COLLISIONAL EVOLUTION OF ROTATING, NON-IDENTICAL PARTICLES

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Abstract. Hämeen-Anttila's (1984) analytical treatment of self-gravitating collisional particle disks is extended to include the particle spin. The equations derived for the coupled evolution of random velocities and spins indicate that friction and surface irregularity typically reduce the local velocity dispersion. Friction, and especially irregularity, also transfer significant amounts of random kinetic energy, E_{kin}, to rotational energy, E_{rot}. The equilibrium ratio $E_{rot}/E_{kin} = 2\beta/(14 - 5\beta)$ if the particles are spherical, and $2(1 + \alpha)/7$ if they are irregular but frictionless, α and β being the coefficients of restitution and friction. These results are not only exact for identical, non-gravitating mass points, but are rather accurate even if finite size, self-gravitational forces, or size distribution are included. Applications to the dynamics of Saturn's rings suggest that the inclusion of rotation is able to reduce the geometrical thickness of the layer of centimeter-sized particles to about one half, at most. Large particles are less affected.

1. Introduction

The collisional evolution of a cloud of particles revolving around a central body has been widely studied during the last fifteen years. The early theoretical studies (Goldreich and Tremaine, 1978; Hämeen-Anttila, 1978) as well as numerical simulations (e.g. Trulsen, 1972a, b; Brahic, 1977; Lukkari, 1978) concentrated on the evolution of Keplerian systems consisting of non-rotating, identical particles. In later analytical studies the effects of gravitational encounters were also taken into account (Cuzzi *et al.*, 1979; Hämeen-Anttila, 1983), as well as the influence of satellite perturbations (Borderies *et al.*, 1983), or an arbitrary axially symmetric potential (Hämeen-Anttila, 1983). Analytical treatments have also been extended to the case of a distribution of particle sizes (Hämeen-Anttila, 1984; Stewart *et al.*, 1984).

These studies have clearly demonstrated the importance of the elasticity model. If the coefficient of restitution, α, is a constant independent of the impact velocity, the system either rapidly disperses through growing eccentricities and inclinations, or flattens to a near monolayer state. On the other hand, if α is a decreasing function of impact velocity, the system may attain a stable equilibrium state, resulting from the balance between energy loss in partially inelastic impacts, and the gain of energy from the viscous shear of differentially rotating particles (Goldreich and Tremaine, 1978; Hämeen-Anttila, 1978). These results have been confirmed by numerical simulations (Hämeen-Anttila and Lukkari, 1980). This more realistic collision model

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has also been used in the recent simulations of collisions and gravitational encounters between non-identical particles (Lukkari and Salo, 1984; Salo and Lukkari, 1984; Salo, 1985).

The collisional evolution of rotating particles has received little attention. Friction enables the transfer of energy between translational and rotational degrees of freedom, thus modifying the collisional balance (Hämeen-Anttila, 1978). Brahic and Hénon (1977) have simulated the effects of friction, but did not take into account the feedback of energy from rotation. Recently Shukmann (1984) has analytically studied the influence of rotation on the collisional evolution of identical particles, while Clairemidi (1984) has reported some simulation results performed with particles having different radii and masses. However, in these simulations α was treated as a constant. The aim of the present study is to derive the equations governing the local evolution of the dispersion of random linear and rotational velocities, for systems of non-identical particles. The treatment is based on Hämeen-Anttila's (1984) more general investigation, and allows for a velocity dependent α, and the gravitational interactions between particles, as well as general axisymmetric central potential. However, external perturbations due to satellites have not been considered. The results of corresponding numerical simulations are published in a separate paper (Salo, 1987).

2. Impact Model

The collisional changes in velocity and spin vectors are assumed to be determined by three factors: elasticity, friction, and irregularity in the impact point. Only binary impacts are considered, and the colliding particles are distinguished by primed and unprimed symbols, while the post-collisional quantities are denoted by subscript 1. The derivation of basic equations is similar to that of Hämeen-Anttila (1978) except that the irregularity is introduced in a more precise manner, and the particles are allowed to have different radii, σ, and masses, μ.

Let $\mathbf{R}$ and $\mathbf{R}'$ be the radius vectors of the particles at the instant of impact (Figure 1a). The pre-collisional velocity difference at the contact point is given by

$$\mathbf{v}_{\text{coll}} = \mathbf{v} - (\sigma\boldsymbol{\omega} + \sigma'\boldsymbol{\omega}') * \mathbf{c}, \tag{1}$$

where $\mathbf{v} = \dot{\mathbf{R}}' - \dot{\mathbf{R}}$ is the relative velocity of the particle centres, ω stands for the spin vector, and $\mathbf{c} = (\mathbf{R}' - \mathbf{R})/(\sigma + \sigma')$ for the unit vector in the direction joining the centres of colliding particles. In the case of smooth, spherical particles, $\mathbf{c}$ would also be perpendicular to the tangent plane of impact. The influence of small irregularities is taken into account by allowing the actual normal vector of the tangent plane, $\mathbf{c}_t$, differ from the vector $\mathbf{c}$ (Figure 1b). At the same time it is assumed that the particles can otherwise be treated as spheres, as was already done in Equation (1). The change in the relative velocity during impact is caused by the pressure and the friction at the impact area. The former reduces the perpendicular component of the

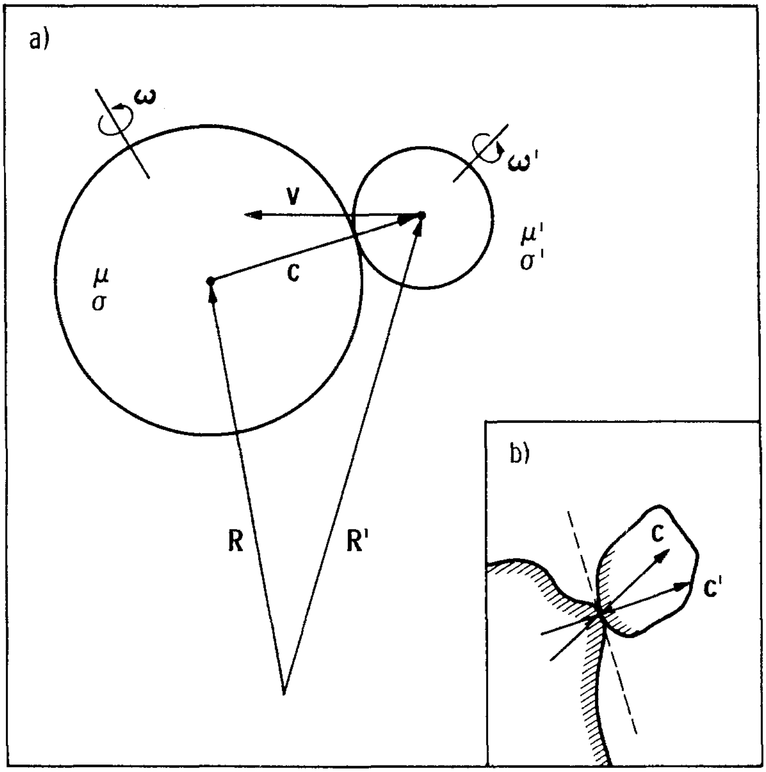

Fig. 1. Schematic representation of the impact model.

relative velocity, while the latter affects the component of $\mathbf{v}_{coll}$ in the tangent plane of impact. Hence,

$$(\mathbf{v}_1)_{coll} = -\alpha\, \mathbf{c}_t\mathbf{c}_t \cdot \mathbf{v}_{coll} + (1-\beta)\mathbf{c}_t * (\mathbf{v}_{coll} * \mathbf{c}_t), \tag{2}$$

where α is the coefficient of restitution and β describes the action of friction. Totally elastic impacts correspond to $\alpha = 1$, while $\alpha = 0$ implies a completely inelastic collision; $\beta = 0$ correspond to frictionless impacts while if $\beta = 1$ the whole tangential component of $\mathbf{v}_{coll}$ is lost.

In order to relate the 6 post-collisional velocity components and 6 spin components to the corresponding quantities before the impact, a similar number of conditions must be specified. The above expression for the $(\mathbf{v}_1)_{coll}$ implies three conditions, and six more are provided by the conservation laws of linear and angular momentum,

$$\mu\dot{\mathbf{R}} + \mu'\dot{\mathbf{R}}' = \mu\dot{\mathbf{R}}_1 + \mu'\dot{\mathbf{R}}'_1, \tag{3}$$

$$\mu\mathbf{R}*\dot{\mathbf{R}} + \mu'\mathbf{R}'*\dot{\mathbf{R}}' + (2/5)(\mu\sigma^2\omega + \mu'\sigma'^2\omega') = \mu\mathbf{R}*\dot{\mathbf{R}}_1 + \mu'\dot{\mathbf{R}}'*\dot{\mathbf{R}}'_1 + 2/5(\mu\sigma^2\omega_1 + \mu'\sigma'^2\omega'_1), \tag{4}$$

where the coefficient 2/5 corresponds to homogeneous spheres. The remaining three relations are obtained by determining how the change in spin is distributed between the particles. The force at the point of contact is opposite but of equal magnitude for both particles, and therefore the torques are proportional to the radii. Since the moment of inertia is proportional to $\sigma^2\mu$, it follows that

$$\mu\sigma(\omega_1 - \omega) = \mu'\sigma'(\omega_1' - \omega'). \tag{5}$$

After some manipulation, Equations (1)–(5) yield

$$\dot{\mathbf{R}}_1 - \dot{\mathbf{R}} = \frac{\mu'}{\mu+\mu'}\left\{\frac{2}{7}[(1+\alpha-\beta)\mathbf{v}_{\text{coll}}\cdot\mathbf{c}_t\mathbf{c}_t + \beta\mathbf{v}_{\text{coll}}] + \right.$$

$$\left. + \frac{5}{7}[(1+\alpha-\beta)\mathbf{v}_{\text{coll}}\cdot\mathbf{c}_t\mathbf{c}_t\cdot\mathbf{cc} + \beta\mathbf{v}_{\text{coll}}\cdot\mathbf{cc}]\right\}, \tag{6}$$

$$\sigma(\omega_1 - \omega) = \frac{\mu'}{\mu+\mu'}\,\frac{5}{7}[(1+\alpha-\beta)\mathbf{v}_{\text{coll}}\cdot\mathbf{c}_t\mathbf{c}*\mathbf{c}_t + \beta\mathbf{c}*\mathbf{v}_{\text{coll}}] \tag{7}$$

The corresponding changes for the other particle are obtained by the multiplication with $-\mu/\mu'$ and μ/μ', respectively. In the case of spherical identical particles these expressions lead to

$$\dot{\mathbf{R}}_1 - \dot{\mathbf{R}} = \frac{1}{2}(1+\alpha)\,\mathbf{v}\cdot\mathbf{cc} + \frac{\beta}{7}[\mathbf{c}*(\mathbf{v}*\mathbf{c}) + \sigma\mathbf{c}*(\omega+\omega')], \tag{8}$$

$$\sigma(\omega_1 - \omega) = \frac{5\beta}{14}[\mathbf{c}*\mathbf{v} - \sigma\mathbf{c}*[(\omega+\omega')*\mathbf{c}]], \tag{9}$$

which are similar to those of Hämeen-Anttila (1978) if we set his $\beta^* = 0$. Shukman's (1984) equations for collisions without slippage correspond to the special case $\beta = 1$. The impact model of Shu *et al.* (1985) assumes that the tangential velocity difference retains its magnitude but is reversed, and therefore it corresponds to the case $\beta = 2$ (notice that their $\mathbf{n}$ corresponds to $-\mathbf{c}$, while $K_e = 2/5$).

The above model for the irregular shape in terms of $\mathbf{c}_t$ is rather crude. Hence the specific choice of $\mathbf{c}_t$ is not very crucial, and we assume that $\mathbf{c}_t$ is obtained from $\mathbf{c}$ by tilting this vector independently in the directions of $\mathbf{c}*(\mathbf{v}*\mathbf{c})/|\mathbf{v}*\mathbf{c}|$ and $(\mathbf{v}*\mathbf{c})/|\mathbf{v}*\mathbf{c}|$ by angles ζ and ξ. To the second order in $k = \sin\zeta$ and $l = \sin\xi$,

$$\mathbf{c}_t = (1 - k^2/2 - l^2/2)\mathbf{c} + k\,\mathbf{c}*(\mathbf{v}*\mathbf{c})/|\mathbf{v}*\mathbf{c}| + l\,(\mathbf{v}*\mathbf{c})/|\mathbf{v}*\mathbf{c}|. \tag{10}$$

If this expression is inserted into Equations (6) and (7) we finally obtain

$$\dot{\mathbf{R}}_1 - \dot{\mathbf{R}} = \frac{\mu'}{\mu+\mu'}\left\{(1+\alpha-\beta)\left\{(1-l^2-k^2)\mathbf{v}\cdot\mathbf{cc} + k\,|\mathbf{v}*\mathbf{c}|\mathbf{c} - \right.\right.$$

$$- \frac{2}{7} [k\, \mathbf{c}*(\mathbf{v}*\mathbf{c})/|\mathbf{c}*\mathbf{v}| + l\, \mathbf{v}*\mathbf{c}/|\mathbf{v}*\mathbf{c}|]\, \mathbf{v} \cdot \mathbf{c} +$$

$$+ \frac{2}{7} [k^2\, \mathbf{c}*(\mathbf{v}*\mathbf{c}) + kl\, \mathbf{v}*\mathbf{c}] +$$

$$\left. + (\sigma\omega + \sigma'\omega') \cdot [k\, \mathbf{c}*\mathbf{v}/|\mathbf{c}*\mathbf{v}| + l\, \mathbf{c}*(\mathbf{v}*\mathbf{c})/|\mathbf{c}*\mathbf{v}|]\, \mathbf{c} \right\} +$$

$$\left. + \frac{2}{7} \beta \mathbf{v} + \frac{5}{7} \beta\, \mathbf{v} \cdot \mathbf{cc} - \frac{2}{7} \beta\, (\sigma\omega + \sigma'\omega')*\mathbf{c} \right\}, \tag{11}$$

$$\sigma(\omega_1 - \omega) = \frac{\mu'}{\mu + \mu'} \frac{5}{7} \left\{ (1 + \alpha - \beta)\, \{\mathbf{v} \cdot \mathbf{c}\, [k\mathbf{c}*\mathbf{v}/|\mathbf{c}*\mathbf{v}| + \right.$$

$$+ l\mathbf{c}*(\mathbf{v}*\mathbf{c})/|\mathbf{c}*\mathbf{v}|] + kl\, \mathbf{c}*(\mathbf{v}*\mathbf{c}) + k^2\, \mathbf{c}*\mathbf{v} -$$

$$- (\sigma\omega + \sigma'\omega') \cdot [k^2(\mathbf{c}*\mathbf{v})(\mathbf{c}*\mathbf{v}) + l^2\mathbf{c}*(\mathbf{v}*\mathbf{c})\mathbf{c}*(\mathbf{v}*\mathbf{c})]/|\mathbf{c}*\mathbf{v}|^2\} +$$

$$\left. + \beta\, [\mathbf{c}*\mathbf{v} - \mathbf{c}*[(\sigma\omega + \sigma'\omega')*\mathbf{c}]] \right\}. \tag{12}$$

3. Construction of Collisional Equations

Hämeen-Anttila (1984) has investigated the general collisional evolution in systems of unequal sized particles, and derived equations for both the evolution of local velocity dispersion and the radial particle flux. In the present study, the simultaneous evolution of rotational states is incorporated into his equations, although attention is restricted to the local evolution of eccentricities and inclinations. Some of the relevant aspects of the general theory are here briefly summarized, for further details, see Hämeen-Anttila (1984).

In an axially symmetric potential field, $U(r, z)$, the orbits of individual particles can be represented with the following truncated series expansions (Hämeen-Anttila, 1983, 1984),

$$\mathbf{R} = \mathbf{r} - \left(\mathbf{rr} + 2\mathbf{r}^*\mathbf{r}^* \sqrt{\frac{U'}{rU'' + 3U'}} \right) \cdot \frac{\mathbf{e}}{r} - \mathbf{N}\, \mathbf{r}^* \cdot \mathbf{l} + \ldots, \tag{13}$$

$$\dot{\mathbf{R}} = \sqrt{\frac{U'}{r}}\left[\mathbf{r}^* + \left(\mathbf{r}^*\mathbf{r} - \mathbf{r}\mathbf{r}^*\,\frac{rU''+U'}{U'}\sqrt{\frac{U'}{rU''+3U'}}\right)\cdot\frac{\mathbf{e}}{r}\right] + $$

$$+\ \mathbf{N}\,\mathbf{r}\cdot\mathbf{l}\,\sqrt{\lambda} + \ldots. \tag{14}$$

The vector $\mathbf{r}$ corresponds to a point moving along a circular reference orbit in the central plane of the system with a constant angular velocity $\sqrt{U'/r}$, while vectors $\mathbf{e}$ and $\mathbf{l}$ are related to the epicyclic deviations of individual particles. For Keplerian systems $\mathbf{e}$ and $\mathbf{l}$ would correspond to vectors pointing to the pericenter and the ascending node, with absolute magnitudes of eccentricity and inclination. The derivatives of the central field, $U' = \partial U/\partial r$, $U'' = \partial^2 U/\partial r^2$, and $\lambda = \partial^2 U/\partial z^2$, are evaluated in the central plane. The unit vector perpendicular to this plane is denoted by $\mathbf{N}$.

The local state of the system can be described by giving the particle density, n, and the dispersion tensors $\mathbf{P} = \overline{\mathbf{ee}} - \bar{\mathbf{e}}\bar{\mathbf{e}}$, and $\mathbf{Q} = \overline{\mathbf{ll}} - \bar{\mathbf{l}}\bar{\mathbf{l}}$ for each r, where the bar refers to the local mean over particles belonging to this coherently moving group. The dispersion of random velocities for pairs of intersecting orbits, $\mathbf{T}$, is related to $\mathbf{P}$ and $\mathbf{Q}$ by

$$\mathbf{T} = \frac{rU''+3U'}{r^3}\left(\frac{\mathbf{r}^*\mathbf{r}}{2}\sqrt{\frac{rU''+3U'}{U'}} - \mathbf{r}\mathbf{r}^*\right)\cdot\mathbf{P}\cdot\left(\frac{\mathbf{r}\mathbf{r}^*}{2}\sqrt{\frac{rU''+3U'}{U'}} - \right.$$

$$\left. -\ \mathbf{r}^*\mathbf{r}\right) + \lambda\,\mathbf{r}\cdot\mathbf{Q}\cdot\mathbf{r}\,\mathbf{N}\mathbf{N}. \tag{15}$$

In the absence of particle accretion or fragmentation, the collisional evolution of dispersion tensors is determined by the equations (Hämeen-Anttila, 1984)

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \int\left[\left(\frac{\partial\bar{\mathbf{e}}}{\partial\dot{\mathbf{R}}}\right)^{\dagger}\cdot W(\dot{\mathbf{R}}\dot{\mathbf{R}})\cdot\left(\frac{\partial\bar{\mathbf{e}}}{\partial\dot{\mathbf{R}}}\right) + \left(\frac{\partial\bar{\mathbf{e}}}{\partial\dot{\mathbf{R}}}\right)^{\dagger}\cdot W(\dot{\mathbf{R}}\mathbf{R})\cdot\left(\frac{\partial\bar{\mathbf{e}}}{\partial\mathbf{R}}\right) + \right.$$

$$\left. +\left(\frac{\partial\bar{\mathbf{e}}}{\partial\mathbf{R}}\right)^{\dagger}\cdot W(\mathbf{R}\dot{\mathbf{R}})\cdot\left(\frac{\partial\bar{\mathbf{e}}}{\partial\dot{\mathbf{R}}}\right)\right]\mathrm{d}p' + \dot{P}, \tag{16}$$

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \int\left[\left(\frac{\partial\bar{\mathbf{l}}}{\partial\dot{\mathbf{R}}}\right)^{\dagger}\cdot W(\dot{\mathbf{R}}\dot{\mathbf{R}})\cdot\left(\frac{\partial\bar{\mathbf{l}}}{\partial\dot{\mathbf{R}}}\right) + \left(\frac{\partial\bar{\mathbf{l}}}{\partial\dot{\mathbf{R}}}\right)^{\dagger}\cdot W(\dot{\mathbf{R}}\mathbf{R})\cdot\left(\frac{\partial\bar{\mathbf{l}}}{\partial\mathbf{R}}\right) + \right.$$

$$\left. \left(\frac{\partial\bar{\mathbf{l}}}{\partial\mathbf{R}}\right)^{\dagger}\cdot W(\mathbf{R}\dot{\mathbf{R}})\cdot\left(\frac{\partial\bar{\mathbf{l}}}{\partial\dot{\mathbf{R}}}\right)\right]\mathrm{d}p' + \dot{\mathbf{Q}}, \tag{17}$$

$$W(\mathbf{XY}) = \frac{\nu}{2}\Bigg[\langle 2(\mathbf{X}_1-\mathbf{X})(\mathbf{Y}_1-\mathbf{Y})-(\mathbf{X}_1-\mathbf{X})(\mathbf{Y}'-\mathbf{Y})-$$

$$-(\mathbf{X}'-\mathbf{X})(\mathbf{Y}_1-\mathbf{Y})\rangle +$$

$$+\langle(\mathbf{X}_1-\mathbf{X})(\mathbf{C}+\mathbf{C}'-2\bar{\mathbf{C}})\rangle\cdot\left(\frac{\partial\bar{\mathbf{Y}}}{\partial\mathbf{C}}\right)+$$

$$+\left(\frac{\partial\bar{\mathbf{X}}}{\partial\mathbf{C}}\right)\cdot\langle(\mathbf{C}+\mathbf{C}'-2\bar{\mathbf{C}})(\mathbf{Y}_1-\mathbf{Y})\rangle\Bigg], \tag{18}$$

$$\nu = \bar{n}'_r\,\pi(\sigma+\sigma')^2 g\sqrt{\frac{8(T+T')}{3\pi}}, \tag{19}$$

$$g = \left[1-\frac{4\pi}{3}\int\bar{n}_r'\sigma'^3\,\mathrm{d}p'\right]^{-1}, \tag{20}$$

$$\bar{n}_r = \frac{n'}{\sqrt{2\pi\mathbf{r}^*\cdot(\mathbf{Q}+\mathbf{Q}')\cdot\mathbf{r}^*}}, \tag{21}$$

where the integrations are carried out over the particle size distribution or some other property distinguishing between particles. The frequency of impacts between different types of particles is denoted with ν, while g represents the filling factor (unity for mass points), and $\bar{n}_r$ is the vertically averaged space density. The quantity T stands for trace $\mathbf{T}$. The vector $\mathbf{C}$ includes the components of $\mathbf{e}$ and $\mathbf{l}$. The operation $\langle\ \rangle$ denotes averaging over impacts,

$$\langle f(\mathbf{v},\mathbf{c})\rangle = \int f(\mathbf{v},\mathbf{c})\,(\mathbf{v}\cdot\mathbf{c})/v\,\mathrm{d}\psi \Big/ \int(\mathbf{v}\cdot\mathbf{c})/v\,\mathrm{d}\psi, \tag{22}$$

where the integration is carried out over those solid angles ψ where impacts are possible, $\mathbf{v}\cdot\mathbf{c}<0$. The derivatives $\partial\mathbf{e}/\partial\mathbf{R}$ etc. follow from Equations (13) and (14),

$$\left(\frac{\partial\bar{\mathbf{e}}}{\partial\mathbf{R}}\right) = -\frac{rU''+U'}{r(rU''+3U')}\frac{\mathbf{rr}}{r^2}-\sqrt{\frac{U'}{(rU''+3U')}}\frac{\mathbf{r}^*\mathbf{r}^*}{r^2},$$

$$\left(\frac{\partial\bar{\mathbf{e}}}{\partial\dot{\mathbf{R}}}\right) = \frac{\sqrt{U'/r'}}{rU''+3U'}\left[\frac{2\mathbf{r}^*\mathbf{r}}{r^2}-\frac{\mathbf{r}^*}{r^2}\sqrt{\frac{rU''+3U'}{U'}}\right], \tag{23}$$

$$\left(\frac{\partial \overline{\mathbf{l}}}{\partial \mathbf{R}}\right) = -\frac{N\mathbf{r}^*}{r^2}, \quad \left(\frac{\partial \overline{\mathbf{l}}}{\partial \dot{\mathbf{R}}}\right) = \frac{N\mathbf{r}}{\sqrt{\lambda}\, r^2},$$

The perturbations corresponding to the precession of apsidal lines and nodes in a general axisymmetric potential are accounted for by Hämeen-Anttila (1984)

$$\dot{\mathbf{P}} = (\mathbf{I}^* \cdot \mathbf{P} - \mathbf{P} \cdot \mathbf{I}^*)(\sqrt{U'/r} - \sqrt{(rU''+3U')/r}), \tag{24}$$

$$\dot{\mathbf{Q}} = (\mathbf{I}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{I}^*)(\sqrt{U'/r} - \sqrt{\lambda}),$$

where $\mathbf{I}^* = (\mathbf{r}^*\mathbf{r} - \mathbf{r}\mathbf{r}^*)/r^2$.

In Hämeen-Anttila's (1984) original work the collisional change of velocity was modelled by Equation (6) with $\mathbf{c}_t = \mathbf{c}$, and $\beta = 0$. In this case, the quantities $W(\dot{\mathbf{R}}\dot{\mathbf{R}})$ etc., consisted of various integrals of $\mathbf{v}$ and $\mathbf{v} \cdot \mathbf{c}$, all of which could be evaluated in terms of $\langle \mathbf{vv} \rangle$ and $\langle v^2 \rangle \mathbf{I}_3$, where $\mathbf{I}_3$ is the three-dimensional unit tensor $(\mathbf{rr} + \mathbf{r}^*\mathbf{r}^*)/r^2 + \mathbf{NN}$. By assuming a Gaussian distribution function for velocities, the mean values were then related to $\mathbf{T}$ and therefore also to $\mathbf{P}$ and $\mathbf{Q}$. Now the rotational state enters into the equation for $\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}$ (Section 2). If we denote $\mathbf{\Omega} = \sigma^2(\overline{\omega\omega} - \overline{\omega}\overline{\omega})$ as the dispersion tensor of spin velocities, it turns out that all the collisional integrals can now be evaluated in terms of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{\Omega}$. The behaviour of $\mathbf{\Omega}$ and $\sigma\bar{\omega}$ can be constructed by generalizing Hämeen-Anttila's (1978) treatment of rotating identical particles to include the dispersion of sizes, giving

$$\frac{d\mathbf{\Omega}}{dt} = \int \nu \left[\langle \sigma^2\omega_1\omega_1 - \sigma^2\omega\omega \rangle - \langle \sigma\omega_1 - \sigma\omega \rangle \overline{\sigma\omega} - \overline{\sigma\omega}\langle \sigma\omega_1 - \sigma\omega \rangle\right] dp', \tag{25}$$

$$\frac{d\sigma\bar{\omega}}{dt} = \int \nu \langle \sigma\omega_1 - \sigma\omega \rangle \, dp'. \tag{26}$$

There is no need to assume any specific functional form for the distribution of spins as long as they are not correlated to random velocities, since there is no difference between $\overline{\omega\omega} - \bar{\omega}\bar{\omega}$ and the average over impacts, $\langle \omega\omega \rangle - \langle \omega \rangle \langle \omega \rangle$.

In the next Section the collisional time derivates are constructed by assuming that the particles can be treated as mass points. In Section 5 the corrections due to finite size are included, while Section 6 considers gravitational interactions between particles.

4. Mass Point Approximation

As long as the velocity difference between colliding particles is dominated by the local dispersion of velocities, and is not greatly affected by the systematic velocity gradient, all the terms proportional to particle radii, except those including $\sigma\omega$ may be ignored. Previous computer simulations of non-rotating particles have shown that this mass point approximation is valid as long as the thickness of the system, measured by $r\sqrt{6\overline{i^2}}$, is larger than a few particle diameters (Hämeen-Anttila and Lukkari, 1980).

Therefore, after setting $\mathbf{R} = \mathbf{R}'$ in Equation (18), the components of W are

$$W(\dot{\mathbf{R}}\dot{\mathbf{R}}) = \frac{\nu}{2}\Bigg[\langle 2(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}) - (\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\dot{\mathbf{R}}' - \dot{\mathbf{R}}) -$$

$$- (\dot{\mathbf{R}}' - \dot{\mathbf{R}})(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\rangle + \langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\mathbf{C} + \mathbf{C}' - 2\overline{\mathbf{C}})\rangle \cdot$$

$$\cdot\left(\overline{\frac{\partial\dot{\mathbf{R}}}{\partial\mathbf{C}}}\right) + \left(\overline{\frac{\partial\dot{\mathbf{R}}}{\partial\mathbf{C}}}\right)^{\dagger} \cdot \langle\mathbf{C} + \mathbf{C}' - 2\overline{\mathbf{C}}\,(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\rangle\Bigg],$$

$$W(\dot{\mathbf{R}}\mathbf{R}) = \frac{\nu}{2}\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\mathbf{C} + \mathbf{C}' - 2\overline{\mathbf{C}})\rangle \cdot \left(\overline{\frac{\partial\mathbf{R}}{\partial\mathbf{C}}}\right), \tag{27}$$

$$W(\mathbf{R}\dot{\mathbf{R}}) = \mathbf{W}(\dot{\mathbf{R}}\mathbf{R})^{\dagger}, \qquad W(\mathbf{R}\mathbf{R}) = 0.$$

By assuming that there is no correlation between rotational and translational velocities the mean values following from Equations (11) and (22) are

$$\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\dot{\mathbf{R}}' - \dot{\mathbf{R}})\rangle = \langle(\dot{\mathbf{R}}' - \dot{\mathbf{R}})(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\rangle =$$

$$= \frac{\mu'}{\mu + \mu'}\,[(1 + \alpha + 2\beta/7) - (1 + \alpha + \beta)(\langle l^2\rangle +$$

$$+ 5/7\,\langle k^2\rangle)]\,\frac{\langle\mathbf{vv}\rangle}{2}, \tag{28}$$

$$\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\mathbf{C} + \mathbf{C}' - 2\overline{\mathbf{C}})\rangle = \langle(\mathbf{C} + \mathbf{C}' - 2\overline{\mathbf{C}})(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\rangle^{\dagger} =$$

$$= \frac{\mu'}{\mu + \mu'}\,[(1 + \alpha + 2\beta/7) - (1 + \alpha + \beta)(\langle l^2\rangle + 5/7\,\langle k^2\rangle)]\,\langle\mathbf{v}(\mathbf{C} + \mathbf{C}' - 2\mathbf{C})\rangle, \tag{29}$$

$$\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\rangle = \left[\frac{\mu'}{\mu + \mu'}\right]^2 \Bigg\{\Bigg[(1 + \alpha + 2\beta/7)^2 - (1 + \alpha - \beta)^2$$

$$\left(\frac{102}{49}\langle l^2\rangle + \frac{6}{7}\langle k^2\rangle\right)$$

$$
- \frac{18}{7}\beta(1+\alpha-\beta)\left(\langle l^2\rangle + \frac{5}{7}\langle k^2\rangle\right)\Bigg]\frac{\langle \mathbf{vv}\rangle}{4} +
$$

$$
+ \Bigg[(1+\alpha-2\beta/7)^2 - (1+\alpha-\beta)^2\left(\frac{86}{49}\langle l^2\rangle + \right.
$$

$$
\left. + \frac{48}{49}\langle k^2\rangle\right) - \frac{10}{7}\beta(1+\alpha-\beta)\left(\langle l^2\rangle + \right.
$$

$$
\left. + \frac{9}{7}\langle k^2\rangle\right)\Bigg]\frac{\langle \mathbf{v}^2\rangle}{12}\mathbf{I}_3 + (1+\alpha-\beta)^2\langle k^2 + l^2\rangle
$$

$$
\times \frac{2\mathbf{I}_3\,\mathrm{tr}(\boldsymbol{\Omega} + \boldsymbol{\Omega}') - (\boldsymbol{\Omega} + \boldsymbol{\Omega}')}{15} +
$$

$$
+ (2\beta/7)^2\,\frac{\mathbf{I}_3\,\mathrm{tr}(\boldsymbol{\Omega} + \boldsymbol{\Omega}') - (\boldsymbol{\Omega} + \boldsymbol{\Omega}')}{3}\Bigg\}. \qquad (30)
$$

Table I lists various collisional integrals, calculated according to Equation (22) needed in the construction of the above, and some later expressions. The above equations also assume that k and l are independent of $\mathbf{c}$ and $\mathbf{v}$, so that all the mean values over k and l can be separated from those over $\mathbf{v}\cdot\mathbf{c}$. Moreover, $\langle k\rangle$, $\langle l\rangle$, and $\langle kl\rangle$ are assumed to vanish.

TABLE I

Collisional integrals calculated according to Equation (22)

$\langle(\mathbf{c}\cdot\mathbf{v})^2\mathbf{cc}\rangle = 1/4\langle\mathbf{vv}\rangle + 1/12\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle\mathbf{v}\cdot\mathbf{ccv}\rangle = 1/2\langle\mathbf{vv}\rangle$
$\langle\mathbf{c}*(\mathbf{v}*\mathbf{c})\rangle = 1/2\langle\mathbf{vv}\rangle$
$\langle\mathbf{c}*\mathbf{v}\,\mathbf{c}*\mathbf{v}\rangle = -1/4\langle\mathbf{vv}\rangle + 1/4\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle\mathbf{c}*(\mathbf{v}*\mathbf{c})\mathbf{v}\cdot\mathbf{cc}\rangle = 1/4\langle\mathbf{vv}\rangle - 1/12\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle\lvert\mathbf{v}*\mathbf{c}\rvert^2\mathbf{cc}\rangle = 1/6\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle(\mathbf{c}\cdot\mathbf{v})^2/\lvert\mathbf{v}*\mathbf{c}\rvert^2\mathbf{c}*(\mathbf{v}*\mathbf{c})\,\mathbf{c}*(\mathbf{v}*\mathbf{c})\rangle = 1/6\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle(\mathbf{c}\cdot\mathbf{v})^2/\lvert\mathbf{v}*\mathbf{c}\rvert^2\mathbf{v}*\mathbf{c}\,\mathbf{v}*\mathbf{c}\rangle = -1/4\langle\mathbf{vv}\rangle + 1/4\langle\mathbf{v}^2\rangle\mathbf{I}_3$
$\langle(\sigma\omega + \sigma'\omega')*\mathbf{c}\,(\sigma\omega + \sigma'\omega')*\mathbf{c}\rangle = 1/3\{\mathrm{tr}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{I}_3 - (\boldsymbol{\Omega} + \boldsymbol{\Omega}')\}$
$\langle(\sigma\omega + \sigma'\omega')\cdot(\mathbf{c}*(\mathbf{v}*\mathbf{c}))/\lvert\mathbf{v}*\mathbf{c}\rvert^2(\sigma\omega + \sigma'\omega')\mathbf{c}*(\mathbf{v}*\mathbf{c})\rangle =$
$= \langle(\sigma\omega + \sigma'\omega')\cdot\mathbf{v}*\mathbf{c}/\lvert\mathbf{v}*\mathbf{c}\rvert^2(\sigma\omega + \sigma'\omega')\mathbf{v}*\mathbf{c}\rangle = 1/15\{2\mathrm{tr}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{I}_3 - (\boldsymbol{\Omega} + \boldsymbol{\Omega}')\}$
$\langle\mathbf{c}*((\sigma\omega + \sigma'\omega')*\mathbf{c})\,\mathbf{c}*((\sigma\omega + \sigma'\omega')*\mathbf{c})\rangle = 1/15\{\mathrm{tr}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{I}_3 + 7(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\}$
$\langle\mathbf{x}*(\sigma\omega*\mathbf{x})\sigma\omega\rangle = 1/3\boldsymbol{\Omega}$, with $\mathbf{x} = \mathbf{c}$ and $\mathbf{x} = \mathbf{c}*(\mathbf{v}*\mathbf{c})/\lvert\mathbf{v}*\mathbf{c}\rvert$

As can be seen from Equation (12) for $\sigma(\omega_1 - \omega)$, $\mathrm{d}\sigma\bar{\omega}/\mathrm{d}t = 0$ for mass point approximation, indicating that randomly oriented particles do not acquire any systematic alignment of spin axis. Therefore, in order to construct $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$, it is sufficient to consider only the term $\sigma^2(\omega_1\omega_1 - \omega\omega)$, which can also be written into form $\sigma\omega(\sigma\omega_1 - \sigma\omega) + (\sigma\omega_1 - \sigma\omega)\sigma\omega + (\sigma\omega_1 - \sigma\omega)(\sigma\omega_1 - \sigma\omega)$. From Equations (12) and (22) it follows that (see Table I)

$$\langle(\sigma\omega_1 - \sigma\omega)(\sigma\omega_1 - \sigma\omega)\rangle = \left[\frac{\mu'}{\mu+\mu'}\right]^2 \left[\frac{5}{7}\right]^2 \left[\left[\beta^2 + \langle k^2\rangle[(1+\alpha)^2 - \beta^2]\right.\right.$$

$$\left.\frac{\langle\mathbf{vv}\rangle}{4} + [3\beta^2 + 2\langle l^2\rangle(1+\alpha-\beta)^2]\frac{\langle\mathbf{v}^2\rangle}{12} + \right.$$

$$\left. + \beta^2\,\frac{7(\boldsymbol{\Omega}+\boldsymbol{\Omega}') + \mathrm{tr}(\boldsymbol{\Omega}+\boldsymbol{\Omega}')\mathbf{I}_3}{15}\right], \tag{31}$$

$$\langle\sigma(\omega_1 - \omega)\sigma\omega\rangle = \langle\sigma\omega\,\sigma(\omega_1 - \omega)\rangle = -\frac{5\mu'}{21(\mu+\mu')}\,[\langle k^2 + l^2\rangle(1+\alpha-\beta) + 2\beta]\,\boldsymbol{\Omega} \tag{32}$$

Before the construction of the final equations for $\mathrm{d}\mathbf{P}/\mathrm{d}t$, $\mathrm{d}\mathbf{Q}/\mathrm{d}t$, and $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$, there are still some expressions which must be approximated. Following Hämeen-Anttila (1984) we take

$$\langle\mathbf{vv}\rangle = \frac{4}{3}(\mathbf{T} + \mathbf{T}'), \tag{33}$$

$$\langle\mathbf{v}(\mathbf{C}+\mathbf{C}' - 2\bar{\mathbf{C}})\rangle \cdot \overline{\left(\frac{\partial\dot{\mathbf{R}}}{\partial\mathbf{C}}\right)} = \frac{4}{3}(\mathbf{T}' - \mathbf{T}) \cdot \left[\mathbf{I}_3 - \right.$$

$$\left. - \frac{(rU'' + U')\mathbf{r}^*\mathbf{r}^* + 2U'\mathbf{rr}}{(rU'' + 3U')r^2}\right]. \tag{34}$$

The above formula for impact velocities would be exact for an isotropic distribution of random velocities, nevertheless it gives a good approximation for more general case.

Equations (15)–(18), (23), and (25)–(34) then disclose that

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \int \left\{\frac{\nu}{9}\left[\frac{\mu'}{\mu+\mu'}\right]^2 \left[\frac{4U'\mathbf{rr} + (rU'' + 3U')\mathbf{r}^*\mathbf{r}^*}{r^3(rU'' + 3U')^2}\left(\alpha_1\,\mathrm{tr}(\mathbf{T}+\mathbf{T}') + \right.\right.\right.$$

$$\left. + \alpha_4\,\mathrm{tr}(\boldsymbol{\Omega}+\boldsymbol{\Omega}')\right) + 3\alpha_2(\mathbf{P}+\mathbf{P}') - \frac{\alpha_5\,U'}{r(rU'' + 3U')^2}\left(\frac{2\mathbf{rr}^*}{r^2} - \frac{\mathbf{r}^*\mathbf{r}}{r^2}\right.$$

$$\sqrt{\frac{rU''+3U'}{U'}}\Bigg) \cdot (\Omega+\Omega') \cdot \left(\frac{2\mathbf{r}^*\mathbf{r}}{r^2} - \right.$$

$$\left. - \frac{\mathbf{r}\mathbf{r}^*}{r^2}\sqrt{\frac{rU''+3U'}{U'}}\right)\Bigg] - \frac{4\nu}{3}\frac{\mu'}{\mu+\mu'}\alpha_3 \mathbf{P}\Bigg\} \mathrm{d}p' + \dot{\mathbf{P}}, \qquad (35)$$

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \int \left\{\frac{\nu}{9}\left[\frac{\mu'}{\mu+\mu'}\right]^2 \left[\left(\alpha_1 \operatorname{tr}(\mathbf{T}+\mathbf{T}') + \alpha_4 \operatorname{tr}(\Omega+\Omega')\right)\frac{\mathbf{rr}}{\lambda r^4} + \right.\right.$$

$$\left.\left. + 3\,\alpha_2 \frac{\mathbf{r}\cdot(\mathbf{Q}+\mathbf{Q}')\cdot\mathbf{r}}{r^2}\right] - \frac{4\nu\mu'}{3(\mu+\mu')}\alpha_3\,\mathbf{r}\cdot\mathbf{Q}\cdot\mathbf{r}\frac{\mathbf{rr}}{\lambda r^2}\right\} \mathrm{d}p' + \dot{\mathbf{Q}},$$

$$(36)$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \int \left\{\nu\left[\frac{5}{7}\right]^2\left[\frac{\mu'}{\mu+\mu'}\right]^2 \left[\frac{-\beta_1}{3}(\mathbf{T}+\mathbf{T}') + \frac{\beta_2}{3}\operatorname{tr}(\mathbf{T}+\mathbf{T}')\mathbf{I}_3 + \right.\right.$$

$$\left.\left. + \beta^2\frac{7(\Omega+\Omega')+\operatorname{tr}(\Omega+\Omega')I_3}{15} - \frac{20\nu\mu'}{21(\mu+\mu')}\beta_3\,\Omega\right\} \mathrm{d}p', \qquad (37)$$

where $\alpha_1-\alpha_5$, $\beta_1-\beta_3$ are shorthand notations for certain expressions of α, β, $\langle l^2\rangle$, and $\langle k^2\rangle$,

$$\begin{aligned}
\alpha_1 &= (1+\alpha-2\beta/7)^2 - (1+\alpha-\beta)^2(86/49\langle l^2\rangle + 48/49\langle k^2\rangle) \\
&\quad - 10/7\beta(1+\alpha-\beta)(\langle l^2\rangle + 9/7\langle k^2\rangle), \\
\alpha_2 &= (1+\alpha+2\beta/7)^2 - (1+\alpha-\beta)^2(102/49\langle l^2\rangle + 6/7\langle k^2\rangle) \\
&\quad - 18/7\beta(1+\alpha-\beta)(\langle l^2\rangle + 5/7\langle k^2\rangle), \\
\alpha_3 &= (1+\alpha+2\beta/7) - (1+\alpha-\beta)(\langle l^2\rangle + 5/7\langle k^2\rangle), \qquad (38)\\
\alpha_4 &= 3[(2\beta/7)^2 + 2/5(1+\alpha-\beta)^2(\langle l^2\rangle + \langle k^2\rangle)], \\
\alpha_5 &= 3[(2\beta/7)^2 + 1/5(1+\alpha-\beta)^2(\langle l^2\rangle + \langle k^2\rangle)], \\
\beta_1 &= \beta^2 + \langle k^2\rangle[(1+\alpha)^2 - \beta^2], \\
\beta_2 &= \beta_1 + 2/3\langle l^2\rangle(1+\alpha-\beta)^2, \\
\beta_3 &= 2\beta + 1/2\langle k^2 + l^2\rangle(1+\alpha-\beta).
\end{aligned}$$

In the construction of these expressions α and β have been treated as constants, and terms containing them have been taken outside the integrals over impacts. Therefore if α and β depend on impact velocity, they must now be considered as certain effective mean values. The same procedure has previously been succesfully used in the case of non-rotating particles (Hämeen-Anttila, 1984), and the comparison to simulations has shown that the resulting accuracy is tolerable, only about 20% at most (Salo, 1985). A simple approximation for the effective α is obtained if its argument is replaced by the mean value over impacts. In the case $\alpha_v = \alpha_v(|\mathbf{c} \cdot \mathbf{v}|)$, the effective α is obtained with

$$\alpha_0 = \alpha_v(\sqrt{\langle(\mathbf{c} \cdot \mathbf{v})^2\rangle}) = \alpha_v(\sqrt{2(T + T')/3}),$$

while if $\alpha_v = \alpha_v(v)$, then

$$\alpha_0 = \alpha_v(\sqrt{\langle v^2 \rangle}) = \alpha_v(\sqrt{4(T + T')/3}).$$

Actually, instead of $\langle(\mathbf{c} \cdot \mathbf{v})^2\rangle$ one should use

$$\langle(\mathbf{c} \cdot \mathbf{v}_{\text{coll}})^2\rangle = 2/3(T + T') + 2/3\langle k^2 \rangle (\Omega + \Omega' - T - T'),$$

and instead of $\langle v^2 \rangle$ the average

$$\langle v_{\text{coll}}^2 \rangle = 4/3(T + T') + 2/3(\Omega + \Omega').$$

5. Corrections Due to Finite Particle Size

The above mass point approximation is valid only if the dispersion of random velocities is much larger than the relative velocity component due to the differential rotation. However, the dense parts of the Saturn's rings are probably very flattened, violating the above assumption. An approximate method was developed by Hämeen-Anttila (1982, 1984) for the inclusion of finite particle size, and his technique is here modified to the case of rotating particles. In order to obtain tolerable expressions irregularity is neglected ($\mathbf{c} = \mathbf{c}_t$).

The velocity difference in impact, caused by the systematic velocity gradient is $\Delta\mathbf{v} = (\sigma + \sigma')\mathbf{c} \cdot \nabla\dot{\mathbf{r}}$, where (Hämeen-Anttila, 1984)

$$\nabla\dot{\mathbf{r}} = [(rU'' + U')\mathbf{r}\mathbf{r}^* - 2U'\mathbf{r}^*\mathbf{r}]/(2r^2\sqrt{rU'}). \tag{39}$$

This contribution to the total impact velocity is assumed to be statistically independent of the velocity difference due to random velocities. Therefore the contributions of finite size to collisional mean values may be calculated from the relation

$$\Delta\langle f(\mathbf{c}, \mathbf{v})\rangle = \int f(\mathbf{c}, \mathbf{v})\Delta\mathbf{v} \cdot \mathbf{c}\,\mathrm{d}\psi \Big/ \int \Delta\mathbf{v} \cdot \mathbf{c}\,\mathrm{d}\psi,\ \Delta\mathbf{v} \cdot \mathbf{c} < 0, \tag{40}$$

corresponding to Equation (22). In the case $\mathbf{c}_t = \mathbf{c}$ Equations (11) and (12) yield for the additional collisional changes

$$\Delta(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}) = \frac{\mu'}{\mu' + \mu}\left[(1 + \alpha - 2\beta/7)\mathbf{c} \cdot \nabla\dot{\mathbf{r}} \cdot \mathbf{c}\mathbf{c} + 2\beta/7\ \mathbf{c} \cdot \nabla\dot{\mathbf{r}}\right], \tag{41}$$

$$\Delta(\sigma\omega_1 - \sigma\omega) = -\frac{\mu'}{\mu' + \mu}\frac{5}{7}\beta(\sigma + \sigma')\,\mathbf{c} \cdot \nabla\dot{\mathbf{r}}{*}\mathbf{c}. \tag{42}$$

TABLE II

Collisional integrals calculated according to Equation (40)

$$\langle \mathbf{c}*\Delta\mathbf{v}\rangle = \frac{\sigma+\sigma'}{5}\,\frac{3U'+rU'}{\sqrt{rU'}}\,\mathbf{N}$$

$$\langle \mathbf{cc}\rangle = \frac{1}{10}\left(4\mathbf{I} + 2\mathbf{NN} - \pi\frac{|rU''-U'|}{rU''-U'}\,\frac{\mathbf{rr}^*+\mathbf{r}^*\mathbf{r}}{r^2}\right)$$

$$\langle \mathbf{c}\cdot\nabla\dot{\mathbf{r}}\cdot\mathbf{c}\,\mathbf{cc}\rangle = \frac{2}{35}\left(\frac{rU''-U'}{\sqrt{rU'}}\,\frac{\mathbf{rr}^*+\mathbf{r}^*\mathbf{r}}{r^2} - \frac{\pi}{8}\,\frac{|rU''-U'|}{rU''-U'}(3\mathbf{I}+\mathbf{NN})\right)$$

$$\langle (\mathbf{c}\cdot\nabla\dot{\mathbf{r}}\cdot\mathbf{c})^2\,\mathbf{cc}\rangle = \frac{1}{35}\,\frac{(rU''-U')^2}{rU'}\left(\frac{4\mathbf{I}+\mathbf{NN}}{9} - \frac{\pi}{8}\,\frac{|rU''-U'|}{rU''-U'}\,\frac{\mathbf{rr}^*+\mathbf{r}^*\mathbf{r}}{r^2}\right)$$

$$\begin{aligned}\langle \mathbf{c}\cdot\nabla\dot{\mathbf{r}}*\mathbf{cc}\cdot\nabla\dot{\mathbf{r}}*\mathbf{c}\rangle = \frac{1}{135rU'}\Big\{&(rU''+U')^2\frac{\mathbf{rr}}{r^2} + 4U'^2\frac{\mathbf{r}^*\mathbf{r}^*}{r^2}\\ &+ 4\,[4U'^2+(rU''+U')^2+2U'(rU''+U')]\,\mathbf{NN}\\ &- \frac{\pi}{2}\,\frac{|rU''-U'\,|U'(rU''+U')}{rU''-U'}\,\frac{\mathbf{rr}^*+\mathbf{r}^*\mathbf{r}}{r^2}\Big\}\end{aligned}$$

Since the calculation of correction factors is based on the omission of random velocities, we can set $\mathbf{C}' = \mathbf{C} = \bar{\mathbf{C}}$ in Equation (18). On the other hand, the difference between $\mathbf{R}$ and $\mathbf{R}'$ must be taken into account. The nonzero corrections for various components of W thus become

$$\begin{aligned}\Delta W(\dot{\mathbf{R}}\dot{\mathbf{R}}) &= \frac{\nu}{2}\,\langle 2\Delta(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\Delta(\dot{\mathbf{R}}_1-\dot{\mathbf{R}}) - \Delta(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\Delta\mathbf{v} - \Delta\mathbf{v}\Delta(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\rangle,\\ \Delta W(\mathbf{R}\dot{\mathbf{R}}) &= \Delta W(\dot{\mathbf{R}}\mathbf{R})^\dagger = -\frac{\nu}{2}\,\langle(\mathbf{R}'-\mathbf{R})\,\Delta(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\rangle.\end{aligned} \tag{43}$$

Equations (16)–(18), (23), and (39)–(43) (see also Table II) then yield the corrections for $\mathrm{d}\mathbf{P}/\mathrm{d}t$ and $\mathrm{d}\mathbf{Q}/\mathrm{d}t$,

$$\begin{aligned}\Delta\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \int \nu\left(\frac{\mu}{\mu+\mu'}\right)^2(\sigma+\sigma')^2\,&[(1+\alpha-2\beta/7)^2\mathbf{S}_1 +\\ &+ 2\beta/7(1+\alpha-2\beta/7)\mathbf{S}_2 + (2\beta/7)^2\,\mathbf{S}\end{aligned}$$

$$\Delta\frac{d\mathbf{Q}}{dt} = \int \left[\nu \left(\frac{\mu}{\mu+\mu'}\right)^2 (\sigma+\sigma')^2 \,[(1+\alpha-2\beta/7)^2 \mathbf{S}_4] - \right.$$

$$\left. - \frac{\nu}{2}\,\frac{\mu'}{\mu+\mu'}\,(\sigma+\sigma')^2\,(1+\alpha-2\beta/7)\mathbf{S}_5 \right] dp'$$

where $\mathbf{S}_1 - \mathbf{S}_5$ are shorthand notations for

$$\mathbf{S}_1 = \frac{1}{35r^2}\left[\frac{rU'-U'}{rU''+3U'}\right]^2 \left[\frac{16}{9}\,\frac{\mathbf{rr}}{r^2} + \frac{4}{9}\,\frac{rU''+3U'}{U'}\,\frac{\mathbf{r^*r^*}}{r^2} - \right.$$

$$\left. - \frac{\pi\,|rU''-U'|}{4(rU''-U')}\sqrt{\frac{rU''+3U'}{U'}}\,\frac{\mathbf{r^*r}+\mathbf{rr^*}}{r^2}\right],$$

$$\mathbf{S}_2 = \frac{4}{35r^2}\,\frac{(rU''-U')(rU''+U')}{(rU''+3U')^2}\left[2\,\frac{\mathbf{rr}}{r^2} - \frac{rU''+3U'}{rU''+U'}\,\frac{\mathbf{r^*r^*}}{r^2} + \right.$$

$$\left. + \frac{3\pi}{16}\,\frac{|U''-U'|}{rU''+U'}\sqrt{\frac{rU''+3U'}{U'}}\,\frac{\mathbf{r^*r}+\mathbf{rr^*}}{r^2}\right],$$

$$\mathbf{S}_3 = \frac{2}{5r^2}\,\frac{(rU''+U')^2}{(rU''+3U')^2}\left[\frac{\mathbf{rr}}{r^2} + \frac{U'(rU''+3U')}{(rU''+U')^2}\,\frac{\mathbf{r^*r^*}}{r^2} - \right.$$

$$\left. - \frac{\pi}{4}\,\frac{rU''+3U'}{rU''+U'}\,\frac{|rU''-U'|}{rU''-U'}\sqrt{\frac{U'}{rU''+3U'}}\,\frac{\mathbf{r^*r}+\mathbf{rr^*}}{2}\right], \qquad (46)$$

$$\mathbf{S}_4 = \frac{1}{35r^2}\,\frac{(rU''-U')^2}{9rU'\lambda}\,\frac{\mathbf{rr}}{r^2},$$

$$\mathbf{S}_5 = \frac{1}{35r^2}\,\frac{|rU''-U'|}{\sqrt{r\lambda U'}}\,\frac{\pi}{8}\,\frac{\mathbf{r^*r}+\mathbf{rr^*}}{r^2},$$

In the case $\beta = 0$ these simplify into the form given by Hämeen-Anttila (1984).

Due to the systematic velocity gradient, $\langle \mathbf{c} * \Delta\mathbf{v} \rangle$ does not vanish (see Table II), and the nonzero expression for $d\sigma\bar{\omega}/dt$ follows from

$$\frac{\mathrm{d}\sigma\bar{\omega}}{\mathrm{d}t} = \int \frac{5\beta}{7} \nu \frac{\mu'}{\mu+\mu'} \left[\frac{\sigma+\sigma'}{5} \frac{3U'+rU''}{\sqrt{rU'}} \mathbf{N} - \right.$$

$$\left. - (\sigma\bar{\omega}+\sigma'\bar{\omega}') \cdot (\mathbf{I}_3 + \langle\mathbf{cc}\rangle) \right] \mathrm{d}p'. \tag{47}$$

The equilibrium solution, $\mathrm{d}\sigma\bar{\omega}/\mathrm{d}t = 0$ is thus determined by

$$\sigma\bar{\omega} \cdot (\mathbf{I}_3 - \langle\mathbf{cc}\rangle) = (3U' + rU'')/\sqrt{rU'}\ \sigma\mathbf{N}/5.$$

In the case of flattened systems, we may use the integral listed in Table II for $\langle\mathbf{cc}\rangle$, yielding

$$\bar{\omega} = (3U' + rU'')/(4\sqrt{rU'})\mathbf{N}. \tag{48}$$

However, this expression for $\langle\mathbf{cc}\rangle$ takes into account only the systematic velocity field. If the system is very thick $\langle\mathbf{cc}\rangle$ might be better approximated by assuming an isotropic distribution of impact velocities, $\langle\mathbf{cc}\rangle = \mathbf{I}_3/3$. This gives a slightly larger estimate for the equilibrium $\bar{\omega}$, namely $\bar{\omega} = 3(3U' + rU'')/(10\sqrt{rU'})\mathbf{N}$.

By assuming that the above equilibrium, $\mathrm{d}\sigma\bar{\omega}/\mathrm{d}t = 0$, is already obtained, the correction terms for $\mathrm{d}\Omega/\mathrm{d}t$ become (Equations (25), (42), and Table II)

$$\Delta\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \int \nu \left[\frac{\mu'}{\mu+\mu'} \right]^2 (\sigma+\sigma')^2 (5\beta/7)^2 \mathbf{S}_6\, \mathrm{d}p',$$

$$\mathbf{S}_6 = \frac{2}{135} \frac{(rU''+U')^2}{rU'} \left\{ \frac{\mathbf{rr}}{r^2} + \left[\frac{2U'}{rU''+U'} \right]^2 \frac{\mathbf{r^*r^*}}{r^2} + \right.$$

$$+ 4\left[1 + \frac{2U'}{rU''+rU'} \left(1 + \frac{2U'}{rU''+U'} \right) \right] \mathbf{NN} - \tag{49}$$

$$\left. - \frac{\pi}{2} \frac{|rU''-U'|}{(rU''-U')} \frac{U'}{rU''+U'} \frac{\mathbf{r^*r} + \mathbf{rr^*}}{r^2} \right\}.$$

These corrections terms, added to the equations for mass point systems give a complete description of the local collisional evolution of rotating particles in the absence of self-gravitational forces. However, a few minor modifications are still needed. In the above formula for ν (Equation (19)) the contribution of the systematic velocity field must be included by adding $3(rU'' - U')^2/70rU'\,(\sigma + \sigma')^2$ to $T + T'$ (Hämeen-Anttila, 1984). The argument of α_0 must undergo the same modification. The proper value of the filling factor g must also be used.

6. Corrections Due to Self-Gravitation

Gravitational encounters correspond to totally elastic impacts, and therefore provide an additional source of energy. Their contribution becomes significant in the case of flattened systems, where the average random velocities are of the same order as the escape velocities of the largest particles (Cuzzi *et al.*, 1979). A quantitative model for the treatment of gravitational encounters was devised by Hämeen-Anttila (1983) who treated them as random fluctuations in the average gravitational field. The inclusion of rotation does not change the gravitational interactions, and the results are not altered in any way. However, since the terms arising from encounters were of the same general form as the collisional terms, Hämeen-Anttila (1984) accounted for them by redefining α, ν, and $(\sigma + \sigma')^2$ in a suitable manner as effective values over both encounters and physical collisions. Although being very elegant, this nomenclature can not be used here because friction and irregularity change the collisional terms. Therefore, we rewrite Hämeen-Anttila's (1984) correction terms in explicit form,

$$\Delta_g \frac{d\mathbf{P}}{dt} = \int \left\{ \frac{4\nu_g}{9} \left[\frac{\mu'}{\mu' + \mu} \right]^2 \left[\frac{4U'\mathbf{rr} + (rU'' + 3U')\mathbf{r}^*\mathbf{r}^*}{r^3(rU'' + 3U')^2} (T + T') + \right.\right.$$
$$\left.\left. + 3(\mathbf{P} + \mathbf{P}') + 9a^2\mathbf{S}_1 \right] - \nu_g \frac{\mu'}{\mu + \mu'} \frac{2\mathbf{P}}{3} \right\} dp', \tag{50}$$

$$\Delta_g \frac{d\mathbf{Q}}{dt} = \int \left\{ \frac{4\nu_g}{9} \left[\frac{\mu'}{\mu + \mu'} \right]^2 \left[\left(\frac{T + T'}{r\lambda} + \frac{3\mathbf{r} \cdot (\mathbf{Q} + \mathbf{Q}')}{r^2} \right) \frac{\mathbf{rr}}{r^2} + \right.\right.$$
$$\left.\left. + 9a^2\mathbf{S}_4 \right] - \nu_g \frac{\mu'}{\mu + \mu'} \left[\frac{2\mathbf{r} \cdot \mathbf{Q} \cdot \mathbf{r}}{3r^2} \frac{\mathbf{rr}}{r^2} + a^2\mathbf{S}_5 \right] \right\} dp', \tag{51}$$

where ν_g is the frequency of encounters, and πa^2 the gravitational cross-section, given by

$$a(T + T' + H_2 a^2) = \frac{3}{2} \pi\gamma(\mu + \mu')^4 \sqrt{3/4\pi}, \tag{52}$$

$$\nu_g = \bar{n}_r' \pi a^2 g \sqrt{\frac{8\pi}{3} (T + T' + H_2 a^2)}, \tag{53}$$

where $H_2 = 3(rU'' - U')^2/70rU'$.

The gravitational forces modify the physical impact processes, since the particles are mutually accelerated during their approach, and correspondingly decelerated after the impact. This may be accounted for by calculating the effective coefficient of restitution as

$$\alpha^2 = \alpha_2^0 - \frac{3\gamma(\mu + \mu')(1 - \alpha_0)^2}{2(\sigma + \sigma')(T + T' + H_2(\sigma + \sigma')^2)} . \tag{54}$$

The same effect must be included in the calculation of the argument of α_0. In the case of $\alpha_v = \alpha_v(|\mathbf{c}\cdot\mathbf{v}|)$ we have

$$\alpha_0 = \alpha_v(\sqrt{2/3(T+T' + H_2(\sigma+\sigma')^2) + \gamma(\mu+\mu')/(\sigma+\sigma')}), \tag{55}$$

where $H_2(\sigma+\sigma')^2$ corresponds to the additional velocity differences due to the finite size (Section 5).

The average gravitational potential is also affected. According to Poisson's equation, written for the equatorial plane of the system,

$$U'' + U'/r + \lambda = 4\pi\gamma \int \mu' n_r' \, \mathrm{d}p'.$$

As long as the sharp radial density variations are not considered, only the vertical component is significantly modified by self-gravitation. If we denote with λ_0 the contribution of external potential, the correction due to self-gravitation, $\Delta\lambda = \lambda - \lambda_0$ is simply

$$\Delta\lambda = 4\pi\gamma \int \mu' \bar{n}_r' \mathrm{d}p'. \tag{56}$$

The vertically averaged space density is used instead of that in the equatorial plane, since it is expected to yield better results for inclined orbits (Hämeen-Anttila, 1984). This approximation also takes into account the possibility that different particle types may occupy layers of different vertical thickness, thus only partially feeling each others contributions.

According to computer simulations of non-rotating particles (Lukkari and Salo, 1984; Salo, 1985) these correction terms give good agreement between the numerical experiments and analytical results as long as the average post-collisional velocities exceed the escape velocities from particle surfaces. Violation of this restriction, equivalent to $\alpha^2 < 0$ means physically an excessive formation of local transient particle groups, not accounted for by the theoretical equations. Inclusion of rotation should not cause any additional limitations for the validity of gravitational terms.

7. Numerical Results

a) *Qualitative Analysis*

The equations derived in the earlier Sections are highly non-linear, and must therefore be numerically integrated. However, before doing this it is useful to study qualitatively the behaviour of random velocities and spin velocities with the inclusion of friction and irregularity. This can be done by calculating the average rate of energy loss due to impacts, and the rate of energy transfer between linear and rotational random velocities.

For identical particles the change in kinetic energy, ΔE_{kin}, and rotational energy, ΔE_{rot}, in each binary impact, are determined by

$$\Delta E_{\mathrm{kin}} = \frac{\mu}{2}\left[\dot{\mathbf{R}}'^{\,2}_1 + \dot{\mathbf{R}}_1^2 - \dot{\mathbf{R}}'^{\,2} - \dot{\mathbf{R}}^2\right],$$

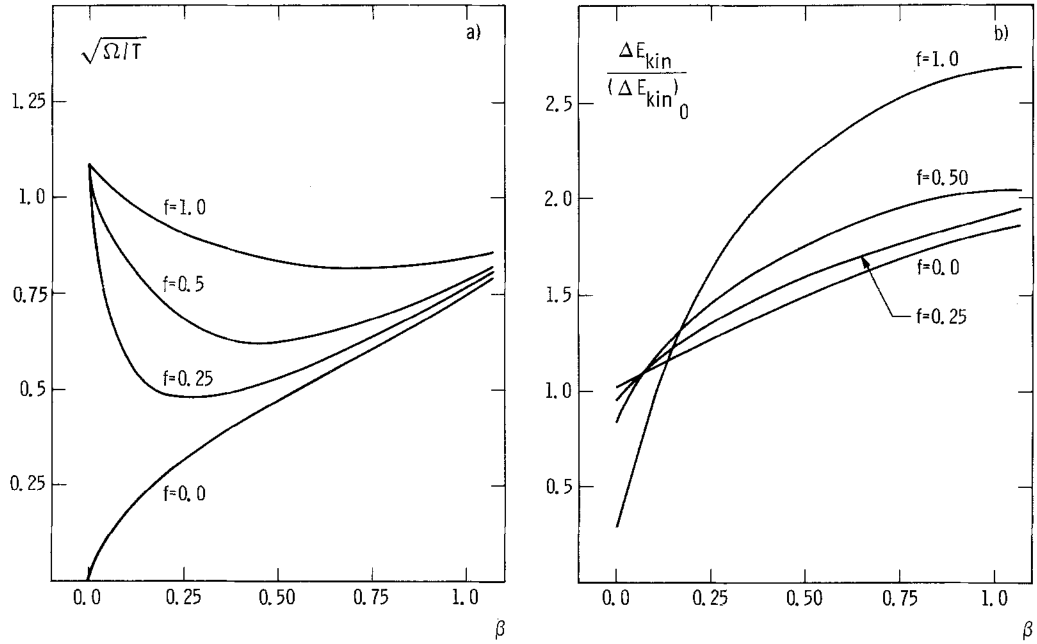

Fig. 2. a) The equilibrium ratio $\sqrt{\Omega/T}$ for identical mass points as a function of β, the coefficient of friction. Different amount of irregularity, described in terms of f, are studied, while α is assumed to have a constant value of 0.68. b) The ratio of average energy loss $\langle \Delta E_{\text{kin}} \rangle$ to that without any friction or irregularity, $\langle \Delta E_{\text{kin}} \rangle_0$.

$$\Delta E_{\text{rot}} = \frac{\mu\sigma^2}{5}\left[\omega'^{\,2}_1 + \omega_1^2 - \omega'^{\,2} - \omega^2\right]. \tag{57}$$

These changes can also be written in the form

$$\begin{aligned} \Delta E_{\text{kin}} &= \mu[(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})^2 - (\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\cdot \mathbf{v}], \\ \Delta E_{\text{rot}} &= 2\mu/5[(\sigma\omega_1 - \sigma\omega)^2 + (\sigma\omega_1 - \sigma\omega)\cdot(\sigma\omega + \sigma\omega')]. \end{aligned} \tag{58}$$

For mass points, the averaging over impacts yields (see Table I)

$$\begin{aligned} \langle \Delta E_{\text{kin}} \rangle = &-\mu T/3\{(1-\alpha^2) + 2\beta/7(2 - 2\beta/7) + \\ &+ \langle k^2\rangle/49[139(1+\alpha-\beta)^2 + \\ &+ 188(1+\alpha-\beta)\beta - 168(1+\alpha-\beta)]\} + \\ &+ \mu\Omega/3[(1+\alpha-\beta)^2\langle k^2\rangle + (2\beta/7)^2], \\ \langle \Delta E_{\text{rot}} \rangle = &\ 4\mu T/15(5/7)^2[\langle k^2\rangle(1+\alpha-\beta)(1+\alpha) + \beta^2/2] - \\ &- 4\mu\Omega/21[\beta + \langle k^2\rangle(1+\alpha-\beta) - 5\beta^2/14], \end{aligned} \tag{60}$$

where it has been assumed that $\langle l^2\rangle = \langle k^2\rangle$. The first term in the expression for ΔE_{kin}, $(1-\alpha^2)$ arises from inelasticity, and is always positive, corresponding to energy loss. The same holds true for $2\beta/7(2 - 2\beta/7)$, so that friction enhances energy dissipation. The third term in brackets, due to irregularity, is also positive for typical values of α. However, there is also a feedback of energy from rotation, described by the term proportional to Ω, so that the net effect of friction, and especially irregularity, depends on the ratio between Ω and T. This ratio in turn is determined by the expression for ΔE_{rot}. In the equilibrium state $\langle \Delta E_{\mathrm{rot}} \rangle$ vanishes, which fixes the ratio Ω/T independently of T (Figure 2a). In Figure 2 we have assumed that in each impact $\mathbf{c}_t$ varies symmetrically around $\mathbf{c}$ so that both k and l attain independent random values from the interval $(-f, f)$. Hence $\langle k^2 \rangle = \langle l^2 \rangle = f^2/3$. As can be seen, both the increased friction and increased irregularity enhence the ratio Ω/T. Figure 2b shows the dependence of $\langle \Delta E_{\mathrm{kin}} \rangle$ on β and f, once the equilibrium ratio of Ω/T is inserted into Equation (59). As expected, both friction and irregularity typically increase energy dissipation for a fixed α, but if β is close to zero irregularity may also reduce $|\langle \Delta E_{\mathrm{kin}} \rangle|$ because of the feedback of energy from large spin velocities.

In the equilibrium state the collisional energy loss is balanced by the viscous shear. In order to keep $\langle \Delta E_{\mathrm{kin}} \rangle$ constant while the dissipation due to friction and irregularity is included, the elasticity term $(1 - \alpha^2)$ must be reduced. Hence larger values of α are required, and due to the relation between α and the impact velocity, equilibrium velocity dispersion must be reduced. Only if β is close to zero, irregularity reduces α and increases T.

For smooth spherical particles the equilibrium ratio $E_{\mathrm{rot}}/E_{\mathrm{kin}} = 2\Omega/5T$ becomes $2\beta/(14 - 5\beta)$, while in the case of irregularity but no friction it is $2(1 + \alpha)/7$. The first result agrees with the calculations of Shukman (1984) who obtained $E_{\mathrm{rot}}/E_{\mathrm{kin}} = 0.23$ for $\beta = 1$. On the other hand the special case $\beta = 2$, as considered by Shu *et al.* (1985), leads to an equipartition between rotational and translational energies. In the latter case the change of kinetic energy is again proportional to $(1 - \alpha^2)$ as shown by Shu *et al.* (1985).

b) *Numerical Solution*

The above simple analysis, although revealing the general effects of friction and irregularity, does not tell how the different components of the velocity tensor behave, or how the equilibrium states of different sized particles are related. It also ignores the effects of finite particle size. The complete information can only be obtained by solving the set of Equations (19)–(21), (24), (35)–(38), and (44)–(56). We shall first shortly describe the numerical method of solution, and then give a few examples of how the local state of collisional systems depends on optical thickness.

The practical calculations are arranged by writing the two-dimensional tensors $\mathbf{P}$ and $\mathbf{Q}$ in the form (Hämeen-Anttila, 1981)

$$\mathbf{Z} = \frac{1}{2}\left[Z\,\frac{\mathbf{rr} + \mathbf{r}^*\mathbf{r}^*}{r^2} + z\,\frac{\mathbf{rr} - \mathbf{r}^*\mathbf{r}^*}{r^2} + z^*\,\frac{\mathbf{rr}^* + \mathbf{r}^*\mathbf{r}}{r^2} \right], \quad \mathbf{Z} = \mathbf{P}, \mathbf{Q}, \qquad (61)$$

while the three-dimensional tensor $\boldsymbol{\Omega}$ has in addition the nonzero component $\Omega_z\mathbf{NN}$. The time-scales of P, Q, Ω, and Ω_z are short, of the order of few tens of impacts per particle, and therefore in principle it is quite easy to follow the asymptotic behaviour of these quantities. However, in the case of low collision frequency some numerical difficulties follow from the fact that the steady evolution of z and z^* towards their equilibrium values is superponed with periodic, slowly decaying oscillations having the time-scale of the orbital period. In dense systems the rapid decrease of T with growing τ causes some difficulties, since for example the importance of self-gravitation depends on both these quantities.

The relation between these quantities and the more commonly used principal axis components of velocity tensor (see e.g. Goldreich and Tremaine, 1978) are easy to derive from Equation (61). The expression for $\mathbf{T}$ (Equation (16)) can be written in the form

$$\mathbf{T} = A\,\mathbf{rr}/r^2 + B\mathbf{r}^*\mathbf{r}^*/r^2 + C(\mathbf{rr}^* + \mathbf{r}^*\mathbf{r})/r^2 + D\mathbf{NN}, \tag{62}$$

with

$$\begin{aligned} A &= r(rU'' + 3U')^2/4U'\,(P+p)/2, \\ B &= r(rU'' + 3U')\,(P - p)/2, \\ C &= -1/4r(rU'' + 3U')\sqrt{(rU'' + 3U')/U'}\,p^*, \\ D &= \lambda r^2(Q + q)/2. \end{aligned} \tag{63}$$

The principal axis components are then obtained by setting

$$\mathbf{T} = c_1^2\mathbf{ss}/s^2 + c_2^2\mathbf{s}^*\mathbf{s}^*/s^2 + c_3^2\mathbf{NN}, \tag{64}$$

giving

$$\begin{aligned} c_1^2 &= 1/2[A + B + \sqrt{(A-B)^2 + 4C^2}]\,, \\ c_2^2 &= 1/2[A + B - \sqrt{(A-B)^2 + 4C^2}]\,, \\ c_3^2 &= D\,, \\ \tan(2\delta) &= 2C/(A - B), \end{aligned} \tag{65}$$

where δ is the angle between $\mathbf{r}^*/r$ and the largest principal axis. The dispersion tensor for rotational velocities can be treated in a similar manner.

For low collision frequency the tensorial equations for the evolution of $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{\Omega}$ can be replaced with scalar equations for $\mathrm{d}P/\mathrm{d}t$, $\mathrm{d}Q/\mathrm{d}t$, $\mathrm{d}\Omega/\mathrm{d}t$, and $\mathrm{d}\Omega_z/\mathrm{d}t$, since all z, $z^* \to 0$ if $\nu \to 0$. In Salo (1987) this simplified set of equations (without self-gravitational terms) is compared to numerical simulations of rarefied Keplerian systems, made with the same impact model including friction and irregularity. In general, good quantitative agreement is found. This confirms the validity of the rotational equations, although due to the smallness of the σ-terms the results are not very sensitive to possible inaccuracy of these terms.

c) *Numerical Models for Identical Particles*

The asymptotic behaviour of $\mathbf{P}$ and $\mathbf{Q}$ depends strongly on the assumed elasticity

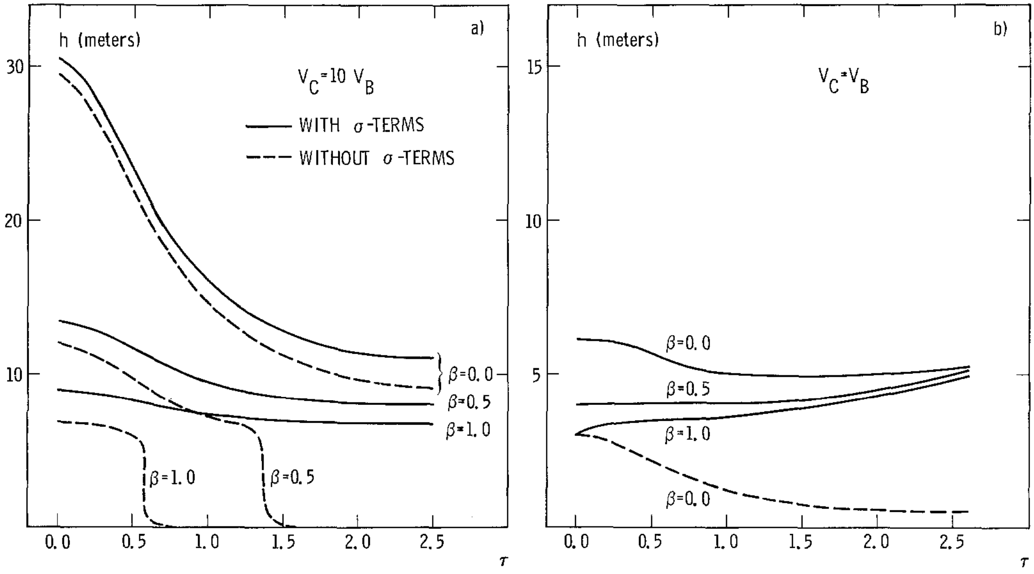

Fig. 3. The dependence between the optical thickness, τ, and the effective geometrical thickness, h, calculated for the Saturnocentric distance of 100000 km. Equation (66) is used for α, both with $v_c = 10v_B$(a), and with $v_c = v_B$ (b). Solid lines are for $\sigma = 1$ m, while dashed lines correspond to the case where σ-terms are insignificant ($\sigma = 0.01$ m).

model (Salo, 1985). If the particles are hard, characterized by $\alpha \gg 0$, and a slowly decreasing $\alpha(v)$-relation, the equilibrium state is a many particle thick multilayer, the thickness of which is determined by the exact dependence between α and v. On the other hand, softer particles, characterized by a sharply decreasing $\alpha(v)$-relation, are likely to flatten to a near monolayer state. For numerical models of Saturn's rings we adopt the function

$$\alpha = \left(\frac{|\mathbf{c} \cdot \mathbf{v}|}{v_c}\right)^{-p}, \qquad 0.25 < \alpha < 1.0, \tag{66}$$

which describes the laboratory measurements of Bridges *et al.* (1984) if $p = 0.234$ and $v_c = v_B = 0.0077$ cm/s. Actually, Bridges *et al.* (1984) experiments were performed for head-on collisions only, so that there was no difference between the absolute magnitude of the impact velocity and its different components. However, like Shu *et al.* (1985), we assume that α depends only on the perpendicular component $|\mathbf{c} \cdot \mathbf{v}|$. In the case of rotating irregular particles one should use $|\mathbf{c}_t \cdot \mathbf{v}_{\text{coll}}|$ (Section 4), but a few numerical calculations showed that the resulting difference is insignificant. Also the use of v or v_{coll} instead of $|\mathbf{v} \cdot \mathbf{c}|$ would not cause any large qualitative differences.

Figure 3 depicts the dependence between the optical thickness, $\tau = n\pi\sigma^2$, and the effective geometrical thickness, defined by $h = r\sqrt{6(Q-q)}$ (Hämeen-Anttila, 1984). These curves, like all the subsequent figures refer to the Saturnocentric distance

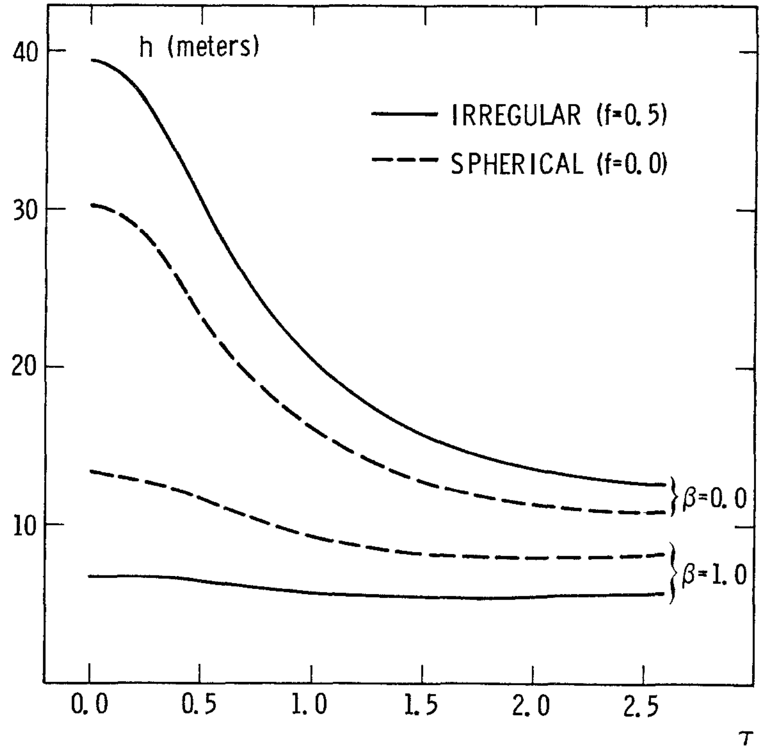

Fig. 4. The influence of irregularity on $h(\tau)$. The parameter $v_c = 10v_B$, while $\sigma = 1$ m. Dashed lines stand for no irregularity, while the solid lines represent the case $f + 0.5$, corresponding to a maximum deviation of about 35° between c and c_t.

$r = 100000$ km. The central potential has been replaced with that of a point mass, $\gamma m = 3.7942 \cdot 10^{16}$ m^3/s^2, so that $U' = \gamma m/r^2$, $U'' = -2\gamma m/r^3$, and $\lambda_0 = \gamma m/r^3$. The above formula for $\alpha(v)$ is used, but instead of taking the measured value for v_c, we have preferred a ten-fold value $v_c = 10v_B = 0.077$ cm/s. The reason for this is that the measured value v_B implies rather soft particles, and therefore the resulting equilibrium thickness would be very small. Indeed, if we neglect the σ-terms (dashed lines in Figure 3), the value of h would be only a few meters. This same result was obtained by Bridges *et al.* (1984) who estimated that $h < 5$ meters for non-rotating mass points. Therefore if we replace the actual distribution of sizes by an effective radius of the order of 1 meter (Shu and Stewart (1985) suggest $\sigma_{\text{eff}} \approx 1.8$ m), it is clear that the average impact velocities are dominated by the systematic velocity field. The resulting geometric thickness would be of the order of few particle radii, and not very sensitive to the impact model. In fact, the flatness of the system would render the effects of extra dissipation due friction almost negligible (Figure 3b). With the use of larger v_c, although perhaps corresponding to unrealistically hard particles, a more general insight into possible types of behaviour should be obtained.

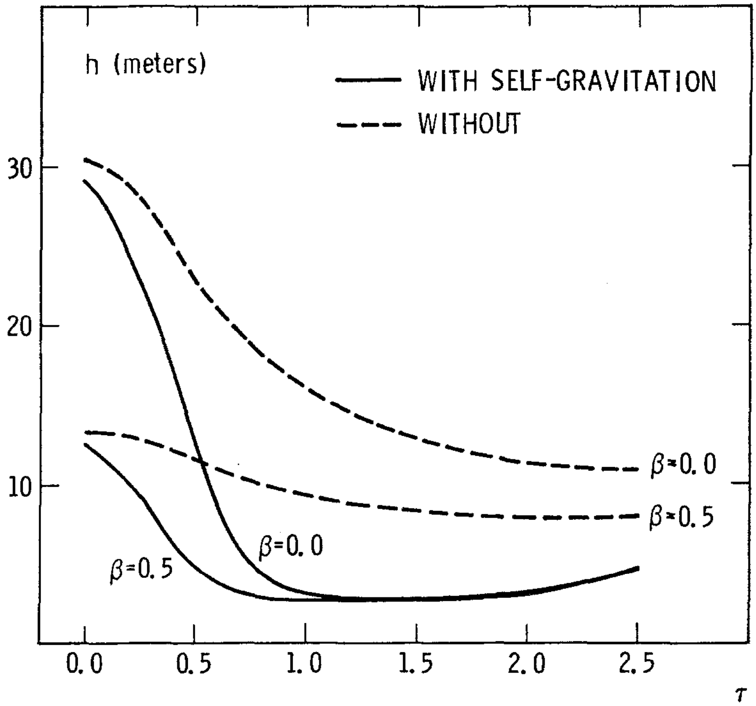

Fig. 5. The influence of gravitational interactions on $h(\tau)$. Two different values of friction ($\beta = 0.0$ and 0.5) are studied both with (solid lines) and without (dashed lines) self-gravitation. The internal density of particles is 0.9 gr/cm^3, while $v_c = 10v_B$ and $\sigma = 1$ m.

The case $\beta = 0$ in Figure 3a describes the typical $h(\tau)$ dependence for non-rotating mass points. The decrease of h with τ is due to the reduced mean free path between impacts, which leads to a less effective gain of energy from the systematic velocity field. At large optical thicknesses the flattening is finally prevented by the finite size of particles. Notice that h is proportional to $\sqrt{T}$ at low τ, while if the system is more dense it takes into account the finite space needed by particles. Therefore for very large τ, while $\sqrt{T}$ is almost constant, h would increase since the particles can not penetrate each other. As was qualitatively shown in Figure 2, random kinetic energy is effectively dissipated by friction, which leads to strongly reduced $\sqrt{T}$ and h. With $v_c = v_B$ (Figure 3b) the behaviour of $h(\tau)$ is much less dramatic.

For frictionless particles the terms arising from finite size are almost insignificant if $v_c = 10v_B$, as can be seen by comparing dashed and solid lines in Figure 3a. However, in the presence of friction the balancing effects of σ-terms are needed even for rather low optical thickness in order to avoid rapid flattening. If $\beta = 1$, the critical τ after which $h \to 0$ without σ-terms would be about 0.6, while for $\beta = 0.5$ it would be about 1.3. The first value is in a close agreement with Shukman's (1984) calculations, who found $\tau_{\text{crit}} \approx 0.73$ for his impact model. However, because of the

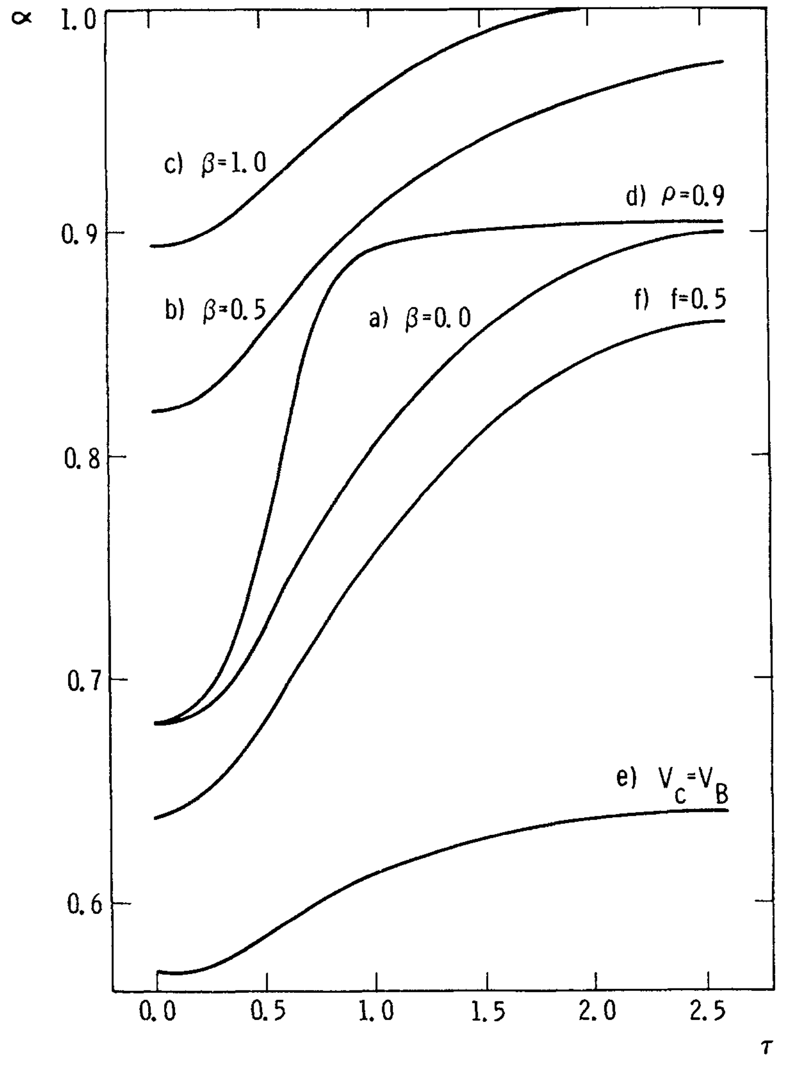

Fig. 6. Equilibrium values of α plotted as function of τ. Unless otherwise indicated $\beta = f = 0$, and $v_c = 10 v_B$. The curves a), b), and c) depict the influence of increasing friction ($\beta = 0.0$, 0.5, and 1.0, respectively), while d) and f) show the effects of self-gravity ($\rho = 0.9$ gr/cm^3) and irregularity ($f = 0.5$). In case e) $v_c = v_B$, so that σ-terms are important.

finite size of particles, these critical values of τ have no practical significance for the local equilibrium state of Saturn's rings.

The influence of irregularity (Figure 4) turns out to depend on the amount of friction in the way anticipated in Section 7a. In general, irregularity enhances the effects of friction, and only if $\beta = 0$, it leads to increased random velocities. Even a very small amount of friction is able to counteract this tendency: for example if $f = 0.5$, all the values $\beta > 0.02$ yield an increased energy dissipation as compared to the case $\beta = f = 0$.

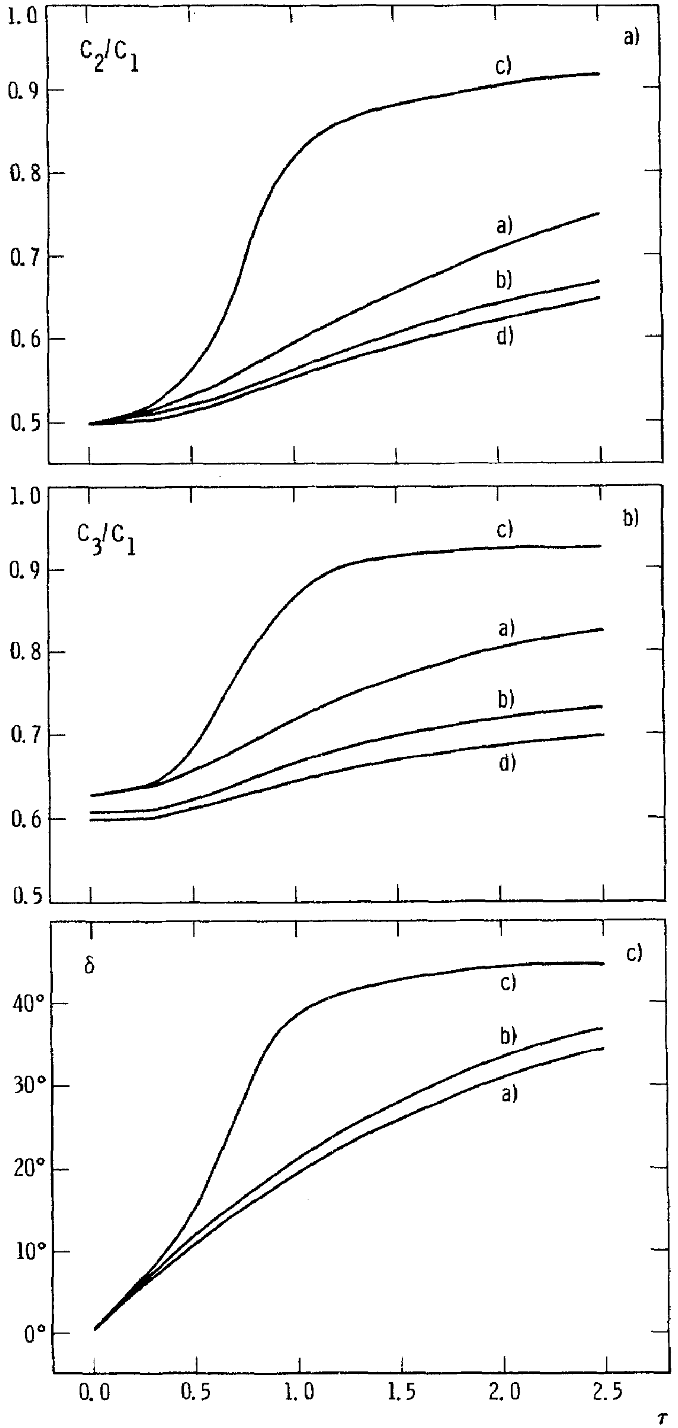

Fig. 7. The behaviour of the velocity ellipsoid as a function of τ. Curve a) approximates non-gravitating mass points ($v_c = 10v_B$), while in b) friction ($\beta = 0.5$), and in c) self-gravitation ($\rho = 0.9$ gr/cm^3) are included. The curve d) represents the influence of finite size ($v_c = v_B$).

The mutual gravitational forces (Figure 5) affect the dynamical state by three different, partially cancelling ways. Firstly, due to the self-gravitation, the vertical component of the average gravitational field is increased, leading to reduced vertical oscillations and increased impact frequency, which both reduce h (Salo and Lukkari, 1982). On the other hand, individual gravitational encounters correspond to totally elastic impacts, and therefore increase velocity dispersion. However, in encounters leading to physical collision the average elasticity is decreased due to the mutual acceleration before impact. Therefore the third factor tends to cancel the second one. According to Figure 5 the factors tending to flatten the system dominate. For small τ this is due to the acceleration before impact, while for large τ the increase in λ is more important. The gravitational scattering is important only if the system is rather flattened without being optically thick. In fact, if $v_c = v_B$, $h(\tau = 0)$ would be slightly (20%) increased due to encounters. However, as will be shown in Section 7d, gravitational scattering has a much more important role if the actual distribution of sizes is taken into account.

The influence of all these factors, friction, irregularity, finite size, and self-gravitation, is possible to interpret in terms of their effects on the relation between α and τ (Figure 6). For mass point systems of identical particles, there is a unique $\alpha(\tau)$ dependence, determined by the local collisional energy balance. Hämeen-Anttila (1978) derived

$$\frac{9\pi^2(19\alpha - 13)}{64(1-\alpha^2)(3-\alpha)^3} = g^2\tau^2, \tag{67}$$

which is very similar to the corresponding result of Goldreich and Tremaine (1978). Our case a) ($v_c = 10v_B$) approximates fairly closely this formula. With friction (b and c), the effective value of α is increased because part of the energy loss is accounted for by friction, and is not due to inelasticity. It is also possible to attain the state $\alpha = 1$ with finite τ if $\beta \neq 0$, to some extent corresponding to τ_{crit} in Figure 3. The inclusion of irregularity alone (*f*) decreases α. The reason for this is the slightly diminished amount of the energy consuming head-on collisions in favor of more grazing impacts. In order to compensate for this, average impact velocities must be increased corresponding to reduced α. In fact, the effects of our irregularity model are rather similar to the effects attributed to rotation in the model by Shu *et al.* (1985). However, they ignore the dissipation due to friction, which is able to reverse the total influence of rotation. The finite size of particles (e) corresponds to an additional source of viscous energy so that it has to be balanced by increased collisional dissipation on lower α. In general, the same should hold true for gravitational encounters, but due to the effects explained above, self-gravitation increases energy dissipation and allows for larger α (case d).

The behaviour of the velocity dispersion tensor is plotted in Figure 7. The case a) corresponds again to the mass point behaviour, and is in a good agreement with Figure 3 of Goldreich and Tremaine (1978). In rarefied systems $\delta \approx 0°$ and

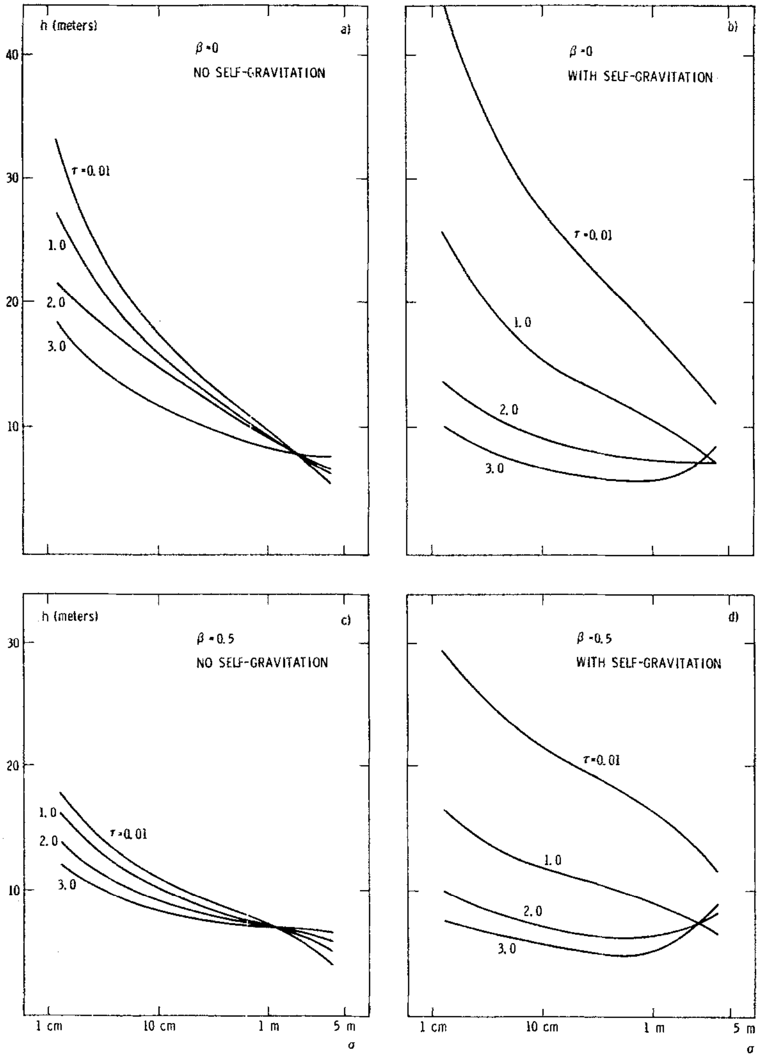

Fig. 8. The geometric thickness of the layer occupied by different sized particles, $h(\sigma)$, shown for different total optical thicknesses. The power-law distribution of radii extends from 1 cm to 5 meters, with $q = 3$. The measured $\alpha(v)$ relation is used, $v_c = v_B$.

$C_1:C_2:C_3 \approx 1:0.5:0.63$, while if $\tau \to \infty$ then $\delta \to 45°$ and $C_1:C_2:C_3 \to 1:1:1$. However, even if $\tau = 2.5$ the system is still quite far from the isotropic velocity dispersion. The inclusion of gravitational forces (c), tends to push the system towards the isotropic limit for smaller values of τ, while friction (b) and finite size (d), both tend to retain anisotropies up to larger τ. The effects of irregularity did not significantly deviate from the case (a).

The dispersion tensor of rotational velocities was found to be almost isotropic for all values of τ. Typically $\Omega_{11} \approx \Omega_{22}$ and $\sqrt{\Omega_{33}/\Omega_{11}} \approx 1.1$ in rarefied systems, which is in agreement with Shukman's (1984) results for $\beta = 1$. It was also found that the ratio $E_{\rm rot}/E_{\rm kin}$ is practically independent of τ, and also of the geometric thickness of the system, whether it is a mass point system, or a monolayer. Therefore the approximate considerations of Section 7a are also more generally valid. This is easy to understand since the correction terms for the influence of finite size on $\mathrm{d}\Omega/\mathrm{d}t$ are much smaller than those in $\mathrm{d}\mathbf{P}/\mathrm{d}t$ and in $\mathrm{d}\mathbf{Q}/\mathrm{d}t$, and since the gravitational forces do not directly affect the rotation.

d) *Models with Size Distribution*

All the above models have assumed that the actual distribution of sizes can be replaced with one effective radius. However, as was shown in the case of non-rotating particles (Salo, 1985), the inclusion of a realistic size distribution gives a rather different equilibrium solution, characterized by the clear differences in the random velocities of the smallest and largest particles. In this Subsection we consider the power-law distribution

$$\frac{\mathrm{d}N}{\mathrm{d}\sigma} \sim \sigma^{-q}, \quad 1 \text{ cm} < \sigma < 5 \text{ m}, \; q = 3, \tag{68}$$

derived from Voyager I radio occultation measurements (Marouf *et al.*, 1983). The density of all particles is assumed to be 0.9 gr/cm^3.

Figure 8 depicts the geometric thickness of layers occupied by different-sized particles $h(\sigma)$ for different total τ. The measured value $v_c = v_B$ is used. As in the case of identical particles, the largest particles are confined almost to a monolayer. However, the centimeter-sized particles have a much larger h than before. Consequently, they are affected by the friction, which is able to reduce the maximum h to about one half. Therefore, the earlier use of the unrealistically large value of v_c for identical particles gave a rather good description of the behaviour of the smallest particles of an extended size distribution. The same holds true for the velocity dispersion tensor: for a given τ the velocity distribution of centimeter-sized particles is more isotropic than that of the largest particles, which are more affected by σ-terms (see Figure 6, case e).

The inclusion of gravitational forces increases the $h_{\rm small}$ by a large fraction. This is due to the gravitational scattering by the largest particles, which dominates over the increased dissipation in physical collisions. However, as τ increases, the increase in the vertical field again becomes important, and h drops with growing τ, faster than in the case of purely collisional interactions. In dense systems the differences in the velocity dispersions of different sized particles are reduced, although h still typically decreases with growing σ.

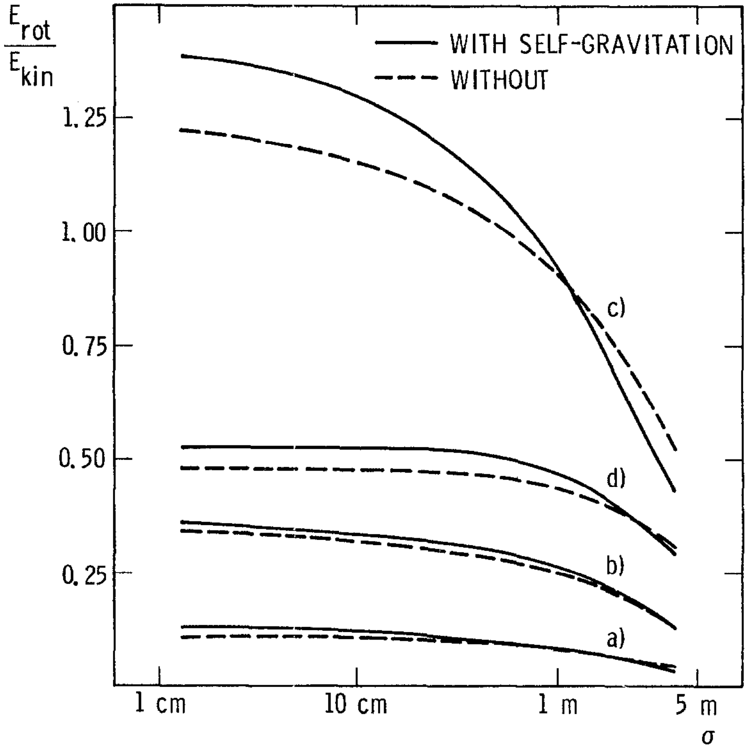

Fig. 9. The equilibrium ratio E_{rot}/E_{kin} plotted as a function of particle radius. Four different combinations of β and f were studied with (solid lines) and without (dashed lines) self-gravitation. In a), b), and c) $f = 0$, while $\beta = 0.5$, 1.0, and 2.0, respectively. In d) $\beta = 0$ and $f = 0.5$.

The behaviour of rotational velocities is somewhat affected by the inclusion of size-distribution. The average value of E_{rot}/E_{kin}, weighted with particle mass, is almost the same as it was for identical particles, but there is a clear difference between the smallest and the largest particles (Figure 9). The dispersion of rotational periods $\approx 2\pi\sigma/\sqrt{\Omega}$, is nevertheless roughly proportional to σ, since $\sqrt{\Omega}$ varies at most by a factor of 4 although σ changes almost by 3 magnitudes. Typical rotation periods are about 4 minutes for a 5 centimeter particle, and about 2 hours for 1 meter particle ($\beta = 0.5$, $f = 0.0$, $\tau = 1$, $\rho = 0.9$ gr/cm^3), but they depend strongly on the parameter values (Table III). For the largest particles the average values of $\sigma\omega_z$ can become comparable to $\sqrt{\Omega_z} \approx \sqrt{\Omega/3}$. For example, in the above typical case with $\beta = 0.5$, $\sqrt{\Omega_z} \approx 5 \cdot 10^{-4}$ m/s for a 5 meter particle, while $\sigma\bar{\omega}_z \approx 10^{-3}$ m/s (Equation (48)). Therefore, if the system is rather flattened, and consequently the dispersion of spins is not very large, the meter-sized particles should have predominantly prograde rotation. This phenomenon was also observed in computer simulations (Salo, 1987). In fact, numerical experiments suggest that $\bar{\omega}_z$ might actually be about a factor of 2 larger than predicted by Equation (48), so that the alignment might be even more pronounced.

TABLE III

Average rotational periods, $\approx 2\pi\sigma/\sqrt{\Omega}$, for different parameter combinations. The distribution of sizes extends from 1 centimeter to 5 meters, with $q = 3$

	τ	without self-gravitation		with self-gravitation	
		$P_{5\text{ cm}}$ (min)	$P_{1\text{ m}}$ (hours)	$P_{5\text{ cm}}$ (min)	$P_{1\text{ m}}$ (hours)
$\beta = 0.5$, $f = 0.0$	0.01	6.5	4.0	3.5	1.9
	1.00	7.2	4.0	3.9	2.0
$\beta = 1.0$, $f = 0.0$	0.01	4.9	2.8	2.3	1.3
	1.00	7.2	2.8	2.3	1.3
$\beta = 2.0$, $f = 0.0$	0.01	0.8	1.2	0.2	0.6
	1.00	1.0	1.2	0.5	0.6
$\beta = 0.0$, $f = 0.5$	0.01	1.7	1.3	1.2	1.0
	1.00	1.9	1.4	1.4	1.0

8. Conclusions

The equations derived for the combined collisional evolution of random velocities and spins indicate that rotation can significantly modify the local equilibrium. Rotation can be induced both by friction and by surface irregularity of impacting bodies. The inclusion of friction, which reduces the tangential component of the relative velocity, increases energy dissipation. Therefore, in the collisional balance between the viscous gain of energy from the systematic velocity field, the losses due to inelasticity can be smaller. Due to the assumed monotonically decreasing relation between α and v, the equilibrium state is attained with a smaller velocity dispersion, and correspondingly smaller geometric thickness. The irregularity tends to enhance the effects of friction, since it favors the more grazing impacts having larger tangential component of relative velocity. Rotation also increases tangential velocity differences. Only if the friction is negligible will the decreased fraction of head-on collisions reduce energy loss.

The exchange of energy between translational and rotational degrees of freedom does not generally lead to energy equipartition, as assumed by Shu *et al.* (1985). Instead, the ratio $E_{\text{rot}}/E_{\text{kin}}$ turns out to depend on the amount of friction and irregularity: for smooth identical particles this ratio becomes $2\beta/(14 - 5\beta)$, while for frictionless irregular particles it is $2(1 + a)/7$. These ratios are practically independent of τ or h. The result for smooth particles is in agreement with the corresponding cases considered by Shukmann (1984) and by Shu *et al.* (1985), namely $\beta = 1$, which leads to $E_{\text{rot}}/E_{\text{kin}} \approx 0.23$, and $\beta = 2$, which leads to equipartition. An interesting point is the behaviour of the mean spin: ω_z is estimated to obtain a positive average of about 1/4 times the orbital angular velocity. For flattened systems $\overline{\sigma\omega_z}$ may exceed the dispersion $\sqrt{\Omega_z}$, leading to clear alignment of spin axes, which was also verified by numerical simulations (Salo, 1987).

Applications to the dynamics of Saturn's rings indicate, however, that the actual importance of rotation depends strongly on the elastic properties of particles. In the case of rather soft particles, as implied by the laboratory measurements of solid ice spheres (Bridges *et al.*, 1984), the equilibrium thickness for identical particles is of the order of a particle diameter, and is not very sensitive to the details of the impact model. Consequently the effects of friction and irregularity are negligible, and they would be important only for considerably harder particles. However, the situation is changed if the distribution of sizes is taken into account, since the small particles occupy a many particle thick layer even if the measured $\alpha(v)$ is assumed. The inclusion of friction is typically able to reduce the maximum thickness of rarefied regions from about 30 meters ($\beta = 0$) to about 15 meters ($\beta = 0.5$), while for large τ it is changed from about 15 meters to 10 meters. The inclusion of gravitational interactions increases these figures to about 45 and 30 meters, if τ is small. For large τ, h is almost independent of β, and is about 6 meters.

The calculated rotation periods are roughly proportional to particle size, typically of the order of 1 minute for 1 cm particles and about 10 hours for the largest 5 meter particles, but again they depend strongly on the optical thickness, and also on the parameter values chosen for friction and irregularity.

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