ROTATIONAL DISTORTION OF STARS HAVING ARBITRARY STRUCTURE DESCRIBED BY FOURTH-ORDER SECTORIAL HARMONICS

*I. Equipotential Surfaces in Sectorial Harmonics**

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Abstract. In a series of papers, the equilibrium configurations of highly rotating fluid bodies have been derived. The deformation of these inhomogeneous self-gravitating fluid, of arbitrary internal structure are due to centrifugation potential. These level surfaces are expressed in terms of fourth-order sectorial harmonics.

In this paper, the main equations of the problem – such as the surface of the distorted body, the gravitational potential at an arbitrary point and the disturbing potential – have been expanded to the fourth-order in terms of the even-order sectorial harmonics.

1. Introduction

Previous investigators expressed the distortion on the figures of equilibrium, due to any disturbing forces, by the individual surface zonal harmonics considering only the corresponding harmonics up to the required degree of approximation. The expression of an equipotential surface is expanded in a series of surface spherical harmonics and it is well-known that any surface spherical harmonic of the two angular variables and may also be expanded in a series of zonal, sectorial and tesseral harmonics (MacRobert, 1948).

It is clear that all the previous investigators, for simplicity, considered only the zonal harmonic terms, but ignored the sectorial and tesseral harmonic terms. Darwin (1910: Vol. III, pp. 145–149, Equation (35), while deriving the centrifugal disturbing potential of two fluid masses, found that one of the terms was a sectorial harmonic of first-order. In considering the limit solution in a closed form for a Roche model density distribution for the case of purely rotational distortion, Kopal (1960) expanded the radius-vector of an equipotential surface in terms of sectorial harmonics of the co-latitude up to the second-order terms (cf. Kopal, 1960, Section II-6, Equation, (6.25); also when he treated the problem of interaction between rotation and tides of two compressible fluid bodies, he expanded the second and fourth sectorial harmonics in a series of zonal harmonics and vice versa (cf. Kopal, 1960, Section IV-I, Equations (1.38)–(1.39)).

In this investigation we are interested in dealing with the distortion of rapidly rotating configuration of self-gravitating compressible fluid bodies of arbitrary

* This work will hereafter be referred to as Paper I.

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structure, by a new approach (cf. Kopal, 1978, Chapter 11.2, pp. 36 and 78; El-Shaarawy, 1975, Chapters 4 and 5), the distortion will be expressed in sectorial harmonics of the co-latitude up to the second-order terms (cf. Kopal, 1960, rotational deformation.

2. Expansion of an Equipotential in Sectorial Harmonics

The associated Legendre function, of degree n and order m, of the first kind, is given (cf. Hobson, 1931; Chapter III, p. 93, Equation (7)) as

$$P_n^m(\cos\theta) = (-1)^m \sin\theta \frac{d^m}{d\cos\theta^m} P_n(\cos\theta) =$$

$$= (-1)^m \frac{(2n!)}{2^n \cdot n!(n-m)!} \sin^m\theta \times \tag{2.1}$$

$$\times \left\{\cos^{n-m}\theta \frac{-(n-m)(n-m-1)}{2\cdot(2n-1)} \cos^{n-m-2}\theta + \ldots\right\}, \tag{2.2}$$

where $P_n(\cos\theta)$ is a Legendre polynomial of degree n, while the tesseral surface harmonics are given in the form

$$P_n^m(\cos\theta) \begin{cases} \cos m\varphi, \\ \sin m\varphi; \end{cases} \tag{2.3}$$

except if $m = n$, when they are termed sectorial surface harmonics (Hobson, 1931, pp. 90–95). If we do so, in Equation (2.2), we obtain

$$P_n^n(\cos\theta) = (-1)^n \frac{(2n)!}{2^n \cdot n!} \sin^n\theta. \tag{2.4}$$

The radius-vector r' of a spheroidal equipotential surface may be defined as

$$r' = a\left\{1 + \sum_{n=0}^{\infty} f_n(a) P_n(\theta', \varphi')\right\} \tag{2.5}$$

where $f_n(a)$ are the radial part, which are responsible for such type of deformation, and $P_n(\theta', \varphi')$ are the surface harmonics of order n. This radius-vector can be expanded, without any loss of generality, in a series of sectorial harmonics in the co-latitude as follows: If we substitute by $\cos\theta$ instead of $\sin\theta$ in r.h.s. of Equation (2.4), we can rewrite it as

$$P_n^n(\cos\theta) = (-1)^n \frac{(2n)!}{2^n \cdot n!} (1 - \cos^2\theta)^{n/2}. \tag{2.6}$$

In the case of purely symmetrical rotation distortion of a fluid body, in hydrostatic equilibrium, only the even order functions are needed. The expansion of

(μ^n) as a linear function of Legendre polynomials (MacRobert, 1948, Chapter V, p. 96) is

$$\mu^n = \frac{n(n-2)\cdots 2}{(2n+1)(2n-1)\cdots(n+1)}(2n+1)P_n(\mu) + \cdots$$

$$+ \cdots + \frac{1}{n+1}P_0(\mu)\,, \tag{2.7}$$

where $\mu = \cos\theta$.

By using these two formulae (Equations (2.6) and (2.7)), we can express $P_n^n(\cos\theta)$ as a linear function of Legendre polynomial $P_n(\cos\theta)$, or vice versa.

If we return to Equation (2.5), so by substituting the sectorial harmonics instead of Legendre polynomials ($P_n(\cos\theta)$), we may also express the radius-vector (r') as a series of sectorial harmonics, in the form

$$r' = a\left\{1 + \sum_{j=0}^{\infty} f_j(a)P_j^j(\cos\theta')\right\} \tag{2.8}$$

so that

$$(r')^n = a^n\left\{1 + \sum_{j=0}^{\infty} f_j(a)P_j^j(\cos\theta')\right\}^n. \tag{2.9}$$

Expanding the r.h.s. of Equation (2.9) up to quantities of fourth-order in surficial distortion; substituting the value of the zero amplitude $f_0(a)$ (using the successive approximation up to fourth order), and if we put $(n+3)$ instead of n, for any value of n, Equation (2.9) becomes

$$(r')^{n+3} = a^{n+3}\left\{1 + (n+3)\sum_{j=0}^{4}[X_{2j} + (n+2)X'_{2j} + \right.$$

$$\left. + (n+2)(n+1)X''_{2j} + (n+2)(n+1)nX'''_{2j}]P_{2j}^{2j}(\cos\theta')\right\}, \tag{2.10}$$

where $j = 4(1)0$; for $n \gtrless 2$,

$$(r')^{2-n} = a^{2-n}\left\{1 + (2-n)\sum_{j=0}^{4}[X_{2j} - (n-1)X'_{2j} + \right.$$

$$\left. + n(n-1)X''_{2j} - n(n-1)(n+1)X'''_{2j}]P_{2j}^{2j}(\cos\theta')\right\}; \tag{2.11}$$

and lastly, for $n = 2$

$$\lim(r'/a) = \lim_{n\to 2}\frac{(r'/a)-1}{(2-n)} \times$$

$$= \sum_{j=0}^{4}[X_{2j} - X'_{2j} + 2X''_{2j} - 6X'''_{2j}]P_{2j}^{2j}(\cos\theta')\,, \tag{2.12}$$

where $f_0(a)$, up to the fourth order is given by

$$f_0 = -\left(f_2 + f_4 + f_6 + f_8 - \frac{2}{32} f^3 + 2 f_2 f_6 + 2 f_2^4\right); \tag{2.11}$$

and the expressions X_j's, for $j = 0, 2, 4, 6$, and 8 can be written in the form

$$X_0 = \frac{2}{3} f_2^3 - 2 f_2 f_6 - 2 f_2^4, \qquad X_0' = \frac{2}{2!} f_2 f_6,$$

$$X_0'' = -\frac{2}{3!} - f_2^3, \qquad X_0''' = \frac{1}{4!} f_2^4,$$

$$X_2 = -\left(\frac{1}{2} f_2 + \frac{5}{3} f_4 + \frac{7}{2} f_6 + 6 f_8\right),$$

$$X_2' = \frac{1}{2!}\left(-7 f_2 f_6 - \frac{2}{3} f_2^4\right),$$

$$X_2'' = -\frac{1}{3!}\left(-\frac{3}{2} f_2^3 + 5 f_2^2 f_4\right),$$

$$X_2''' = -\frac{1}{4!}(2 f_2^4),$$

$$X_4 = \frac{1}{24}\left(f_4 + \frac{27}{5} f_6 + \frac{594}{35} f_8\right),$$

$$X_4' = \left(\frac{1}{24}\right)\left(\frac{1}{2!}\right)\left[\frac{24}{7} f_2 f_4 + \frac{40}{7} f_4^2 + 18 f_2 f_6 + \frac{18}{35} f_2^2\right],$$

$$X_4'' = \left(\frac{1}{24}\right)\left(\frac{1}{3!}\right)\left[\frac{93}{7} f_2^2 f_4\right],$$

$$X_4''' = \left(\frac{1}{24}\right)\left(\frac{1}{4!}\right)\left[\frac{108}{7} f_2^4\right],$$

$$X_6 = -\frac{1}{720}\left[f_6 + \frac{52}{7} f_8\right],$$

$$X_6' = \left(-\frac{1}{720}\right)\left(\frac{1}{2!}\right)\left[\frac{100}{33} f_4^2 + \frac{10}{11} f_2 f_4 + \frac{76}{11} f_2 f_6\right],$$

$$X_6'' = \left(\frac{-1}{720}\right)\left(\frac{1}{3!}\right)\left[\frac{18}{77} f_2^3 + \frac{390}{77} f_2^2 f_4\right],$$

$$X_6''' = \left(-\frac{1}{720}\right)\left(\frac{1}{4!}\right)\left(\frac{216}{385} f_2^4\right),$$

$$X_8 = \left(\frac{1}{40320}\right) f_8 ,$$

$$X_8' = \left(\frac{1}{40320}\right)\left(\frac{1}{2!}\right)\left[\frac{56}{65} f_2 f_6 + \frac{490}{1287} f_4^2\right],$$

$$X_8'' = \left(\frac{1}{40320}\right)\left(\frac{1}{3!}\right)\left(\frac{84}{143} f_2^2 f_4\right),$$

$$X_8''' = \left(\frac{1}{40320}\right)\left(\frac{1}{4!}\right)\left(\frac{72}{715} f_2^4\right). \tag{2.14}$$

3. Interior and Exterior Potentials

The total potential of our configuration arising from its mass are the interior (U), and the exterior (V). It is well-known that the interior potential for $n > 2$ is given by (cf., e.g., Kopal, 1973; Section 3, p. 155, Equations (3.17)–(3.30))

$$U = \sum_{n=0}^{\infty} r^n U_n , \tag{3.1}$$

where

$$U_n = \frac{G}{2-n} \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left\{ \int_0^{\pi} \int_0^{2\pi} (r')^{2-n} P_n(\cos\gamma) \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \right\} \mathrm{d}a, \quad n \neq 2; \tag{3.2}$$

and for $n = 2$

$$U_2 = G \int_{a_o}^{a_1} \rho \frac{\partial}{\partial a} \left\{ \int_0^{\pi} \int_0^{2\pi} (\log r') P_2(\cos\gamma) \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \right\} \mathrm{d}a; \tag{3.3}$$

Also the exterior potential, for any value n, is

$$V = \sum_{n=0}^{\infty} (r)^{-(n+1)} Vn , \tag{3.4}$$

where

$$V_n = \frac{G}{n+3} \int_0^{a_0} \rho \frac{\partial}{\partial a} \left\{ \int_0^{\pi} \int_0^{2\pi} (r')^{n+3} P_n(\cos\gamma) \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \right\} \mathrm{d}a, \tag{3.5}$$

If $n = 0$,

$$U_0 = 4\pi G \int_{a_0}^{a_1} \rho a \, \mathrm{d}a \tag{3.6}$$

and

$$V_0 = G \int_0^{a_0} \mathrm{d}m' = Gm_0 \, , \tag{3.7}$$

where

$$m_0 = 4\pi \int_0^{a_0} \rho a^2 \, \mathrm{d}a \, ; \tag{3.8}$$

m_0 being the mass interior to a_0, and U_0 is independent of θ and φ

$$\mathrm{d}m' = \rho r'^2 \, \mathrm{d}r' \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \, , \tag{3.9}$$

and

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi') \, . \tag{3.10}$$

One can obtain $(r')^{n+3}/(n+3)$ from Equation (2.10) in the form

$$\frac{(r')^{n+3}}{n+3} = a^{n+3} \sum_{j=0}^{8} \mathscr{L}_j(a) P_j^j(\cos\theta') \, , \tag{3.11}$$

where $(n, j) = 0, 2, 4, 6$, and 8. Inserting Equation (3.11) into Equation (3.5) we find that

$$V_n = G \sum_{j=0}^{8} \int_0^{a_0} \rho \frac{\partial}{\partial a} \{a^{n+3} \mathscr{L}_j(a) \, \mathrm{d}a\} \times$$

$$\times \left\{ \int_0^{\pi} \int_0^{2\pi} P_j^j(\cos\theta') P_n(\cos\gamma) \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \right\} \mathrm{d}a \, . \tag{3.12}$$

Similarly, it is possible to substitute for $\{(r')^{2-n}/(2-n)\}$, using Equation (2.11) we can find that

$$\frac{(r')^{2-n}}{2-n} = a^{(2-n)} \sum_{j=0}^{8} G_j(a) P_j^j(\cos\theta') \, . \tag{3.13}$$

Inserting Equation (3.13) into Equation (3.2) we obtain

$$U_n = G \sum_{j=0}^{8} \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \{a^{(2-n)} G_j(a)\, \mathrm{d}a\} \times$$

$$\times \int_0^{\pi} \int_0^{2\pi} P_j^j(\cos \theta') P_n(\cos \gamma) \sin \theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \Big\} \, \mathrm{d}a \,. \tag{3.14}$$

From both Equations (3.12) and (3.14), it is found that it is necessary to evaluate the double integral

$$I = \int_0^{\pi} \int_0^{2\pi} P_j^j(\cos \theta') P_n(\cos \gamma) \sin \theta' \, \mathrm{d}\theta' \, \mathrm{d}\varphi' \,. \tag{3.15}$$

In order to evaluate this double-integral, we can use the additional theorem for expanding the surface spherical harmonic ($P_n(\cos \gamma)$) as a function of zonal, tesseral and sectorial harmonics (MacRobert, 1948, p. 138)

$$P_n(\cos \gamma) = P_n(\cos \theta) P_n(\cos \theta') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \cos m(\varphi - \varphi') \times$$

$$\times P_n^m(\cos \theta) P_n^m(\cos \theta') \,, \tag{3.16}$$

where the angle (γ) and $\cos \gamma$ are defined before (Equation (3.10)).

For $m = n$, Equation (3.16) becomes

$$P_n(\cos \gamma) = P_n(\cos \theta) P_n(\cos \theta') + \frac{2}{(2n)!} \cos n(\varphi - \varphi') \times$$

$$\times P_n^n(\cos \theta) P_n^n(\cos \theta') \,. \tag{3.17}$$

Substituting by (3.17) in (3.15) we obtain

$$I = I_1 + I_2 \,, \tag{3.18}$$

where

$$I_1 = P_n(\cos \theta) \int_0^{2\pi} \mathrm{d}\varphi' \int_0^{\pi} P_j^j(\cos \theta') P_n(\cos \theta') \sin \theta' \, \mathrm{d}\theta' \tag{3.19}$$

and

$$I_2 = \frac{2}{(2n)!} P_n^n(\cos \theta) \int_0^{2\pi} \cos n(\varphi - \varphi') \, \mathrm{d}\varphi' \int_0^{\pi} P_j^j(\cos \theta') \times$$

$$\times P_n^n(\cos \theta') \sin \theta' \, \mathrm{d}\theta' \,. \tag{3.20}$$

We are now in a position to evaluate I_1, by putting

$$I_1 = 2\pi P_n(\cos\theta) I_{11} \tag{3.21}$$

and

$$I_{11} = \int_0^{\pi} P_j^j(\cos\theta') P_n(\cos\theta') \sin\theta' \,\mathrm{d}\theta' , \tag{3.22}$$

we may evaluate this integral using Equation (2.6), in sectorial harmonics, and using the orthogonality condition of Legendre polynomials (MacRobert, 1948, Chapter V; Rainville, 1960; Theorem 63, p. 174)

$$\left.\begin{aligned} I_{11} &= \frac{2}{2n+}(-1)^{n/2} n! \\ &= 0 \end{aligned}\right\} \begin{aligned} &\text{for } j = n \\ &\quad j < n \end{aligned} \tag{3.23}$$

The substitution of Equation (3.23) into Equation (3.21), gives

$$I_1 = \frac{4\pi}{2n+1}(-1)^{n/2} n! P_n(\cos\theta) . \tag{3.24}$$

Turning to the second integral, I_2 (Equation (3.20)), we can put

$$I_2 = \frac{2}{(2n)!} P_n^n(\cos\theta) \cdot I_{21} \cdot I_{22} , \tag{3.25}$$

where

$$\begin{aligned} I_{21} &= \int_0^{2\pi} \cos n(\varphi - \varphi') \,\mathrm{d}\varphi' , \\ I_{22} &= \int_0^{\pi} P_j^j(\cos\theta') P_n^n(\cos\theta') \sin\theta' \,\mathrm{d}\theta' , \end{aligned} \tag{3.26}$$

The evaluation of I_{22} is straightforward from the orthogonality property,

$$\left.\begin{aligned} I_{22} &= 0 \\ &= (-1)^n (2n)! \cdot \frac{2}{2n+1} \end{aligned}\right\} \text{for} \begin{aligned} &j \neq n , \\ &j = n ; \end{aligned} \tag{3.27}$$

and the evaluation of I_{21} requires the use of simple trigonometrical formulae; integrating them and using the limits

$$I_{21} \to 0 . \tag{3.28}$$

Therefore, from Equations (3.27) and (3.28), it follows that

$$I_2 = 0 . \tag{3.29}$$

From both Equations (3.24) and (3.29) we obtain

$$\begin{aligned} I &= \int_0^{\pi}\int_0^{2\pi} P_j^j(\cos\theta')P_n(\cos\gamma)\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' = \\ &= \left.\begin{array}{ll} \dfrac{4\pi}{2n+1}(-1)^{n/2}\cdot n!P_n^0(\cos\theta) & j=n , \\ = 0 & j<n ; \end{array}\right\} \end{aligned} \tag{3.30}$$

where $P_n^0(\cos\theta)$ is a Legendre polynomials of degree n, but since we require only sectorial harmonics, we can apply the transformations for changing Legendre polynomials into sectorial harmonics, this will be done in Paper II.

Inserting Equation (3.30) in both Equations, (3.2) and (3.5) we find that

$$V_n(r) = \frac{4\pi G}{2n+1}(-1)^{n/2}(n!)P_n^0(\cos\theta)\int_0^{a_0}\rho\frac{\partial}{\partial a}\{a^{n+3}F_n(a)\}\,\mathrm{d}a , \tag{3.31}$$

we can put

$$F_n(a) = (-1)^{n/2}n!\int_0^{a_0}\rho\frac{\partial}{\partial a}\{a^{n+3}\mathscr{L}_n(a)\}\,\mathrm{d}a . \tag{3.32}$$

Equation (3.4) then becomes

$$V(r) = 4\pi G\sum_{n=0}^{\infty}\frac{F_n(a)}{(2n+1)}r^{-(n+1)}P_n^0(\cos\theta) . \tag{3.33}$$

Similarly,

$$\begin{aligned} U_n(r) = \frac{4\pi G}{(2n+1)}(-1)^{n/2}n!P_n^0(\cos\theta)\times \\ \times\int_{a_0}^{a_1}\rho\frac{\partial}{\partial a}\{a^{2-n}G_n(a)\}\,\mathrm{d}a ; \end{aligned} \tag{3.34}$$

and

$$E_n(a) = (-1)^{n/2}(n!)\int_{a_0}^{a_1}\rho\frac{\partial}{\partial a}\{a^{2-n}G_n(a)\}\,\mathrm{d}a . \tag{3.35}$$

Therefore, Equation (3.1) becomes

$$U(r) = 4\pi G \sum_{n=0}^{\infty} \frac{E_n(a)}{(2n+1)} (r)^n P_n(\cos\theta) . \tag{3.36}$$

The amplitudes of the interior ($E_n(a)$) and the exterior ($F_n(a)$) potentials can be evaluated as follows: Inserting Equation (2.11) into Equation (3.35), for $n = 0$ we have

$$E_0(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} [a^2(\tfrac{1}{2} + X_0 + X_0')] \, \mathrm{d}a , \tag{3.37}$$

and for $n > 0$

$$E_n(a) = (-1)^{n/2}(n!) \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \{a^{2-n}[X_n - (n-1)X_n' + $$

$$+ n(n-1)X_n'' - n(n-1)(n+1)X_n''']\} \, \mathrm{d}a . \tag{3.38}$$

Substituting for $n = 0, 2, 4, 6,$ and 8; and for the $X_n'^s$ from Equation (2.14), we have

$$E_0 = \int_{a_0}^{a_1} \rho \frac{2}{2a} \left[a^2 \left(\frac{1}{2} + \frac{2}{3} f_2^3 - f_2 f_6 - 2 f_2^4 \right) \mathrm{d}a , \tag{3.39}$$

$$E_2 = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left(f_2 + \frac{10}{3} f_4 + 7 f_6 + 12 f_8 - 7 f_2 f_6 - f_2^3 + \right.$$

$$\left. + \frac{10}{3} f_2^2 f_4 - \frac{5}{3} f_2^4 \right) \mathrm{d}a , \tag{3.40}$$

$$E_4(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[\frac{1}{a^2} \left(f_4 + \frac{27}{5} f_6 + \frac{594}{35} f_8 - \frac{36}{7} f_2 f_4 - \right. \right.$$

$$\left. - \frac{27}{35} f_2^2 - \frac{60}{7} f_4^2 - 27 f_2 f_6 + \frac{186}{7} f_2^2 f_4 - \frac{270}{7} f_2^4 \right] \mathrm{d}a , \tag{3.41}$$

$$E_6(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[\frac{1}{a^4} \left(f_6 + \frac{52}{7} f_8 - \frac{250}{33} f_4^2 - \frac{25}{11} f_2 f_4 - \right. \right.$$

$$\left. \left. - \frac{380}{11} f_2 f_6 + \frac{90}{77} f_2^3 + \frac{1950}{77} f_2^2 f_4 - \frac{54}{11} f_2^4 \right) \right] \mathrm{d}a \tag{3.42}$$

and

$$E_8(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a}\left[\frac{1}{a^6}\left(f_8 - \frac{196}{65} f_2 f_6 - \frac{1715}{1287} f_4^2 + \frac{784}{143} f_2^2 f_4 - \frac{1512}{713} f_2^4\right]\mathrm{d}a\,. \tag{3.43}$$

Similarly, by an insertion of Equation (2.10) into Equation (3.32), we get

For $n = 0$

$$F_0(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a}\left[a^3\left(\frac{1}{3} + X_0 + 2X_0' + 2X_0''\right)\right]\mathrm{d}a\,, \tag{3.44}$$

for $n > 0$

$$F_n(a) = (-1)^{n/2}(n!) \int_0^{a_0} \rho \frac{\partial}{\partial a}\{a^{n+3}[X_n + (n+2)X_n' + $$

$$+ (n+2)(n+1)X_n'' + n(n+2)(n+1)X_n''']\}\,\mathrm{d}a\,, \tag{3.45}$$

where $X_n'^2$ are still defined by Equation (2.14). Confining our attention to $n = 0$, 2, 4, 6 and 8, we obtain, retaining only quantities of fourth order or less,

$$F_0(a) = \int_0^{a_0} \rho\, a^2\,\mathrm{d}a\,, \tag{3.46}$$

$$F_2(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a}\left\{a^5\left(\frac{f}{2} + \frac{10}{3} f_4 + 7f_6 - 6f_2^3 + 12f_8\right.\right.$$

$$\left.\left. + 28 f_2 f_6 + \frac{20}{3} f_2^4 + 20 f_2^2 f_4\right)\right\}\mathrm{d}a\,, \tag{3.47}$$

$$F_4(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a}\left\{a^7\left(f_4 + \frac{54}{35} f_2^2 + \frac{27}{5} f_6 + \frac{72}{7} f_2 f_4 + \frac{594}{35} f_8 + \right.\right.$$

$$\left.\left. + \frac{120}{7} f_4^2 + 54 f_2 f_6 + \frac{465}{7} f_2^2 f_4 + \frac{540}{7} f_2^4\right)\right\}\mathrm{d}a\,, \tag{3.48}$$

$$F_6(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a}\left\{a^9\left(f_6 + \frac{40}{11} f_2 f_4 + \frac{24}{11} f_2^3 + \frac{52}{7} f_8 + \frac{304}{11} f_2 f_6 + \right.\right.$$

$$\left.\left. + \frac{520}{11} f_2^2 f_4 + \frac{432}{55} f_2^4\right)\right\}\mathrm{d}a\,, \tag{3.49}$$

and

$$F_8(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a} \left\{ a^{11} \left(f_8 + \frac{56}{13} f_2 f_6 + \frac{2450}{1287} f_4^2 + \frac{1260}{143} f_2^2 f_4 + \frac{432}{143} f_2^4 \right) \right\} \mathrm{d}a \,. \tag{3.50}$$

4. Total Potential

It is required to define the total potential, i.e., the sum of the potential arising from the total mass of the configuration, the internal and external potentials, and the disturbing potential which distorts this configuration.

The distortion of our body is due to centrifugal force arising from the axial rotation, and we found that the disturbing potential is given by

$$V'(r') = \tfrac{1}{2} w^2 r'^3 \sin^2 \theta \tag{4.1}$$

but from Equation (2.4), for $n = 2$,

$$P_2^2(\cos \theta) = 3 \sin^2 \theta \,. \tag{4.2}$$

Substituting Equation (4.2) into Equation (4.1), the disturbing potential, in a series of sectorial harmonics is given as

$$V'(r') = \tfrac{1}{6} w^2 (r'^2) P_2^2(\cos \theta) \tag{4.3}$$

or more generally,

$$V'(r') = \sum_{n=0}^{\infty} D_n r'^n P_n^n(\cos \theta) \,, \tag{4.4}$$

where

$$D_0 = 0, \qquad D_2 = \tfrac{1}{6} w^2 (1 + 8 f_2^3) \,,$$

$$D_4 = \frac{1}{6} \frac{w^2}{a^2} \left(-\frac{18}{35} f_2^3 \right), \qquad D_6 = 0 \,, \tag{4.5}$$

and

$$D_8 = \frac{1}{6} \frac{w^2}{a^6} \frac{-3 f_2^3}{25025} \,.$$

The potential of a self-gravitating configuration is the sum of the potentials (3.33), (3.36) and (4.4). Inserting (r') instead of (r), we find it to be given by

$$\psi(r', \theta) = 4\pi G \sum_{n=0}^{\infty} \left[\frac{(r')^n E_n(a) + (r')^{-(n+1)} E_n(a)}{(n+1)} \right] P_n^0 (\cos \theta) + $$

$$+ \sum_{n=0}^{\infty} D_n r'^n P_n^n(\cos \theta) \,, \tag{4.6}$$

or

$$\psi(r', \theta) = \psi_1(r', \theta) + \psi_2(r', \theta) , \tag{4.7}$$

where

$$\psi_1(r', \theta) = 4\pi G \sum_{n=0}^{\infty} \left\{\frac{(r')^n E_n(a) + (r)^{-(n+1)} F_n(a)}{2n+1}\right] P^0(\cos\theta) , \tag{4.8}$$

and

$$\psi_2(r', \theta) = \sum_{n=2}^{\infty} D_n r'^n P_n^n(\cos\theta) . \tag{4.9}$$

We may expand the r.h.s. of Equation (4.6) or (4.7) in a Neumann series of the form

$$\psi(r', \theta) = \sum_{n=0}^{\infty} \alpha_n(a) P_n^0(\cos\theta) + \sum_{n=2}^{\infty} B_n(a) P_n^n(\cos\theta) , \tag{4.10}$$

where

$$\alpha_n(a) = \frac{2n+1}{2} \int_0^{\pi} \psi_1(r', \theta) P_n^0(\cos\theta) \sin\theta \, d\theta , \tag{4.11}$$

or

$$\alpha_n(a) = 4\pi G \left[\frac{(r')^n E_n(a) + (r')^{-(a+1)} F_n(a)}{(2n+1)}\right] . \tag{4.12}$$

Similarly

$$B_n(a) = \frac{2n+1}{2} \frac{1}{(-1)^n (2n)!} \int_0^{\pi} \psi_2(r', \theta) P_n^n(\cos\theta) \sin\theta \, d\theta , \tag{4.13}$$

or

$$B_n(a) = D_n r'^n , \tag{4.14}$$

where the coefficients D_n are defined by Equation (4.5).

Over an equipotential surface the total potential must be constant. It is satisfied only if all terms of the r.h.s. of Equation (4.6) or Equation (4.7) factored by $P_n(\cos\theta)$ and $P_n^n(\cos\theta)$, for $n > 0$, vanish, and this happens only if both

$$\alpha_n(a) = 0 \quad \text{and} \quad \beta_n(a) = 0, \quad \text{for} \quad n > 0 . \tag{4.15}$$

So the only remaining term is $\alpha_0(a)$: the constant values which defines the required potential for an equipotential level surface of mean radius a.

References

Darwin, G. H.: 1910, *Scientific Papers*, Vol. 3 (Figures of Equilibrium of Rotating Liquid), Cambridge.
El-Shaarawy, M. B.: 1974, Ph.D. Thesis, University of Manchester (unpublished).
Hobson, E. W.: 1931, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press.
Kopal, Z.: 1960, *Figures of Equilibrium of Celestial Bodies*, University of Wisconsin Press, Madison.
Kopal, Z.: 1973, *Astrophys. Space Sci.* **24**, 145–174.
MacRobert, T. M.: 1948, *Spherical Harmonics*, Macmillan, New York.
Rainville, E. D.: 1960, *Special Functions*, MacMillan, New York.