THERMOELASTIC DEFORMATIONS OF THE EARTH'S LITHOSPHERE: A MATHEMATICAL MODEL

PAOLO LANZANO

U.S. Naval Research Laboratory, Washington, D.C., U.S.A.

(Received 12 March, 1985)

Abstract. We examine the problem of the thermoelastic deformation of a spherical Earth with constant elastic parameters heated from within by the spontaneous decay of radiogenic elements.

The problem consists of the simultaneous solution of the Navier–Stokes equation and the heat conduction equation. We reach an integrodifferential equation which we solve by means of the Laplace transform and the Green's function approach.

We obtain analytic solutions for the temperature distribution and radial deformation as infinite series of functions of the radial distance and time, depending also on a sequence of eigenvalues. We provide particular solutions for the case when the two specific heats C_p and C_v are approximately equal.

We believe that our analytic results are applicable to the study of the oceanic lithosphere deformations. Our approach could be successfully applied to ascertain the deformation according to other regimes of internal heating.

1. Introduction

When Earth accumulated from the primeval cloud from which the solar system was formed about 4.5×10^9 years ago it brought within its mass a certain fraction of radioactive elements. The radiogenic heat released by their decay was bound to spread throughout the entire mass by conduction and convection and eventually escape by radiation through its surface.

We are referring here to the radioactive elements of the Uranium class (K^{40}, Th^{232}, U^{235}, U^{238}) which possess a half-life of the order of 10^9 to 10^{10} years. This is comparable with the age of the solar system and it means that the interior of the Earth has been warming up secularly and more likely that it will do so for several thousand million years (see Urey (1955) and Alfvén and Arrhenius (1974)).

Both stars and planets derive their internal energy from nuclear sources: the stars from the fusion of light elements, the planets from the disintegration of heavy radioactive elements. Whereas the physical conditions prevailing in the stellar interiors are sufficiently extreme to influence the rate of energy production, the rate of spontaneous disintegration of the radioactive material in the planetary interiors is totally unaffected by the prevailing conditions due to the insulation of thick layers of silicate mantles exhibiting the properties of viscoelastic bodies. Planets do not shine because of their silicate mantles, however they do radiate thermal energy into space.

The two processes, radiative cooling through the surface and radiogenic heating at the interior, must entail certain mechanical consequences. Any secular change in their temperature profile must bring about a corresponding contraction or dilatation of the

Earth, Moon, and Planets **34** (1986) 283–304.

material. Radial strain will arise that might exceed the strength of the rocks and thus affect the structure of the respective layers. Stresses arising from heating and cooling of the planetary masses are large enough to alter even their size.

In many earlier studies, the temperature field has been considered independent of the corresponding strains. This is of course a rough approximation. In fact, any change in the amount of heat in a volume element of the body will give rise to strain and stress; conversely, loading of the body and corresponding strain will produce a temperature field. The coupling of temperature and strain fields constitutes the essence of thermoelasticity.

The purpose of this paper is to study the thermoelastic stress generated in the Earth's lithosphere because of the heat of the radiogenic sources and to take into account the compression caused by the self-gravitation of the various layers in contrasting the thermal expansion.

To obtain an analytic solution for our problem, we make a number of simplifying assumptions. We are assuming: (1) a spherical Earth, (2) constant density ϱ, (3) constant elastic parameters λ, μ, (4) that the sources of radiogenic heat are concentrated at the center of the configuration, (5) hydrostatic equilibrium before heating, and (6) that both the displacement vector $\mathbf{u}$ and the temperature T are functions of the radial distance r and time t alone; i.e., $\mathbf{u}(r, t)$ and $T(r, t)$.

We make use of the following equations: (1) Navier–Stokes equation safeguarding the conservation of momentum, and (2) the equation of heat flow in solids which ensures the conservation of energy. We are ignoring possible convection currents which at present are considered to be plausible whenever the conductive temperature gradient is larger than the adiabatic gradients of silicate rocks.

A number of analytic solutions exist in the form of infinite series that provide temperature distribution within a sphere according to certain regimes of heat generation (see Carslaw and Jaeger, 1959, pp. 242–246). Two such solutions, one due to Allan (1956) the other to Lowan (1935) have been used by Kopal (1963) to examine the problem of lunar and terrestrial planets interiors. In a more recent publication (Kopal, 1966), a new solution was formulated by Kopal which he applied to the lunar case. In a later paper Kopal (1968) provided a general formulation of the problem for the case of variable elastic parameters, but no further explicit solutions were furnished there.

2. Navier–Stokes Equations

The thermodynamic variables which describe a volume element of the body are: temperature T_1 and components ε_{ij} of the strain tensor.

The temperature consists of the sum of two terms

$$T_1(r, t) = T_0(r) + T(r, t). \tag{1}$$

$T_0(r)$ represents the initial state of the body in which both strains and stresses vanish; it is also called the steady state equilibrium temperature. Since we assume that before stressing our configuration is in hydrostatic equilibrium, $T_0(r)$ will be balanced by the radial pressure $P(r)$. $T(r, t)$ represents the changes in temperature because of deformations, internal heat sources and secular cooling due to the escape of internal heat into space. This is the temperature which will appear within any linearized equation of motion which describes a variation from an initial state.

In the presence of thermal effects, the stress–strain relations can be written as

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \left[\lambda(\nabla \cdot \mathbf{u}) - \frac{\alpha}{\beta} T\right] \delta_{ij}, \tag{2}$$

see, e.g., Boley and Weiner (1960), p. 249 or Nowacki (1962), p. 39; where σ_{ij} are the components of the stress tensor, ε_{ij} are the components of the strain tensor, δ_{ij} are the Kronecker deltas, λ, μ are the Lamé parameters for an isothermal deformation, which we assume to be constant, and $\mathbf{u}(r, t)$ is the displacement vector;

$$\alpha = -\frac{1}{\varrho}\left(\frac{\partial \varrho}{\partial T}\right)_P \tag{3}$$

is the coefficient of volume thermal expansion in $(\mathrm{deg})^{-1}$, and

$$\beta = \frac{1}{k} = \frac{1}{\varrho}\left(\frac{\partial \varrho}{\partial P}\right)_T = \frac{3}{3\lambda + 2\mu} \tag{4}$$

is the coefficient of isothermal compression in $\mathrm{cm}^2/\mathrm{dyne}$, which is also the inverse of the bulk parameter k.

From Equation (2) we can determine the equation governing the momentum of the displacement vector $\mathbf{u}$. This is the Navier–Stokes equation

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u} + \mathbf{F} - \frac{\alpha}{\beta}(\nabla T) = \varrho\ddot{\mathbf{u}}, \tag{5}$$

which takes into account the temperature field (see Nowacki, 1962, p. 41), where ϱ is the density of the material, and $\mathbf{F}$ is the resultant of the applied forces.

Since we are assuming hydrostatic equilibrium, we have

$$\mathbf{F} = -\varrho\mathbf{g} = \nabla P, \tag{6}$$

where $P(r)$ is the purely radial pressure.

Both the temperature profile $T(r, t)$ and displacement vector $\mathbf{u}(r, t)$ are assumed to depend only on radial distance r and time t. No dependence on latitude and longitude will be considered in the present formulation.

Because of the previous assumptions, it will be

$$\nabla \times \mathbf{u} \equiv 0,$$

and since in our considerations we can safely neglect the secular variation of the acceleration ($\ddot{\mathbf{u}} \equiv 0$), the Navier–Stokes equation can be written as

$$(\lambda + 2\mu)\beta\nabla(\nabla \cdot \mathbf{u}) = \alpha\nabla T - \beta\nabla P.$$

Introducing the non-dimensional radial displacement

$$\xi = \frac{1}{r} u(r, t), \tag{7}$$

we reach the final form

$$\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^3\xi)\right] = \frac{\alpha}{\beta(\lambda + 2\mu)}\nabla T - \frac{1}{\lambda + 2\mu}\nabla P. \tag{8}$$

The solution of this equation must remain finite at $r = 0$ and satisfy the boundary condition

$$(1 - \sigma)r\frac{\partial \xi}{\partial r} + (1 + \sigma)\xi = 0 \tag{9}$$

at the outermost surface $r = R$; this represents the vanishing of the purely radial component of the stress tensor. In Equation (9), σ is the non-dimensional Poisson ratio

$$\sigma = \frac{\lambda}{2(\lambda + \mu)}. \tag{10}$$

Due to the linearity of Equation (8), we can represent its solution as the sum of two terms

$$\xi(r, t) = \xi_1(r, t) + \xi_2(r). \tag{11}$$

The first term is due to the temperature gradient, the second to the pressure gradient.

Let us first deal with the term $\xi_2(r)$. It satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^3\xi_2)\right] = -\frac{1}{\lambda + 2\mu}\nabla P = Cr, \tag{12}$$

where

$$C = \frac{4\pi G\bar{\varrho}^2}{3(\lambda + 2\mu)} \tag{13}$$

is a constant depending on the mean density of the sphere and the constant Lamé parameters. The solution of Equation (12) is readily obtained as

$$\xi_2(r) = \frac{C}{10}r^2 + C_1 + \frac{C_2}{r^3},$$

where C_1, C_2 are two arbitrary constants. We choose $C_2 = 0$ because $\xi_2(0)$ must remain finite, and the boundary condition at $r = R$, Equation (9), gives rise to

$$C_1 = -\frac{CR^2}{10}\left(\frac{3 - \sigma}{1 + \sigma}\right).$$

We get

$$\xi_2(x) = \frac{CR^2}{10}\left(x^2 - \frac{3 - \sigma}{1 + \sigma}\right), \tag{14}$$

where

$$x = r/R \tag{15}$$

is the fractional radius. Since for any known material $\sigma \leqslant 1/2$, it will be

$$\xi_2(x) < 0,$$

i.e., a compression due to self-gravitation. However, the component

$$\varepsilon_{rr} = \frac{\mathrm{d}}{\mathrm{d}r}(r\xi_2)$$

of the strain tensor can be seen to vanish within the body at the fractional radius

$$\bar{x}^2 = \frac{3 - \sigma}{3(1 + \sigma)} < 1. \tag{16}$$

ε_{rr} is negative between zero and $\bar{x}$, positive beyond that point.

We next consider the non-dimensional radial displacement ξ_1 due to heating. It satisfies the equation

$$\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^3\xi_1)\right] = \frac{\alpha}{\beta(\lambda + 2\mu)}\frac{\partial T}{\partial r}. \tag{17}$$

If we neglect the forcing term, we can find a solution to the homogeneous equation

$$r\frac{\partial^2\xi_1}{\partial r^2} + 4\frac{\partial\xi_1}{\partial r} = 0$$

in the form

$$\xi_1 = r^c,$$

with the parameter c satisfying the indicial equation $c(c + 3) = 0$. The solution of the homogeneous equation is then

$$\xi_1 = C_1 + C_2 r^{-3}.$$

The solution of Equation (17) with the boundary conditions given by Equation (9) can be obtained by the variation of parameters method, whereby the arbitrary constants C_1 and C_2 will be replaced by appropriate functions of r for the purpose of accommodating the forcing term and boundary condition.

The procedure easily yields

$$\xi_1(r, t) = \frac{\alpha}{\beta(\lambda + 2\mu)}\left[\frac{1}{r^3}\int_0^r T(r, t)r^2\,\mathrm{d}r + \right.$$

$$\left. + \frac{2}{R^3}\left(\frac{1 - 2\sigma}{1 + \sigma}\right)\int_0^R T(r, t)r^2\,\mathrm{d}r\right]. \tag{18}$$

This is the fundamental relation which describes the radial deformation once the temperature distribution has been ascertained. ξ_1 remains finite at $r = 0$. We must note here that in obtaining Equation (18) we have assumed

$$T(R, t) = 0, \tag{19}$$

that is to say the vanishing of the temperature at the outer surface of the sphere. This assumption does not constitute any restriction with respect to the class of models for which

$$T(R, t) = \text{constant},$$

because in our case the temperature profile $T(r, t)$ should then be augmented by the constant temperature at $r = R$.

For physical reasons, we can safely assume that $T(R, t)$ should remain constant.

3. Heat Conduction Equation

We proceed by considering the fundamental equation of heat conduction

$$\frac{\partial h}{\partial t} = \kappa \nabla^2 T + W, \tag{20}$$

where h is the absorbed heat per unit volume in erg/cm^3, κ is the heat conduction coefficient in erg/(s · cm · deg), and W is the amount of heat generated in a unit volume per unit time in erg/(cm^3 · s).

Conservation of energy dictates that any increment in the absorbed heat be balanced by a variation of the internal energy and stress related energy

$$\mathrm{d}h = \mathrm{d}U - \sigma_{ij}(\mathrm{d}\varepsilon_{ij}), \tag{21}$$

where

$$U(T_1, \varepsilon_{ij})$$

is the internal energy per unit volume; it is a function of the total temperature and the strain field. The summation convention applies to the repeated indices in Equation (21) and in subsequent equations.

The specific entropy of the body can then be written as

$$\mathrm{d}S = \frac{\mathrm{d}h}{T_1} = \frac{1}{T_1}\left[\frac{\partial U}{\partial T_1}\,\mathrm{d}T_1 + \left(\frac{\partial U}{\partial \varepsilon_{ij}} - \sigma_{ij}\right)(\mathrm{d}\varepsilon_{ij})\right].$$

Entropy being a total differential, it will entail the equality of the two following partial derivatives

$$\frac{\partial}{\partial T_1}\left[\frac{1}{T_1}\left(\frac{\partial U}{\partial \varepsilon_{ij}} - \sigma_{ij}\right)\right] = \frac{1}{T_1}\frac{\partial^2 U}{\partial T_1 \partial \varepsilon_{ij}}.$$

Expanding the above expression and simplifying we get

$$\frac{1}{T_1}\left(\frac{\partial U}{\partial \varepsilon_{ij}} - \sigma_{ij}\right) = -\frac{\partial \sigma_{ij}}{\partial T_1} = \frac{\alpha}{\beta}\,\delta_{ij}.$$

The last equality is a consequence of Equation (2). Thus we have

$$dS = \frac{1}{T_1}\frac{\partial U}{\partial T_1}\,dT_1 + \frac{\alpha}{\beta}\,d(\varepsilon_{ij}\delta_{ij}).$$

Now

$$\varepsilon_{ij}\delta_{ij} = \varepsilon_{ii} = \nabla \cdot \mathbf{u}$$

is the divergence of the displacement vector, and when $\varepsilon_{ij} = 0$ (i.e., when no strain exists), Equation (21) makes it clear that

$$\frac{\partial U}{\partial T_1} = \frac{\partial h}{\partial T_1} = \varrho C_v,$$

where C_v is the specific heat per unit mass when no strain field exists. This is equivalent to saying that a constant volume be maintained. The dimension of C_v is erg/(g · deg). The entropy becomes

$$dS = \varrho C_v \frac{dT_1}{T_1} + \frac{\alpha}{\beta}\,d(\nabla \cdot \mathbf{u}).$$

Assuming ϱ and C_v to be constant, we can integrate the above expression starting from the state corresponding to $T = 0$ and $\nabla \cdot \mathbf{u} = 0$ up to a generic state and get

$$S = \varrho C_v \ln (T_1/T_0) + \frac{\alpha}{\beta}(\nabla \cdot \mathbf{u}).$$

Since $T_1 = T_0 + T$, expansion of the expression

$$\ln\left(\frac{T_1}{T_0}\right) = \ln\left(\frac{T_0 + T}{T_0}\right) = \ln\left(1 + \frac{T}{T_0}\right) \cong \frac{T}{T_0} + \ldots,$$

gives rise to an approximate representation for the specific entropy

$$S \cong \varrho C_v \frac{T}{T_0} + \frac{\alpha}{\beta}\nabla \cdot \mathbf{u}.$$

The amount of absorbed heat will be

$$h = T_0 S \cong \varrho C_v T + \frac{\alpha}{\beta} T_0(r)(\nabla \cdot \mathbf{u}).$$

Substituting into Equation (20), we can finally write

$$\varrho C_v \frac{\partial T}{\partial t} = \kappa \nabla^2 T - \frac{\alpha}{\beta} T_0(r) \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) + W, \tag{22}$$

see also Biot (1956). When the heat sources consist of the spontaneous decay of a number of radioactive elements, we can specifically write

$$W = \sum_j \varrho \varepsilon_j \exp(-\lambda_j t). \tag{23}$$

Here, the subscript j extends to the four elements of the Uranium family, ε_j is the thermal energy of the source in erg/(g · s), and λ_j is the decay constant of the radioactive element in s^{-1}.

The term containing the divergence of the displacement vector is the only link to the Navier–Stokes equation. Since we take into accounts its time derivative, the only term that matters is the $\xi_1(r, t)$, given by Equation (18).

Let us combine Equations (22) and (23) and rewrite them as

$$\frac{\partial T}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) - \frac{\gamma - 1}{\alpha}\frac{\partial}{\partial t}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \xi_1)\right] + \sum_j \phi_j \exp(-\lambda_j t), \tag{24}$$

where, $K = \kappa/\varrho C_v$ is the thermal diffusivity in cm^2/s, $\gamma = C_p/C_v$ is the ratio of the two specific heats, $\phi_j = \varepsilon_j/C_v$ is the rate of temperature increase in deg/s, and

$$\frac{\gamma - 1}{\alpha} = \frac{\alpha T_0(r)}{\varrho \beta C_v}$$

is a relation that can be proved to be valid in thermodynamics (see, e.g., Ubbelohde, 1952, p. 13); it is also related to $\gamma_1 = \alpha/\varrho\beta C_v$ which is the Grüneisen parameter.

In solving the above equation, we must assume an initial temperature distribution

$$T(r, 0) \equiv f(r), \tag{25}$$

and a distribution at the outer surface which, with physical confidence, we can assume to be constant. As mentioned earlier, there is no loss of generality by taking

$$T(R, t) \equiv 0, \tag{26}$$

because to our result $T(r, t)$ we can add

$$T(R, t) \equiv \text{constant} \neq 0.$$

Note that should it be plausible to assume $\gamma = 1$, the displacement term ξ_1 will not appear in the energy equation.

When we replace within Equation (24) the expression for $\xi_1(r, t)$ provided by Equation (18) we get

$$A^2 r \frac{\partial T}{\partial t} + B^2 r \int_0^R r^2 \frac{\partial T}{\partial t}\,\mathrm{d}r = \frac{\partial^2 (rT)}{\partial r^2} + \sum_j \frac{r}{K}\phi_j \exp(-\lambda_j t), \tag{27}$$

where we have denoted

$$A^2 = \frac{1}{K}\left[1 + \frac{\gamma - 1}{\beta(\lambda + 2\mu)}\right],$$

$$B^2 = \frac{6}{KR^3}\left(\frac{1 - 2\sigma}{1 + \sigma}\right)\frac{(\gamma - 1)}{\beta(\lambda + 2\mu)}. \tag{28}$$

These expressions should be considered functions of $T_0(r)$ via the γ; however, due to the limited variation of this quantity, both A^2 and B^2 shall be considered constants in the future developments, greatly facilitating our quest for an analytic solution.

4. Solution of an Integrodifferential Equation

Equation (27) is an integro-differential equation for a function of two variables. Its dependence upon the time variable can be characterized as a Fredholm integral equation. We shall solve Equation (27) by making use of the Laplace transform in order to eliminate the time variable and concentrate on the resulting equation in the radial variable.

To simplify the execution of the Laplace transform, we shall introduce a new variable which leads to a vanishing initial value. This goal can be achieved by switching from T to a new variable V defined by

$$rT(r, t) = V(r, t) + F(r), \tag{29}$$

where

$$F(r) = rT(r, 0) = rf(r). \tag{30}$$

The equation will transform into

$$A^2 \frac{\partial V}{\partial t} + rB^2 \int_0^R r \frac{\partial V}{\partial t} \mathrm{d}r = \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 F}{\partial r^2} + \sum_j \frac{r}{K} \phi_j \exp(-\lambda_j t).$$

The initial and boundary conditions become:

$$V(0, t) = 0 \qquad \text{because } F(0) = 0,$$

$$V(r, 0) = 0$$

because $rT(r, 0) = F(r)$ as given by Equation (30), and

$$V(R, t) = R[T(R, t) - f(R)];$$

and this will also vanish because

$$f(R) = T(R, 0) = T(R, t) \equiv 0.$$

We apply next the Laplace transform

$$L\{V(r, t)\} = \int_0^\infty V(r, t) \exp(-\eta t) \,\mathrm{d}t = Y(r, \eta)$$

(see Widder, 1946). Recalling that

$$L\left\{\frac{\partial V}{\partial t}\right\} = \eta Y(r, \eta) - V(r, 0)$$

and

$$L\{\exp(-\lambda_j t)\} = \frac{1}{\eta + \lambda_j},$$

we reach the equation

$$Y'' + \alpha^2 Y = -\alpha^2 \frac{B^2}{A^2} r \int_0^R rY(r) \,\mathrm{d}r - S(r, \eta). \tag{31}$$

Here primes denote derivatives with respect to r and we have set

$$\alpha^2 = -\eta A^2$$

$$S(r, \eta) = \frac{1}{\eta} F'' + \frac{r}{K} \sum_j \frac{\phi_j}{\eta + \lambda_j}. \tag{32}$$

This parameter α^2, which will give rise to a sequence of eigenvalues α_n, should not be confused with the variable α defined by Equation (3). The boundary conditions are easily obtainable as

$$Y(0, \eta) = Y(R, \eta) = 0. \tag{33}$$

Equation (31) is still an integrodifferential equation but contains only one variable; the term S is the forcing function.

Let us first study the homogeneous equation obtained by neglecting the forcing function

$$Y'' + \alpha^2 Y = -\alpha^2 \frac{B^2}{A^2} r \int_0^R rY(r)\, \mathrm{d}r, \tag{34}$$

with same boundary conditions $Y(0) = Y(R) = 0$. We wish to investigate for which values of the parameter α we get solutions not identically zero.

For this purpose we write

$$C = \frac{B^2}{A^2} \int_0^R rY(r)\, \mathrm{d}r \tag{35}$$

and solve

$$Y'' + \alpha^2 Y = -\alpha^2 Cr.$$

The primitive function is

$$Y = -Cr + M \sin(\alpha r) + N \cos(\alpha r).$$

The conditions $Y(0) = 0$ leads immediately to $N = 0$; the other two conditions $Y(R) = 0$ and Equation (35) give rise to the homogeneous linear system

$$-CR + M \sin(R\alpha) = 0$$

$$\left(\frac{1}{3} + \frac{A^2}{B^2 R^3}\right)(R\alpha)^2 (CR) - M \sin(R\alpha)[1 - (R\alpha)\cot(R\alpha)] = 0$$

for the determination of CR and $M \sin(R\alpha)$. The determinant of the coefficients is

$$\Delta_1 \equiv 1 - (R\alpha)\cot(R\alpha) - (R\alpha)^2 \left(\frac{1}{3} + \frac{A^2}{B^2 R^3}\right).$$

We shall obtain only the zero solution corresponding to $C = 0$, $M = 0$ unless we choose the parameter α to satisfy the transcendental equation $\Delta_1 = 0$, i.e.,

$$1 - (R\alpha)\cot(R\alpha) = (R\alpha)^2 \left(\frac{1}{3} + \frac{A^2}{B^2 R^3}\right), \tag{36}$$

where A, B are the constants given by Equation (28). If this be the case, the only valid condition is

$$M = CR/\sin(R\alpha).$$

Denoting by α_n the solutions of Equation (36), the eigensolutions of Equation (34) can be expressed as

$$Y_n(r, \alpha_n) = C_n\left[-r + \frac{R\sin(r\alpha_n)}{\sin(R\alpha_n)}\right]. \tag{37}$$

Our next objective will be to attain the general solution of Equation (31) by means of the Green function approach. We first must investigate how to extend to the integrodifferential equation in question the classical Green function theory which has only been developed for ordinary linear differential equations.

We are going to show that for any ξ such that

$$0 \leqslant \xi \leqslant R$$

we can obtain a unique continuous function $K(r, \xi)$ which satisfies the integro-differential equation

$$K''(r, \xi) + \alpha^2 K(r, \xi) = -\alpha^2 \frac{B^2}{A^2} r \int_0^R rK(r, \xi)\,\mathrm{d}r, \tag{38}$$

and boundary conditions

$$K(0, \xi) = 0, \qquad K(R, \xi) = 0$$

for values of α which do not belong to a subset of eigenvalues. Furthermore, in each of the two subintervals (0 to ξ) and (ξ to R), the function $K(r, \xi)$ is of class C'' (i.e., has continuous second derivatives); and at $r = \xi$, K is continuous but its derivative K' is discontinuous in such a way that

$$\lim_{\varepsilon\to 0} \left.\frac{\partial K(r, \xi)}{\partial r}\right|_{\xi-\varepsilon}^{\xi+\varepsilon} = -1.$$

To prove the above statement, let us denote by

$$K_1(r, \xi) = -C_1 r + M_1\sin(\alpha r) + N_1\cos(\alpha r)$$

the Green function in the interval 0 to ξ; similarly by

$$K_2(r, \xi) = -C_2 r + M_2\sin(\alpha r) + N_2\cos(\alpha r)$$

the Green function for the interval ξ to R. We must prove that for generic values of ξ and α we can determine the six coefficients C_1, C_2; M_1, M_2; N_1, N_2 in a unique way from the conditions which serve as the definition of the Green function:

$$K_1(0, \xi) = 0, \qquad K_2(R, \xi) = 0;$$

$$K_1(\xi, \xi) = K_2(\xi, \xi) \quad \text{(continuity of } K \text{ at } r = \xi);$$

$$\lim_{\varepsilon\to 0}\,[K_2'(\xi + \varepsilon, \xi) - K_1'(\xi - \varepsilon, \xi)] = -1.$$

Because K_1 and K_2 must be solutions of Equation (38) in each subinterval, we must add the two conditions

$$\int_0^{\xi} rK_1(r, \xi)\,\mathrm{d}r = \frac{A^2}{B^2}\,C_1;$$

$$\int_{\xi}^{R} rK_2(r, \xi)\,\mathrm{d}r = \frac{A^2}{B^2}\,C_2;$$

according to Equation (35).

For a better understanding of the situation, we can define K_1 to vanish identically in (ξ to R) and K_2 to vanish identically in (0 to ξ). Then we are dealing with an equivalent equation

$$(K_1 + K_2)'' + \alpha^2(K_1 + K_2) = -\alpha^2\frac{B^2}{A^2}\,r\left[\int_0^{\xi} rK_1\,\mathrm{d}r + \int_{\xi}^{R} rK_2\,\mathrm{d}r\right].$$

The condition $K_1(0, \xi) = 0$ yields immediately $N_1 = 0$. The other five conditions will ultimately yield a linear system in the remaining five unknowns (C_1, C_2; M_1, M_2; N_2) of the nonhomogeneous type since the right-hand sides consist of the set $(0, 0, 0, 0, -1)$.

Recalling the elementary integral expressions

$$\int x\sin x\,\mathrm{d}x = \sin x - x\cos x,$$

$$\int x\cos x\,\mathrm{d}x = \cos x + x\sin x,$$

it is a simple matter to write down the linear system in question. We do not think it is necessary to reproduce it here. For further discussion of our results it is however necessary to examine the determinant of the coefficient matrix. Our computations show that this determinant is

$$\begin{aligned}
\Delta(\alpha, \xi) = {} & -\frac{1}{\alpha^2}[\sin(\alpha\xi) - (\alpha\xi)\cos(\alpha\xi)]\times \\
& \times\left\{\left(\frac{R}{\alpha^2} + \xi R^2 - \frac{R^3}{3} - 2\frac{A^2}{B^2}\right)\sin(\alpha R - \alpha\xi) - \right. \\
& - \frac{R}{\alpha}(R - \xi)\cos(\alpha R - \alpha\xi) + \\
& + 2\frac{A^2}{B^2}[(\alpha R) - (\alpha\xi)\cos(\alpha R - \alpha\xi)] + \\
& \left. + \tfrac{1}{3}\alpha R\xi[2\xi^2 - R^2\cos(\alpha R - \alpha\xi)]\right\} + \\
& + \frac{R}{\alpha}\left(\frac{A^2}{B^2} + \frac{\xi^3}{3}\right)[\sin(\alpha R) - (\alpha R)\cos(\alpha R)] - \\
& - \alpha\sin(\alpha R)\left(\frac{A^2}{B^2} + \frac{\xi^3}{3}\right)\left(\frac{R^3}{3} - \frac{\xi^3}{3} + \frac{A^2}{B^2}\right).
\end{aligned} \tag{39}$$

This expression clearly does not vanish identically in α and ξ. This proves that for values of α and ξ which are not roots of the equation

$$\Delta(\alpha, \xi) = 0,$$

we obtain a unique Green function $K_1 + K_2$ for our integrodifferential equation.

Before leaving this topic, let us briefly discuss here the particular case of the Green function when we choose $\xi = R$, because it will be useful in future developments. The previous conditions yield

$$K_1(0, R) = 0; \qquad K_1(R, R) = K_2(R, R) = 0;$$

$$C_2 = 0; \qquad \int_0^R rK_1(r, R)\,\mathrm{d}r = \frac{A^2}{B^2} C_1;$$

$$\Delta(\alpha, R) = \frac{A^2}{B^2}\frac{R}{\alpha}\sin(R\alpha)\left[1 - (R\alpha)\cot(R\alpha) - (R\alpha)^2\left(\frac{1}{3} + \frac{A^2}{B^2R^3}\right)\right].$$

Clearly, $K_1(r, R)$ must coincide with the continuous solution $Y(r, \alpha)$. However, since we are choosing $\alpha \neq \alpha_n$ so that $\Delta(\alpha, R) \neq 0$, $K_1(r, R)$ will reduce identically to zero in the interval between zero and R. The same can be said about its derivative. Thus, we can write

$$K_1(r, R) \equiv 0; \qquad \frac{\partial K_1(r, R)}{\partial r} \equiv 0. \tag{40}$$

Next we want to prove that the function

$$Y(r, \alpha) = \int_0^R K(r, \xi)S(\xi, \eta)\,\mathrm{d}\xi \tag{41}$$

satisfies the original integro-differential equation and its boundary conditions. For this purpose, let us rewrite the above equation as

$$Y(r, \alpha) = \int_0^r K_2(r, \xi)S(\xi, \eta)\,\mathrm{d}\xi + \int_r^R K_1(r, \xi)S(\xi, \eta)\,\mathrm{d}\xi. \tag{42}$$

This is so because in the first integral we have $0 \leqslant \xi \leqslant r$ and thus K_2 is the appropriate representation of K in that interval; similarly, for the second integral it will be $r \leqslant \xi \leqslant R$ and consequently K_1 is the appropriate representation for K. From Equation (42), it follows that $Y(0, \alpha) = 0$ because $K_1(0, \xi) = 0$ and that $Y(R, \alpha) = 0$ because $K_2(R, \xi) = 0$.

To establish the differential relation which is satisfied by Equation (41), let us differentiate Equation (42) twice with respect to the radial distance and find that

$$\begin{aligned} Y''(r, \alpha) &= \int_0^r K_2''S\,\mathrm{d}\xi + \int_r^R K_1''S\,\mathrm{d}\xi + [K_2'(r, r) - K_1'(r, r)]S(r, \eta) \\ &= \int_0^R K''(r, \xi)S(\xi, \eta)\,\mathrm{d}\xi - S(r, \eta) \end{aligned}$$

where

$$\eta = -\alpha^2/A^2.$$

It follows then, using Equation (41), that

$$Y''(r, \alpha) + \alpha^2 Y(r, \alpha) = \int_0^R [K''(r, \xi) + \alpha^2 K(r, \xi)] S(\xi, \eta)\, \mathrm{d}\xi - S(r, \eta).$$

Because $K(r, \xi)$ satisfies Equation (38), the integral which appears in the right-hand side of the above expression can be written as

$$-\alpha^2 \frac{B^2}{A^2} r \int_0^R \left[\int_0^R r K(r, \xi)\, \mathrm{d}r \right] S(\xi, \eta)\, \mathrm{d}\xi.$$

Interchanging the order of integration we obtain

$$-\alpha^2 \frac{B^2}{A^2} r \int_0^R r \left[\int_0^R K(r, \xi) S(\xi, \eta)\, \mathrm{d}\xi \right] \mathrm{d}r = -\alpha^2 \frac{B^2}{A^2} r \int_0^R r Y(r, \alpha)\, \mathrm{d}r,$$

the last step being a consequence of Equation (41). Thus it has been proved that the function defined by Equation (41) satisfies the equation

$$Y''(r, \alpha) + \alpha^2 Y(r, \alpha) = -\alpha^2 \frac{B^2}{A^2} r \int_0^R r Y(r, \alpha)\, \mathrm{d}r - S(r, \eta)$$

which indeed is the original Equation (31) with $\alpha^2 = -\eta A^2$.

It is clear then that it is legitimate to proceed and use the Green function approach in the solution to our integrodifferential equation. Although the procedure which was exemplified above for the existence and uniqueness of the Green function could be used to obtain an analytical representation of the same, we shall use a more expeditious approach based on the theory of integral equations. This we are going to discuss briefly in what follows.

We consider two sequences of functions

$$Y_n(r, \alpha_n^2); \qquad Z_n(r, \alpha_n^2)$$

defined in the interval $(0, R)$ and depending upon the parameter α_n^2, and assume that these two sequences:

(1) are orthogonal in said interval: i.e.,

$$\int_0^R Y_n(r, \alpha_n^2) Z_m(r, \alpha_m^2)\, \mathrm{d}r = 0 \quad \text{for } n \neq m,$$

(2) are normalized to unit: i.e.,

$$\int_0^R Y_n(r, \alpha_n^2) Z_n(r, \alpha_n^2)\, \mathrm{d}r = 1, \quad \text{and}$$

(3) satisfy differential systems with boundary conditions, which we symbolize respectively by $S_y = 0$ and $S_z = 0$.

Under the above assumptions, it can be proved that the Green function $K(r, \xi; \alpha^2)$ of the system $S_y = 0$ and the Green function $G(r, \xi\ \alpha^2)$ for $S_z = 0$, evaluated for a value of $\alpha^2 \neq \alpha_n^2$, can be written as

$$K(r, \xi; \alpha^2) = \sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2) Z_n(\xi, \alpha_n^2)}{\alpha_n^2 - \alpha^2};$$

$$G(r, \xi; \alpha^2) = \sum_{n=1}^{\infty} \frac{Z_n(r, \alpha_n^2) Y_n(\xi, \alpha_n^2)}{\alpha_n^2 - \alpha^2} \equiv K(\xi, r; \alpha^2). \tag{43}$$

This result is a direct consequence of Hilbert's "bilinear formula" and has been formulated in equivalent forms by Kowalewski (1930), Ince (1927) and Tricomi (1963). In order to be able to use these results, we must ascertain a new set of functions depending on the same eigenvalues and which turns out to be orthogonal to the original set.

Let us consider, as an initial trial, the set

$$Z = D \sin (r\beta), \quad \text{with } \beta \neq \alpha.$$

We get

$$\int_0^R Y(r, \alpha^2) Z(r, \beta^2)\, \mathrm{d}r = CD \int_0^R \left[-r + \frac{R \sin (r\alpha)}{\sin (R\alpha)} \right] \sin (r\beta)\, \mathrm{d}r$$

$$= CD \frac{\sin (R\beta)}{\beta^2(\alpha^2 - \beta^2)} \{\beta^2[1 - (R\alpha) \cot (R\alpha)] - $$

$$- \alpha^2[1 - (R\beta) \cot (R\beta)]\}.$$

We know that

$$1 - (R\alpha) \cot (R\alpha) = (R\alpha)^2 \left(\frac{1}{3} + \frac{A^2}{B^2 R^3} \right),$$

which is Equation (36); if we impose that

$$1 - (R\beta) \cot (R\beta) = (R\beta)^2 \left(\frac{1}{3} + \frac{A^2}{B^2 R^3} \right),$$

i.e., that $\beta \neq \alpha$ is also a solution of Equation (36), or equivalently that $\beta = \alpha_m$ is also an eigenvalue but $\neq \alpha_n$, then the orthogonality condition will be identically satisfied.

Thus we have shown that the two sequences of functions

$$Y_n(r, \alpha_n^2) = C_n \left[-r + \frac{R \sin (r\alpha_n)}{\sin (R\alpha_n)} \right],$$

$$Z_m(r, \alpha_m^2) = D_m \sin (r\alpha_m) \tag{44}$$

are orthogonal provided that α_n, α_m be two different roots of the same Equation (36).

In order to normalize these two sets to unity, i.e. have

$$\int_0^R Y_n(r, \alpha_n^2) Z_n(r, \alpha_n^2)\, \mathrm{d}r = 1,$$

we must choose the constants C_n and D_n in such a way that

$$(C_n D_n)^{-1} = \int_0^R \left[-r + \frac{R \sin (r\alpha_n)}{\sin (R\alpha_n)} \right] \sin (r\alpha_n) \, \mathrm{d}r =$$

$$= \frac{R^2}{2 \sin (R\alpha_n)} \left[1 - \frac{2 \sin^2(R\alpha_n)}{(R\alpha_n)^2} + \frac{\sin (2R\alpha_n)}{2R\alpha_n} \right]. \tag{45}$$

To conclude, we can say that the Green function

$$K(r, \xi; \alpha^2)$$

valid for Equation (31) with a value of α differing from an eigenvalue is obtainable by using Hilbert's bilinear formula and is furnished by Equations (43), (44), (36), and (45). Let us also note that the differential system with boundary conditions satisfied by Z_n is simply

$$Z_n'' + \alpha_n^2 Z_n = 0, \qquad Z_n(0) = 0. \tag{46}$$

5. Analytic Expressions for Temperature Distribution and Radial Deformation

We now make use of Hilbert's formula for the Green function according to the formulation developed in the previous section. Recalling Equations (41), (43) and (32) we are in a position to write

$$Y(r, \eta) = \sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2)}{\alpha_n^2 - \alpha^2} \left\{ \frac{1}{\eta} \int_0^R F''(\xi) Z_n(\xi, \alpha_n^2) \, \mathrm{d}\xi + \right.$$

$$\left. + \frac{1}{K} \left(\sum_j \frac{\phi_j}{\eta + \lambda_j} \right) \int_0^R \xi Z_n(\xi, \alpha_n^2) \, \mathrm{d}\xi \right\}, \tag{47}$$

where K is the thermal diffusivity and

$$\eta = -\alpha^2 / A^2$$

is the parameter introduced through the Laplace transform. We evaluate by parts the first integral which appears in the above formula to obtain

$$\int_0^R F''(\xi) Z_n(\xi) \, \mathrm{d}\xi = (Z_n F')_0^R - (Z_n' F)_0^R + \int_0^R F Z_n'' \, \mathrm{d}\xi$$

$$= Z_n(R) F'(R) - Z_n'(R) F(R) -$$

$$- \alpha_n^2 \int_0^R F(\xi) Z_n(\xi) \, \mathrm{d}\xi.$$

This is so because: $Z_n(0) = 0$; $F(0) = 0$ due to the fact that $F(r) = rf(r)$; finally $Z_n'' = -\alpha_n^2 Z_n$ due to Equation (46).

We substitute the above expression into Equation (47) and note that because of Equations (43) and (40) we have

$$\sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2) Z_n(R, \alpha_n^2)}{\alpha_n^2 - \alpha^2} = K(r, R; \alpha) \equiv 0,$$

similarly

$$\sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2) Z_n'(R, \alpha_n^2)}{\alpha_n^2 - \alpha^2} = K'(r, R; \alpha) \equiv 0.$$

We are then left with

$$Y(r, \eta) = -\sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2)}{\alpha_n^2 + \eta A^2} \frac{\alpha_n^2}{\eta} \int_0^R F(\xi) Z_n(\xi, \alpha_n^2)\, \mathrm{d}\xi$$
$$+ \frac{1}{K} \sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2)}{\alpha_n^2 + \eta A^2} \left(\sum_j \frac{\phi_j}{\eta + \lambda_j} \right) \int_0^R \xi Z_n(\xi, \alpha_n^2)\, \mathrm{d}\xi.$$

The two fractional expressions in η can be decomposed by a partial fraction procedure. It is easy to verify that

$$\frac{\alpha_n^2}{\eta} \frac{1}{\alpha_n^2 + \eta A^2} = \frac{1}{\eta} - \frac{1}{\eta + (\alpha_n/A)^2},$$

$$\frac{1}{\alpha_n^2 + \eta A^2} \frac{1}{\eta + \lambda_j} = \frac{1}{A^2} \left[\frac{1}{\eta + \lambda_j} - \frac{1}{\eta + (\alpha_n/A)^2} \right] \frac{1}{(\alpha_n/A)^2 - \lambda_j}.$$

We therefore obtain

$$Y(r, \eta) = -\frac{1}{\eta} \sum_{n=1}^{\infty} Y_n(r, \alpha_n^2) \int_0^R F(\xi) Z_n(\xi, \alpha_n^2)\, \mathrm{d}\xi +$$
$$+ \sum_{n=1}^{\infty} \frac{Y_n(r, \alpha_n^2)}{\eta + (\alpha_n/A)^2} \int_0^R F(\xi) Z_n(\xi, \alpha_n^2)\, \mathrm{d}\xi$$
$$+ \frac{1}{KA^2} \sum_{n=1}^{\infty} \left[\sum_j \left(\frac{1}{\eta + \lambda_j} - \frac{1}{\eta + (\alpha_n/A)^2} \right) \frac{\phi_j Y_n(r, \alpha_n^2)}{(\alpha_n/A)^2 - \lambda_j} \right] \times$$
$$\times \int_0^R \xi Z_n(\xi, \alpha_n^2)\, \mathrm{d}\xi.$$

Let us remark here that the first summation appearing in the above equation is simply a representation of $F(r)$ in terms of the orthogonal sets of functions Y_n, Z_n. In fact, assume the representation

$$F(r) = \sum_{n=1}^{\infty} C_n Y_n(r)$$

of $F(r)$ in terms of Y_n. Using the property of the orthogonal sets

$$\int_0^R Y_n(r) Z_m(r)\, \mathrm{d}r = \delta_{n,m},$$

we can write

$$\int_0^R F(r) Z_m(r)\, \mathrm{d}r = \sum_{n=1}^{\infty} C_n \int_0^R Y_n(r) Z_m(r)\, \mathrm{d}r = C_m.$$

It then follows that

$$F(r) = \sum_{n=1}^{\infty} Y_n(r) \int_0^R F(\xi) Z_n(\xi)\, d\xi. \tag{48}$$

Next we note that

$$L^{-1}\left(\frac{1}{\eta + \alpha}\right) = \exp(-\alpha t); \qquad L^{-1}\left(\frac{1}{\eta}\right) = 1.$$

We are now ready to perform the inverse Laplace transform to retrieve the temperature distribution:

$$T(r, t) = f(r) + \frac{1}{r} V(r, t) = f(r) + \frac{1}{r} L^{-1}\{Y(r, n)\}.$$

Upon recalling the explicit expressions for the functions Y_n, Z_n and the condition imposed on their multipliers C_n, D_n by the unit normalization, we can write our final result as follows:

$$T(r, t) = \sum_{n=1}^{\infty} F_n(r) G_n(t), \tag{49}$$

where

$$F_n(r) = \frac{2\left[\frac{R}{r}\sin(r\alpha_n) - \sin(R\alpha_n)\right]}{1 - 2\frac{\sin^2(R\alpha_n)}{(R\alpha_n)^2} + \frac{\sin(2R\alpha_n)}{2R\alpha_n}} \tag{50}$$

is a non-dimensional function of the fractional radius

$$x = r/R,$$

and

$$G_n(t) = \frac{1}{R^2}\left(\int_0^R rf(r)\sin(r\alpha_n)\, dr\right)\exp\left[-\left(\frac{\alpha_n}{A}\right)^2 t\right] + \\ + \sum_j \left(\frac{B^2}{3} + \frac{A^2}{R^3}\right)\frac{\sin(R\alpha_n)}{B^2}\frac{\phi_j}{(K\alpha_n^2 - KA^2\lambda_j)} \times \\ \times \left\{\exp(-\lambda_j t) - \exp\left[-\left(\frac{\alpha_n}{A}\right)^2 t\right]\right\}, \tag{51}$$

is a function of time whose dimension is deg.

A few remarks are appropriate at this stage of our considerations:

(1) $T(0, t)$ is finite because $T(r, t)$ depends on the limit of $\sin(\alpha r)/r$ as r approaches zero:

(2) $T(R, t) = 0$ because $F_n(R) = 0$;

(3) $T(r, 0)$ reduces to $f(r)$; this can be seen by using the representation of $rf(r)$ in terms of the two sets of orthogonal functions as expressed by Equation (48);

(4) the dependence of $T(r, t)$ on time is of an exponential character.

The numerical difficulties that must be overcome in order to reach results from the above expression consist of: (1) evaluation of the eigenvalues α_n as solutions of

Equation (36), and (2) evaluation of the integral of $rf(r)\sin(r\alpha)$ once an initial temperature profile $f(r)$ has been agreed upon. In this regard, we deem it adequate to choose as initial temperature distribution a quadratic function in r/R:

$$T(r, 0) = f(r) \equiv f_0 + f_1\left(1 - \frac{r^2}{R^2}\right), \tag{52}$$

where f_0, f_1 are given constants whose dimension is deg. Recalling that

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x,$$

we can easily evaluate our integral and obtain

$$\begin{aligned} \frac{1}{R^2}\int_0^R rf(r)\sin(r\alpha_n)\,dr &= \left[f_0 - 2f_1 + \frac{6f_1}{(R\alpha_n)^2}\right]\frac{\sin(R\alpha_n)}{(R\alpha_n)^2} - \\ &\quad - \left[f_0 + \frac{6f_1}{(R\alpha_n)^2}\right]\frac{\cos(R\alpha_n)}{R\alpha_n}. \end{aligned} \tag{53}$$

Now that we have attained an analytical representation for the temperature, we use Equation (18) to retrieve an expression for the non-dimensional radial deformation ξ_1. The operations required are only integration with respect to r, thus the functions $G_n(t)$ will not be affected.

Elementary integrations give:

$$\begin{aligned} \xi_1(r, t) &= \frac{2\alpha}{\beta(\lambda + 2\mu)}\sum_{n=1}^{\infty}\left\{\frac{(R/r)^3}{(R\alpha_n)^2}[\sin(r\alpha_n) - (r\alpha_n)\cos(r\alpha_n)] - \right. \\ &\quad \left. - \tfrac{1}{3}\sin(R\alpha_n) + 2\left(\frac{1 - 2\sigma}{1 + \sigma}\right)\frac{\sin(R\alpha_n)}{B^2}\frac{A^2}{R^3}\right\} \times \\ &\quad \times \frac{G_n(t)}{1 - 2\dfrac{\sin^2(R\alpha_n)}{(R\alpha_n)^2} + \dfrac{\sin(2R\alpha_n)}{2R\alpha_n}}. \end{aligned} \tag{54}$$

One can easily verify that $\xi_1(r, t)$ remains finite at $r = 0$.

6. Approximate Solution for the Earth's Mantle

For the silicates that constitute the Earth's mantle the average value of the quantity $\alpha^2/\varrho\beta C_v$ can be taken to be about 2×10^{-5} (deg)$^{-1}$, see, e.g., Birch (1961) and Kopal (1963). Assuming then the steady state temperature $T_0(r)$ to be about 2000°, we can write

$$\gamma \cong 1 + 4 \times 10^{-2} = 1.04.$$

As a first order of approximation, we shall take $\gamma = 1$ which, due to Equation (28), entails the specific relations:

$$B^2 = 0; \qquad A^2 K = 1.$$

The conduction equation, Equation (24), will then simplify to

$$\frac{\partial T}{\partial t} = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial r}\right) + \sum_j \phi_j \exp\left(-\lambda_j t\right),$$

and does not depend on the displacement ξ_1. We shall also consider that the initial temperature distribution $T(r, 0)$ be given by Equation (52) as a quadratic expression in the fractional radius.

Let us rewrite Equation (36) which defines the eigenvalues as

$$\frac{\sin(R\alpha_n) - (R\alpha_n)\cos(R\alpha_n)}{(R\alpha_n)^2} = \frac{\sin(R\alpha_n)}{B^2}\left(\frac{B^2}{3} + \frac{A^2}{R^3}\right). \tag{55}$$

We see that the limit of this equation when B^2 approaches zero is $\sin(R\alpha_n) = 0$, whose solutions are

$$R\alpha_n = n\pi, \quad (n = 0, 1, 2, \ldots).$$

We can now evaluate our particular solution by taking the limit of Equations (49) and (54) when B^2 approaches zero and $R\alpha_n$ approaches $n\pi$. For this purpose, let us note first that

$$\sin(R\alpha_n) = 0; \qquad \cos(R\alpha_n) = (-1)^n$$

$$r\alpha_n = \frac{r}{R}(R\alpha_n) = n\pi x.$$

Next, from Equation (53) we get

$$\frac{1}{R^2}\int_0^R r f(r)\sin(r\alpha_n)\,\mathrm{d}r = \frac{(-1)^{n+1}}{n\pi}\left[f_0 + \frac{6f_1}{(n\pi)^2}\right],$$

and from Equation (55), we see that

$$\lim_{\substack{B^2\to 0\\(R\alpha_n)\to n\pi}}\left[\frac{\sin(R\alpha_n)}{B^2}\right] = \frac{R^3}{A^2}\lim_{(R\alpha_n)\to n\pi}\left[\frac{\sin(R\alpha_n) - (R\alpha_n)\cos(R\alpha_n)}{(R\alpha_n)^2}\right] =$$

$$= \frac{R^3}{A^2}\frac{(-1)^{n+1}}{n\pi}.$$

The temperature distribution in degrees and the non-dimensional deformation can be written in terms of time and fractional radius $x = r/R$ as

$$T(x, t) = \sum_{n=1}^{\infty} 2(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi x} H_n(t);$$

$$\xi_1(x, t) = \frac{2\alpha}{\beta(\lambda + 2\mu)} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}}{(n\pi x)^3} [\sin(n\pi x) - (n\pi x)\cos(n\pi x)] + \right.$$

$$\left. + \frac{2}{(n\pi)^2} \left(\frac{1 - 2\sigma}{1 + \sigma} \right) \right\} H_n(t); \tag{56}$$

where

$$H_n(t) = \left(f_0 + \frac{6f_1}{n^2\pi^2} \right) \exp\left[-K \left(\frac{n\pi}{R} \right)^2 t \right] +$$

$$+ \sum_j \frac{\phi_j}{K(n\pi/R)^2 - \lambda_j} \left[\exp(-\lambda_j t) - \exp\left(-K \frac{n^2\pi^2 t}{R^2} \right) \right] \tag{57}$$

has the dimension of deg.

7. Discussion of the Results and Conclusions

We have obtained a solution for the thermoelastic deformations of a spherical Earth with constant elastic parameters when the heat sources consist of the spontaneous decay of various radioactive elements emanating from the center of the configuration.

We have dealt with the Navier–Stokes equation and the heat conduction equation and have shown that these two equations are generally interrelated unless the two specific heats (C_p and C_v) of the material have the same value. We have solved the Navier–Stokes equation and ascertained the radial contraction of the spherical Earth due to its self-gravitation, and its radial expansion caused by a known temperature profile.

We have solved simultaneously for the two fundamental equations and, having eliminated the time variable by the use of the Laplace transform, we have reached an integrodifferential equation (IDE) for the temperature distribution as function of the radial distance. The forcing terms of this IDE consist of the initial temperature distribution within the sphere and the radiogenic sources. We have found solutions to the IDE without forcing terms in the form of simple functions depending on a sequence of eigenvalues; these eigenvalues can be obtained by solving a transcendental equation depending on the thermal diffusivity of the material.

We have reached the general solution of our problem by integrating the product of the Green function and forcing function over the whole radial distance. To do that, we first have shown that a unique Green function exists for our problem for each point of discontinuity and secondly we have made recourse to classical results of analysis to express this Green's function as an infinite series of the product of two orthogonal sets of functions.

The final results of our investigation are analytic expressions for the temperature distribution and radial deformations as infinite series of the radial distance and of the time. Each term of these infinite series depends on one of the eigenvalues.

The temperature decreases outwardly as $1/r$, the radial deformations as $1/r^2$; the decay in time is of the exponential type depending on the half-lives of the radiogenic elements and the thermal diffusivity of the material. The numerical summation of these series, once the eigenvalues have been ascertained, should not present any difficulty because of the decreasing character of the constituent functions.

We have applied this analytical model to the Earth's mantle. The two specific heats of the constituent silicates are here close enough to be considered approximately equal. This has allowed us to obtain at once the sequence of the eigenvalues and has given rise to simpler infinite series for temperature and deformations.

Numerical evaluation of our analytical results remains to be implemented; this will be done in a future work to be considered complementary to the present one. We have, however, compared our solution to Kopal's fundamental treatment of heat conduction through the lunar interior and deformations of the lunar crust (cf. Kopal, 1966; pp. 116–121). Our solution appears to be more comprehensive than Kopal's results, but compatible with Kopal's formulation; and most of Kopal's numerical work presented there can be verified using our equations for appropriate choices of the parameters.

To conclude. We believe that the continuous radiation of thermal energy due to the spontaneous decay of radiogenic material should be taken into account in the overall study of the oceanic lithosphere, on top of which local disturbances like magma chambers and hot spots should be added. Hence our new model for evaluating heating rates and related stresses should be of interest in many practical problems of geological deformation. Our analytical solutions should be considered a guideline to the overall problem of numerical integration over a layered Earth. Furthermore, our approach can be applied with slight modifications to models where the heat sources are of a nature different from the one considered in this paper.

References

Alfvén, H. and Arrhenius, G.: 1974, *Astrophys. Space Sci.* **29**, 63.
Allan, D. W.: 1956, *Amer. Math. Monthly* **63**, 315.
Biot, M. A.: 1956, *J. Appl. Phys.* **27**, 250.
Birch, F.: 1961, *Geophys. J.R. Astron. Soc.* **4**, 295.
Boley, B. A. and Weiner, J. H.: 1960, *Theory of Thermal Stresses*, Wiley, New York.
Carslaw, H. S. and Jaeger, J. C.: 1959, '*Conduction of Heat in Solids*', Second Edition, Oxford Univ. Press.
Ince, E. L.: 1927, '*Ordinary Differential Equations*', Longmans Green, London.
Kopal, Z.: 1963, *Icarus* **2** 376.
Kopal, Z.: 1966, '*An Introduction to the Study of the Moon*, D. Reidel Publishing Company, Dordrecht, Holland.
Kopal, Z.: 1968, *Icarus* **9**, 231.
Kowalewski, G.: 1930, '*Integralgleichungen*, W. de Gruyter, Berlin.
Lowan, A. N.: 1935, *Amer. J. Math.* **57**, 174.
Nowacki, W.: 1962, *Thermoelasticity*, Pergamon Press, London.
Tricomi, F. G.: 1957, *Integral Equations*, Interscience Publishers, New York.
Ubbelohde, A. R.: 1952, *Thermodynamical Principles*, Oxford Univ. Press.
Urey, H. C.: 1955, *Proc. U.S. Nat. Acad. Sci.* **41**, 127
Widder, D. V.: 1946, *Laplace Transform*, Princeton Univ. Press.