. Let P be a Markov transition kernel with property (N). We define a set $\tilde{\mathscr{F}}$ of positive measurable functions on E by setting

$$f \in \tilde{\mathscr{F}} \Leftrightarrow P^*(f) \geqq f, \qquad \underset{a, b \in \mathbb{R}^+}{\exists} \ \underset{g \in \mathscr{F}_0^+}{\exists} : f \leqq a + b\, U^*(g). \tag{49}$$

We say that a positive function f is *slowly increasing* with respect to P, if $P^*(f)$ is not much bigger than k on the set $\{x \mid f(x) \leqq k\}$ for large k.

More precisely, define

$$a_k(f) := \sup \frac{\langle \lambda P,\ 1_{\{f \geqq k\}} \cdot f\rangle}{k \langle \lambda P, 1_{\{f \geqq k\}}\rangle} \tag{50}$$

where the supremum is extended over all finite m-continuous measures λ concentrated on $\{f \leqq k\}$.

We say f is slowly increasing, if

$$\overline{\lim_{k \to \infty}}\, a_k(f) \leqq 1. \tag{51}$$

Definition 12. "The property N^*".

We say that a Markov kernel P has the property (N^*) if in addition to (N) the following holds.

For every $g \in \mathscr{F}_0^+$ the function $U^*(g)$ is slowly increasing.

Example. A random walk on $\mathbb{Z}^1$ with finite mean has property (N^*): Let us finish this chapter with the most important consequence of property (N^*).

Proposition 3. *Let P be a Markov transition kernel with property (N^*). Consider a minimal recurrence time R of a probability measure $\nu \in \mathscr{E}_3$. Let $(\tilde{A}_m)$ be a characteristic sequence of P, generated by $g \in \mathscr{F}_0^+$;* (10). *Then we have, when we denote by a_m the quantity* Prob(X_k hits $\complement \tilde{A}_m$ before R)

$$\lim_{m \to \infty} (a_m m) = c < \pi, g \rangle \qquad \textit{with } c := \inf(b \mid \nu \leqq b\,\pi). \tag{53}$$

d) A Criterion for $\mathscr{E}_2 = \mathscr{E}_3$

With the notion introduce so far we can complete our knowledge about the classes $\mathscr{E}_i$ $(i = 1, 2, 3)$ of measures and their natural relations.

Proposition 4. *Let P be a Markov transition kernel with property (N). Now if the following conditions hold.*

(i) $E_n := \{U^*(g) \leqq n\} \in \tilde{\mathscr{A}} \;\; \forall n \in \mathbb{N}$ (for $g \in \mathscr{F}_0^+$). (54)

(ii) *There exists a partition of E: $E = \sum B_k$ and real-valued functionals $\alpha_k(\nu, \mu)$ defined for $\nu, \mu \in \mathscr{E}_0$ such that*

$$(\alpha) \quad |(\nu - \mu)\, U - \sum_k \alpha_k(\nu, \mu)\, \pi\, 1_{B_k}|\, 1_{\complement E_n} \leqq a_n \pi, \qquad \lim_{n \to \infty} a_n = 0 \tag{55}$$

$$(\beta) \quad \lim_{n \to \infty} \langle \pi\, 1_{B_k}, P^{*n}(1_B) \rangle \geqq \beta \quad \text{for some } B \in \mathscr{B}$$

$$\text{with } m(B) > 0 \text{ and } \beta > 0. \tag{56}$$

then we can conclude

$$\mathscr{E}_2 = \mathscr{E}_3. \tag{57}$$

Example. A symmetric random walk on $\mathbb{R}^1$ with finite variance and absolutely continuous transition density. $(B^1 = (-\infty, 0]$, $B^2 = (0, +\infty])$.

e) Some Basic Relations Concerning Subharmonic Functions

We start with a very basic combinatoric relation for stopping sequences. As in Part I we use the fact that all problems concerning randomized stopping times can be translated into problems concerning measures on the statespace via the notion of stopping sequences (compare Part I, Sect. 1).

Lemma 1a. *Let $(\nu_n)_{n \in \mathbb{N}}$ be a stopping sequence with initial distribution ν and final distribution ρ, such that $\|\nu\| = \|\rho\|$. Furthermore let f be a subharmonic function with $\langle \nu, f \rangle < \infty$, $f \geqq 0$. Then*

$$\text{(a)} \quad \langle \nu_n, f \rangle = \langle \nu, f \rangle - \langle \Gamma_0 + \dots + \Gamma_{n+1}, f \rangle + \langle \nu_0 + \dots + \nu_n, g \rangle \tag{58}$$

$$\textit{where } g := P^*(f) - f; \;\; \Gamma_{k+1} := \nu_k P - \nu_{k+1}. \tag{59}$$

In order to formulate the second part of this lemma we need to introduce some more notations.

Let f be a slowly increasing subharmonic function in $\tilde{\mathscr{F}}$. Define the set $A_m := \{f \leqq m\}$, d_n: $v_{n+1} = v_n P d_n$.
Then we define

$$\bar{v}^{(m)} := v\left(d_0 \sum_{j=1}^{\infty} \prod_{k=1}^{j} (1_{A_m} P d_k)_j 1_{\complement A_m}\right) \tag{60}$$

$$\bar{\rho}^{(m)} := v(1-d_0)\left(\sum_{j=1}^{\infty} \prod_{k=1}^{j} (1_{A_m} P(1-d_k))\right) \tag{61}$$

$$\bar{\eta}^{(m)} := \sum_{j=0}^{\infty}\left({}_m d_0 \cdot \prod_{k=1}^{j} (P_m d_k)\right). \quad {}_m d_k = 1_{A_m} d_k. \tag{62}$$

Note: If a stopping sequence $(v_n)_{n\in\mathbb{N}}$ is derived from a randomized stopping time T of $({}_vX_n)_{n\in\mathbb{N}}$ then with $H_m := \inf(k \mid {}_vX_k \in \complement \tilde{A}_m)$ we have

$$\bar{v}^m(A) := \operatorname{Prob}({}_vX_{H_m} \in A;\, T > H_m) \tag{63}$$

$$\bar{\rho}^m(A) := \operatorname{Prob}({}_vX_T \in A;\, T \leqq H_m)$$

Lemma 1b.

$$\langle \bar{v}^n, f\rangle = \langle v, f\rangle - \langle \bar{\rho}^n, f\rangle + \langle \bar{\eta}^n, g\rangle. \tag{65}$$

Proof of Lemma 1. a) Is shown by induction over n (details omitted).

b) This follows from a) by considering the stopping sequences $(v_n^k)_{n\in\mathbb{N}}$ defined by

$$v_{n+1}^k := (v_n^k) P d_n 1_{E_k} \quad \text{with } d_n\colon\ v_{n+1} = v_n P d_n \tag{66}$$

Part a) gives for each k

$$\langle v_{n+1}^k, f\rangle = \langle v, f\rangle - \langle \Gamma_0^k + \ldots + \Gamma_{n+1}^k, f\rangle + \langle v_0^k + \ldots + v_n^k, g\rangle \tag{67}$$

Now by construction

$$\langle v_n^k, f\rangle \leqq k\,\|v_n^k\|, \quad \langle \Gamma_0^k + \ldots + \Gamma_{n+1}^k, f\rangle \nearrow \langle \bar{\rho}^k + \bar{v}^k, f\rangle < \infty \tag{68}$$

$$\left\langle \sum_{j=0}^{n} v_j^k, g\right\rangle \nearrow \langle \bar{\eta}^k, g\rangle \tag{69}$$

which leads together with (67) immediately to b).

2. The Proof of Theorem 3

a) Proof of Theorem 3a

1. (i) $\Rightarrow$ (ii)

Now, if $((v_n)(\mu_n))_{n\in\mathbb{N}}$ is a short coupling of v, μ, we have $n\,\|\bar{v}^n\| = o(1)$ and by the fact that P has property (N^*) that: $\langle \bar{v}^n, f\rangle \sim n \cdot \|\bar{v}^n\|$. Therefore we conclude with (65):

$$\langle \rho - v, U^*(g)\rangle = \langle \eta, g\rangle. \tag{70}$$

From this we obtain with part a) of Lemma 1

$$\lim_{n\to\infty} \langle \nu_n, U^*(g)\rangle = 0; \qquad \langle \rho, U^*(g)\rangle < \infty. \tag{71}$$

This gives for a harmonic function h with $|h| \leqq a + b\, U^*(g)$

$$\lim_{n\to\infty} \langle \nu_n, h\rangle = 0 \quad (\text{similar } \lim_{n\to\infty} \langle \mu_n, h\rangle = 0). \tag{72}$$

Now define $(\Gamma_n^1)_{n\in\mathbb{N}}$, $(\Gamma_n^2)_{n\in\mathbb{N}}$ by

$$\nu_n P = \Gamma_{n+1}^1 + \nu_{n+1}, \qquad \mu_n P = \Gamma_{n+1}^2 + \mu_{n+1} \tag{73}$$

and conclude from (58) by using $\Gamma_0^i + \ldots + \Gamma_{n+1}^i \uparrow \rho$ $(i=1,2)$ and $\langle \rho, U^*(g)\rangle < \infty$ that

$$\langle \nu - \mu, h\rangle = \langle \nu_n - \mu_n, h\rangle + \langle (\Gamma_0^1 + \ldots + \Gamma_{n+1}^1) - (\Gamma_0^2 + \ldots + \Gamma_{n+1}^2), h\rangle \xrightarrow[n\to\infty]{} 0. \tag{74}$$

q.e.d.

2. (ii) $\Rightarrow$ (i)

The proof of this part relies on the following facts, which allow us to reduce our problem to one dealing with a kernel of a type much easier to handle.

Lemma 2. *Under the assumptions of Theorem* 3, *the following holds:*

(a) For the implication (ii) $\Rightarrow$ (i) *it is sufficient to consider only kernels with the additional property*

$$\underset{x\in E}{\exists}\, m(\{x\}) > 0. \tag{75}$$

(b) If P fulfills the assumptions of the theorem and has property (75) *it is sufficient for the proof of the implication* (ii) $\Rightarrow$ (i) *to show that this holds for* $\nu, \mu \in \mathscr{E}^1(P)$ *with*

$$\langle \nu - \mu, U^*(1_{\{x\}})\rangle = 0. \tag{76}$$

In the following we will in view of Lemma 2 (a) assume, that our kernel has the property (75).

Now in order to prove (ii) $\Rightarrow$ (i), assume that for $\nu, \mu \in \mathscr{E}^1(P)$ exists no short coupling. This implies that for an exact coupling with effects $(\nu - \mu)\, U^+$, $(\nu - \mu)\, U^-$ we have: (upper index with respect to $\{U^*(1_{\{x\}}) \leqq n\}$, see (60)–(62))

$$\lim_{n\to\infty} (\|\bar{\nu}^n\| + \|\bar{\mu}^n\|)\, n = a > 0. \tag{77}$$

By part (b) of our Lemma 2 we can furthermore assume without loss of generality

$$\langle \nu - \mu, U^*(1_{\{x\}})\rangle = 0. \tag{78}$$

We will construct a harmonic function h with

$$\langle \nu - \mu, h\rangle = a \neq 0 \qquad |h| \leqq b + c \cdot U^*(1_{\{x\}}). \tag{79}$$

The construction proceeds as follows:

(i) Consider the sets ($\tilde{\nu}$ denotes the m-density of ν!)

$$A_n^+ := \{(\tilde{\nu}-\tilde{\mu})\,\tilde{U}^+ > 0\} \cap \complement E_n \qquad A_n^- := \{(\tilde{\nu}-\tilde{\mu})\,\tilde{U}^- \geqq 0\} \cap \complement E_n \tag{80}$$

where

$$E_n := \{U^*(1_{\{x\}}) \leqq n\}$$

and define functions h_n^+, h_n^- as follows:

$$h_n^+(z) := \begin{cases} n & z \in A_n^+ \\ 0 & z \in A_n^- \cup \{x\} \\ \text{harmonic continued} \\ \text{elsewhere} \end{cases} \qquad h_n^-(z) := \begin{cases} n & z \in A_n^- \\ 0 & z \in A_n^+ \cup \{x\} \\ \text{harmonic continued} \\ \text{elsewhere} \end{cases}. \tag{81}$$

Now we define a sequence $(h_n)_{n\in\mathbb{N}}$ of bounded functions on E by

$$h_n := h_n^+ - h_n^-. \tag{82}$$

In order to apply a compactness argument to this sequence $(h_n)_{n\in\mathbb{N}}$ we will consider h_n as element of $(L^1(E,\tau))^*$, with the measure τ defined as follows:

$$\tau := (U^*(1_{\{x\}})+1)\,dm(\cdot) \quad (\text{assume } m(E)=1), \tag{83}$$

Here we associate $g \leftrightarrow [\cdot, g] \in (L^1(E,\tau))^*$ with

$$[f,g] := \int f \cdot g\,dm \qquad f \in L^1(E,\tau),\ |g| \leqq a' + b'\,U^*(1_{\{x\}}). \tag{84}$$

From the general theory of normed spaces it is known that a sequence in $(L^1(E,\tau))^*$ is weak-$*$-compact if the sequence is bounded in the norm. Therefore we show

$$|h_n| \leqq a + b\,U^*(1_{\{x\}}) \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{85}$$

For that purpose consider

$$\bar{h}_n := h_n^- + h_n^+ \qquad (\text{Note } |h_n| \leqq \bar{h}_n\ \forall n \in \mathbb{N}). \tag{85}$$

From the construction of $(\bar{h}_n)_{n\in\mathbb{N}}$ we know that there exist numbers a_n, $b_n \geqq 0$ and functions k_n, $l_n \geqq 0$ such that

$$P^*(\bar{h}_n) - \bar{h}_n = a_n\,1_{\{x\}} - k_n, \qquad P^*(U^*(g) \wedge n) - U^*(g) \wedge n = b_n\,1_{\{x\}} - l_n. \tag{87}$$

This implies

$$P^*(\bar{h}_n - U^*(g) \wedge n) - (\bar{h}_n - U^*(g) \wedge n) = (a_n - b_n)\,1_{\{x\}} - k_n + l_n. \tag{88}$$

Now observe that $1_{E_n} \cdot k_n = 0$ and conclude that

$$\bar{h}_n - (U^*(g) \wedge n) \qquad \text{is subharmonic and bounded on } E_n \setminus \{x\}. \tag{89}$$

Since by definition $\bar{h}_n - U^*(g) \wedge n = 0$ on $\complement E_n \cup \{x\}$, we can conclude

$$\bar{h}_n \leqq U^*(1_{\{x\}}) \wedge n \leqq U^*(1_{\{x\}}) \qquad \forall n \in \mathbb{N}. \tag{90}$$

Now select a subsequence $(n_j)\subseteq\mathbb{N}$ such that $(h_{n_j})_{j\in\mathbb{N}}$ converges in the weak-*-topology of $(L^1(E,\tau))^*$ and define

$$\tilde{h}:=w\text{-}*\text{-}\lim_{j\to\infty} h_{n_j}. \tag{91}$$

This function $\tilde{h}$ has the properties

$$|\tilde{h}|\leqq U^*(1_{\{x\}}); \qquad P^*(\tilde{h})-\tilde{h}=c\,1_{\{x\}} \quad \text{for some } c\in\mathbb{R}. \tag{92}$$

(ii) Our harmonic function h is now defined as

$$h:=\tilde{h}-c\,U^*(1_{\{x\}}). \tag{93}$$

It remains to show that $\langle\nu-\mu,h\rangle\neq 0$. To see this note first we have by construction of $\tilde{h}$ and with $\langle\nu+\mu,\ U^*(1_{\{x\}})\rangle<\infty$

$$\langle\nu-\mu,\tilde{h}\rangle=\lim_{j\to\infty}\langle\nu-\mu,h_{n_j}\rangle \tag{94}$$

and since we assumed that $\langle\nu-\mu,U^*(1_{\{x\}})\rangle=0$ we also have

$$\langle\nu-\mu,h\rangle=\lim_{j\to\infty}\langle\nu-\mu,h_{n_j}\rangle. \tag{95}$$

But from Lemma 1 b we obtain

$$\langle\nu-\mu,h_{m_j}\rangle=\langle\bar{\nu}^{m_j}-\bar{\mu}^{m_j},h_{m_j}\rangle+\langle\bar{\rho}_1^{m_j}-\bar{\rho}_2^{m_j},h_{m_j}\rangle-\langle\bar{\eta}^{m_j}-\bar{\xi}^{m_j},1_{\{x\}}\rangle\,c_{m_j} \tag{96}$$

and this gives us since $\langle(\nu-\mu)\,U,\ 1_{\{x\}}\rangle=\langle\nu-\mu,\ -U^*(g)\rangle=0$ that

$$\langle\nu-\mu,h_{m_j}\rangle=\langle\bar{\nu}^{m_j}-\bar{\mu}^{m_j},h_{m_j}\rangle+\langle\bar{\rho}_1^{m_j}-\bar{\rho}_2^{m_j},h_{m_j}\rangle \tag{97}$$

and so we get by noting $\langle\rho,U^*(g)\rangle<\infty$ and $\rho_{1,2}^{m_j}\nearrow\rho$ that

$$\begin{aligned}\langle\nu-\mu,h\rangle&=\lim_{j\to\infty}\langle\nu-\mu,h_{m_j}\rangle=\lim_{j\to\infty}\langle\bar{\nu}^{m_j}-\bar{\mu}^{m_j},h_{m_j}\rangle\\&=\lim_{j\to\infty}m_j(\|\bar{\nu}^{m_j}\|+\|\bar{\mu}^{m_j}\|)=a>0.\quad\text{q.e.d.}\end{aligned} \tag{98}$$

3. (i) $\Leftrightarrow$ (iii).
This follows immediately from the following lemma.

Lemma ([8]). *Consider a coupling* $[\nu,(\nu_n);\ \mu,(\mu_n)]$ *for* $\nu,\mu\in\mathscr{E}^1(P)$ *with effects* $(\nu-\mu)\,U^+$, $(\nu-\mu)\,U^-$.

$$\operatorname{Prob}(T_n=+\infty)+\operatorname{Prob}(S_n=+\infty)\leqq\|\bar{\nu}^n\|+\|\bar{\mu}^n\| \tag{99}$$

$$\|\bar{\nu}^n\|+\|\bar{\mu}^n\|\leqq\operatorname{Prob}(T_n=+\infty)+\operatorname{Prob}(S_n=+\infty)+o(n^{-1}). \tag{100}$$

b) Proof of Theorem 3b

From Lemma 1 b we conclude for the coupling $(\nu,(\nu_n);\mu,(\mu_n))$

$$\langle\bar{\nu}^n,U^*(g)\rangle=\langle\nu,U^*(g)\rangle-\langle\bar{\rho}_1^n,U^*(g)\rangle+\langle\bar{\eta}^n,g\rangle \tag{101}$$

$$\langle\bar{\mu}^n,U^*(g)\rangle=\langle\mu,U^*(g)\rangle-\langle\bar{\rho}_2^n,U^*(g)\rangle+\langle\bar{\xi}^n,g\rangle \tag{102}$$

so that by using the following consequence of the property (N^*)

$$\langle \bar{v}^n, U^*(g)\rangle = \|\bar{v}^n\| \qquad n + o(1) \tag{103}$$

we obtain that for a short coupling with effects η, ξ

$$\langle v-\mu, U^*(g)\rangle = -\langle \eta-\xi, g\rangle. \tag{104}$$

With Theorem 2b of Part I it follows that there is at most one pair of disjoint effects η, ξ for a coupling of v and μ with the identity (104), namely $(v-\mu)\,U^{+,-}$. q.e.d.

c) Proof of Lemma 2

We proof this lemma by constructing a new transition kernel $\bar{Q}$ which satisfies the conditions of our lemma but also preserves the fact that v, μ have no short coupling. We will not prove that the new kernel has property (N^*), we will just show that we have all the properties we used in the proof of Theorem 3, Part 2.

1. Define the following Markov transition kernel Q on $(E, \mathscr{B})$.

$$Q := 1/2 \sum_{n=0}^{\infty} (1/2\,P)^n. \tag{105}$$

Furthermore, we define the operator U_Q^* by setting

$$U_Q^*(g) = U_*(g) - g + 1\,\|g\|^{\infty} \qquad \forall\, g \in \mathscr{F}_0^+(P). \tag{106}$$

Now U_Q^* has on $\mathscr{F}_0(P)$ all the properties stated in Proposition 2b and furthermore $U_Q^*(g)$ is slowly varying with Q. We omit the straightforward but lengthy calculations here (see [8]).

The following fact is essential to show that for our problem Q is equivalent to P: suppose for $v, \mu \in \mathscr{E}_1(P)$ exists a function h with

$$Q^*(h) = h, \qquad |h| \leqq a + b\,U_Q^*(g) \qquad \text{for some } g \in \mathscr{F}_0^+(P) \quad \text{and for suitable } a, b \in \mathbb{R}^+. \tag{107}$$

Then an elementary calculation shows that (compare [8])

$$P^*(h) = h, \qquad |h| \leqq a' + b'\,U_P^*(g) \qquad \text{for some } a', b' \in \mathbb{R}^+. \tag{108}$$

Now, a lemma of Neveu in [4] shows that Q is m-Harris recurrent and has the following property which will be important for our next step

$$\underset{A_i \uparrow E}{\exists}\ \underset{b_i > 0}{\exists}: \ Q \geqq 1_{A_i} \otimes (b_i\,\pi) \tag{109}$$

where π is a positive σ-finite invariant measure of P (and also of Q!).

2. To proceed further we have to obtain a transition kernel with an atom. So we consider a statespace $E \cup \{*\}$ and "extend" Q to a Markov transition

kernel $\bar{Q}$ on $(E\cup\{*\},\tilde{\mathscr{B}})$ in such a way that we have

$$\langle \nu-\mu, U^*_{\bar{Q}}(1_{\{*\}})\rangle=0, \quad \bar{m}\bar{Q}\leqq\bar{m} \quad \text{with} \quad \bar{m}:=m+\delta_{\{*\}}. \tag{110}$$

$\bar{Q}$ is constructed as follows: choose $A\in\tilde{\mathscr{A}}(Q)\cap\tilde{\mathscr{A}}(P)$ and a function $k(\cdot)$ on E such that

$$\text{(i) } 0\leqq k(x)\leqq 1 \ \forall x\in E,\ 1_{\complement A}k\equiv 0,\ 1>\int k\,d\pi>0 \tag{111}$$

$$\text{(ii) } Q\geqq 1_A\otimes(a\pi 1_A) \text{ for some } a>0. \tag{112}$$

$$\text{(iii) } \langle \nu-\mu, U^*_Q(k)\rangle=0 \tag{113}$$

$$\text{(iv) } (\pi 1_A)Q\in\mathscr{E}_1(Q), \quad \operatorname*{supess}_{x\in A}(U^*_Q(1_A)_{(x)})<\infty. \tag{114}$$

Fix a number $b\in(0,a)$. Then define $\bar{Q}$ according to the following list:

$$\bar{Q}(x,B)=Q(x,B); \quad \forall:\ x\in\complement A, \quad B\in\mathscr{B} \tag{115}$$

$$\bar{Q}(x,B)=Q(x,B); \quad \forall:\ x\in A, \quad B\subseteq\complement A \tag{116}$$

$$\bar{Q}(x,B)=Q(x,B)-b\,k(x)\,\pi(B); \quad \forall:\ x\in A, \quad B\subseteq A, \tag{117}$$

$$\bar{Q}(x,\{*\})=b\,k(x)\,\pi(A); \quad \forall:\ x\in A \tag{118}$$

$$\bar{Q}(\{*\},B)=\Big(\int_A Q(\cdot,B)\,d\pi\Big)\pi^{-1}(A); \quad \forall:\ B\in\mathscr{B}. \tag{119}$$

We define the operator $U^*_{\bar{Q}}$ on $\mathscr{F}_0^+(P)$ now as follows:

$$U^*_{\bar{Q}}(g)=\begin{cases} U^*_Q(g) & \text{on } E\\ (\pi(A))^{-1}\langle \pi 1_A, U^*_Q(g)\rangle & \text{on } \{*\}\end{cases} \tag{120}$$

$$U^*_{\bar{Q}}(1_{\{*\}})=\begin{cases} c_1\,U^*_Q(k)+c_2 1_E & \text{on } E \qquad c_1:=b(1+b\langle\pi,k\rangle)^{-1}\\ 0 \quad \text{on } \{*\} & \qquad c_2:=1-b\langle\pi,k\rangle(1+b\langle\pi,k\rangle)^{-1}.\end{cases} \tag{121}$$

One again verifies (using (i)–(iv) above) that $U^*_{\bar{Q}}(1_{\{*\}})$ has the properties (ii)–(iii) in Proposition 2b and is slowly varying with $\bar{Q}$. For details of this straightforward calculation we refer the reader to [8]. The following statement is now crucial for our considerations: suppose we have a function $\bar{h}$ on $E\cup\{*\}$ such that

$$\bar{Q}^*(\bar{h})=\bar{h}, \quad \langle \nu-\mu,\bar{h}\rangle\neq 0, \quad |h|\leqq a+b\,U^*_{\bar{Q}}(1_{\{*\}}) \tag{122}$$

then we obtain by setting

$$h:=\bar{h}|_E+c\,U^*_Q(k) \qquad c:=-b(\langle \pi 1_A,\bar{h}\rangle-\bar{h}(\{*\})) \tag{123}$$

a function with the properties

$$\langle \nu-\mu,h\rangle\neq 0, \quad Q^*(h)=h, \quad |h|\leqq a_1+a_2\,U^*_Q(1_A) \tag{124}$$

for some $a_1,a_2\in\mathbb{R}^+$
(this is obtained by an elementary calculation).

3. Observe now that $\bar{Q}$ is a Markov transition kernel for whom the arguments in (75)–(98) apply. Therefore, putting Part 1 and Part 2 ((107), (108),

(123), (124)) together allows us to conclude, that if ν, μ have no short coupling with respect to $\bar{Q}$ then there exists a P-harmonic function h with $|h| \in \tilde{\mathscr{F}}(P)$ and $\langle \nu - \mu, h \rangle \neq 0$. In order to prove our Lemma 2 it remains now to show, that if ν, μ have no short coupling with respect to P, then they have no short coupling with respect to $\bar{Q}$.

To prove this observe that Lemma 1b implies

$$\begin{array}{lcl} \nu, \mu \text{ have a short coupling} & & \langle \nu, f \rangle \leqq \langle \rho, f \rangle \\ \text{with respect to } P \text{ and with} & \Leftrightarrow & \langle \mu, f \rangle \leqq \langle \rho, f \rangle \quad \forall f \in \tilde{\mathscr{F}}^+(P) \\ \text{final distribution } \rho & & \langle \rho, f \rangle < \infty \end{array} \tag{125}$$

Here we used that $U^*(g)$ is slowly varying for the "$\Rightarrow$" direction. But with the definition given in (120) and (121) this equivalence holds also for $\bar{Q}$. Now observe that if f is a positive P-subharmonic function then the function

$$\tilde{f}\colon \tilde{f}(x) = f(x) \quad \text{for } x \in E, \quad \tilde{f}(\{*\}) = \langle \pi 1_A, f \rangle (\pi(A))^{-1} \tag{126}$$

is $\bar{Q}$ subharmonic and lies in $\tilde{\mathscr{F}}(Q)$. Therefore the equivalence (125) and $\bar{Q}(\{*\}, E) = 1$ imply our assertion.

3. The Proof of Theorem 4

a) Proof of Theorem 4b

1. We will show that by the $(\nu, (\nu - \mu) U^+)$, $(\mu, (\nu - \mu) U^-)$ flooding schemes a regular exact coupling is defined, which has final distribution ρ. For this purpose we will apply Theorem 2b (Part I) and therefore we are going to show that the assumptions made there, are met in our situation. Denote by $\eta_{\nu,\mu}$ the effect of the (ν, μ)-filling scheme. Since

$$\eta_{\nu,\mu} \geqq 0 \qquad \eta_{\nu,\mu} - \eta_{\nu,\mu} P = \nu - \mu \tag{127}$$

we have:

$$\sum_0^n (\nu - \mu) P^k = \eta_{\nu,\mu} - \eta_{\nu,\mu} P^{n+1} \leqq \eta_{\nu,\mu} \tag{128}$$

$$\sum_0^n (\nu - \mu) P^k = -\eta_{\mu,\nu} + \eta_{\mu,\nu} P^{n+1} \geqq -\eta_{\mu,\nu} \tag{129}$$

and furthermore since $\nu, \mu \in \mathscr{E}^1(P)$ we also know

$$\eta_{\nu,\mu} + \eta_{\mu,\nu} = a\pi \qquad \text{for some } a \in \mathbb{R}^+. \tag{130}$$

By definition of U we get from (128), (129), (130)

$$|(\nu - \mu) U| \leqq a\pi. \tag{131}$$

This gives immediately that the assumptions of Theorem 2b are fulfilled for $|(\nu - \mu) U|$ and that therefore there exists a coupling with this effect and final distribution ρ. (It can be obtained by the $(\nu,(\nu-\mu)U^+)$, $(\mu,(\nu-\mu)U^-)$-flooding schemes as shown in the proof of Theorem 2b in Part I). To show regularity of a coupling $[\nu,(\nu_n);\ \mu,(\mu_n)]$ obtained as described above, we use the following

Lemma 3. *Under the assumption of Theorem 4b, we have for a coupling* $[\nu,(\nu_n);\ \mu,(\mu_n)]$ *with σ-finite effects (if we denote by* $(\hat\rho_n^1,\hat\rho_n^2)$ *the continuation to a space-time coupling* (22, 23)) *that*

$$V_n(B)=\left(\sum_0^n(\nu-\mu)P^k\right)(B)-\left(\sum_0^n\nu_k-\mu_k\right)(B)\qquad\forall B\in\tilde{\mathscr{A}}\tag{132}$$

where

$$V_n(B):=\sum_{j,k=0}^n(\hat\rho_k^1(j,B)-\hat\rho_k^2(j,B)).\tag{133}$$

By the definition of U the right side of (132) converges to 0 for every $B\in\tilde{\mathscr{A}}$, and therefore our coupling is regular.

2. The uniqueness of the effects and final distribution of a regular exact coupling with disjoint effects is an immediate consequence of our Lemma 3.

Proof of Lemma 3. From the definition of $\hat\rho^1$, $\hat\rho^2$ and the time-space operator $\hat P$ it follows

$$\sum_0^n\nu P^k=\sum_0^n\nu_k+\sum_{j=0}^n\sum_{k=0}^n(\hat\rho^1)\hat P^k(j,\cdot)\qquad\forall:\ n\in\mathbb{N}\tag{134}$$

and therefore with the analogous relation for (μ_n), we conclude

$$\sum_0^n(\nu-\mu)P^k=\sum_0^n(\nu_k-\mu_k)+\sum_{j=0}^n\sum_{k=0}^n(\hat\rho^1-\hat\rho^2)\hat P^k(j,\cdot)\qquad\forall:\ n\in\mathbb{N}.\tag{135}$$

On the other hand, we get from relation (32), Part I since $\hat P$ is transient

$$\sum_{k=0}^\infty(\hat\rho^1-\hat\rho^2)\hat P^k(j,\cdot)=\sum_{k=0}^\infty\hat\rho_k^1(j,\cdot)-\hat\rho_k^2(j,\cdot).\tag{136}$$

Now note that by construction of $(\hat\rho_k^1)_{k\in\mathbb{N}}$, $(\hat\rho_k^2)_{k\in\mathbb{N}}$:

$$\sum_{k=0}^\infty(\hat\rho^1-\hat\rho^2)\hat P^k(j,\cdot)=\sum_{k=0}^j(\hat\rho^1-\hat\rho^2)\hat P^k(j,\cdot)\qquad\forall j\in\mathbb{N}\tag{137}$$

$$\sum_{k=0}^\infty(\hat\rho_k^1-\hat\rho_k^2)(j,\cdot)=\sum_{k=0}^j(\hat\rho_k^1-\hat\rho_k^2)(j,\cdot)\qquad\forall j\in\mathbb{N}.\tag{138}$$

With the two last equations, we obtain the assertion immediately from Eq. (135).

b) Proof of Theorem 4a

O. The main ingredients of our proof are the following lemmata (the straightforward proof of Lemma 4 will be omitted, Lemmas 5 and 6 are proved in part c) and d) of this section).

Lemma 4. *Let P be a Markov transition kernel with the property (N^*). Consider a successful coupling $[v,(v_n);\ \mu,(\mu_n)]$ with effects given by $(v-\mu)\,U^+$, $(v-\mu)\,U^-$. Here we assume $v,\mu\in\mathscr{E}_1$. Let g be a function in $\mathscr{F}_0^+$. Denote by ρ^m, η^m, ξ^m final distributions respectively effects of the couplings for v, μ with disjoint effects with respect to the transition kernel $1_{E_m}\cdot P+1_{\complement E_m}$ where $E_m:=\{U^*(g)\leqq m\}$. Then $\eta^m\,[\xi^m]$ is dominated by the effect of the $(v,\mu)\,[(\mu,v)]$-filling scheme.*

Lemma 5. *Under the assumptions of Theorem 4b we have for all $g\in L^1_+(E,\pi)$*

$$\begin{aligned}&\text{(i)}\ \lim_{n\to\infty}\langle\eta^n,g\rangle=\langle\eta,g\rangle\qquad \eta:=(v-\mu)\,U^+\\&\quad\ \lim_{n\to\infty}\langle\xi^n,g\rangle=\langle\xi,g\rangle\qquad \xi:=(v-\mu)\,U^-\end{aligned}\tag{139}$$

$$\begin{aligned}\text{(ii)}\ \lim_{n\to\infty}\langle\rho^n,g\rangle=\langle\rho,g\rangle\qquad -\rho:&=(v-\mu)\,U^+(I-P)-v\\&=(v-\mu)\,U^-(I-P)-\mu.\end{aligned}\tag{140}$$

The following notation will be useful in the sequel

$$\lambda_B:=(\lambda)\left(\sum_{n=0}^{\infty}(1_{\complement B}P)^n\,1_B\right);\qquad v^n:=((v-\mu)_{\complement E_n})^+.\tag{141}$$

Lemma 6. *Under the assumptions of Theorem 4b we have*

$$\text{(i)}\ (v^n-\mu^n)\,U+(\eta^n-\xi^n)=(v-\mu)\,U\tag{142}$$

$$\text{(ii)}\ \underset{B\in\tilde{\mathscr{A}}}{\forall}\ \underset{\theta_n\in\mathbb{R}}{\exists}:\ (v^n-\mu^n)\,U\,1_B=((v^n_B-\mu^n_B)\,U+\theta_n\,\pi)\,1_B\tag{143}$$

$$\text{(iii)}\ \lim_{n\to\infty}(v^n_B-\mu^n_B)\,U(B)=0\ \ \forall\!:\ B\in\tilde{\mathscr{A}}(P).\tag{144}$$

$$\lim_{n\to\infty}(\tilde v^n_B-\tilde\mu^n_B)\,\tilde U=0\qquad m(\cdot)\text{-a.e.},\ \forall\!:\ B\in\tilde{\mathscr{A}}(P)\tag{145}$$

where $\tilde v$ denotes the m-density of v.

1) Let us first study the behaviour of $(\eta^n)_{n\in\mathbb{N}}$, $(\xi^n)_{n\in\mathbb{N}}$ and strengthen (139) to an a.s. statement. By using Lemma 6 (ii), (i) and Lemma 5 (i), we can conclude with Lemma 6 (iii) that

$$\lim_{n\to\infty}[(\tilde v^n-\tilde\mu^n)\,\tilde U]=0\qquad m(\cdot)\text{-a.e.}\tag{146}$$

Now by using the equations in Lemma 6 (i) and the fact $\eta^m\wedge\xi^m=0$, we obtain ($\tilde\tau$ denotes the π-density of a measure τ)

$$\lim_{n\to\infty}\tilde\eta^n=\tilde\eta\qquad\lim_{n\to\infty}\tilde\xi^n=\tilde\xi.\tag{147}$$

The dominated convergence theorem can be applied now (see Lemma 4) to get

$$\eta=(\tilde{\mathscr{A}},m)-\lim_{n\to\infty}\eta^n;\qquad\xi=(\tilde{\mathscr{A}},m)-\lim_{n\to\infty}\xi^n.\tag{148}$$

2) Now let us consider the sequence $(\rho^n)_{n\in\mathbb{N}}$. (149)

From the Poisson equation for η^n we obtain

$$\rho^n = \eta^n P - \eta^n + \nu - \nu^n. \tag{150}$$

We know already that

$$\lim_{n\to\infty} \tilde{\eta}^n = \tilde{\eta} \tag{151}$$

and by the dominated convergence theorem together with Lemma 4 we can also conclude from (151)

$$\lim_{n\to\infty} \tilde{\eta}^n P = \tilde{\eta} P. \tag{152}$$

Now, by construction $\nu^{(m)}$ converges a.e. to 0. Putting these statements all together leads to

$$\lim_{n\to\infty} \tilde{\rho}^n = \tilde{\eta} P - \tilde{\eta} + \nu = \tilde{\rho} \qquad m(\cdot)\text{-a.e.} \tag{153}$$

Now we have the following situation:

$$\tilde{\rho}^n \geqq 0, \quad \tilde{\rho} \geqq 0; \quad \lim_{n\to\infty} \tilde{\rho}^n = \tilde{\rho} \quad \text{a.e.};$$

$$\langle \rho^n, 1_A \rangle \xrightarrow[n\to\infty]{} \langle \rho, 1_A \rangle \qquad \forall A \in L^1(E, \pi). \tag{154}$$

With standard techniques from measure theory one concludes from (154)

$$\lim_{n\to\infty} (\|\tilde{\rho}^n - \tilde{\rho}\|_\pi^1) = 0. \quad \text{q.e.d.} \tag{155}$$

c) Proof of Lemma 5

1. First we analyze the behaviour of $(\langle \eta^n, g \rangle)_{n\in\mathbb{N}}$, $(\langle \xi^n, g \rangle)_{n\in\mathbb{N}}$. By using Lemma 4 we obtain the following inequalities (denote by $\eta_{\nu,\rho}$ the effect of the (ν, ρ)-filling scheme, (38) Part I) (156)

$$\eta^n \leqq \eta_{\nu,\mu} \leqq \eta_{\nu,\mu} + \eta_{\mu,\nu} = a\pi \qquad \text{for some } a \in \mathbb{R}^+ \tag{157}$$

$$\xi^n \leqq \eta_{\mu,\nu} \leqq \eta_{\mu,\nu} + \eta_{\nu,\mu} = a\pi \qquad \text{for some } a \in \mathbb{R}^+. \tag{158}$$

The inequalities (157) and (158) show that the π-densities of η^n, ξ^n, (we will denote them by $\tilde{\eta}^n$, $\tilde{\xi}^n$) form a weak-$*$-compact sequence in the space

$$(L^1(E, \pi))^* \cong L^\infty(E, \pi). \tag{159}$$

Now let τ^+, τ^- be weak-$*$-limit points of $(\tilde{\eta}^n)_{n\in\mathbb{N}}$, respectively, $(\tilde{\xi}^n)_{n\in\mathbb{N}}$ and choose $S \subseteq \mathbb{N}$ such that $(\tilde{\eta}^m)_{m\in S}$, $(\tilde{\xi}^m)_{m\in S}$ converge to $\tilde{\tau}^+$ respectively $\tilde{\tau}^-$ in the weak-$*$-topology of $(L^1(E, \pi))^*$. We will show that this implies

$$\tilde{\tau}^+ = (\nu - \mu)\, \tilde{U}^+ \tag{160}$$

$$\tilde{\tau}^- = (\nu - \mu)\, \tilde{U}^- \tag{161}$$

which means that $(\tilde{\eta}^m)_{m\in S}$; $(\tilde{\xi}^m)_{m\in S}$ are convergent in the weak-$*$-topology and the limit points are $(\nu - \mu)\, \tilde{U}^+$ respectively $(\nu - \mu)\, \tilde{U}^-$ and that is just our assertion.

Now in order to show (160) and (161) note that it is sufficient to show

$$\langle \nu-\mu, U^*(g)\rangle+\langle \tilde{\tau}^+-\tilde{\tau}^-, g\rangle=0 \qquad \forall g\in L^1(E,\pi). \tag{162}$$

To show this identity observe that by Lemma 1 and Theorem 1b (Part I)

$$\langle \nu^m-\mu^m, U^*(g)\rangle=\langle \nu-\mu, U^*(g)\rangle+\langle \eta^m-\xi^m, g\rangle \qquad \forall m\in S. \tag{163}$$

It remains therefore to show that

$$\lim_{m\to\infty} (\langle \nu^m-\mu^m, U^*(g)\rangle)=0. \tag{164}$$

But we assumed that P has the property (N^*). Using (N^*) ((52)) and the fact $\langle \nu+\mu, U^*(g)\rangle<\infty$ we obtain $(E_n:=\{U^*(g)\leqq n\})$

$$\langle \nu^m+\rho^m 1_{\complement E_m}, U^*(g)\rangle=(\|\mu^m\|+\|\rho^m 1_{\complement E_m}\|)(m+o(m)) \tag{165}$$

$$\langle \mu^m+\rho^m 1_{\complement E_m}, U^*(g)\rangle=(\|\nu^m\|+\|\rho^m 1_{\complement E_m}\|)(m+o(m)). \tag{166}$$

Now apply Lemma 1 and get

$$\|\nu^m+\mu^m\|=O(m^{-1}), \qquad \|\rho^m 1_{\{U^*(g)\geqq m;\}}\|=O(m^{-1}) \tag{167}$$

(165), (166) and (167) the desired result (102) q.e.d.

2. Now we analyze the behaviour of $(\langle\rho^n, g\rangle)_{n\in\mathbb{N}}$. By Theorems 2a and 3a we have the following Poisson equations:

$$\eta^n-\eta^n P=\nu-(\nu^m+\rho^m) \tag{168}$$

$$\eta-\eta P=\nu-\rho. \tag{169}$$

Furthermore, we know that $\nu^n\leqq\eta_{\nu,\mu}+\eta_{\mu,\nu}=a\pi$ and $\lim_{n\to\infty}\nu^n=0$ a.e. Therefore we have

$$\lim_{n\to\infty}\langle \nu^n, g\rangle=0 \qquad \forall g\in L^1(E,\pi). \tag{170}$$

Since $g\in L^1(E,\Pi)$ implies $P^*(g)\in L^1(E,\pi)$; the relation (139) has the consequence

$$\lim_{n\to\infty}\langle \eta^n P, g\rangle=\langle \eta P, g\rangle \qquad \forall g\in L^1(E,\pi). \tag{171}$$

Putting (170), (171) together gives now

$$\lim_{m\to\infty}\langle \rho^m, g\rangle=\langle \rho, g\rangle. \quad \text{q.e.d.} \tag{172}$$

d) Proof of Lemma 6

1. First we need some results on properties of the extension of the operator U to the following set M

$\nu-\mu\in M \Leftrightarrow \nu, \mu$ are probability measures on $(E, \mathscr{B})$ with

(i) $\left|\sum_0^n (\nu-\mu) P^k\right| \leqq a\pi \quad \forall n\in\mathbb{N}$, for some $a\in\mathbb{R}^+$. (173)

(ii) The $m(\cdot)$-densities of $\sum_0^m (\nu-\mu) P^k$ converge m-a.e. (174)

Define $(\nu-\mu)\, U := (\tilde{\mathscr{A}}, m) - \lim_{n\to\infty} \left(\sum_0^n (\nu-\mu)\right) P^k$ for $(\nu-\mu)\in M$. (175)

Next we show the following facts, using the definition of U and M:

(α) $(\tilde{\mathscr{A}}, m) - \lim_{n\to\infty} [(\nu-\mu) U P^n] = 0$ (176)

(β) $U(I-P) = I$ on M (177)

(γ) Let $(\nu-\mu)\in M$. If for ν', μ' the effects η, ξ of the (ν, ν'), (μ, μ')-filling schemes have the property

$$(\tilde{\mathscr{A}}, m) - \lim_{n\to\infty} (\eta+\xi) P^n = 0 \tag{178}$$

then

$$(\nu'-\mu')\in M. \tag{179}$$

2. Here we prove Lemma 6(i). From the fact that

$$(\tilde{\mathscr{A}}, m) - \lim_{m\to\infty} (\eta^n+\xi^n) P^m = 0 \tag{180}$$

we conclude by using (178) above, that $(\nu^m-\mu^m)\in M\ \forall m\in\mathbb{N}$. We have from the construction of η^m, ξ^m the following Poisson equations

$$(\nu^m-\mu^m)\, U(I-P) = \nu^m-\mu^m \tag{181}$$

$$(\eta^m-\xi^m)(I-P) = (\nu-\mu)-(\nu^m-\mu^m). \tag{182}$$

An elementary calculation shows then

$$\sum_0^n (\nu-\mu) P^k = ((\nu^m-\mu^m)\, U + (\eta^m-\xi^m)) - ((\nu^m-\mu^m)\, U + (\eta^m-\xi^m)) P^{n+1} \tag{183}$$

Let n go to infinity and get by using (176)

$$(\nu-\mu)\, U = (\nu^m-\mu^m)\, U + (\eta^m-\xi^m) \qquad \forall m\in\mathbb{N}. \quad \text{q.e.d.} \tag{184}$$

3. We omit the proof of the Balayage relation (ii) for the operator U. It can be carried out completely in our framework.

4. The proof of the assertion (iii) of our Lemma 6 uses the following facts: there exists a system of sets $\mathscr{D}\subset\mathscr{A}$ which contains an exhaustion of E with (denote by $\tilde{\nu}$ the m-density of ν)

(*) $$\forall_{B\in\mathscr{D}} \lim_{m\to\infty} (\tilde{\nu}_B^m - \tilde{\mu}_B^m) P\tilde{U} = 0 \qquad m\text{-a.e.} \tag{185}$$

(**) $$\forall_{B\in\mathscr{D}} \lim_{m\to\infty} (\tilde{\nu}_B^m - \tilde{\mu}_B^m) = 0 \quad m\text{-a.e.} \tag{186}$$

Before we show (*) and (**), let us show how to finish the proof by these means. By the Poisson equation for U (we get from (*) and (**))

$$\lim_{m\to\infty} (\tilde{v}_B^m - \tilde{\mu}_B^m)\,\tilde{U} = 0 \qquad m\text{-a.e.}, \qquad \forall B \in \mathscr{D}. \tag{187}$$

We will show in the proof of (*) that there exists a function $q(\cdot)\colon E \to \mathbb{R}^+$ such that for all $B \in \mathscr{D}$

$$\langle v_B^m + \mu_B^m, q\rangle \leqq N < \infty \qquad \forall m \in \mathbb{N}. \tag{188}$$

$$|(v_B^m - \mu_B^m)\,U| \leqq \langle v_B^m + \mu_B^m, q\rangle \quad \pi + |v_B^m - \mu_B^m|, \; q > \pi_+! \tag{189}$$

At this point it is clear that we can conclude from (187), (188) and (189) with the dominated convergence theorem that

$$(\tilde{\mathscr{A}}, m) - \lim_{m\to\infty} [(v_B^m - \mu_B^m)\,U] = 0 \qquad \forall B \in \mathscr{D}. \quad \text{q.e.d.} \tag{190}$$

In order to show (*) we establish the following: there exists a kernel $U(x,y)$, a function $q(x)$ and a system of sets $\mathscr{D}$ such that

(i) $A \in \mathscr{D} \Rightarrow A \subseteq \tilde{\mathscr{A}}(P);\ \underset{N}{\exists}\ \underset{x\in A}{\forall}\ q(x) \leqq N;\ \underset{\bar{N}}{\exists}\ \underset{x\in A}{\forall}:\ U^*(g)_{(x)} \leqq \bar{N}$ (191)

(ii) $\mathscr{D}$ contains an exhaustion of E (192)

(iii) $\lim_{n\to\infty} \left(\sum_{k=1}^{n} (\tilde{v} - \tilde{\mu})\,P^k\right)_{(y)} = \int_E (\tilde{v} - \tilde{\mu})(x)\,U(x,y)\,dm(x) \quad m\text{-a.e.},$ (193)

for all measures $v, \mu \in \mathscr{E}_0(P)$, which are concentrated on some set $A \in \mathscr{D}$.

(iv) $|U(x,y)| \leqq q(x)\,\tilde{\pi}(y); \quad m \otimes m$ a.e. (194)

Now apply this to a measure $(v_B^m - \mu_B^m)$ with $B \in \mathscr{D}$ and get

$$((\tilde{v}_B^m - \tilde{\mu}_B^m)\,P)\,\tilde{U}(y) = \int (\tilde{v}_B^m - \tilde{\mu}_B^m)_{(x)}\,U(x,y)\,dm(x) \qquad m\text{-a.e.}, \tag{195}$$

$$|(\tilde{v}_B^m - \tilde{\mu}_B^m)(x)\,U(x,y)| \leqq \text{const} \cdot |\tilde{v}_B^m - \tilde{\mu}_B^m|_{(x)} \qquad \tilde{\pi}(y). \tag{196}$$

Then from the fact that $\lim_{n\to\infty} \|v_B^n - \mu_B^n\| = 0$ one concludes (*).

We can define $U(x,y)$, $q(x)$, $\mathscr{D}$ as follows: choose some $\lambda \in \mathscr{E}_0(P)$ with $\|\lambda\| = 1$, and note that by property (N) (iv) we have $(\delta_x P) \in \mathscr{E}_1(P)$.

$$U(x,A) := \lim_{n\to\infty} \left(\sum_{k=0}^{n} (\delta_x P - \lambda)\,P^k_{(A)}\right) \qquad \text{for all } A \in \tilde{\mathscr{A}}(P). \tag{197}$$

$U(x,y)$ is a version of the $m \otimes m$ density of the measure U on $(E \times E,\ \mathscr{B} \otimes \mathscr{B})$, which is defined by setting for all $A, B \in \tilde{\mathscr{A}}(P)$

$$U(A,B) := \int_A U(x,B)\,dm(x) \qquad (|U| \text{ has finite variation on sets in } \mathscr{D} \otimes \mathscr{D}). \tag{198}$$

$$q(x) := \inf(a \mid a\,\pi - \eta_{\delta_x P, \lambda} \geqq 0) \qquad (\eta_{\delta_x P, \lambda} := \text{effect of the } (\delta_x P;\ \lambda)\text{-filling scheme}). \tag{199}$$

It is now straightforward to show (191)–(194) and we omit the details.

6. Now in order to show (**) choose λ in the construction of $U(x,y)$ to be $(\pi\, 1_B)$. Consider the stopping sequences $(\nu_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$ generated by the $(\nu,\pi\, 1_B)$, $(\mu,\pi\, 1_B)$-filling schemes. Denote the effects by η respectively ξ. Then one obtains (details are left to the reader)

$$|\nu_B^m-\mu_B^m|=|((\nu-\mu)_{\complement E_m})_B| \tag{200}$$

$$=(\nu^m-\mu^m)_B+((\rho_1^m-\rho_2^m)_{\complement E_m})_B \tag{201}$$

$$\leqq(\eta-\eta^m)+(\xi-\xi^m)+|\pi\, 1_B-\rho_1^m|+|\pi\, 1_B-\rho_2^m| \tag{202}$$

$$+\|\rho_1^m-\rho_2^m|\cdot\pi\, 1_B+|\rho_1^m-\rho_2^m|+|\int U(x,\cdot)\,d|\rho_1^m-\rho_2^m|(x)|$$

$$\leqq(\eta-\eta^m)+(\xi-\xi^m)+|\pi\, 1_B-\rho_1^m|+|\pi\, 1_B-\rho_2^m|$$

$$+\|\rho_1^m-\rho_2^m\|\,\pi\, 1_B+|\rho_1^m-\rho_2^m|+\langle|\rho_1^m-\rho_2^m|,q\rangle\,\pi\, 1_B. \tag{203}$$

From the last inequality (**) is an immediate consequence observing that by construction for $i=1,2$

$$\eta^m\uparrow\eta,\ \xi^m\uparrow\xi;\qquad \rho_m^1+\rho_m^2\leqq 2\pi\, 1_B,\ \rho_i^m\uparrow\pi\, 1_B,\ \langle\pi\, 1_B,q\rangle<\infty. \tag{204}$$

4. Appendix: Proof of Proposition 2b

Since Proposition 2b is heavily used in the proof of our main theorem we sketch the proof here. The uniqueness proof is straightforward, so we concentrate here on showing that if P has the property (N), there exists a positive finite function f with the properties (45)-(47). We will need the following fact which also appears in different formulations in [3] and [4]. For the proof we refer to Part I, Sect. 5b.

Lemma 7 ([8]). *Let P be a m-Harris recurrent and ergodic transition kernel. Consider a measure μ which is supported by a set $A\in\tilde{\mathscr{A}}$ and dominated by a positive σ-finite invariant measure of P. Then the effect $\eta_{\nu,\mu}$ of the (ν,μ)-filling scheme has the property that $\eta_{\nu,\mu}(A)$ is bounded by a constant for all probability measures ν on $(E,\mathscr{B})$.*

a) We will define f by a twofold limit procedure.
First we choose a sequence $(B_k)\uparrow E$ with $B_k\in\tilde{\mathscr{A}}$ and a measure $\lambda\in\mathscr{E}_1$ with support in B_0. Denote by D the map which assigns to a finite positive function h the measure $h\cdot d\pi$. (π is a positive σ-finite invariant measure of P). Now we define continuous linear functionals F_n^k on $L^\infty(B_k,\pi)$ by setting

$$F_n^k(l)=\left\langle D(l)-\|D(l)\|\,\lambda,\ -\sum_{k=0}^{n}P^{*k}(g)\right\rangle\qquad\forall:\ l\in L_+^\infty(B_k,\pi). \tag{205}$$

Property (N) (iii) allows us now to define

$$F^k:=w\text{-}*\text{-}\lim_{n\to\infty}F_n^k. \tag{206}$$

Since the functionals $(F_n^k)_{n\in\mathbb{N}}$ can be represented by π-continuous measures on B_k, the same is true for F^k by the theorem of Vitali-Hahn-Saks. So we have

$$F^k(l)=\int f_k\cdot l\,d\pi \qquad \text{with } f_k\in L^1(B_k,\pi). \tag{207}$$

Now, define

$$\tilde{f}:=\sum_{k=0}^{\infty} f_{k+1}\cdot 1_{B_{k+1}\setminus B_k}+f_0. \tag{208}$$

Since we have $\sum_{k=0}^{n}(\nu-\mu)P^k\geqq -\eta_{\nu,\mu}$, with $\eta_{\nu,\mu}:=$effect of the (ν,μ)-filling scheme, we can conclude from Lemma 7, that

$$F_n^k(l)\geqq -C\,\|D(l)\| \tag{209}$$

or in other words $\tilde{f}$ is bounded below. This allows us to define

$$f:=\tilde{f}+|\operatorname*{infess}_{x\in E}(\tilde{f}(x))|. \tag{210}$$

It remains now to show that f has the properties (45)–(47)

(i) (45) is a consequence of the fact that $f_k\in L^1(B_k,\pi)$ ([8]).

(ii) (46) holds by construction for ν, $\mu\in\mathscr{E}_0$. So we have to show it holds for ν, $\mu\in\mathscr{E}_1$. We will prove that

$$\lim_{n\to\infty}\left\langle \sum_{k=0}^{n}(\nu-\lambda)P^k, g\right\rangle=\langle \nu, f\rangle \qquad \forall\, \nu\in\mathscr{E}_1. \tag{211}$$

By definition of $\mathscr{E}_1$ we have for a measure $\nu\in\mathscr{E}_1$ a decomposition $\nu=\sum_0^\infty \nu\,1_{A_k}$ with $A_k\in\tilde{\mathscr{A}}$ so that the effects η^k, ξ^k of the $(\nu\,1_{A_k}P, \|\nu\,1_{A_k}\|\,\lambda)$ respectively $(\|\nu\,1_{A_k}\|\,\lambda, \nu\,1_{A_k})$-filling schemes have the property that $\sum_0^\infty \eta^k+\xi^k$ is a σ-finite measure. Therefore we can write (define for abbreviation $\lambda_k=\lambda\cdot\|\nu\,1_{A_k}\|$)

$$\begin{aligned}\sum_{j=0}^{n}(\nu-\lambda)P^j&=\sum_{j=0}^{n}\sum_{k=0}^{\infty}(\nu\,1_{A_k}-\lambda_k)P^j\\&=\sum_{k=0}^{\infty}\sum_{k=0}^{n}((\nu\,1_{A_k})-\lambda_k)P^j.\end{aligned} \tag{212}$$

Now observe that (rewrite the sum by using the Poisson equations for η^k, ξ^k)

$$\xi^k\leqq\sum_0^n(\nu\,1_{A_k}-\lambda_k)P^j\leqq\eta^k+\nu\,1_{A_k} \qquad \forall\colon\ n\in\mathbb{N}. \tag{213}$$

Since $\nu+\sum_{k=0}^{\infty}\eta^k+\xi^k$ is a σ-finite measure and effect of a recurrence time it is of the form $a\pi$. This now allows to conclude from (212) with the dominated convergence theorem, that (211) holds.

(iii) In order to show (47) (the Poisson equation $(P^*-I)(f)=g$ holds), consider a measure $v\in\mathscr{E}_0$ such that $vP\in\mathscr{E}_1$. As above we can show that

$$\langle v,f\rangle=\lim_{n\to\infty}\left\langle\sum_0^n(v-\lambda)P^k,g\right\rangle,\qquad \langle vP,f\rangle=\lim_{n\to\infty}\left\langle\sum_0^n(vP-\lambda)P^j,g\right\rangle. \tag{214}$$

This means

$$\langle vP,f\rangle-\langle v,f\rangle=\lim_{n\to\infty}\langle v-vP^{n+1},g\rangle=\langle v,g\rangle \quad ([2]). \tag{215}$$

We rewrite this statement now as follows:

$$\langle v,(P^*-I)(f)\rangle=\langle v,g\rangle \quad \forall v \quad \text{with } v\in\mathscr{E}_0,\ vP\in\mathscr{E}_1. \tag{216}$$

One can show that there exists a sequence $B_k\uparrow E$ such that

$$\begin{aligned}(v\leqq a\pi \quad &\text{for some } a\in\mathbb{R}^+,\ v \text{ is supported by } B_k\\ &\text{for some } k\in\mathbb{N})\Rightarrow(v\in\mathscr{E}_0,\ vP\in\mathscr{E}_1).\end{aligned} \tag{217}$$

(217) finishes of course the proof of (47).

The proof of the assertion (217) proceeds as follows.

Choose a sequence $B_k^1\uparrow E$ with $B_k^1\in\tilde{\mathscr{A}}$ (possible by the "lemma of Harris").

$$B_k^2:=\{x\,|\,q(x)+r(x)\leqq k\} \qquad B_k^3=\{x\,|\,t(x)\leqq k\}. \tag{218}$$

Here

$$q(x)=\inf(a\,|\,\eta_{\delta_xP,\lambda}\leqq a\pi) \qquad r(x)=\inf(a\,|\,P(x,\cdot)\leqq a\cdot\pi(\cdot))$$

$$t(x)=\sum_{k=0}^{\infty}\left\|\frac{dP(x,\cdot)}{d\pi}1_{A_k}\right\|^{\infty}. \tag{220}$$

Here, (A_k) is the partition of E whose existence is required in (N)(iv).

By property, (N)(iii) $q(x)$, $r(x)$ are finite, by property (N)(iv) $t(x)$ is finite. Therefore, we can define an exhaustion $(B_k)_{k\in\mathbb{N}}$ of E by setting

$$B_k=B_k^1\cap B_k^2\cap B_k^3. \tag{221}$$

In order to show that with this (B_k) (217) holds, we have to prove that $vP\in\mathscr{E}_1$ for v with $v\leqq a\pi$ and which are supported by some B_k. We show

$$\sum_{k=0}^{\infty}(vP)1_{A_k}+\eta^k+\xi^k\leqq 2\langle v,q+r\rangle\pi+\langle v,t\rangle\cdot\pi\leqq C\cdot\pi \qquad C\in\mathbb{R}^+. \tag{222}$$

Here η^k denotes the effect of the $((vP1_{A_k})P,\lambda\cdot\|v1_{A_k}\|)$-filling scheme and ξ^k the effect of the $(\lambda\cdot\|v1_{A_k}\|,\ vP1_{A_k})$-filling scheme. The (A_k) are the same as in (220). A careful look at the definition of $q(\cdot)$, $v(\cdot)$, $t(\cdot)$ shows now that the proof of (222) is straightforward (compare [8]) for details).

The proofs of Propositions 1, 3 and 4 can be found in [8] and are omitted here.

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