EXPONENTIAL DISTANCE RELATIONS IN PLANETARY-LIKE SYSTEMS GENERATED AT RANDOM

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Abstract. It is often thought that the exponential distance relations that can be found amongst the distances of the planets and of the satellites of Jupiter, Saturn, and Uranus, of the form $a_n = \alpha\beta^n$, with a_n being the semi-major axis of the n^{th} body, can be similarly represented by sequences of sorted random numbers generated with some constraints corresponding to certain physical processes. We give in this paper some indications showing that pure chance or random processes only cannot explain the planetary and satellite distance distributions, in particular the exponential spacings, by comparing the distance relations of the real systems to these of planetary-like systems generated at random.

Exponential distance relations for the present planetary and satellites systems of Jupiter, Saturn and Uranus are described, considering the two cases without and with introduction of 'holes' for the large spacings observed in the sequences of bodies.

Random systems are created by generating distances at random following uniform, normal and exponential distributions, with no consideration for other orbital elements or masses as we are only interested in distance relations. Random systems without constraints are first compared to the real systems. In a following step, random systems with a corresponding number of bodies to that of the real systems and with the constraint of having a number of large spacings equivalent to that of the real systems are investigated. In a later step, we impose on the generation process the additional constraint of the 'closeness not too close' condition, i.e. for the random systems to have distances between adjacent 'bodies' greater than critical attraction distances calculated by considering the present masses of the real main planets and satellites.

Comparisons of the regression coefficients means of the exponential distance relations of random systems to the characteristics of the real systems show that there are significant differences, in particular the coefficients β of the random systems are on average smaller than for the corresponding real systems, except for some particular cases which are shown not to be significant.

It is concluded that distance relations observed in the present real systems can not be compared to sequences of sorted random numbers. Furthermore, additional physical processes other than the 'closeness not too close', have to be considered to explain the observed distance relations and in particular the exponential spacings.

1. Introduction

The problem of distance distributions in planetary and satellites systems has always been a matter of controversy among astronomers. Two major points of controversy exist: first, the way to represent the distribution of planetary and satellite distances, or how to find a simple mathematical relation between distances of natural bodies revolving around a central body; secondly, is there any physical reason for a distance distribution or is such a distribution at random, subject to certain physical constraints?

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In 1766, Titius made the first attempt of representation of planetary distances with a relation giving empirically the semi-major axis a_n in AU of the n-th planet

$$a_n = 0.4 + (0.3 \times 2^n) \quad (1)$$

for the values of n: $-\infty$ for Mercury, 0 for Venus, 1 for Earth and so on. Representations of planetary distances are numerous. A description of most of them can be found elsewhere (Nieto, 1972, Melchior, 1947 and references therein). Relations similar to (1) can also be found for the satellite systems of Jupiter, Saturn, and Uranus (Nieto, 1970). It is striking that most of the distance representations have a semi-exponential form (a constant added to a pure exponential). It was shown (Basano and Hugues, 1979) that all planet distances accurately fit a pure exponential relation

$$a_n = \alpha\beta^n \quad (2)$$

with $\alpha = 0.2853$AU and $\beta = 1.5226$ for successive integral values of n (1 for Mercury, 2 for Venus, etc.) when considering three 'holes' or 'missing planets'. Two 'holes' correspond to the asteroidal belt; Chiron is found at the third 'hole' location. In a previous paper (Pletser, 1986), we adapted the hypothesis of 'missing bodies' to the satellite systems of Jupiter, Saturn and Uranus. All the 'holes' in the internal part of the Jupiter and Saturn systems were filled by rings and small satellites. The new Uranian rings and small satellites discovered by Voyager 2 (Stone and Miner, 1986, Lane *et al.*, 1986) fit well into the two remaining 'holes' of the Uranian system. The simplicity of the pure exponential form (2) has to be outlined as no arbitrary constant is needed in obtaining an accurate representation. These two factors, simplicity and accuracy of the representation, are of prime importance.

The question concerning the physical or cosmogonic reasons to explain the observed distances distributions is more difficult to answer. Some authors do believe that there is no evidence for physical reasons. It was argued (Lecar, 1973) that the spacing ratio expressed in the Titius-Bode's relation can be generated by a sequence of random numbers, subject to the constraint that adjacent values cannot be "too close to each other". The physical reason for this constraint is that if two planets were too close to each other during the accretion process, they would coalesce or cease to grow because they were competing for the same accretion material. Although the physical reason of the 'closeness not too close' condition is perfectly correct, we feel that the conclusion drawn is too speculative. Nevertheless, several theories of the solar system formation have attempted to account for these distance relations; references can be found in (Nieto, 1972). Dermott (1972) showed that the orbit distributions of the Jovian and Saturnian systems are non-random considering the resonances and the preference for near-commensurabilities among the satellites, while the planetary orbit distribution can be regarded as non-random. In the same respect, the Uranian system is dubious (Dermott, 1973).

The purpose of this paper is to give some indications showing that pure chance or random processes can not be invoked to explain the planetary and satellite distance distributions. It is intended to see if exponential distance relations of the form (2) can be found for distances generated at random following uniform, normal and exponential distributions and to compare the random systems to the real systems with respect to the regression coefficient β, the accuracy of the representation and the number of 'holes' introduced in the systems. In Section 2, the distance relations for the real systems are described. An algorithm to generate random systems and to deduce relations (2) without and with 'holes' is given in Section 3. Random systems unconstrained and constrained to have a certain number of large spacings between bodies and to have bodies 'not too close to each other' are searched for in Section 4. The results are discussed in Section 5.

2. 'Holes' in Planetary and Satellite Systems

We consider a system of m bodies revolving around a primary at distances a_i, with i being the integer body rank, $1 \leqslant i \leqslant m$ and $i = 1$ for the first body, closest to the primary. An exponential distance relation (2) for this system is found by a linearized exponential regression of the distances of classification integer numbers n_i attributed to each revolving body of rank i. In attributing the n_i to the bodies, particular attention must be paid to the comparison of distance ratios of two successive bodies

$$r_i = a_i / a_{i-1} \tag{3}$$

as the accuracy of the representation (2) depends on the value of β, which can be approximated roughly by the geometrical mean of the ratios r_i. If all the ratios r_i are similar or at least of the same order of magnitude, the classification numbers n_i are taken as being equal to the body ranks i. If some ratios r_i in a given system are much larger than others, we can classify the system in two ways: either ignore the discrepancies and take the classification numbers n_i equal to the ranks i with the possible consequence of obtaining a poor distance representation; or postulate that one or several gaps exist in the sequence of bodies and introduce a certain number of 'holes'. The number of 'holes' is determined by comparing the order of magnitude of the ratios r_i. A magnitude order is defined as the number of time a small ratio or a geometrical mean of small ratios, has to be multiplied by itself to roughly equal a large ratio. A number k of 'holes' is introduced between two bodies having a large distance ratio, such as the $(k+1)$th root of the large ratio becomes of the same magnitude order as the small ratios. When attributing a classification number to the body of rank 1, the ratio of its distance to the primary radius is compared to the other ratios of the system. A number h of 'holes' may be introduced between the body of rank 1 and the primary without affecting the coefficient β and the linear correlation coefficient (LCC) of a regression, as linear transformations of distances, e.g. shift of all classification numbers affect only the distance coefficient α.

The main objection against this empirical method is that the number of introduced 'holes' influences the final value of β and the LCC; for increasing number k of 'holes' between the first and the last body, β decreases and the LCC increases, both towards unity. The right compromise is found, first, by keeping the number of 'holes' to a minimum; secondly, by comparing the value of β to the geometrical mean of the small ratios and if a too large discrepancy is found, the number k of 'holes' is modified and third, for real planetary and satellite systems, only the main bodies are considered as a first step.

For the four real systems listed in Table I, two groupings are considered: systems-1 consist of only the main bodies (terrestrial and giant planets, large and medium size satellites); systems-2 include the bodies of systems-1 and the small bodies (Asteroids, small satellites and rings).

For the systems-1, the first distance ratio r_{11} is the ratio of the first body semi-major axis and the primary equatorial radius. For the planetary systems, the proto-Sun radius, assumed to be 0.25 AU, is considered. For the large ratios r_{i1}, the exponent $(k + 1)$ or $(h + 1)$ determines the number k or h of 'holes' to be introduced between two successive bodies or between the primary and the first body. In the systems-2, among the small bodies having close semi-major axis, the largest is chosen as these can not be considered separately (being too numerous). Though it is always possible to find two asteroids in the asteroidal belt matching closely the two 'holes' location suggested by the large distance ratio between Mars and Jupiter, we select Vesta and Hygiea among the four largest asteroids having diameters greater than 400 km (distance ratios: $r_{\text{Ceres/Vesta}} = 1.171$; $r_{\text{Pallas/Ceres}} = 1.001$; $r_{\text{Hygiea/Pallas}} = 1.135$; data from Bowell *et al.*, 1982, Williams, 1982). We consider Himalia and Pasiphae, the largest amongst each external groups of four satellites of Jupiter (data from Morrison *et al.*, 1977) and 1986U1, the largest amongst the inner small satellites of Uranus (data from Stone and Miner, 1986). Mean distances for the rings grouped with their shepherd satellites were calculated (Pletser, 1986). The distance ratios r_{i2} are listed in Table I with the corresponding classification numbers n_i. The small bodies and rings fit very well into all the 'hole' locations of the planetary and Uranus systems and in the inner part of the Jupiter and Saturn systems. The empty 'holes' are located in the outer part of the Jupiter system, between Callisto and the two groups of irregular satellites, and in the outer part of the Saturn system, between Rhea and Titan and between Hyperion and the irregular Iapetus and Phoebe.

The following characteristics of the real systems are given in Table II:

- the β_{r0} and LCC_0 of exponential regressions of semi-major axis on integer ranks i, without 'holes';
- the numbers h and k of 'holes' between primary and first body and between first and last body;
- the β_{rk} and LCC_k of regressions of semi-major axis on classification numbers n_i, with k 'holes'.

All the β_{rk} are close to the systems small ratios of Table I and, as expected, the relation $\beta_{r0} \geqslant \beta_{rk}$ holds for all the real systems. The hypothesis of introducing 'holes' in the distance relations is justified first, by the discrepancies observed between the β_{r0} of systems-1 (main bodies only, without 'holes') and the β_{r0} of systems-2 (main and small bodies, without 'holes'), as for the planetary and Saturnian systems, or when the

TABLE I

Distance ratios, rank and classification numbers of real systems

Sun-1 [-2]	i_1	i_2	r_{i1}	r_{i2}	n_i
Mercury	1	1	1.548		1
Venus	2	2	1.868		2
Earth	3	3	1.383		3
Mars	4	4	1.524		4
[Vesta]		5		1.550	5
[Hygiea]		6		1.331	6
Jupiter	5	7	$3.415 = (1.506)^3$	1.655	7
Saturn	6	8	1.837		8
[Chiron]		9		1.433	9
Uranus	7	10	$2.011 = (1.418)^2$	1.403	10
Neptune	8	11	1.567		11
Pluto	9	12	1.310		12

Jupiter-1 [-2]	i_1	i_2	r_{i1}	r_{i2}	n_i
[Ring Sec. 1]		1		1.415	1
[Ring Prim.]		2		1.256	2
Amalthea	1	3	$2.539 = (1.364)^3$	1.429	3
[Ring Sec. 2]		4		1.710	4
Io	2	5	$2.326 = (1.525)^2$	1.360	5
Europe	3	6	1.591		6
Ganymede	4	7	1.595		7
Callisto	5	8	1.757		8
[Himalia]		9		$6.101 = (1.572)^4$	12
[Pasiphae]		10		$2.031 = (1.425)^2$	14

Saturn-1 [-2]	i_1	i_2	r_{i1}	r_{i2}	n_i
[Ring D]		1		1.159	1
[Ring C]		2		1.196	2
[Ring B]		3		1.261	3
[Ring A]		4		1.254	4
Janus	1	5	$2.510 = (1.202)^5$	1.146	5
Mimas	2	6	1.225		6
Enceladus	3	7	1.283		7
Tethys	4	8	1.238		8
Dione	5	9	1.281		9
Rhea	6	10	1.397		10
Titan	7	11	$2.318 = (1.234)^4$		14
Hyperion	8	12	1.212		15
Iapetus	9	13	$2.404 = (1.245)^4$		19
Phoebe	10	14	$3.638 = (1.240)^6$		25

Uranus-1 [-2]	i_1	i_2	r_{i1}	r_{i2}	n_i
[Rings]		1		1.655	1
[1986U1]		2		1.560	2
1985U1	1	3	$3.359 = (1.498)^3$	1.378	3
Miranda	2	4	1.511		4
Ariel	3	5	1.470		5
Umbriel	4	6	1.393		6
Titania	5	7	1.640		7
Oberon	6	8	1.337		8

System-1: system considering only main bodies.
System-2: system considering main and small bodies (between brackets).
i_2, i_2: rank numbers of system-1 and system-2.
r_{i1}, r_{i2}: ratios of successive semi-major axis for systems-1 and 2.
n_i: classification numbers.

TABLE II

Characteristics of the real systems

System	m	β_{r0}	LCC_0	h	k	β_{rk}	LCC_k
Sun-1	9	1.866	0.9906	0	3	1.523	0.9988
Sun-2	12	1.525	0.9984	0	0	=	=
Jupiter-1	5	1.752	0.9927	2	1	1.590	0.9984
Jupiter-2	10	1.793	0.9608	0	4	1.547	0.9984
Saturn-1	10	1.570	0.9527	4	11	1.251	0.9995
Saturn-2	14	1.412	0.9459	0	11	1.247	0.9995
Uranus-1	6	1.472	0.9986	2	0	=	=
Uranus-2	8	1.456	0.9988	0	0	=	=

=: repeat of previous values for $k = 0$.
System: as listed in Table I.
m: number of bodies considered in the real system.
β_{r0}, LCC_0: coefficient β and linear correlation coefficient of exponential regression without 'holes'.
h, k: numbers of 'holes' introduced between primary and first body and between first and last body.
β_{rk}, LCC_k: coefficient β and linear correlation coefficient of exponential regression with k 'holes'.

β_{r0}'s are similar, by the discrepancy between the LCC_0's, as for the Jupiter systems, and second, by the closeness of the β_{rk} and LCC_k for the systems-1 and systems-2. The closeness of the β_{rk} shows also that the number k of introduced 'holes' is adequately chosen.

3. Generation of Random Systems

3.1. Algorithm for Generation of Random Systems

Random systems of m bodies are created with orbit radii generated at random. No concern is made for other orbital elements or masses as we are only interested in distance relations. Sequence of $m + 1$ random variables are generated and sorted in ascending order. The m last variables are divided by the smallest, in order to obtain m random orbit radii a_i in units similar to primary radius units.

The 'holes' are introduced after comparison of the ratios r_i to a mean ratio $\bar{r}$ of small ratios, as in Section 2. A criterion to select $\bar{r}$ is chosen as follows. First, for $1 \leqslant i, j \leqslant m$, we compute the ratios

$$e_{ij} = |(r_i - r_j)/r_i| \tag{4}$$

with vertical bars denoting the absolute value. We select the ratio r_j for which appears the largest occurrence of e_{ij} smaller than a certain limit L_1. If several r_j have the same occurrence of $e_{ij} < L_1$, the smallest r_j is selected. Secondly, for the selected ratio r_j, all the ratios r_i^* such as e_{ij} is smaller than a second limit L_2, are determined. If no such ratios r_i^* can be found for the selected ratio r_j, we take the next r_j for which appears the second largest occurrence of $e_{ij} < L_1$ and another set of r_i^* is calculated. If no ratios r_i^* can be determined at all, the smallest ratio r_i is taken as the mean ratio $\bar{r}$. Otherwise, the mean ratio $\bar{r}$ is the geometrical mean of all the ratios r_i^*. The values of L_1 and L_2 are found by applying this criterion to the four real systems-1 of Table I in order to obtain the characteristics of Table II, giving $L_1 = 0.2$ and $L_2 = 0.091$.

The numbers h and k_i of 'holes' to be introduced between the primary and the first body and between two successive bodies of rank $i-1$ and i are computed as in Section 2 using the mean ratio $\bar{r}$. Some of the k_i or h may obviously be nil. The classification numbers are attributed to each body: $n_1 = h + 1$ to the first body and the following, $n_i = n_{i-1} + k_i + 1$, to the bodies of rank i for $1 < i \leqslant m - 1$ and with $k = \Sigma\, k_i$. An exponential regression of the distances a_i on the body ranks i give the exponential distance relation without 'holes' and a second regression on the classification numbers n_i give the distance relation with 'holes'.

3.2. Random variable generators

The random variables are generated from three distributions: uniform, normal and exponential. Uniform random integer variables X_i are generated by the linear mixed-congruential method

$$X_i = (pX_{i-1} + q) \text{ modulo } M \tag{5}$$

where p, q, and M are integers, with $p, q < M$ (Knuth, 1969, Kennedy and Gentle, 1980). Normal random variables are generated by the method of acceptance-rejection of uniform random variables (Kinderman and Ramage, 1976, Marsaglia, 1964, Ahrens and Dieter, 1972). Exponential random variables E_i are generated by the inversion method $E_i = -\log(X_i/M)$ (Ahrens and Dieter, 1972).

The randomness of a sequence of numbers can be tested but it must be outlined that a sequence may exhibit a global randomness and local non-randomnesses. Therefore subsequences extracted from a main sequence may not be guaranteed perfect randomness, if this ever exists. One of the most efficient tests to detect departure from non-randomness in a sequence is the run test (Levene and Wolfowitz, 1944, Knuth, 1969). The three random generators were successfully run tested with the following two sets of parameters:

$$p = 5^{15} = 30517578125; \quad q = 1; \quad M = 2^{35} = 34359738368 \tag{6}$$

$$p = (2^{12} - 3) = 4093; \quad q = 1; \quad M = 2^{24} = 16777216. \tag{7}$$

The first set can be considered as "good" values as they pass other efficient tests (Knuth, 1969).

4. Random Systems

4.1. Unconstrained random systems

Random systems without constraints are first investigated. 500 systems of m bodies (with $m = 4$, 6, 8, 10, 12, and 14) are generated by the three random generators, with

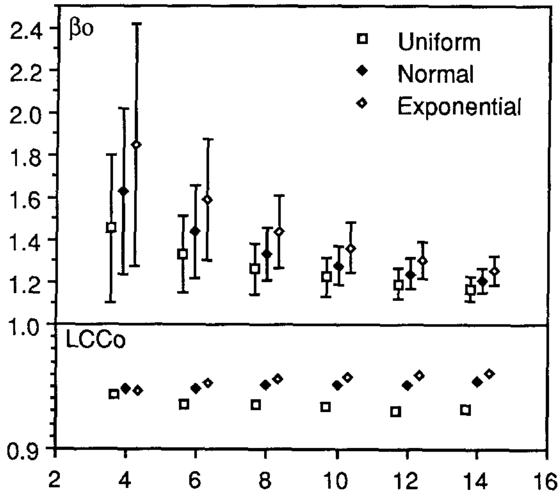

Fig. 1. The mean regression coefficients $\bar{\beta}_0$ and $\bar{L}CC_0$ of sets of 500 unconstrained random systems of 4, 6, 8, 10, 12, and 14 bodies for the uniform, normal and exponential generators (the error bars represent the standard deviations).

the parameters (6). For each system, the regression coefficients and the numbers of 'holes' are computed. The means $\bar{\beta}_0$, $\bar{L}CC_0$, $\bar{\beta}_k$ and $\bar{k}$ for each set of 500 random systems are shown in Figures 1 and 2, where the error bars represent the standard deviations σ. For the three generators, the means $\bar{\beta}$ and the σ decrease for increasing number m of bodies and, for each value of m, the inequality

$$\bar{\beta}_{0,\,E} > \bar{\beta}_{0,\,N} > \bar{\beta}_{0,\,U} > \bar{\beta}_{k,\,E} > \bar{\beta}_{k,\,N} > \bar{\beta}_{k,\,U} \tag{8}$$

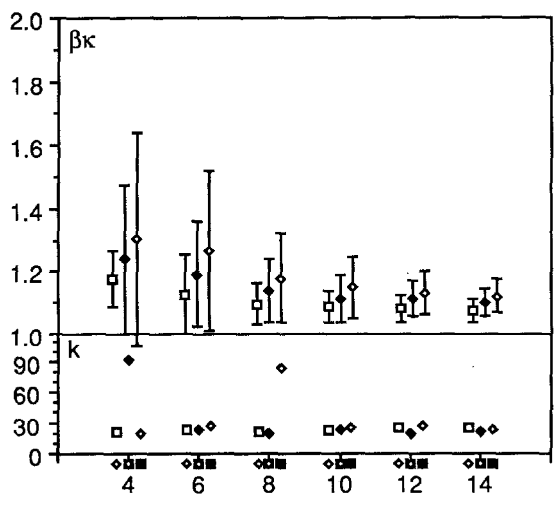

Fig. 2. The mean regression coefficients $\bar{\beta}_k$ and means of number $\bar{k}$ of 'holes' between the first and the last body of sets of 500 unconstrained random systems (same notations as Figure 1).

holds among the six means $\bar{\beta}$, where the indexes E, N, and U refer to the generator type. For increasing m, there is a slight decrease of the means $\overline{\mathrm{L}}\mathrm{CC}_{0,\,U}$ and a slight increase of $\overline{\mathrm{L}}\mathrm{CC}_{0,\,N}$ and $\overline{\mathrm{L}}\mathrm{CC}_{0,\,E}$. The means $\bar{k}$ of number of 'holes' introduced between the first and last body vary between 19 and 27, except in two cases (84 and 91), corresponding to extremely high values of k in two random systems; recall that k is calculated for each random system in the same way as for the real systems. All the means $\overline{\mathrm{L}}\mathrm{CC}_k$ are greater than 0.995. If we compare these means to the values of the real systems with corresponding number of bodies, the means $\bar{\beta}_0$, $\overline{\mathrm{L}}\mathrm{CC}_0$ are smaller than the corresponding real β_{r0} and LCC_{r0}, except for systems having a small number of bodies: Jupiter-1 and Uranus-1 ($m = 5$ and 6). For all corresponding systems, the means $\bar{\beta}_k$ and $\overline{\mathrm{L}}\mathrm{CC}_k$ are smaller than the β_{rk} and LCC_{rk} while the means $\bar{k}$ of 'holes' are much larger than for the real systems.

4.2. Constrained Random Systems with Constraint on Number of Large Spacings

In Table II, the large ratios between some bodies of systems-1 or between first bodies and primaries suggest that large spacings between main bodies can be chosen as a first constraint in generating random systems corresponding to real systems. We generate constrained random systems of m bodies having predetermined numbers H and K of large spacings, without imposing their locations, with m, H and K corresponding to the real systems m, h, and k. A number N of unconstrained random systems are generated with the parameters (7), until 500 systems are found with H and K large spacings. More than 4 million random systems are investigated in this way. The Figure 3 shows the means $\bar{\beta}$ and $\overline{\mathrm{L}}\mathrm{CC}$ for sets of 500 constrained random systems, designated by the corresponding real system name between quotes. Comparing these means to the real systems values, the means $\bar{\beta}_0$ and $\bar{\beta}_k$ are smaller than the β_r's for all systems, except for the ⟨Uranus-1-2⟩systems, while the means $\overline{\mathrm{L}}\mathrm{CC}_0$ and $\overline{\mathrm{L}}\mathrm{CC}_k$ are smaller than the real LCC's, except the $\overline{\mathrm{L}}\mathrm{CC}_0$ for the ⟨Jupiter-2⟩and⟨Saturn-1-2⟩systems.

4.3. Random Systems with the 'Closeness not too Close' Condition

In a remarkable Monte-Carlo computer simulation, Dole (1970) generated coplanar planetary systems by injecting into a Laplace-type nebula of gas and dust small nuclei, one at a time, with orbits semi-major axis and eccentricities determined by random numbers. The nuclei would grow into protoplanets by accreting dust and gas, if their mass were large enough and their temperature low enough. The condition of 'closeness but not too close' was expressed by allowing the coalescence of bodies on crossing orbits or coming within a certain gravitational interaction distance x, function of the nucleus semi-major axis a and instantaneous mass M, relative to the primary unit mass: i.e.,

$$x = a[M/(1 + M)]^{1/4} = a\mu^{1/4}. \tag{9}$$

We consider upper limit cases where the nuclei masses are replaced by the present planets mass (Levy, 1979, except Pluto-Charon: Reinsch and Pakull, 1986) and present satellites mass (Jupiter and Saturn satellites: Morrison *et al.*, 1977, except

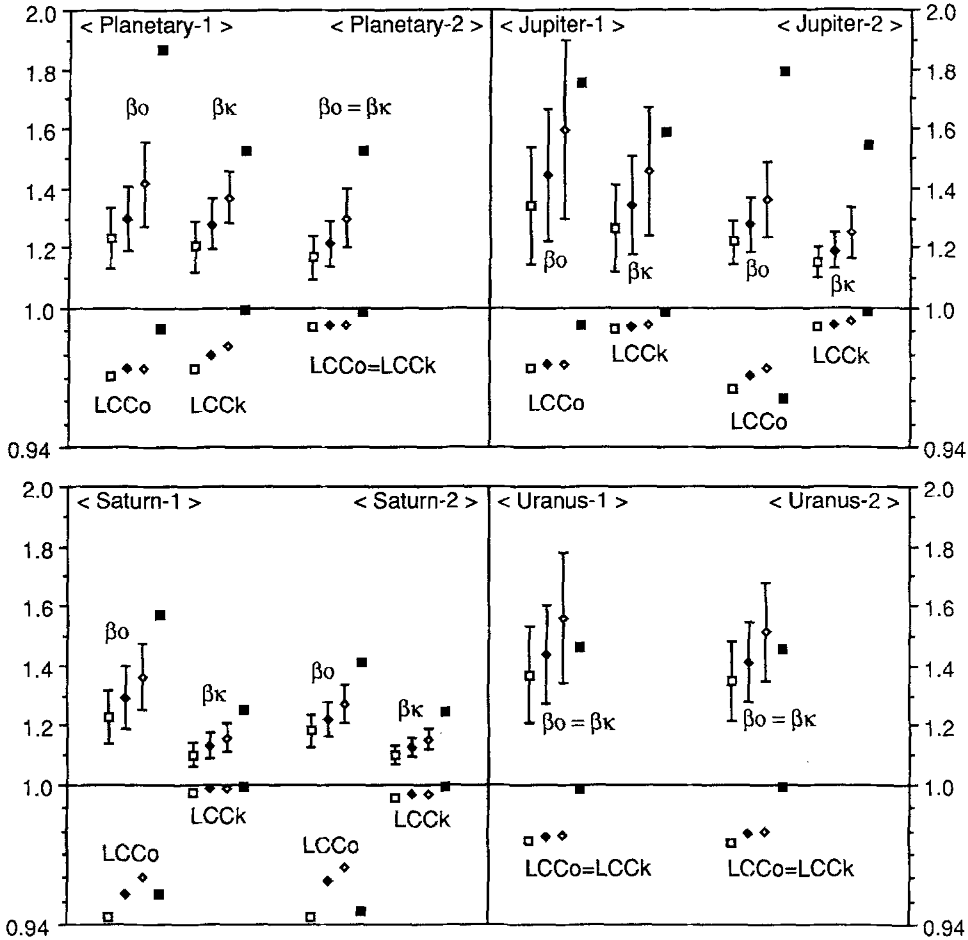

Fig. 3. The mean regression coefficients $\bar{\beta}$ and $\overline{\text{LCC}}$ of sets of 500 random ⟨systems-1-2⟩ constrained to have numbers of large spacings equivalent to that of corresponding real systems, compared to regression coefficients β and LCC of real systems (same notations as in Figure 1; real systems: ■).

Janus: Aksnes, 1985; Uranus satellites: Stone and Miner, 1986, except satellite 1985U1, approximated by the mass of a sphere of 85 km radius and density of 1.2). Random systems are generated with bodies not 'too close to each other'. This condition is introduced in the above algorithm by constraining the orbits radii of two successive bodies, such as

$$a_i - a_{i-1} > x_i^*, \tag{10}$$

where x_i^* is the largest of the critical distances x_{i-1} and x_i, giving the condition on the distance ratios

$$r_i > 1 + (x_i^*/a_{i-1}) = R \tag{11}$$

with R being $(1 + \mu_{i-1}^{1/4})$ or $(1 - \mu_i^{1/4})^{-1}$ if the largest critical distance corresponds to the inner or to the outer body. Constrained random systems corresponding to real

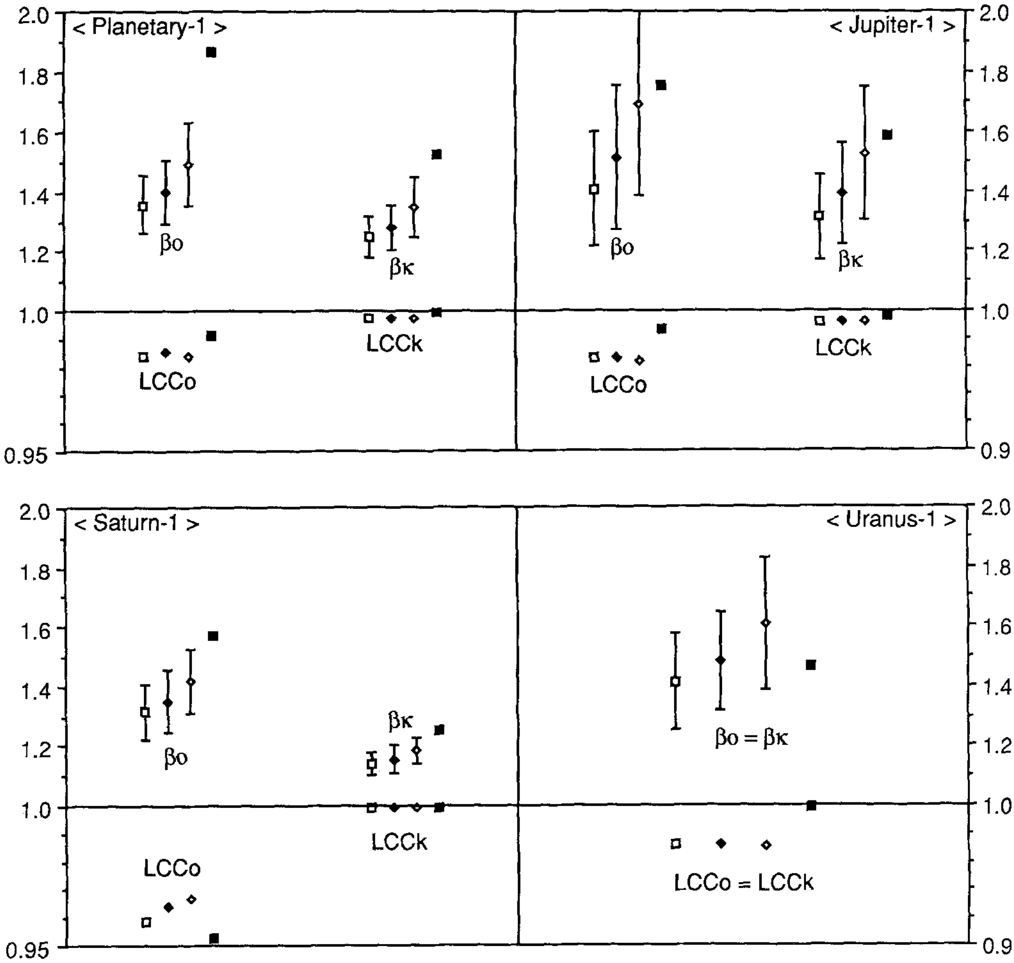

Fig. 4. Means of regression coefficients $\bar{\beta}$ and $\bar{L}CC$ of sets of 500 random ⟨systems-1⟩ with the constraints of 'large spacings' and of 'closeness not too close', compared to regression coefficients β and LCC of real systems (same notation as in Figure 3).

systems-1 are generated with the parameters (7) with the large spacings constraint and the additional constraint (11). More than eleven million random systems are investigated in this way. The Figure 4 shows the means $\bar{\beta}$ and $\bar{L}CC$ for sets of 500 such constrained random systems. Comparing these to the real systems values, the means $\bar{\beta}_0$ and $\bar{\beta}_k$ are smaller than the β_r's for all cases, except for the ⟨Uranus-1⟩ systems with the three generators and the ⟨Jupiter-1⟩ systems with the exponential generator, while the means $\bar{L}CC_0$ and $\bar{L}CC_k$ are smaller than the real LCC, except for the $\bar{L}CC_0$ of the ⟨Saturn-1⟩ systems.

5. Discussion

Although the random systems characteristics means differ from the real systems ones, their range includes values similar to the real systems characteristics. The difference between real systems and random systems becomes more apparent for systems having a large number of bodies and for increasing number of constraints in the generating process. It is important to note that the differences between real and random systems are not related to the introduction of 'holes' in the computation of the distance distributions, as it can be seen by comparing the β_{r0} and LCC_{r0} to the means $\bar{\beta}_0$ and $\overline{\mathrm{LCC}}_0$ (Figures 3 and 4). Another difference is the discrepancy, even in first approximation, between the means $\bar{\beta}_0$ and $\bar{\beta}_k$ of constrained random systems, while the real systems-1 β_{rk} are similar to the systems-2 β_{ro} (see Table II), due to the fact that all the 'holes' of the real systems-1 are filled by small bodies of systems-2 (see Table I), except for the outer part of the Jupiter and Saturn systems. A further difference, except for the ⟨Saturn-1⟩ systems with the 'large spacings' and 'closeness but not too close' constraints, is a poor representation of constrained random systems distance distributions with 'holes', less good on the average than for the real systems despite using the same criterion for the introduction of 'holes'.

To compare the spacing ratios of random and real systems, a constrained random system is defined to be similar to a real system for the same numbers of bodies, if its coefficient β (β_0 or β_k) is in the arbitrarily chosen range around β_r (β_{r0} or β_{rk})

$$\beta_r - [(\beta_r - 1)/10] \leqslant \beta \leqslant \beta_r + [(\beta_r - 1)/10]. \tag{12}$$

Among the N random systems generated for each case, $N^* = 500$ systems are found compliant with the specified constraints and N_0, N_k, and N_{0k} are the numbers of systems having their β_0, β_k and simultaneously β_0 and β_k in the ranges (12) respectively. One defines the following probabilities of generating a random system compliant to the specified constraints:

- $P_S = N^*/N$;
- or similar to a real system without 'holes' in the distance relation (β_0 in range (12)): $P_0 = N_0/N$;
- or similar to a real system with k 'holes' in the distance relation (β_k in range (12)): $P_k = N_k/N$;
- or similar to a real system simultaneously with and without 'holes' in the distance relation (β_0 and β_k simultaneously in the ranges (12)): $P_{0k} = N_{ok}/N$.

For the three generators, the indicative magnitude orders of these four probabilities are shown in the Figure 5 for the ⟨systems-1⟩ and ⟨systems-2⟩ with the 'large spacings' constraints and for ⟨systems-1⟩ with the 'large spacings' and 'closeness not too close' constraints. The probabilities P_S are of the order of 10^{-2} or less. The probabilities P_0, P_k, and P_{0k} are around 10^{-3} for systems with small numbers of bodies (⟨Jupiter-1⟩ and ⟨Uranus⟩ systems) and between several 10^{-4} and 10^{-7} for systems with large numbers of bodies (⟨Planetary⟩, ⟨Jupiter-2⟩ and ⟨Saturn⟩ systems).

Generating with specified constraints a random system having its coefficient β of distance relation with or without the introduction of 'holes', close to the coefficient β_r of a corresponding real system has therefore little chance to occur. Generating with specified constraints simultaneously four random systems of the same numbers of

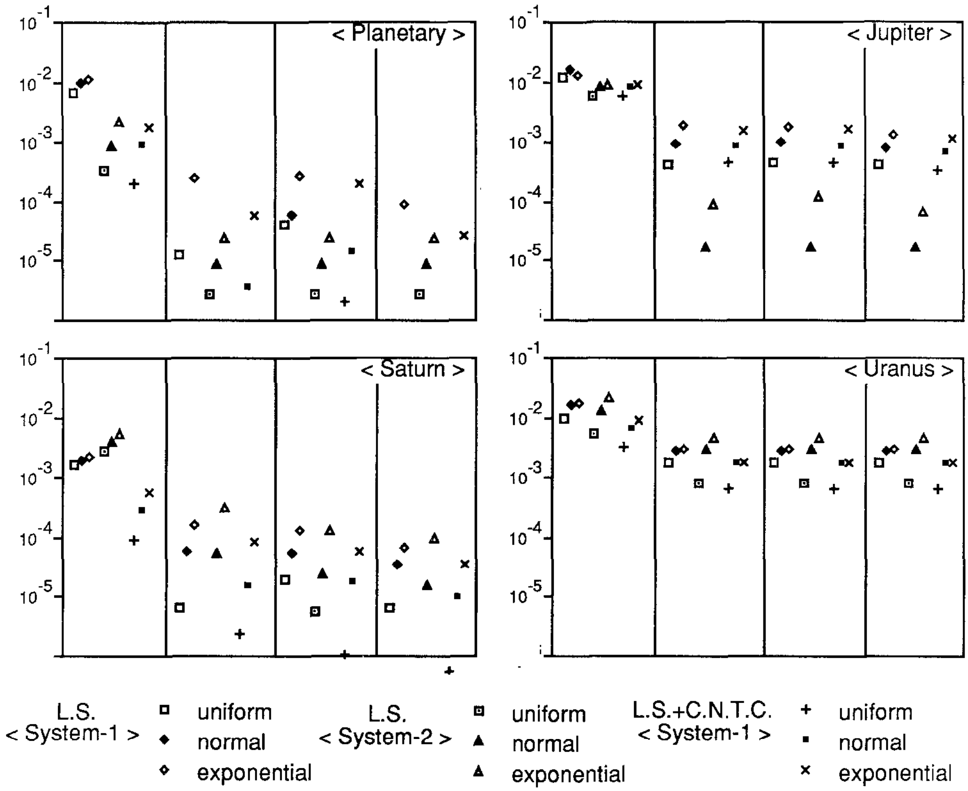

Fig. 5. Semi-logarithmic diagrams of four probabilities (in each diagram, columns from left to right: P_S, P_0, P_k, P_{0k}) defined in text, of generating random ⟨systems-1-2⟩ having their coefficients β close to that of corresponding real systems for different constraints (L.S.: 'Large spacings', C.N.T.C.: 'Closeness but not too close') on the random generation process and for the uniform, normal and exponential generators. The values not appearing could not be found, the corresponding number of systems being nil.

bodies as the four real systems and having their coefficients β close to the four real systems β_r has an even smaller probability of occurring, given by

$$P_{a,\,4\text{syst}} = P_{a,\,\text{Sun-}b} \times P_{a,\,\text{Jupiter-}b} \times P_{a,\,\text{Saturn-b}} \times P_{a,\,\text{Uranus-b}} \tag{13}$$

with $a = (0)$, (k) or $(0k)$, $b = 1$ or 2 and where it is assumed that there is no causal relation between the distance relations of the four generated systems, i.e. the probabilities are independent. These are of the order of 10^{-13} to 10^{-17} for the 'large spacings' constraints and of 10^{-14} to 10^{-19} for the 'large spacings' and 'closeness but not too close' constraints.

One can argue that, instead of comparing the real systems β_r to the random systems means $\bar{\beta}$, it would have been better to compare the β_r to the modes of the frequency distributions of the random systems β. As the frequency distributions of the random systems β_0 and β_k have positive skewness, the modes and medians are smaller than the arithmetical means. The real systems β_r are greater than most of the corresponding random systems means $\bar{\beta}$ and the comparison with the modes would have been worse.

For cases where the random systems means $\bar{\beta}$ are close to the β_r, statistical decision tests are useful. Supposing that a real system distance relation can be found by a random process, its β_r should be representative of the β's of the population of all systems of the same number of bodies generated by the same random process, i.e. the β_r should be an estimate of this population mean $\langle\beta\rangle$. Considering that each set of $N^* = 500$ random systems is large enough to assume normality with mean $\bar{\beta}$ and standard deviation σ_s and taking these as estimates of the population mean $\langle\beta\rangle$ and standard deviation σ, a two-tailed test on the Z-score z_N give the significance level α_N of rejecting the null hypothesis: β_r is respresentative of the population mean $\langle\beta\rangle$

$$H_0\colon \langle\beta\rangle = \beta_r;\ H_1\colon \langle\beta\rangle \neq \beta_r \quad \text{with} \quad z_N = (\bar{\beta} - \beta_r)/(\sigma_s/\sqrt{N^*}). \tag{14}$$

Only the systems of a small number of bodies are worth examining: ⟨Jupiter-1⟩ and ⟨Uranus-1-2⟩ with random generators and constraints as indicated in Table III with the Z-scores z_N and the significance levels α_N. These are nil or too small to be significative, leading to the rejection of the hypothesis H_0 in all cases, except for the ⟨Uranus-1⟩ system one with the 'large spacings' and 'closeness not too close' constraints and the normal generator. All other systems have higher z_N values. As the normal assumption of the β distributions is not exactly correct, two non-parametric distribution-free tests are additionally performed. For the sign test (Bailey, 1983), the differences $(\beta - \beta_r)$ are computed. If the hypothesis H_0 is founded, we should expect a similar number D of positive and negative differences, i.e. the difference distribution should be binomial with $p = 0.5$. Using the large-sample normal approximation to the binomial, the normal test variable is

$$z_S = (D - N^*p)\,[N^*p(1-p)]^{-1/2}. \tag{15}$$

A more powerful test is the Wilcoxon's signed rank sum test (Bailey, 1983). The sums T of the ranks of the positive and negative differences are computed. The normal test variable is

$$z_W = [|T - N^*(N^*+1)/4| - 1/2]\,[N^*(N^*+1)(2N^*+1)/24]^{-1/2}. \tag{16}$$

The levels of significance for both tests, shown in Table III, are nil or two small to be significative, leading again to the rejection of the null hypothesis, except with the Wilcoxon test for the same case as with the normal test. For this latter case, its tendency toward a possible random-like generation can be considered as marginal for the following reasons: first, only two of the three tests show agreement; second, it is not reflected in any other Uranus-like system configurations, only for the particular configuration of 6 bodies with two large spacings between the primary and first body, no large spacings between the first and the last body and the 'closeness but not too close' constraint; third, it is only valid for this particular kind of generation from a normal random distribution. Note also that the corresponding mean $\overline{\text{LCC}}$ for the normal generator (0.986) is well below the real Uranus LCC (0.9986) and that, among the four real systems, it would be surprising that only the Uranus system exhibits this tendency of a possible random-like generation.

TABLE III

Z-scores and levels of significance for random systems of small number of bodies

Constraint	System	m	β_r		$\bar{\beta}$	$\lvert z_N \rvert$	α_N	$\lvert z_S \rvert$	α_S	$\lvert z_W \rvert$	α_W
Unconstrained	Uranus-1	6	1.472	N	1.436	3.672	0.0002	18.872	–	18.293	–
				E	1.590	9.283	–	4.562	–	7.438	–
	Uranus-2	8	1.456	E	1.439	2.174	0.030	4.562	–	4.147	–
Large Spacings	Uranus-1	6	1.472	N	1.435	4.948	–	7.245	–	6.286	–
				E	1.559	8.818	–	4.293	–	7.171	–
	Uranus-2	8	1.456	N	1.411	7.684	–	9.749	–	8.992	–
				E	1.511	7.435	–	3.935	–	6.003	–
Large Spacings	Uranus-1	6	1.472	U	1.414	7.952	–	7.960	–	8.268	–
+ Not too close.				N	1.482	1.390	0.165	2.952	0.003	0.427	0.669
				E	1.609	13.838	–	9.570	–	12.087	–
	Jupiter-1	5	1.752	E	1.691	4.432	–	5.456	–	5.689	–
	Jupiter-1	5	1.590	E	1.524	6.696	–	7.334	–	7.440	–

m: number of bodies in a random system.
β_r: coefficient β of a corresponding real system.
$\bar{\beta}$: mean of β for a set of 500 random systems.
$\lvert z_N \rvert$, $\lvert z_S \rvert$, $\lvert z_W \rvert$: absolute value of the Z-scores defined in text.
α_N, α_S, α_W: levels of significance for two-tailed tests.
U, N, E: uniform, normal and exponential random generators.

Therefore, the real systems β_r are not representative of the β's of the corresponding random systems population for all systems considered except for the above possibly dubious Uranus-1 case.

For the planetary system, one can argue that, instead of the present planets mass reflecting the present planetary system state, the proto-planets higher mass at the end of the accretion phase should be used in Dole's critical distance expression in the 'closeness but not too close' constraint. This was performed (Pletser, 1987) by considering the present planets mass increased up to the solar abundance and similar conclusions were found for the planetary system.

Other additional constraints could have been used on the random generation process. These can be deduced from the particularities of the real systems distributions and distance ratios (see Table I). Considering three groups of bodies in each real system (inner, giant and outer groups), first, the large spacings occur more often among the bodies of the outer groups or between the outer and the giant groups; second, no large spacings occur in the inner groups of the real systems-2 while for the real systems-1, large spacings are observed in the inner groups or between the giant and the inner groups or the primary; third, no large spacings occur in the giant groups, except for the pairs Saturn-Uranus and Rhea-Titan (Titan seams to be a special case as it has so many different features in respect of the other Saturn satellites; see e.g. Prentice, 1984). The sequences Io to Ganymede and Janus to Dione have to be outlined as, inside a sequence, the distance ratios are all nearly equal (this can be easily explained in first approximation by Kepler's third law and the existing resonances or near-commensurabilities amongst their mean motions). As large spacings and repetitive similar ratios seem to occur somehow at particular locations in the real systems, additional constraints could be deduced for generating random systems resembling more closely to the real systems. These new constraints are not investigated as it seems obvious that they would reduce even more the probabilities of generating random systems similar to real systems. Also, increasing the number of constraints would eventually lead to random systems increasingly similar to real systems, but with the consequence of losing totally the random aspect of the generation process.

Random generators other than from uniform, normal and exponential distributions could also have been used, but these three main distributions have the advantage of natural simplicity. Other random distributions would tend to loose this natural simplicity and transform the random aspect of generation into an ad hoc generation kind.

6. Conclusions

Although the arrangements of the real systems as in Table I can be discussed, they can be regarded as representative of the present real systems. Their distance relations without and with introduction of 'holes' are compared to these of unconstrained random systems and of constrained random systems having more specific similarities with the real systems. The characteristics of the unconstrained random systems are on average different to that of the corresponding real systems, in particular the coefficients β are smaller on average to that of the real systems.

In a following step, we imposed on the random generation process two constraints corresponding to features existing in the distance distributions of the real systems, namely an equivalent number of large spacings and the 'closeness not too close' condition, i.e. for the random systems to have distances between adjacent bodies greater than critical attraction distances calculated by considering the present masses of main planets and satellites. More than 16 million constrained random systems were analysed. Very few are similar to the real systems, specially for systems with a large number of bodies. Magnitude orders of probabilities of generating constrained random systems with distance relations having their coefficients β in the range (12) around the β_r values of real systems were deduced. These showed that the chances of obtaining a random system similar to a real system are very small. The means $\bar{\beta}$ of the $\langle$Jupiter-1$\rangle$, $\langle$Uranus-1-2$\rangle$ random systems, with small numbers of bodies, were found roughly close to the corresponding real systems β_r. Three statisical tests showed that we can reject, with very low significance levels, the hypothesis that the real system β_r is representative of the population of random systems generated with the same number of bodies and constraints, except for one dubious case.

Therefore, the distance distributions, and in particular the exponential spacings, observed in the real systems can not be compared to distributions of sorted random numbers sequences and the distance relations of the real systems can not be explained by a random generation process, in particular with the simultaneous constraints of having an equivalent number of large spacings and of 'closeness not too close'. This last condition is certainly necessary but not sufficient to explain the observed exponential spacings. Other physical and dynamical processes, for example drag in a gaseous nebula, resonances or tides for satellites, have to be considered to account for the observed distance distributions of the planets and satellites. Random processes are believed to be important for planetesimals accretion but these would act only locally and not for the overall distance distributions.

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