FUTURE DYNAMICAL EVOLUTION OF THE NEPTUNE–TRITON SYSTEM. A NEW SYNTHETIC METHOD OF ANALYSIS

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Abstract. A new approach to the study of future dynamical evolution of the Neptune–Triton system is here presented. After the analytical development of the model, the final results are compared with those drawn by previous works on the same subject.

1. General

The dynamical evolution of Neptune–Triton system was studied following a general path of analysis, concerning the planet–satellite system, by some authors (Goldreich and Soter, 1966). Furthermore, McCord (1966), starting from a previous study of Mc Donald (1964), worked out the differential equations which gave the changes of orbital elements of the satellite as functions of the perturbing force field arising from tidal effects. Hence, the solution of the problem provided the complete dynamical history of the system.

In the present paper a quite different approach is proposed in order to obtain the analysis of the system evolution; the results partially confirm those of McCord and partially provide some improvements in the description of this astronomical phenomenon.

2. First Approach to the Model

The system under study is schematically outlined in Figure 1. The two-body problem is perturbated by tidal effects on Neptune. The planet is to be considered as an extended deformable body; and its shape will be regarded as that of a prolate spheroid an account of tidal action exerted by Triton (a mass-point in the model). The motion of this last mass-point T will be considered around the mass centre of Neptune 0. The polar coordinates of T, in the orbital plane, are r and α. The motion of Triton is retrograde and the osculating orbit is assumed circular, owing to its present very low eccentricity. Neptune's own prograde rotation motion is described by the angle ϕ between the spheroid semi-major axis and x-axis. In fact the plane of Triton's orbit is not perpendicular to Neptune's spin axis; this will be considered later.

Since the satellite's revolution period is longer than the planet's rotation period, the lagging tide is carried ahead of the satellite by an angle ϵ. The relation of the tidal dissipation parameter Q to this lag angle ϵ is of MacDonald, given by

$$Q = \frac{1}{\tan 2\epsilon} \cong \frac{1}{2\epsilon}. \tag{1}$$

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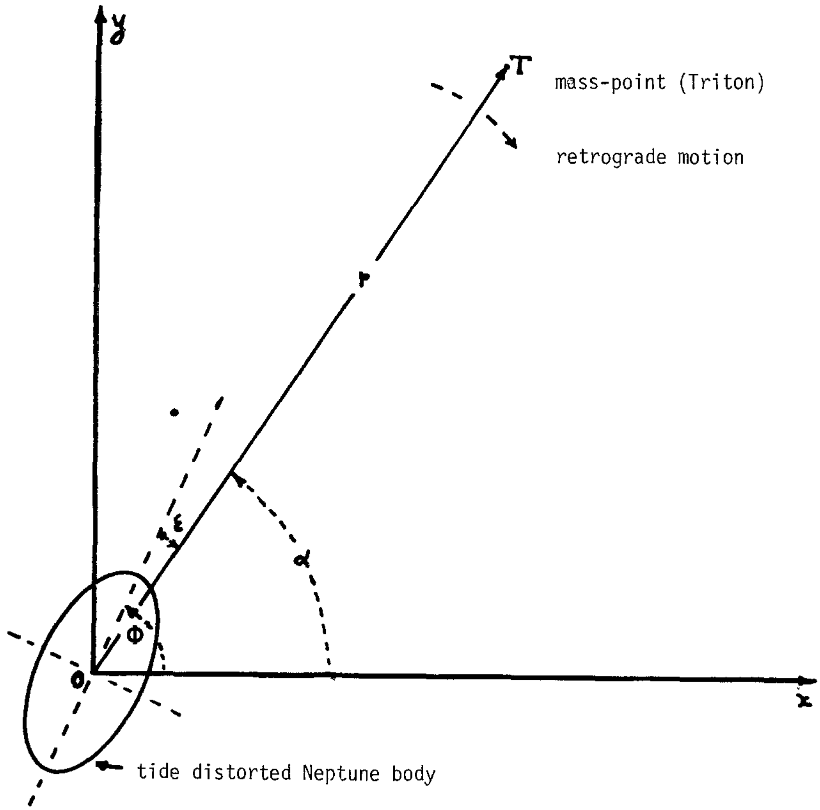

Fig. 1. Qualitative sketch of the Neptune–Triton system geometry.

Starting from the to-day system situation there is an action of the tidal torque to retard Neptune's rotation rate and to exert a tangential force (of resisting type) on Triton. The calculation of this tidal torque C provides the formula (cf. Goldreich, 1966)

$$C = \frac{9}{4} G m_2^2 \frac{A^5}{a^6} \frac{1}{Q} = \frac{K_1}{a^6} \frac{1}{Q}, \tag{2}$$

in which
G = constant of gravitation
m_2 = Triton's mass,
A = Neptune's average radius,
a = Triton's osculating orbit radius.

A simple differential equation is established expressing the energy balance. If v_α is the tangential component of Triton's velocity, in the unperturbed Keplerian motion, i.e., $v_\alpha = \sqrt{G(m_1 + m_2)/a} \cong \sqrt{Gm_1/a}$ (m_1 = Neptune's mass), the product $F_{RT} v_\alpha \, dt$ is the work done by resisting force $F_{RT} = C/a$ (exerted on Triton by tidal effects) during the time interval dt. This work is done at the expense of the binary system bounding energy.

Therefore, the equation is

$$\frac{\mathrm{d}a}{dt} = -\frac{9}{2}\left(\frac{G}{m_1}\right)^{1/2}\frac{m_2}{Q}\frac{A^5}{a^{11/2}}. \tag{3}$$

Differentiation of Kepler's third law $n^2a^3 = Gm_1$ gives the expression

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{3}{2}\frac{n}{a}\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{4}$$

which, combined with (3), provides the differential equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{27}{4}G\frac{A^5}{Q}\frac{m_2}{a^8}. \tag{5}$$

Equations (3) and (5) represent the simplest approximate relationships which contain secular changes of osculating radius a and of satellite mean motion n.

The present model endeavours to improve considerably Equation (3). More precisely, the angle ϵ is considered, instead constant during the evolution, variable and also Q^{-1} which becomes a certain function $f(\epsilon)$. The choice of function $f(\epsilon)$ is achieved *by means of the calculus of variations*. In other words, it is assumed that during a complete evolution of the motion of the system – till Triton plunges into Neptune – the energy dissipation into heat, because of tidal effects, will be a minimum. On the basis of this principle, a fundamental differential equation, which describes the secular variations of a, is obtained. The equation solution allows to point out the basic features of the evolution under study.

3. Analytical Development of the Model

At any instant or epoch t the power dissipation of the system is given by

$$p(t) = C\frac{\mathrm{d}\phi}{\mathrm{d}t} + F_{RT}v_\alpha, \tag{6}$$

in which

ϕ = rotation angle of Neptune (Figure 1),
C = tidal torque,
F_{RT} = tangential force on Triton (in modulus),
v_α = Triton tangential velocity (in modulus).

Using the (2), since $v_\alpha = \sqrt{Gm_1/a}$,

$$F_{RT}v_\alpha = \frac{C}{a}\sqrt{Gm_1/a} = K_1\sqrt{Gm_1}\frac{1}{a^{15/2}}f(\epsilon) = K_1\sqrt{Gm_1}\frac{1}{a^{15/2}}f(\phi-\alpha)$$

where $\epsilon = \phi - \alpha$. Then the term $C(\mathrm{d}\phi/\mathrm{d}t)$ becomes

$$C\frac{\mathrm{d}\phi}{\mathrm{d}t} = C\frac{\mathrm{d}}{\mathrm{d}t}(\alpha+\epsilon) = C\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + C\frac{\mathrm{d}\alpha}{\mathrm{d}t}.$$

Furthermore, since $v_\alpha = a(\mathrm{d}\alpha/\mathrm{d}t)$, we will have

$$C\frac{\mathrm{d}\phi}{\mathrm{d}t} = C\sqrt{Gm_1}\frac{1}{a^{3/2}} + C\frac{\mathrm{d}\epsilon}{\mathrm{d}t}. \tag{7}$$

Writing $\mathrm{d}\epsilon/\mathrm{d}t$ so that

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \sqrt{Gm_1}a^{-3/2}\frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha} \tag{8}$$

and substituting (8) into (7) we have

$$C\frac{\mathrm{d}\phi}{\mathrm{d}t} = C\sqrt{Gm_1}\frac{1}{a^{3/2}} + C\sqrt{Gm_1}\frac{1}{a^{3/2}}\frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha} = C\frac{\sqrt{Gm_1}}{a^{3/2}}\left(1 + \frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha}\right).$$

The dissipated power p, as a function of α, will be given by

$$p(\alpha) = C\sqrt{Gm_1}\frac{1}{a^{3/2}}\left(2 + \frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha}\right) = K_1\sqrt{Gm_1}\frac{1}{a^{15/2}}\left[2f(\epsilon) + f(\epsilon)\frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha}\right]. \tag{9}$$

This remains substantially constant during a unique revolution. Let $\mathscr{N}$ be the total number of Triton's revolutions till the future encounter with Neptune; furthermore p_i is the dissipation power relevant to the i-th revolution having period T_i. The total dissipated energy will be

$$\sum_1^{\mathscr{N}} {}_ip_iT_i = E_d.$$

The time interval of the complete evolution of the system is

$$\tau = \sum^{\mathscr{N}} {}_iT_i.$$

Then we will have

$$P_m = \frac{1}{\tau}\sum_1^{\mathscr{N}} {}_ip_iT_i \tag{10}$$

in which P_m is the *average dissipated power during the motion complete evolution*. From (10) we deduce that

$$P_m\tau = E_d. \tag{11}$$

Let us apply not the principle just said. Among all the evolution configurations, logically possible and of time duration τ, the natural one is characterized by the minimum of E_d, i.e., following the (11), of P_m. This last quantity is expressed by

$$P_m = K_1\sqrt{Gm_1}\frac{1}{2\pi\mathscr{N}}\int_0^{2\pi\mathscr{N}}\left[-f(\epsilon)\frac{1}{a^{15/2}}\frac{\mathrm{d}\epsilon}{\mathrm{d}\vartheta} + 2\frac{f(\epsilon)}{a^{15/2}}\right]\mathrm{d}\vartheta, \tag{12}$$

where $\mathrm{d}\vartheta = -\mathrm{d}\alpha$ owing to retrograde motion of Triton.

The statement is very similar of that in many other fields of physical sciences. In fact, for instance, in dynamics there is *the Gauss' minimum constraint principle*. Also in the theory of electricity this principle is demonstrated: "When a steady current flows through a network of conductors, the currents are distributed in such a way that the rate of generation of heat in the network is a minimum" (Jeans, 1948).

The integral I, which will be minimized choosing the extremal function $f(\epsilon)$, is provided by the expression

$$I = \int_0^{2\pi\mathcal{N}} \left[-\frac{f(\epsilon)}{a^{15/2}} \frac{\mathrm{d}\epsilon}{\mathrm{d}\vartheta} + 2\frac{f(\epsilon)}{a^{15/2}} \right] \mathrm{d}\vartheta = \int_0^{2\pi\mathcal{N}} F(\epsilon, \dot{\epsilon}, \vartheta)\, \mathrm{d}\vartheta, \tag{13}$$

because a is a function of ϑ and $\dot{\epsilon} = (\mathrm{d}\epsilon/\mathrm{d}\vartheta)$.

The Euler–Lagrange equation yields

$$\frac{\partial F}{\partial \epsilon} = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{\partial F}{\partial \dot{\epsilon}}\right). \tag{14}$$

Let us calculate the relative terms:

$$\frac{\partial F}{\partial \epsilon} = -\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{f(\epsilon)}{a^{15/2}}\dot{\epsilon} + 2\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon}\frac{1}{a^{15/2}}, \tag{15}$$

$$\frac{\partial F}{\partial \dot{\epsilon}} = -\frac{f(\epsilon)}{a^{15/2}}, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{\partial F}{\partial \dot{\epsilon}}\right) = -\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon}\dot{\epsilon}\frac{1}{a^{15/2}} - f(\epsilon)\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{1}{a^{15/2}}\right). \tag{17}$$

Equation (14) becomes

$$-\frac{1}{a^{15/2}}\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon}\dot{\epsilon} + 2\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon}\frac{1}{a^{15/2}} = -\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon}\dot{\epsilon}\frac{1}{a^{15/2}} - f(\epsilon)\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{1}{a^{15/2}}\right),$$

and then

$$\frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\epsilon} = \frac{15}{4}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\vartheta}. \tag{18}$$

The extremal function $f(\epsilon)$ satisfies the differential Equation (17). Coming back to (3) in paragraph 2 and putting $f(\epsilon)$ in place of Q^{-1} and also $n\,\mathrm{d}t = \mathrm{d}\vartheta$, $K_0 = \frac{9}{2}(m_2/m_1)A^5$, we obtain

$$f(\epsilon) = -\frac{a^4}{K_0}\frac{\mathrm{d}a}{\mathrm{d}\vartheta}. \tag{19}$$

We can establish then the following system of differential equations:

$$\left.\begin{aligned} \frac{\mathrm{d}f(\epsilon)}{\mathrm{d}\vartheta} &= \frac{15}{4}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\vartheta}\frac{\mathrm{d}\epsilon}{\mathrm{d}\vartheta}, \\ f(\epsilon) &= -\frac{a^4}{K_0}\frac{\mathrm{d}a}{\mathrm{d}\vartheta}, \end{aligned}\right\} \tag{20}$$

in which the unknown functions $f(\epsilon)$ and $a(\vartheta)$ appear. By elimination of function $f(\epsilon)$, we deduce this differential equation in the unknown function $a(\vartheta)$, in terms of the derivative $\mathrm{d}\epsilon/\mathrm{d}\vartheta$ as

$$\frac{\mathrm{d}^2 a}{\mathrm{d}\vartheta^2} + \frac{4}{a}\left(\frac{\mathrm{d}a}{\mathrm{d}\vartheta}\right)^2 + \frac{15}{4}\frac{K_0}{a^5}\frac{\mathrm{d}\epsilon}{\mathrm{d}\vartheta}\frac{\mathrm{d}a}{\mathrm{d}\vartheta} = 0. \tag{21}$$

Let us study now the behaviour of the term $\mathrm{d}\epsilon/\mathrm{d}\vartheta$ to eliminate; and consider the angle ϵ and its nature. The relative mean motion n_r, resulting from Triton's retrograde motion and from angular velocity of Neptune's rotation is simply

$$n_r = n + \omega_N, \tag{22}$$

in which
n = Triton's mean motion,
ω_N = angular velocity of Neptune's rotation.
Instead of considering the angular lag, after which the maximum of tide is felt on Neptune, let us point out the corresponding time delay Δt. Then we shall have

$$\epsilon = n_r \Delta t. \tag{23}$$

Thus a new concept is introduced namely, as a first approximation, *Δt is considered constant during the entire evolution of motion*. Then we shall write

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \Delta t \frac{\mathrm{d}}{\mathrm{d}t}(\omega_N + n) = \Delta t \left(\frac{\mathrm{d}\omega_N}{\mathrm{d}t} + \frac{\mathrm{d}n}{\mathrm{d}t}\right). \tag{24}$$

Using the following formula which provides the rate of despin of a planet due to tidal torque from its satellite (Goldreich, 1966)

$$\frac{\mathrm{d}\omega_N}{\mathrm{d}t} = -\frac{9}{4} G \frac{m_2^2}{m_1} \frac{A^3}{\beta Q a^6}, \tag{25}$$

in which the numerical coefficient β is assumed to be $\frac{1}{3}$, we get

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \Delta t \left(\frac{27}{4} G \frac{A^5 m_2}{Q\ a^8} - \frac{9}{4} \frac{m_2^2}{m_1} \frac{A^3}{\beta Q a^6}\right). \tag{26}$$

In the expression (26) we also used the (5) of section 2. From the (26) it should be very easy to deduce $\mathrm{d}\epsilon/\mathrm{d}\vartheta$, however it is better, in view of the successive calculations, to transform (21) in order to mean t as independent variable. The final result is

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} + \frac{11}{2}\frac{1}{a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 + \frac{15}{4} K_0 (G m_1)^{1/2} \left(\frac{k_1}{a^{13}} - \frac{k_2}{a^{11}}\right) \frac{\mathrm{d}a}{\mathrm{d}t} = 0 \tag{27}$$

in which

$$k_1 = \frac{27}{4}\frac{\Delta t}{Q} G^{1/2} m_1^{-1/2} m_2 A^5,$$

$$k_2 = \frac{9}{4}\frac{\Delta t}{Q} G^{1/2} \frac{m_2^2}{m_1^{3/2}} \frac{A^3}{\beta};$$

and other symbols have already been defined.

Equation (27) is the desired fundamental equation which describes the secular variation of a. The assumptions underlying its derivation are:

(1) the condition of minimum energy dissipation during the entire evolution of motion,

(2) the time delay Δt is considered constant as well as Q (equal to inital value), within the expressions of k_1 and k_2.

As regards to point (2) it is assumed Q constant, instead of equal to $1/f(\epsilon)$ as it should be rigorously. It is to be noted that the form of Equation (27), and the initially high value of Q (order of 10^4)allow to consider the resulting error to be small, owing to the minor influence of third term or (27). In any case this is the risk for the present model; in fact there is a compromise between a not too complicated differential equation and a very rigorous analytical description of the physical problem.

4. Solution of Equation (27) and Discussion of the Obtained Results

Let us integrate (27), after the substitution of numerical values concerning Triton–Neptune system. Then we have:

$$m_1 = 10.6 \times 10^{28}\,\text{g} \qquad A = 22.4 \times 10^8\,\text{cm}$$

$$m_2 = 3.4 \times 10^{26}\,\text{g} \qquad \Delta t = 57 \times 10^{-3}\,\text{s}$$

$$G = 6.67 \times 10^{-8}\,\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2} \qquad Q = 7.2 \times 10^4.$$

It was assumed, as central value, Neptune's Q equal to 7.2×10^4 (Goldreich, 1966). Putting

$$x = \frac{t}{t_0}, \qquad y = \frac{a}{A},$$

in which $t_0 = 10^8\,\text{yr} = 31.54 \times 10^{16}\,\text{s}$, the (27) is rewritten in the normalized form

$$\frac{d^2y}{dx^2} + 5.5\frac{1}{y}\left(\frac{dy}{dx}\right)^2 + \frac{1.76}{y^{13}}\frac{dy}{dx} - \frac{5.9}{y^{11}}10^{-3}\frac{dy}{dx} = 0. \tag{28}$$

For $x = 0$, we have $y = 15.8$ and also $dy/dx = -0.127$ [related to before said central value Q in connection to formula (3)]. The analytical details of solution method are reported in Appendix. The results of integration are plotted in Figure 2 (assuming the central value of Q). It should be noted that some authors consider this central value low (Farinella *et al.*, 1980). In the same Figure 2 there is the graph deriving from McCord calculations (1966), with the same Q value. Furthermore the Table I collectes the time intervals τ necessary to Triton to plunge into Neptune, corresponding to the following values: $\frac{1}{10}Q$, Q and $10Q$.

As a first comment, it is to be noted that the present model provides lower values of the time interval τ; as a second comment there is an exact proportionality between Q's and τ's in McCord model, whereas we note a certain deviation from it in the present model (more deviation when dissipation is higher).

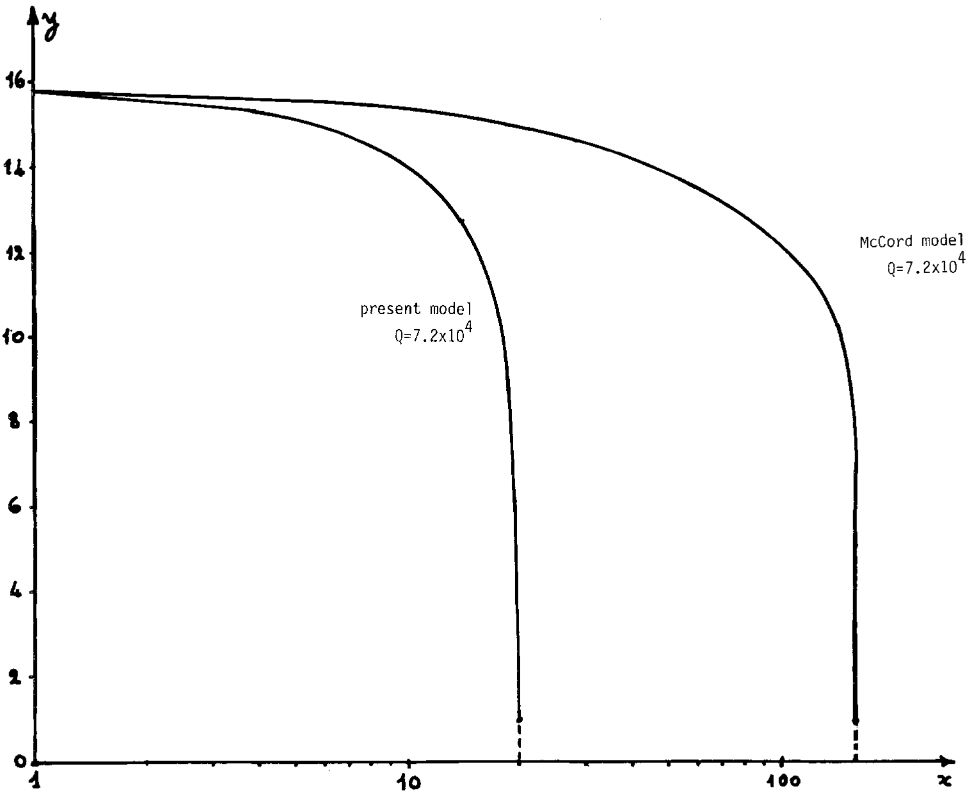

Fig. 2. The future variation of semimajor axis a of Triton's orbit (in normalized form). A comparison between the models.

TABLE I

Values of Q	Present model	McCord model
7.2×10^3	$\tau = 2.8 \times 10^8$ yr	$\tau = 1.11 \times 10^9$ yr
7.2×10^4	$\tau = 2 \times 10^9$ yr	$\tau = 1.11 \times 10^{10}$ yr
7.2×10^5	$\tau = 2.2 \times 10^{10}$ yr	$\tau = 1.11 \times 10^{11}$ yr

Since Equation (27) was written with minimal dissipation condition, McCord results seems to be less reasonable than those of the present model. Indeed the dissipation, during the complete evolution, cannot be less than minimum previously fixed.

As a final observation let us come back to a matter seen in Section 2. Neptune's spin axis is not perpendicular to Triton orbital plane; in fact there is an inclination angle of about 21° (Figure 3). The vector $\bar{\omega}_N$ is split into the two components $\bar{\omega}_{NN}$ and $\bar{\omega}_{NS}$. The former is the fundamental vector in the considered dynamical phenomenon; the latter has no influence in the process. In fact the planetary body, which is distorted by tidal effects, is of spheroidal shape (in the first approximation); furthermore this prolate spheroid has a symmetry (or revolution) axis directed as $\bar{\omega}_{NS}$. The demonstration of this

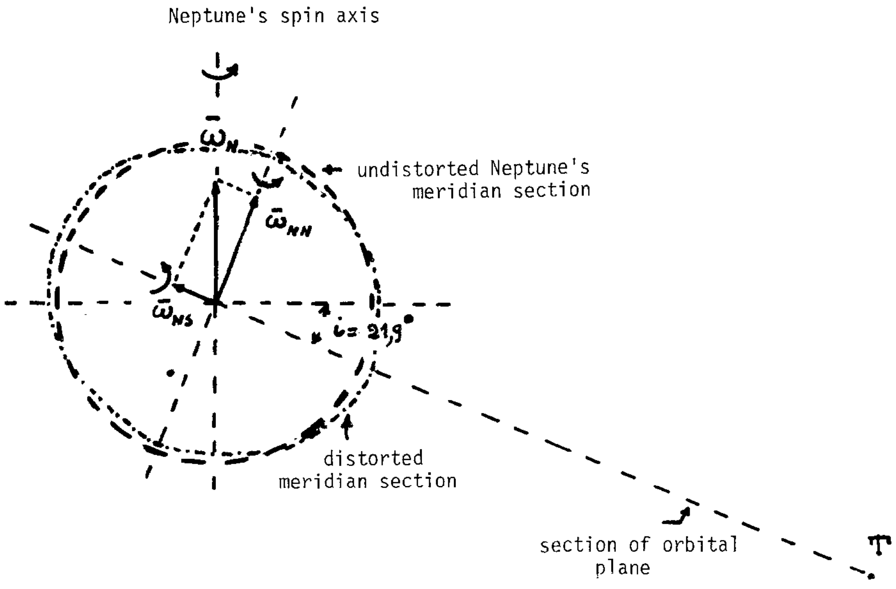

Fig. 3. Qualitative sketch to demonstrate the time-independance of inclination angle i during the complete evolution motion.

fact follows also, starting from another analytical point of view, from McCord paper; indeed in this study the inclination angle is proved to be time-independent during the complete evolution motion (McCord, 1966).

Appendix

The purpose of this appendix is to illustrate the solution method of Equation (28). Introducing the auxiliary variable p defined by

$$p = \frac{dy}{dx},$$

$$\frac{d^2y}{dx^2} = \frac{dp}{dy}p,$$

the (28) becomes

$$p\frac{dp}{dy} + \frac{5.5}{y}p^2 + \frac{1.76}{y^{13}}p - \frac{5.9 \times 10^{-3}}{y^{11}}p = 0.$$

If $p \neq 0$, dividing both members by p we have

$$\frac{dp}{dy} + \frac{5.5}{y}p = -\frac{1.76}{y^{13}} + \frac{5.9 \times 10^{-3}}{y^{11}}. \tag{A1}$$

This is an equation readily integrable, in the unknown function $p = p(y)$, of this type (Loiero, 1982)

$$\frac{dp}{dy} + M(y)p = N(y). \tag{A2}$$

Assuming a solution of the following type

$$p = K(y)\, e^{-\int M(y)\,dy}$$

we deduce that

$$\frac{dp}{dy} = K(y)\, e^{-\int M(y)\,dy}[-M(y)] + \frac{dK(y)}{dy} e^{-\int M(y)\,dy}$$

Inserting dp/dy in (A2) we have

$$-M(y)\, e^{-\int M(y)\,dy} K(y) + \frac{dK(y)}{dy} e^{-\int M(y)\,dy} + M(y)K(y)\, e^{-\int M(y)\,dy} = N(y),$$

and then

$$\frac{dK(y)}{dy} = N(y)\, e^{\int M(y)\,dy},$$

$$K(y) = \int N(y)\, e^{\int M(y)\,dy}\,dy + C_0; \tag{A3}$$

with C_0 arbitrary constant of integration. Finally we have

$$p = \left[\int N(y)\, e^{\int M(y)\,dy} dy + C_0\right] e^{-\int M(y)\,dy}$$

Since $M(y) = 5.5/y, N(y) = -1.76/y^{13} + 5.96/y^{11}\, 10^{-3}$, from (A3) we obtain

$$p = \frac{dy}{dx} = 0.27y^{-12} - 1.31 \times 10^{-3}y^{-10} + C_0 y^{-5.5}. \tag{A4}$$

Therefore we get the $p = dy/dx$ versus y, in the field $1 \leqslant y \leqslant 15.8$. The constant C_0 is got putting $dy/dx = -0.127$ for $x = 0$; the result for C_0 is -0.5×10^6. Then (A4) becomes

$$\frac{dy}{dx} = \frac{0.27 - 1.31 \times 10^{-3}y^2 - 0.5 \times 10^6 y^{6.5}}{y^{12}}; \tag{A5}$$

and this equation can be integrated by numerical or graphical methods.

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