A. W. MARRIS & W. F. AMES

Addendum: On Complex-Lamellar Motions

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Introduction

In the paper cited (1975) MARRIS sought to delimit the steady rotational universal complex-lamellar motions* of Navier-Stokes fluids. The conclusions of the analysis were summarized by the following theorem.

Theorem I.5. *Let* $\boldsymbol{v}(\boldsymbol{x})$ *be a steady rotational universal complex-lamellar motion of a Navier-Stoken fluid. The motion must be one of the following:*

(1) *A plane or axi-symmetric motion (i.e. a motion in which the stream-lines are meridians and the vortex-lines are lines of latitude on a family of surfaces of revolution).*

(2) *A motion whose stream-lines are parallel straight lines.*

(3) *A motion whose Lamb surfaces** are general helicoids. The stream-lines are geodesics on the helicoids, while the vortex-lines, the geodesic parallels, are circular helices. The stream-lines are normal to a family of helicoids. The surfaces of constant vorticity are circular cylinders whose axis is the axis of the helicoids. The vorticity magnitude is inversely proportional to the square root of the stream-line torsion.*

It was pointed out in the paper that the complex-lamellar circular helical motion of STRAKHOVITCH (1963, p. 102) was a special case of the motion of Theorem I.5, Part 3. For this motion the physical components of the velocity and vorticity are given in cylindrical co-ordinates as

$$\begin{aligned} v_r &= 0, \qquad v_\theta = \frac{d}{r} + \frac{k\,a\,r}{2}, \qquad v_z = -\frac{r}{a}\,v_\theta = -\frac{d}{a} - \frac{k\,r^2}{2}, \\ \omega_r &= 0, \qquad \omega_\theta = k\,r, \qquad \omega_z = k\,a, \end{aligned} \tag{I.1}$$

where d, k and a are constants.

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* A complex-lamellar motion is a motion in which the velocity and vorticity are perpendicular vectors.

** *I. e.* surfaces containing the stream-lines and vortex-lines.

Since the publication of that work it was discovered that the conditions associated with Case (3) of Theorem I.5 lead to a clearer and stronger statement than that originally given. It could be proved that the vorticity field for the complex-lamellar motions that were not plane, rectilinear or axi-symmetric, was the same as the vorticity field for the circular helical motion given by (I.1).

Accordingly Theorem I.5 should be replaced by the following:

Main Theorem. *The only steady, rotational, universal complex-lamellar motions of a Navier-Stokes fluid are*

1) *Plane or axi-symmetric motions,*
2) *Motions whose stream-lines are parallel straight lines.*
3) *Motions obtained by superposing a steady isochoric irrotational motion on the circular helical motion given by* (I.1).

It was then necessary to determine whether Case (3) offered any viable solutions. Definite conditions are required for the superposition to be possible.* In Chapter 3 we construct the following example:

$$\begin{aligned} v_r &= -\frac{\lambda}{a}\frac{(z-a\theta)}{r}+\frac{\varepsilon}{r}, \\ v_\theta &= \frac{\lambda \log r}{r}+\frac{d}{r}+\frac{k a r}{2}, \\ v_z &= -\frac{r}{a}v_\theta = -\frac{\lambda \log r}{a}-\frac{d}{a}-\frac{k r^2}{2}, \end{aligned} \tag{I.2}$$

where v_r, v_θ and v_z are physical components of the velocity referred to cylindrical co-ordinates.

1. Summary of Background Material

The velocity $\boldsymbol{v}=v\,\boldsymbol{s}$, where $\boldsymbol{s}$ is the unit vector tangent to the stream-line, and the vorticity $\boldsymbol{\omega}=\operatorname{curl}\boldsymbol{v}$ must satisfy the conditions,

$$\boldsymbol{v}\cdot\boldsymbol{\omega}=0, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{v}=0, \tag{1.2}$$

$$\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v})=\mathbf{0}, \tag{1.3}$$

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\omega}=\mathbf{0}. \tag{1.4}$$

Since $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are not parallel or zero, the condition (1.3) guarantees the existence of Lamb surfaces $\psi=$constant containing the stream-lines and vortex-lines in accordance with

$$\boldsymbol{v}\times\boldsymbol{\omega}=\operatorname{grad}\psi. \tag{1.5}$$

It was shown that (1.1) and (1.3) require that the stream-lines are geodesics in the Lamb surfaces. The vortex-lines are geodesic parallels in the Lamb surfaces.

* For an irrotational motion $\boldsymbol{v}_2$ to be superposed upon a circulation-preserving motion $\boldsymbol{v}_1$ with vorticity $\boldsymbol{\omega}_1$, one must have (1960, p. 396)

$$\operatorname{curl}(\boldsymbol{\omega}_1\times\boldsymbol{v}_2)=\mathbf{0}.$$

The unit tangent $\boldsymbol{b}$ to the vortex-line points along the bi-normal to the stream-line; thus

$$\boldsymbol{\omega}=\omega\,\boldsymbol{b} \tag{1.6}$$

where ω is the vorticity magnitude.

We denote the principal normal and bi-normal to the vortex-line by $\boldsymbol{n}_b$ and $\boldsymbol{b}_b$, respectively. It was shown that (1975, p. 146)

$$\boldsymbol{b}_b=\frac{(\kappa+\operatorname{div}\boldsymbol{n})\,\boldsymbol{s}-\theta\,\boldsymbol{n}}{\kappa_b} \tag{1.7}$$

and

$$\boldsymbol{n}_b=\frac{-\theta\,\boldsymbol{s}-(\kappa+\operatorname{div}\boldsymbol{n})\,\boldsymbol{n}}{\kappa_b}, \tag{1.8}$$

where κ is the stream-line curvature, div $\boldsymbol{n}$ is the first curvature of the Lamb surface and $\theta=\boldsymbol{b}\cdot\operatorname{grad}\boldsymbol{s}\cdot\boldsymbol{b}$ is the geodesic curvature of the $\boldsymbol{b}\cdot$ lines on the Lamb surfaces. Two further vector field parameters appeared, the torsion τ of the stream-line and the quantity $\psi=\boldsymbol{n}\cdot\operatorname{grad}\boldsymbol{s}\cdot\boldsymbol{n}$. The quantity

$$\kappa_b=[(\kappa+\operatorname{div}\boldsymbol{n})^2+\theta^2]^{\frac{1}{2}} \tag{1.9}$$

is the curvature of the $\boldsymbol{b}$-line (vortex-line).

It was shown that the torsion τ_b of the vortex-line was minus the stream-line torsion τ,

$$\tau_b=-\tau. \tag{1.10}$$

It was also shown that

$$\operatorname{div}\boldsymbol{b}=0, \tag{1.11}$$

and it was proved that each of the quantities θ, ψ, τ, κ, κ_b, τ_b, div $\boldsymbol{n}$, v, and ω bears a constant value along a $\boldsymbol{b}$-line.

One has the representations

$$\operatorname{curl}\boldsymbol{s}=\kappa\,\boldsymbol{b}, \tag{1.12}$$

$$\operatorname{curl}\boldsymbol{n}=\psi\,\boldsymbol{b}, \tag{1.13}$$

$$\operatorname{curl}\boldsymbol{b}=(\kappa+\operatorname{div}\boldsymbol{n})\,\boldsymbol{s}-\theta\,\boldsymbol{n}-2\tau\,\boldsymbol{b}, \tag{1.14}$$

formulae which exhibit κ and ψ as the curvatures of the vector-lines of $\boldsymbol{s}$ and $\boldsymbol{n}$ and verify that the curvature of the vector-lines of $\boldsymbol{b}$ is as given by (1.9). One notes from (1.14) that the abnormality of the vector-lines of $\boldsymbol{b}$ has the value -2τ. Hence the vortex-lines will be the orthogonal trajectories of a family of surfaces if and only if the motion is plane.

The following compatibility relations among the vector field parameters have to be satisfied:

$$\frac{\delta\tau}{\delta n}+2\tau(\kappa+\operatorname{div}\boldsymbol{n})=0, \tag{1.15}$$

$$\frac{\delta\theta}{\delta n}+(\theta-\psi)(\kappa+\operatorname{div}\boldsymbol{n})=0, \tag{1.16}$$

$$\frac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n})+\theta(2\kappa+\operatorname{div}\boldsymbol{n})=0, \tag{1.17}$$

$$\frac{\delta\tau}{\delta s}+2\theta\tau=0, \tag{1.18}$$

$$\frac{\delta\theta}{\delta s}+\theta^2-\kappa(\kappa+\operatorname{div}\boldsymbol{n})-\tau^2=0, \tag{1.19}$$

$$\frac{\delta\kappa}{\delta n}-\frac{\delta\psi}{\delta s}-\kappa^2-\psi^2-3\tau^2=0, \tag{1.20}$$

$$\frac{\delta}{\delta n}(\kappa+\operatorname{div}\boldsymbol{n})+\theta\psi+(\kappa+\operatorname{div}\boldsymbol{n})^2-\tau^2=0. \tag{1.21}$$

In these formulae the symbols $\frac{\delta}{\delta s}, \frac{\delta}{\delta n}$ denote the directional derivatives $\boldsymbol{s}\cdot\operatorname{grad}$ and $\boldsymbol{n}\cdot\operatorname{grad}$.

Finally it was shown that the vorticity magnitude ω must satisfy the conditions

$$\frac{\delta}{\delta s}\log\omega=\theta \tag{1.22}$$

and

$$\frac{\delta}{\delta n}\log\omega=\kappa+\operatorname{div}\boldsymbol{n}. \tag{1.23}$$

2. Proof of Main Theorem

It is evident from (1.22) and (1.23) that if the vorticity magnitude ω is spatially constant, then θ and $(\kappa+\operatorname{div}\boldsymbol{n})$ are zero. It then follows from (1.21) that τ is also zero. These conditions suffice to ensure that $\operatorname{grad}\boldsymbol{b}=0$ (see equation (1.8) of (1975)). In this case the vector-lines of $\boldsymbol{b}$ are parallel straight lines and one has the case of plane motion. We discount this case and require that ω be not constant.

From (1.8), (1.22), and (1.23), and the fact that ω is constant along a $\boldsymbol{b}$-line, we have

$$\boldsymbol{n}_b=-\frac{1}{\kappa_b}\operatorname{grad}\log\omega. \tag{2.1}$$

The representation (2.1) shows that the vector-lines of $\boldsymbol{n}_b$ are the orthogonal trajectories of the family of surfaces $\omega=$constant.

Since $\boldsymbol{n}_b$ is the principal normal to the vector-lines of $\boldsymbol{b}$ (the vortex-lines), it follows that the $\boldsymbol{b}$-lines are geodesics on the surfaces $\omega=$constant. The relation (1.7) checks also that the vector-lines of $\boldsymbol{b}_b$, namely the bi-normals to the vortex-lines, lie on the surfaces $\omega=$constant.

Since $\frac{\delta\kappa_b}{\delta b}$ and $\frac{\delta\tau_b}{\delta b}$ are both zero, the $\boldsymbol{b}$-lines, being curves of constant curvature and torsion, must be circular helices. It follows that the surfaces $\omega=$constant are circular cylinders.

From (1.8) we have

$$\operatorname{curl} \boldsymbol{n}_b = -\operatorname{grad}\left(\frac{\theta}{\kappa_b}\right) \times \boldsymbol{s} - \frac{\theta}{\kappa_b} \operatorname{curl} \boldsymbol{s} - \operatorname{grad}\left(\frac{\kappa + \operatorname{div} \boldsymbol{n}}{\kappa_b}\right) \times \boldsymbol{n} - \frac{\kappa + \operatorname{div} \boldsymbol{n}}{\kappa_b} \operatorname{curl} \boldsymbol{n}. \tag{2.2}$$

Using the expressions (1.12) and (1.13) for curl $\boldsymbol{s}$ and curl $\boldsymbol{n}$, the expression (1.9) for κ_b and the compatibility conditions (1.15) to (1.21) we verify that

$$\operatorname{curl} \boldsymbol{n}_b = \boldsymbol{0}. \tag{2.3}$$

It follows that the vector-lines of $\boldsymbol{n}_b$ are rectilinear. We conclude that the surfaces $\omega = \text{constant}$ must be concentric circular cylinders.

One may write

$$\boldsymbol{b} = \frac{r}{(r^2 + a^2)^{\frac{1}{2}}} \boldsymbol{e}_\theta + \frac{a}{(r^2 + a^2)^{\frac{1}{2}}} \boldsymbol{e}_z, \tag{2.4}$$

where $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ are unit vectors perpendicular to and parallel to the generators of the cylindrical surfaces $\omega = \text{constant}$. In the representation (2.4) the curvature and the torsion of the vortex-line are given by

$$\kappa_b = \frac{r}{r^2 + a^2}, \tag{2.5}$$

$$\tau_b = \frac{a}{r^2 + a^2}, \tag{2.6}$$

and we see that

$$\frac{\tau_b^2 + \kappa_b^2}{\tau_b} = \frac{1}{a}. \tag{2.7}$$

The parameter a in (2.4) bears a constant value on a particular surface of the family $\omega = \text{constant}$. We claim that a is, in fact, spatially constant. Indeed, by (1.9) and (1.10) we can rewrite (2.7) in the form

$$\frac{\tau^2 + (\kappa + \operatorname{div} \boldsymbol{n})^2 + \theta^2}{\tau} = -\frac{1}{a}. \tag{2.8}$$

Taking the directional derivative of the left hand side of (2.8) with respect to $\boldsymbol{s}$ and $\boldsymbol{n}$ and using the formulae (1.15) to (1.21), we verify that $\frac{\delta a}{\delta s}$ and $\frac{\delta a}{\delta n}$ are both zero. Hence a must be spatially constant.

Again from (1.18) and (1.22) one has

$$\frac{\delta}{\delta s} [\omega^2 \tau] = 0, \tag{2.9}$$

while from (1.15) and (1.23)

$$\frac{\delta}{\delta n} [\omega^2 \tau] = 0. \tag{2.10}$$

Since ω and τ maintain a constant value along a $\boldsymbol{b}$-line, it follows that $\omega^2 \tau$ is spatially constant. We write

$$\omega^2 \tau = -c \tag{2.11}$$

so that by (1.10)

$$\omega^2 \tau_b = c. \tag{2.12}$$

From (1.6) and (2.4) and by using (2.6) and (2.12), for the vorticity vector we have

$$\omega = \frac{\omega r}{(r^2+a^2)^{\frac{1}{2}}} \boldsymbol{e}_\theta + \frac{\omega a}{(r^2+a^2)^{\frac{1}{2}}} \boldsymbol{e}_z = k\,[r\,\boldsymbol{e}_\theta + a\,\boldsymbol{e}_z], \tag{2.13}$$

where

$$k = \frac{c^{\frac{1}{2}}}{a^{\frac{1}{2}}}. \tag{2.14}$$

Thus the vorticity vector (2.13) is the same as the vorticity vector (I.1) for the complex-lamellar circular helical motion of Strakhovitch.

Consequently, the motion under consideration can only differ from the circular helical motion by a superposable steady isochoric irrotational motion.

This proves the main theorem.

3. A Particular Solution

The vortex-lines, whose unit tangent is the bi-normal $\boldsymbol{b}$ to the stream-lines, are circular helices on the family of concentric cylinders $\omega = \text{constant}$. The vector-lines of the bi-normal $\boldsymbol{b}_b$ to the vortex-lines, being the orthogonal trajectories of the vortex-lines on the surface, are also circular helices.

If we write

$$\Omega_b = \boldsymbol{b} \cdot \operatorname{curl} \boldsymbol{b}, \qquad \Omega_{b_b} = \boldsymbol{b}_b \cdot \operatorname{curl} \boldsymbol{b}_b, \qquad \Omega_{n_b} = \boldsymbol{n}_b \cdot \operatorname{curl} \boldsymbol{n}_b \tag{3.1}$$

for the abnormalities of the vortex-lines, their bi-normals and principal normals, respectively, then it is known that these abnormalities are connected by the formula

$$\Omega_{n_b} + \Omega_{b_b} = \Omega_b - 2\tau_b. \tag{3.2}$$

By (1.10) and (1.14) one has

$$\Omega_b = -2\tau = 2\tau_b. \tag{3.3}$$

Also by (2.1), one has

$$\Omega_{n_b} = 0. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$\Omega_{b_b} = 0. \tag{3.5}$$

Hence the circular helical vector-lines of $\boldsymbol{b}_b$ are the orthogonal trajectories of a family of surfaces. The vector-lines of $\boldsymbol{b}_b$ are the stream-lines for the complex-lamellar circular helical motion of Strakhovitch given by (I.1). Our theorem asserts that the remaining complex-lamellar motions of the class under consideration are obtained by superposing an isochoric irrotational motion on this flow.

Using cylindrical co-ordinates, we write

$$\boldsymbol{v}=v_r\,\boldsymbol{e}_r+v_\theta\,\boldsymbol{e}_\theta+v_z\,\boldsymbol{e}_z, \tag{3.6}$$

where $\boldsymbol{e}_\alpha$ are unit vectors so that physical components are implied. Since

$$\boldsymbol{\omega}=k\,[r\,\boldsymbol{e}_\theta+a\,\boldsymbol{e}_z], \tag{2.15}$$

the condition

$$\boldsymbol{v}\cdot\boldsymbol{\omega}=0 \tag{1.1}$$

requires that

$$\frac{v_\theta}{v_z}=-\frac{\omega_z}{\omega_\theta}=-\frac{a}{r}. \tag{3.7}$$

Since

$$\boldsymbol{\omega}=\operatorname{curl}\boldsymbol{v},$$

we obtain from (2.15) and (3.7)

$$\frac{\partial v_\theta}{\partial\theta}+a\frac{\partial v_\theta}{\partial z}=0, \tag{3.8}$$

$$a\frac{\partial v_r}{\partial z}+\frac{\partial}{\partial r}[r\,v_\theta]=k\,a\,r, \tag{3.9}$$

and

$$-\frac{\partial v_r}{\partial\theta}+\frac{\partial}{\partial r}(r\,v_\theta)=k\,a\,r. \tag{3.10}$$

From (3.9) and (3.10) one has

$$\frac{\partial v_r}{\partial\theta}+a\frac{\partial v_r}{\partial z}=0. \tag{3.11}$$

The equations (3.8) and (3.11) are integrated directly to give

$$v_\theta=F(u,r), \tag{3.12}$$

$$v_r=G(u,r), \tag{3.13}$$

where

$$u=z-a\,\theta. \tag{3.14}$$

It follows from (3.7) that

$$v_z=-\frac{r}{a}\,v_\theta=-\frac{r}{a}\,F(u,r). \tag{3.15}$$

It is evident from (2.15) and (3.14) that $\boldsymbol{\omega}\cdot\operatorname{grad}u=0$. Hence the surfaces $u=$ constant contain the vortex-lines. These surfaces are the right helicoids orthogonal to the vector-lines of $\boldsymbol{b}_b$. Their existence is guaranteed by (3.5).

The condition

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1.2}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial v_\theta}{r\,\partial\theta} + \frac{\partial v_z}{\partial z} = 0$$

gives, by (3.7),

$$\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial}{\partial\theta}\,v_\theta - \frac{r^2}{a}\frac{\partial v_\theta}{\partial z} = 0,$$

so that by (3.8)

$$\frac{\partial}{\partial r}(r\,v_r) + \frac{a^2+r^2}{a^2}\frac{\partial v_\theta}{\partial\theta} = 0. \tag{3.16}$$

From (2.15) we verify that

$$\operatorname{curl} \boldsymbol{\omega} = 2\,k\,\boldsymbol{e}_z,$$

so that

$$\operatorname{curl}\operatorname{curl} \boldsymbol{\omega} = \boldsymbol{0}. \tag{1.4}$$

Similarly, from (2.15), (3.6), and (3.7), we obtain

$$\boldsymbol{\omega}\times\boldsymbol{v} = k\left[-\frac{a^2+r^2}{a}\,v_\theta\,\boldsymbol{e}_r + a\,v_r\,\boldsymbol{e}_\theta - r\,v_r\,\boldsymbol{e}_z\right], \tag{3.17}$$

and we verify that

$$\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v}) = \boldsymbol{0}, \tag{1.3}$$

by virtue of (3.8), (3.11) and (3.16).

We conclude that the velocity field given by (3.12) to (3.15) has to satisfy the two conditions

$$-\frac{\partial v_r}{\partial\theta} + \frac{\partial}{\partial r}(r\,v_\theta) = k\,a\,r \tag{3.10}$$

and

$$\frac{\partial}{\partial r}(r\,v_r) + \frac{a^2+r^2}{a^2}\frac{\partial v_\theta}{\partial\theta} = 0. \tag{3.16}$$

Writing

$$\alpha = r\,v_r, \tag{3.18}$$

$$\beta = r\,v_\theta - \frac{k\,a\,r^2}{2}, \tag{3.19}$$

and using

$$u = z - a\,\theta, \tag{3.14}$$

we transform (3.10) and (3.16) respectively to

$$\frac{\partial\beta}{\partial r} + \frac{a}{r}\frac{\partial\alpha}{\partial u} = 0, \tag{3.20}$$

$$\frac{\partial\alpha}{\partial r} - \frac{(a^2+r^2)}{a\,r}\frac{\partial\beta}{\partial u} = 0. \tag{3.21}$$

These are the equations to be satisfied by the motion.

A simple particular solution is given by

$$\alpha = -\frac{\lambda}{a} u + \varepsilon, \tag{3.22}$$

$$\beta = \lambda \log r + d, \tag{3.23}$$

where λ, ε, and d are constants. We obtain

$$\begin{aligned} v_r &= -\frac{\lambda}{a} \frac{(z - a\theta)}{r} + \frac{\varepsilon}{r}, \\ v_\theta &= \lambda \frac{\log r}{r} + \frac{d}{r} + \frac{k a r}{2}, \\ v_z &= -\frac{r}{a} v_\theta = -\frac{\lambda \log r}{a} - \frac{d}{a} - \frac{k r^2}{2}. \end{aligned} \qquad \text{(I.2), (3.24)}$$

This motion reduces to the complex-lamellar circular helical motion (I.1) when ε and λ are zero. When λ is zero, the motion consists of the circular helical motion superimposed by the isochoric irrotational motion of a source or a sink on the axis.

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School of Engineering Science & Mechanics
Georgia Institute of Technology
Atlanta

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