NUMERICAL DETERMINATION OF ASYMMETRIC PERIODIC SOLUTIONS IN THE PLANAR GENERAL THREE BODY PROBLEM AND THEIR STABILITY

T. TSOUROPLIS and C. G. ZAGOURAS
Dept of Appl. Mathematics, University of Patras, Greece

(Received 7 November, 1983)

Abstract. A numerical study of asymmetric periodic solutions of the planar general three body problem is presented. The equations of variation are integrated numerically and the algorithms for the numerical determination of families of such periodic orbits are given. These orbits refer to a rotating frame of reference. The linear isoenergetic stability is examined through the stability parameters while the results are given in tables and figures.

1. Introduction

The significance of periodic solutions in the study of nonintegrable dynamical systems has been pointed out by many investigators since Poincaré.

The most work on periodic solutions has been done on the Restricted Three-Body Problem, a simplified version of the Three-Body Problem. But whilst there have been numerous investigations into the symmetric periodic solutions only a few articles have dealt with asymmetric periodic solutions (e.g. Message, 1970; Hénon, 1965; Taylor, 1983). This is due to the greater complexity in the numerical procedures and the large amount of computing time required to determine asymmetric periodic solutions.

In the more complicated general three-body problem families of periodic solutions were computed only when it was proved that families of periodic solutions exist with fixed masses of all the bodies, with respect to a rotating frame (Hénon, 1974; Hadjidemetriou, 1975). Since then, several families of symmetric periodic solutions of this problem have been determined.

In this paper the algorithms for the numerical determination of periodic solutions of the general 3-body problem which are not symmetric with respect to the axis joining the two more massive bodies, are developed. Then, five families of such solutions are computed and their linear isoenergetic stability is examined. The study of the stability character of each orbit contains the numerical integration of the variational equations simultaneously with the equations of motion. The results are presented in tables and figures.

2. Numerical Determination of Asymmetric Periodic Solutions

We use a rotating system of dimensionless coordinates with origin at the center of mass of the two more massive bodies P_1 and P_2.

Earth, Moon, and Planets 30 (1984) 79–94. 0167–9295/84/0301–0079$02.40.

The position of the three-body system is fully determined in terms of the coordinates x, y of the third body P_3, the distance x_2 of P_2 from the origin and the angle θ between the rotating and non-rotating system.

In the rotating coordinate system the Equations of motion of the planar general three body problem are

$$\begin{aligned} \ddot{x} &= Bx + x\dot{\theta}^2 + 2\dot{\theta}\dot{y} + \ddot{\theta}y + \mu A x_2, \\ \ddot{y} &= (B + \dot{\theta}^2)y - x\ddot{\theta} - 2\dot{x}\dot{\theta}, \\ \ddot{x}_2 &= (m_3 B^* + \dot{\theta}^2)x_2 - (1-m_3)(1-\mu)^3/x_2^2 + m_3(1-\mu)Ax, \\ \ddot{\theta} &= -2\dot{\theta}\dot{x}_2/x_2 + m_3(1-\mu)Ay/x_2; \end{aligned} \tag{1}$$

or, in first-order form,

$$\begin{aligned} &\frac{dX_1}{dt} = X_4 \triangleq f_1, \qquad \frac{dX_2}{dt} = X_5 \triangleq f_2, \qquad \frac{dX_3}{dt} = X_6 \triangleq f_3 \\ &\frac{dX_4}{dt} = BX_1 + X_1 X_8^2 + 2X_8 X_5 + \dot{X}_8 X_2 + \mu A X_3 \triangleq f_4, \\ &\frac{dx_5}{dt} = (B + X_8^2)X_2 - X_1\dot{X}_8 - 2X_4 X_8 \triangleq f_5, \\ &\frac{dX_6}{dt} = (m_3 B^* + X_8^2)X_3 - (1-m_3)(1-\mu)^3/X_3^2 + m_3(1-\mu)AX_1 \triangleq f_6, \\ &\frac{dX_7}{dt} = X_8 \triangleq f_7, \\ &\frac{dX_8}{dt} = -2X_8 X_6/X_3 + m_3(1-\mu)AX_2/X_3 \triangleq f_8, \end{aligned} \tag{2}$$

where

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = (x, y, x_2, \dot{x}, \dot{y}, \dot{x}_2, \theta, \dot{\theta}).$$

A periodic solution $\mathbf{X}(\mathbf{X}_0; t)$ of the above Equations will satisfy

$$X_i(\mathbf{X}_0; t + T) = X_i(\mathbf{X}_0; t), \qquad i \neq 7 \tag{3}$$

where T is the period and $X_0 = (X_{01}, \ldots X_{08})$ is the initial-conditions vector. Further, without loss of generality, we shall fix initial values of y, θ and $\dot{\theta}$ as follows: $y_0 = 0$, $\theta_0 = 0, \dot{\theta}_0 = 1$. The periodicity conditions are written in the form:

$$\begin{aligned} x(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= x_0, && \text{(a)} \\ y(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= y_0, && \text{(b)} \\ x_2(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= x_{20}, && \text{(c)} \\ \dot{x}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= \dot{x}_0, && \text{(d)} \\ \dot{y}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= \dot{y}_0, && \text{(e)} \\ \dot{x}_2(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= x_{20}, && \text{(f)} \\ \dot{\theta}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20}; T) &= \dot{\theta}_0. && \text{(g)} \end{aligned} \tag{4}$$

In practice, the condition (4b) is satisfied 'by force' since we start and terminate the numerical integration when the orbit crosses the Ox axis. Further, due to the integrals of the problem only four of the remaining six periodicity conditions are truly independent. Essentially, therefore, the periodicity conditions are only four and in this work we have used the conditions (4a, c, d, f).

From these periodicity conditions corrector–predictor algorithms can be established for the numerical determination of entire series of asymmetric periodic solutions. In the corrector phase we assume an initial state vector $\mathbf{X}_0$ which approximately leads to a periodic orbit of (approximate) period T, and seek to adjust this state vector by differential corrections to improve iteratively the accuracy of periodicity.

If we integrate the Equations of motion and stop at the second crossing with the Ox-axis (after one full revolution) we have, in general,

$$\mathbf{X}(\mathbf{X}_0; T) \neq \mathbf{X}_0.$$

We seek corrections $\delta\mathbf{X}_0 = (\delta x_0, 0, \delta x_{02}, \delta\dot{x}_0, \delta\dot{y}_0, \delta\dot{x}_{20}, 0, 0)$ such that

$$\mathbf{X}(\mathbf{X}_0 + \delta\mathbf{X}_0; T + \delta T) = \mathbf{X}_0 + \delta\mathbf{X}_0. \tag{5}$$

Expanding in Taylor series and neglecting terms of order higher than the first, we have

$$X_i + \frac{\partial X_i}{\partial X_{01}}\,\delta X_{01} + \frac{\partial X_i}{\partial X_{03}}\,\delta X_{03} + \frac{\partial X_i}{\partial X_{04}}\,\delta X_{04} + \frac{\partial X_i}{\partial X_{05}}\,\delta X_{05} + \frac{\partial X_i}{\partial X_{06}}\,\delta X_{06}$$

$$+ \frac{\partial X_i}{\partial T}\,\delta T = X_{0i} + \delta X_{0i}, \qquad (i = 1, 2, 3, 4, 6). \tag{6}$$

For $i = 2$ we obtain, in particular,

$$\frac{\partial X_2}{\partial X_{01}}\delta X_{01} + \frac{\partial X_2}{\partial X_{03}}\delta X_{03} + \frac{\partial X_2}{\partial X_{04}}\delta X_{04} + \frac{\partial X_2}{\partial X_{05}}\delta X_{05} + \frac{\partial X_2}{\partial X_{06}}\delta X_{06} +$$

$$\frac{\partial X_2}{\partial T}\delta T = 0; \tag{7}$$

since, for $t = T$, $x_2 = y = 0$ while $\delta X_{02} = \delta y_0 = 0$. Solving now Equations (7) for δT and substituting into relations (6) we get

$$X_i + u_{i1}\delta X_{01} + u_{i3}\delta X_{03} + u_{i4}\partial X_{04} + u_{i5}\delta X_{05} + u_{i6}\delta X_{06}$$

$$= X_{0i} + \delta X_{0i}, \quad i = 1, 3, 4, 6. \tag{8}$$

where

$$u_{ij} = \frac{\partial X_i}{\partial X_{0j}} - \frac{\partial X_2}{\partial X_{0j}}\frac{f_i}{f_2}, \qquad i = 1, 3, 4, 6, \tag{9}$$

('variations at the crossing'; Markellos, 1977)

If we assume X_{04} constant or equivalently $\delta X_{04} = 0$, Equations (8) become

$$\begin{aligned}(u_{11}-1)\delta X_{01}+u_{13}\delta X_{03}+u_{15}\delta X_{05}+u_{16}\delta X_{06} &= X_{01}-X_1,\\ u_{31}\delta X_{01}+(u_{33}-1)\delta X_{03}+u_{35}\delta X_{05}+u_{36}\delta X_{06} &= X_{03}-X_3,\\ u_{41}\delta X_{01}+u_{43}\delta X_{03}+u_{45}\delta X_{05}+u_{46}\delta X_{06} &= X_{04}-X_4,\\ u_{61}\delta X_{01}+u_{63}\delta X_{03}+u_{65}\delta X_{05}+(u_{66}-1)\delta X_{06} &= X_{06}-X_6.\end{aligned} \tag{10}$$

This system is the corrector of the algorithm. It is solved for the corrections δX_{01}, δX_{03}, δX_{05}, δX_{06}, which are then added to the corresponding components of the initial state vector to obtain a better approximation to the periodic orbit with period $T+\delta T$. After repeated applications to the corrector we find (assuming convergence) the periodic (to the desired accuracy) solution characterized by the value X_{04} which is kept constant during the correction process. We then proceed to a single application of the predictor:

$$\begin{aligned}(u_{11}-1)\Delta X_{01}+u_{13}\Delta X_{03}+u_{15}\Delta X_{05}+u_{16}\Delta X_{06} &= -u_{14}\Delta X_{04},\\ u_{31}\Delta X_{01}+(u_{33}-1)\Delta X_{03}+u_{35}\Delta X_{05}+u_{36}\Delta X_{06} &= -u_{34}\Delta X_{04},\\ u_{41}\Delta X_{01}+u_{43}\Delta X_{03}+u_{45}\Delta X_{05}+u_{46}\Delta X_{06} &= (1-u_{44})\Delta X_{04},\\ u_{61}\Delta X_{01}+u_{63}\Delta X_{03}+u_{65}\Delta X_{05}+(u_{66}-1)\Delta X_{06} &= -u_{64}\Delta X_{04}.\end{aligned} \tag{11}$$

This predictor is designed to obtain the approximate initial state vector $\mathbf{X}_0+\Delta\mathbf{X}_0$ corresponding to another periodic orbit (along the family), characterized by the value $X_{04}^* = X_{04}+\Delta X_{04}$, where the 'increment' $\Delta \mathbf{X}_{04}$ is arbitrary but small so that convergence of the subsequent application of the corrector is secured. The values of the 'sensitivities' u_{ij} involved in Equations (10) and (11) are computed from relations (9), where the 'variations' $\partial X_i/\partial X_{0j}$ are known through numerical integration of the linear variational Equations:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = PV,$$

where

$$V = (v_{ij}) = (\partial X_i/\partial X_{0j})$$

and

$$P = \left(\frac{\partial f_i}{\partial x_j}\right), \qquad i,j = 1,\ldots,8.$$

3. Stability

If $\mathbf{X}_0$ is the vector, in phase space, corresponding to a periodic orbit and $\mathbf{X}_0+\delta\mathbf{X}_0$ is the vector of a neighboring orbit corresponding to the same value of the energy and angular momentum integrals, then a transformation T is constructed which transforms the initial state $\mathbf{X}_0$ to the state $\mathbf{X}$ when the orbit crosses the surface of section $X_2 = Y = 0$ for the second time (simple orbits). This transformation is expressed as

$$\mathbf{X} = \boldsymbol{\sigma}(\mathbf{X}_0), \tag{13}$$

where

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_3, \sigma_4, \sigma_6). \tag{14}$$

After linearization, the transformation (13) is written as

$$\delta \mathbf{X} = A\delta \mathbf{X}_0 \tag{15}$$

where

$$\delta \mathbf{X} = (\delta X_1, \delta X_3, \delta X_4, \delta X_6)^T,$$

$$\delta \mathbf{X}_0 = (\delta X_{01}, \delta X_{03}, \delta X_{04}, \delta X_{06})^T, \tag{16}$$

and A is the 4×4 matrix with elements the first partial derivatives of the functions (σ_1, σ_3, σ_4, σ_6) with respect to the initial conditions – i.e.,

$$A = (\alpha_{ij}) = \left(\frac{\partial \sigma_i}{\partial X_{0j}}\right), \qquad i,j = 1, 3, 4, 6. \tag{17}$$

The conditions for stability are:

$$\Delta > 0, \qquad |p| < 2, \qquad |q| < 2, \tag{18}$$

where

$$\Delta = \alpha^2 - 4(\beta - 2), \qquad p = \tfrac{1}{2}(\alpha + \sqrt{\Delta}), \qquad q = \tfrac{1}{2}(\alpha - \sqrt{\Delta}) \tag{19}$$

and

$$\alpha = -(\alpha_{11} + \alpha_{33} + \alpha_{44} + \alpha_{66}) \tag{20}$$

$$\beta = \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{14} \\ \alpha_{41} & \alpha_{44} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{16} \\ \alpha_{61} & \alpha_{66} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{vmatrix} + \begin{vmatrix} \alpha_{44} & \alpha_{46} \\ \alpha_{64} & \alpha_{66} \end{vmatrix}, \tag{21}$$

(Hadjidemetriou, 1975). The elements a_{ij} can be determined as functions of the elements v_{ij} of the 'variational' matrix from the expressions

$$a_{1i} = \left(v_{1i} - \frac{x_4}{x_5} v_{2i}\right) + \left(v_{15} - \frac{x_4}{x_5} v_{25}\right) D_{i5} + \left(v_{18} - \frac{x_4}{x_5} v_{28}\right) D_{i8},$$

$$a_{3i} = \left(v_{3i} - \frac{x_6}{x_5} v_{2i}\right) + \left(v_{35} - \frac{x_6}{x_5} v_{25}\right) D_{i5} + \left(v_{38} - \frac{x_6}{x_5} v_{28}\right) D_{i8},$$

$$a_{4i} = \left(v_{4i} - \frac{\dot{x}_4}{x_5} v_{2i}\right) + \left(v_{45} - \frac{\dot{x}_4}{x_5} v_{25}\right) D_{i5} + \left(v_{48} - \frac{\dot{x}_4}{x_5} v_{28}\right) D_{i8},$$

$$a_{6i} = \left(v_{6i} - \frac{\dot{x}_6}{x_5} v_{2i}\right) + \left(v_{65} - \frac{\dot{x}_6}{x_5} v_{25}\right) D_{i5} + \left(v_{68} - \frac{\dot{x}_6}{x_5} v_{28}\right) D_{i8},$$

$$(i = 1, 3, 4, 6) \tag{22}$$

where

$$D_{i5} = -(F_{1i}F_{28} - F_{2i}F_{18})/D,$$

$$D_{i8} = -(F_{2i}F_{15} - F_{1i}F_{25})/D, \tag{23}$$

$$D^{\cdot} = F_{15}F_{28} - F_{18}F_{25},$$

and

$$F_{1j} = \frac{\partial F_1}{\partial x_j} = \frac{\partial E}{\partial x_j}, \qquad F_{2j} = \frac{\partial F_2}{\partial x_j} = \frac{\partial P}{\partial x_j}, \qquad j = 1, 3, 4, 6 \tag{24}$$

with $F_1 = E$ and $F_2 = P$ denoting, respectively, the energy and angular momentum integrals.

TABLE I

The series A^g_{20} ($\mu = 0.25$, $\dot{x}_0 = -0.172\,92$)

A/A	m_3	x_{01}	x_{03}	x_{05}	x_{06}	E	T	p	q
1	0.000 103	−2.330 48	0.749 905	1.901 39	−0.000 337	−0.093 803	12.4773	−1.998	−39.19
2	0.001 203	−2.326 72	0.749 145	1.898 76	−0.002 430	−0.094 241	12.3959	−2.037	−36.25
3	0.007 607	−2.316 80	0.746 023	1.893 17	−0.008 955	−0.096 140	12.1764	−2.316	−35.33
4	0.014 009	−2.312 58	0.743 557	1.892 64	−0.013 446	−0.097 700	12.0441	−2.629	−36.98
5	0.021 809	−2.310 58	0.740 923	1.894 86	−0.017 850	−0.099 396	11.9264	−3.075	−39.80
6	0.036 509	−2.311 38	0.736 489	1.902 84	−0.024 301	−0.102 142	11.7723	−3.983	−46.22
7	0.050 629	−2.315 01	0.732 556	1.913 94	−0.029 892	−0.104 733	11.6526	−5.030	−54.27
8	0.060 149	−2.318 33	0.729 997	1.921 95	−0.033 118	−0.106 298	11.5880	−5.759	−60.14
9	0.078 649	−2.325 98	0.725 141	1.938 60	−0.038 778	−0.109 138	11.4803	−7.234	−73.04
10	0.087 649	−2.333 00	0.722 809	1.947 04	−0.041 305	−0.110 434	11.4339	−7.996	−80.06
11	0.100 269	−2.336 09	0.719 556	1.959 11	−0.044 647	−0.112 165	11.3737	−9.119	−90.75
12	0.111 689	−2.341 71	0.716 622	1.970 17	−0.047 492	−0.113 645	11.3230	−10.16	−101.3
13	0.119 689	−2.345 74	0.714 556	1.978 01	−0.049 404	−0.114 642	11.2891	−10.93	−109.3
14	0.134 089	−2.35307	0.710 838	1.992 17	−0.052 676	−0.116 338	11.2313	−12.37	−124.7
15	0.150 109	−2.361 31	0.706 676	2.007 97	−0.056 093	−0.118 088	11.1706	−14.07	−143.9
16	0.160 569	−2.366 71	0.703 940	2.018 29	−0.058 211	−0.119 155	11.1327	−15.25	−157.6
17	0.170 189	−2.371 67	0.701 409	2.027 77	−0.060 084	−0.120 083	11.0989	−16.38	−171.2
18	0.179 809	−2.376 62	0.698 862	2.037 23	−0.061 891	−0.120 961	11.0659	−17.56	−185.5
19	0.185 809	−2.379 70	0.697 265	2.043 12	−0.062 985	−0.121 484	11.0457	−18.32	−195.0
20	0.190 009	−2.381 85	0.696 143	2.047 23	−0.063 737	−0.121 838	11.0318	−18.87	−201.8
21	0.194 849	−2.384 32	0.694 846	2.051 96	−0.064 589	−0.122 234	11.0159	−19.52	−209.9
22	0.2000	−2.386 94	0.693 459	2.056 99	−0.065 480	−0.122 642	10.9992	−20.21	−218.9

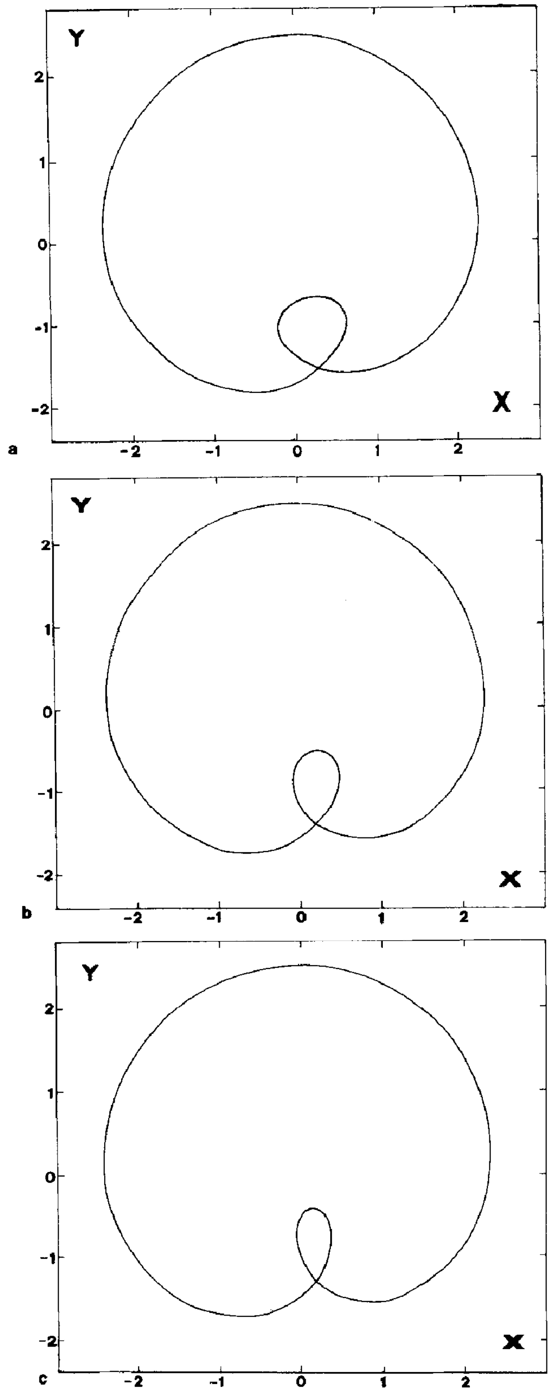

Fig. 1a–c. Typical orbits of the series A^{g}_{20} with the initial conditions: 1(a). $x_{01} = -2.330\,48$, $x_{03} = 0.749\,905$, $E = -0.093\,803$; 1(b). $x_{01} = -2.336\,09$, $x_{03} = 0.719\,556$, $E = -0.112\,165$; 1(c). $x_{01} = -2.386\,94$, $x_{03} = 0.693\,459$, $E = -0.122\,642$.

4. Results

Following the considerations and techniques described in the previous paragraphs, we started the computation of asymmetric periodic solutions of the general three body problem using initial conditions of such solutions of the restricted problem given by Markellos (1977) and Hénon (1965) for values of the mass parameter $\mu = 0.25$ and $\mu = 0.5$. This can be done since Hadjidemetriou (1975) has proved that almost all the periodic orbits of the restricted problem can be continued to the general problem by increasing the mass of the originally massless body.

(i) *Series* A^g_{20} *for* $\dot{x}_0 = -0.17292$.

The starting point ($x_0 = -2.3310, \dot{x}_0 = -0.17292, \dot{y}_0 = 1.9017$) belongs to the bifurcations series A_{20} given in Markellos (1977). The series presented here is formed by gradual increase of the mass m_3 of the third body, while μ is kept constant ($\mu = 0.25$). In Table I we list the initial conditions, the energy constant E, the period T and the stability parameters p and q of the selected orbits of this series.

The period of the orbits changes along the family decreasing from $T \simeq 12.4773$ to $T \simeq 10.9992$. Also the energy constant varies along the family from $E \simeq -0.093\,80$ to $E \simeq -0.122\,64$. The values of the stability parameter p and q in the last two columns imply that no stable orbits exist in the part of the series A^g_{20} we computed. In Figures 1(a)–(c) selected orbits of this series are presented in the (x, y) plane. In Figures 3 and 4 projections of the series A^g_{20} characteristic curve, in the (m_3, x_{01}) and (m_3, x_{05}) planes, are given.

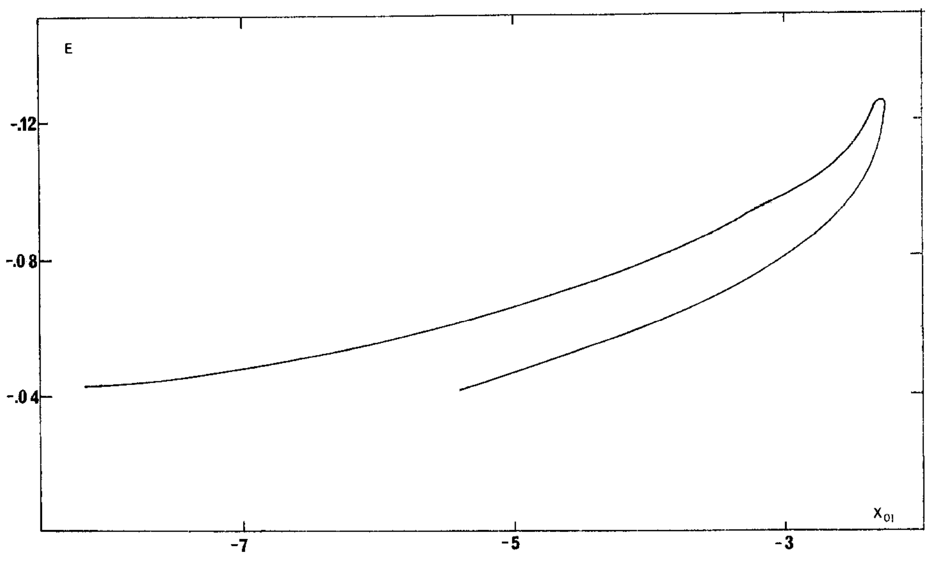

Fig. 2. Projection of the characteristic of the family A^g_{20} on the (E, x_{01}) plane.

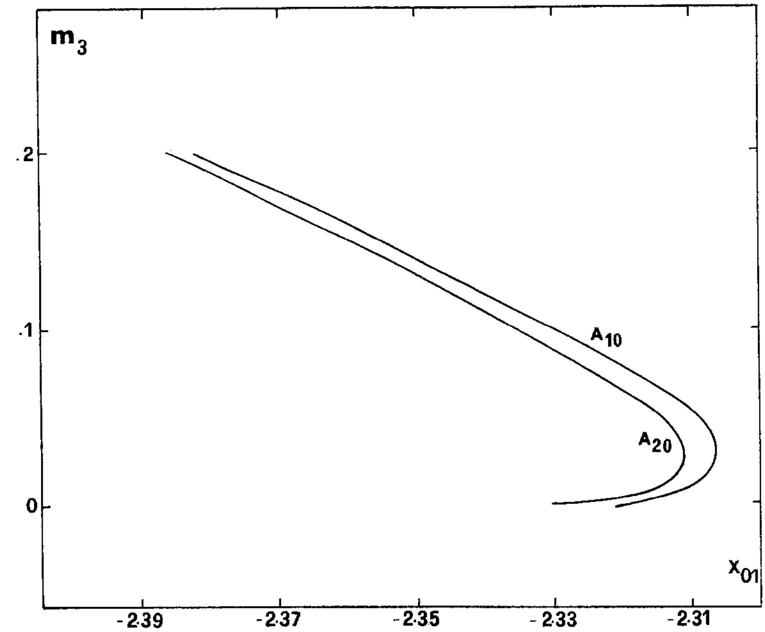

Fig. 3. Projections of the characteristics of the series A^g_{10} and A^g_{20} on the (m_3, x_{01}) plane.

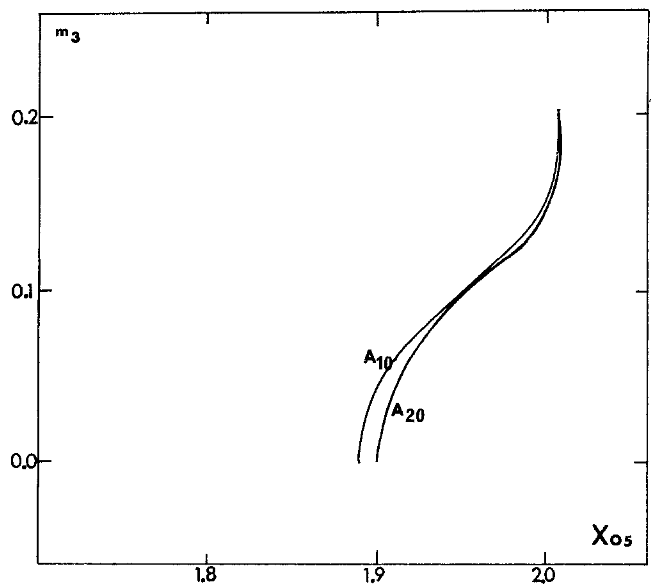

Fig. 4. Projections of the characteristics of the series A^g_{10} and A^g_{20} on the (m_3, x_{05}) plane.

(ii) *Family A_2^g for $m_3 = 0.2$.*

This is a family of asymmetric periodic orbits with starting point the member of the series A_{20}^g which corresponds to $m_3 = 0.2$. The initial condition which varies in the predictor step and is kept constant during the corrector step of the algorithms is $\dot{x}_0$. The initial velocity of the third particle decreases, reaches a local minimum and then increases to the infinity.

Numerical data for this family are given in Table II. As it is seen in this table the period T varies in a large interval. All orbits are unstable since there is no member of the family for which both inequalities $|p| < 2$, $|q| < 2$ hold.

In Figure 2 we illustrate the characteristic curve of this family in the (E, x_{01}) plane. The shape of the orbits of this family are very similar to the shape of the corresponding orbits of the series A_{20}^g.

(iii) *Series A_{10}^g for $\dot{x} = 0.174\,30$.*

This series is generated from an orbit ($x_0 = -2.3214$, $\dot{x}_0 = 0.174\,30$, $\dot{y}_0 = 1.8922$), member of the bifurcation series A_{10} of the restricted problem given in Markellos (1977). The series is also formed by gradual increase of the mass m_3 while μ is kept constant ($\mu = 0.25$).

In Table III the initial conditions, the energy constant, the period T and the values of the stability parameters p and q of selected orbits of this series are listed. All orbits are unstable.

In Figures 3 and 4 projections in the (m_3, x_{01}) and (m_3, x_{05}) planes of the characteristic curve of this series are presented.

(iv) *Series b_{10}^g for $\dot{x}_0 = -0.312$ and family b_1^g for $m_3 = 0.0052$.*

We used as starting point an orbit of the restricted problem given by Hénon (1965) for $\mu = 0.5$. This orbit is a member of the family bifurcated from the family b of symmetric periodic orbits originated from the collinear equilibrium point L_3. The initial conditions we started from, are $x_0 = -1.7156$, $\dot{x}_0 = -0.0312$, $E = 2.03$ for $m_3 = 0$. Increasing the value of m_3 from zero to about 0.0052 we obtain a series of periodic solutions. Numerical data for this series are given in Table IV. In Table V part of a family of periodic orbits which is continuation of the orbit with initial conditions given in the last entry of the Table IV, is given for $m_3 = 0.0052$. This series contains stable and unstable members. Illustration of an orbit of this family is given in Figure 5.

(v) *Series c_{10}^g for $\dot{x}_0 = 0.0245$ and family c_1^g for $m_3 = 0.05$.*

The starting point is an orbit of the restricted problem given by Hénon (1965) for $\mu = 0.5$. This orbit is a member of the bifurcated from the family c of symmetric periodic orbits originated from the collinear equilibrium point L_1.

The generating initial conditions for this series are: $x_0 = -0.3057$, $\dot{x}_0 = 0.0245$,

TABLE II

The family A_2^g ($\mu = 0.25, m_3 = 0.2$)

A/A	x_{01}	x_{03}	x_{04}	x_{05}	x_{06}	E	T	p	q
1	−5.423 88	0.804 888	−0.093 180	5.064 04	0.063 152	−0.043 549	59.3443	3.957	−205.5
2	−4.942 83	0.798 672	−0.099 679	4.570 29	0.068 935	−0.047 732	51.4879	2.192	−191.7
3	−4.536 15	0.792 384	−0.106 680	4.151 91	0.074 868	−0.051 992	45.0386	0.629	−179.7
4	−3.742 35	0.776 020	−0.127 679	3.332 77	0.094 311	−0.063 166	32.9453	−2.232	−152.4
5	−3.013 28	0.752 251	−0.166 679	2.582 70	0.106 695	−0.079 678	22.2347	−4.125	−112.5
6	−2.824 01	0.743 318	−0.181 679	2.391 51	0.106 607	−0.086 081	19.4884	−4.821	−102.6
7	−2.716 63	0.737 352	−0.190 679	2.284 77	0.103 845	−0.090 496	17.9356	−5.455	−99.42
8	−2.554 64	0.7265 63	−0.203 679	2.128 18	0.092 689	−0.098 554	15.5751	−7.023	−100.8
9	−2.427 06	0.715 509	−0.210 809	2.013 12	0.070 922	−0.107 148	13.6167	−9.230	−112.8
10	−2.399 47	0.712 525	−0.211 237	1.990 52	0.062 586	−0.109 506	13.1536	−9.963	−118.2
11	−2.371 12	0.708 974	−0.210 620	1.699 33	0.050 786	−0.112 315	12.6368	−10.93	−128.3
12	−2.33072	0.699 861	−0.200 420	1.956 89	0.054 135	−0.119 303	11.4916	−14.40	−160.1
13	−2.364 18	0.694 373	−0.180 020	2.024 00	−0.048 940	−0.122 500	11.0215	−18.75	−204.4
14	−2.386 95	0.693 459	−0.172 920	2.056 99	−0.065 480	−0.122 041	10.9992	−20.22	−218.9
15	−2.390 50	0.693 356	−0.171 920	2.061 96	−0.067 760	−0.122 634	10.9993	−20.43	−220.9
16	−2.405 17	0.693 004	−0.168 020	2.082 06	−0.076 561	−0.122 547	11.0103	−21.26	−228.9
17	−2.422 49	0.692 675	−0.163 220	2.108 40	−0.087 221	−0.122 320	11.0399	−22.31	−238.7
18	−2.455 15	0.692 387	−0.156 620	2.147 51	−0.101 647	−0.121 810	11.1085	−23.82	−252.5
19	−2.615 18	0.692 775	−0.130 020	2.342 53	−0.158 522	−0.117 693	11.6983	−31.03	−312.5
20	−3.047 62	0.697 430	−0.088 020	2.83131	−0.250 358	−0.104 777	13.9447	−49.98	−442.6
21	−3.82 257	0.705 469	−0.050 200	3.666 80	−0.339 299	−0.085 830	18.8456	−88.30	−659.4
22	−5.270 09	0.715 245	−0.025 620	5.158 53	−0.420 948	−0.063 478	29.6942	−176.6	−1107
23	−6.923 07	0.721 596	−0.014 220	6.840 28	−0.467 523	−0.048 778	44.1405	−300.5	−1726
24	−7.672 75	0.723 798	−0.011 120	7.689 48	−0.482 906	−0.043 628	52.2050	−371.4	−2090
25	−8.186 70	0.724 738	−0.009 920	8.117 46	−0.489 381	−0.041 418	56.4487	−409.1	−2287

TABLE III

The series A^{g}_{10} ($\mu = 0.25, \dot{x}_0 = 0.174\,30$)

A/A	x_{01}	x_{03}	x_{05}	x_{06}	x	E	T	p	q
1	0.000 103	−2.321 50	0.749 959	1.892 29	0.000 042	−0.093 750	12.3018	−1.938	−15.07
2	0.0005 03	−2.321 60	0.749 793	1.892 43	0.000 138	−0.093 880	12.3183	−1.960	−17.70
3	0.001 503	−2.321 01	0.749 331	1.892 07	0.000 018	−0.094 166	12.3235	−2.003	−22.48
4	0.007 625	−2.313 36	0.746 429	1.887 62	−0.004 704	−0.095 946	12.1895	−2.254	−30.96
5	0.010 425	−2.310 97	0.745 272	1.886 82	−0.006 772	−0.096 672	12.1287	−2.383	−32.49
6	0.015 945	−2.30802	0.743 209	1.886 92	−0.010 358	−0.097 985	12.0285	−2.663	−35.07
7	0.024 965	−2.306 29	0.740 215	1.890 01	−0.015 255	−0.099 916	11.9025	−3.178	−39.23
8	0.034 965	−2.306 89	0.737 195	1.895 77	−0.019 833	−0.101 863	11.7952	−3.809	−44.20
9	0.059 650	−2.310 76	0.732 697	1.907 67	−0.026 062	−0.104 704	11.6622	−4.929	−53.15
10	0.069 650	−2.31423	0.730 002	1.916 09	−0.029 506	−0.106 352	11.5937	−5.661	−59.43
11	0.075 980	−2.320 34	0.726 044	1.929 54	−0.034 223	−0.108 679	11.5041	−6.840	−69.83
12	0.085 985	−2.324 81	0.723 443	1.938 85	−0.037 127	−0.110 141	11.4508	−7.669	−77.49
13	0.100 005	−2.331 42	0.719 818	1.952 22	−0.040 938	−0.112 080	11.3822	−8.886	−89.18
14	0.110 050	−2.336 32	0.717 239	1.961 90	−0.043 499	−0.113 390	11.3368	−9.796	−98.30
15	0.120 005	−2.341 31	0.714 659	1.971 66	−0.045 944	−0.114 641	11.2936	−10.74	−108.1
16	0.130 050	−2.346 36	0.712 073	1.981 42	−0.048 285	−0.115 835	11.2524	−11.72	−118.6
17	0.140 425	−2.351 69	0.709 370	1.991 73	−0.050 621	−0.117 019	11.2112	−12.79	−130.3
18	0.150 025	−2.356 60	0.706 869	2.001 20	−0.052 690	−0.118 056	11.1746	−13.82	−141.9
19	0.160 245	−2.361 85	0.704 192	2.011 28	−0.054 808	−0.119 104	11.1369	−14.90	−155.2
20	0.170 445	−2.367 09	0.701 505	2.021 32	−0.056 841	−0.120 094	11.1004	−16.15	−169.3
21	0.185 465	−2.374 78	0.697 515	2.036 06	−0.059 697	−0.121 448	11.0484	−18.01	−191.9
22	0.195 085	−2.379 68	0.694 936	2.045 47	−0.061 445	−0.122 250	11.0160	−19.26	−207.7
23	0.2	−2.382 17	0.693 612	2.050 26	−0.062 315	−0.122 641	11.9999	−19.93	−216.1

TABLE IV
The series b^g_{10} ($\mu = 0.5, \dot{x}_0 = -0.0312$)

A/A	m_3	x_{01}	x_{03}	x_{05}	x_{06}	E	T	p	q
1	0.000 001	−1.715 57	0.500	1.478 55	0.0	−0.124 999	5.8040	0.611	−1.775
2	0.000 203	−1.716 84	0.500 100	1.479 74	0.000 026	−0.124 947	5.8129	0.608	−1.777
3	0.001 001	−1.722 09	0.500 490	1.484 48	0.000 130	−0.124 737	5.8479	0.596	−1.786
4	0.003 600	−1.739 40	0.501 769	1.500 14	0.000 464	−0.124 041	5.9645	0.570	−1.817
5	0.005 207	−1.750 22	0.502 549	1.509 94	0.000 666	−0.123 601	6.0379	0.560	−1.835

TABLE V
The family b^g_1 ($\mu = 0.5, m_3 = 0.0052$)

A/A	x_{01}	x_{03}	x_{04}	x_{05}	x_{06}	E	T	p	q
1	−1.750 22	0.502 549	−0.0312	1.509 94	0.000 666	−0.123 601	6.0378	0.560	−1.835
2	−1.750 69	0.502 608	−0.0433	1.507 60	0.009 623	−0.123 578	6.0450	1.537	−1.831
3	−1.751 86	0.502 761	−0.6290	1.502 30	0.001 533	−0.123 367	6.0957	1.759	−1.854
4	−1.753 91	0.503 038	−0.0842	1.494 70	0.002 380	−0.123 067	6.1618	1.516	−1.888
5	−1.755 73	0.503 289	−0.0970	1.489 55	0.003 078	−0.122 797	6.2148	2.081	−1.914
6	−1.756 89	0.503 448	−0.1034	1.486 94	0.003 501	−0.122 628	6.2459	2.390	−1.927

TABLE VI

The series C^g_{10} ($\mu = 0.5, \dot{x}_0 = 0.0245$)

A/A	m_3	x_{01}	x_{03}	x_{05}	x_{06}	E	T	p	q
1	0.000 1	−0.262 428	0.500 098	1.792 96	−0.000 007	−0.125 026	5.2664	−0.432	−1.068
2	0.001 025	−0.263 136	0.500 976	1.792 56	−0.000 067	−0.125 265	5.2753	−0.237	−1.186
3	0.010 027	−0.270 154	0.509 564	1.789 25	−0.000 698	−0.127 526	5.3642	0.880	−1.588
4	0.021 229	−0.279 209	0.520 295	1.786 68	−0.001 605	−0.130 147	5.4791	1.857	−1.769
5	0.030 031	−0.286 670	0.528 785	1.786 19	−0.002 449	−0.132 053	5.5745	2.485	−1.852
6	0.05	−0.305 990	0.548 563	1.794 75	−0.005 289	−0.135 782	5.8335	3.440	−1.968
7	0.058 437	−0.319 484	0.557 989	1.819 88	−0.008 631	−0.136 739	6.0517	2.845	−2.006

TABLE VII

The family C^g_1 ($\mu = 0.5, m_3 = 0.05$)

A/A	x_{01}	x_{03}	x_{04}	x_{05}	x_{06}	E	T	p	q
1	0.306 010	0.548 580	0.026	1.794 73	−0.005 623	−0.135 767	5.8335	3.629	−1.966
2	0.306 160	0.548 703	0.035	1.794 55	−0.007 668	−0.135 655	5.8495	4.999	−1.955
3	0.306 269	0.548 789	0.040	1.794 46	−0.008 845	−0.135 577	5.8594	5.946	−1.952
4	0.306 555	0.549 006	0.050	1.794 31	−0.011 794	−0.135 381	5.8841	8.242	−1.953
5	0.306 966	0.549 289	0.060	1.794 35	−0.014 009	−0.135 123	5.9165	11.087	−1.964
6	0.307 545	0.549 657	0.070	1.794 74	−0.017 036	−0.134 787	5.9591	14.512	−1.984
7	0.308 438	0.550 150	0.080	1.796 02	−0.020 634	−0.134 337	6.0163	18.523	−2.013
8	0.309 637	0.550 705	0.088	1.798 68	−0.024 348	−0.133 832	6.0815	22.153	−2.045

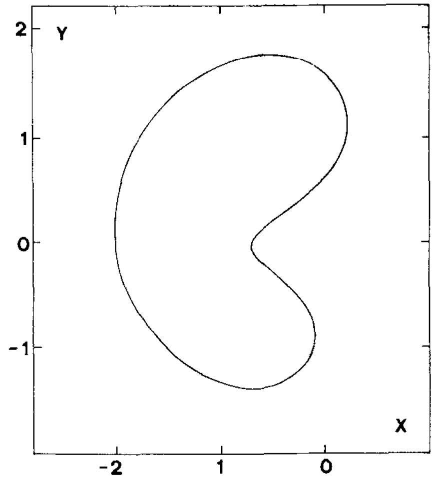

Fig. 5. Typical orbit member of the family b_1^g.

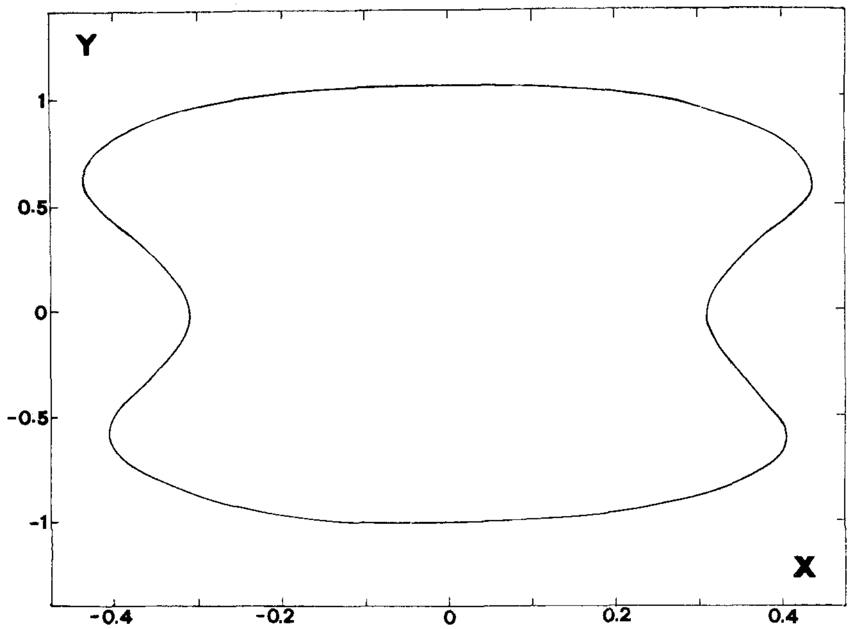

Fig. 6. Typical orbit member of the family c_1^g.

$E = 2.373$. For $\dot{x}_0 = x_{04} = 0.245$ = const. we vary gradually the value of the third body m_3 and compute a number of assymmetric periodic orbits for different values of m_3. The initial conditions thus obtained, are listed in Table VI. The last periodic solution included in Table VI is continued for m_3 = constant, to a family of such solutions. The numerical results obtained are listed in Table VII. In Figure 6 illustration of an orbit of the family c_1^g is given.

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