THE MOTION OF ARTIFICIAL SATELLITES IN THE SET OF EULERIAN REDUNDANT PARAMETERS

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Abstract. In this paper, the connections between orbit dynamics and rigid body dynamics are established throughout the Eulerian redundant parameters, the perturbation equations for any conic motion of artificial satellites are derived in terms of these parameters. A general recursive and stable computational algorithm is also established for the initial-value problem of the Eulerian parameters for satellites prediction in the Earth's gravitational field with axial symmetry. Applications of the algorithm are considered for the two cases of short and long term predictions. For the short-term prediction, we consider the problem of the final state prediction of some typical ballistic missiles in the geopotential model with zonal harmonic terms up to J_{36}, while for the long-term prediction, we consider the perturbed J_2 motion of Explorer 28 over 100 revolutions.

1. Introduction

Orbit computations of artificial satellites become one of the most important problems at present, this due to their wide applications in scientific researches, mission planning and military purposes, etc. As far as the computation techniques are concerned, the applications of the special perturbation methods to the equations of motion in terms of the redundant variables, provide the most powerful and accurate techniques that have been devised recently for satellite ephemeris with respect to any type of perturbing forces (cf. e.g., Sharaf *et al.*, 1987a, b; Sharma and Raj, 1988; Awad, 1988).

Despite the many advantages of the Eulerian redundant parameters (cf., e.g., Carrington and Junkins, 1984; Vadali, 1988; Cid and Saturio, 1988) by which they have gained popularity in recent years in the rigid body dynamics and in the analysis of rotational motion of artificial and natural satellites, they have not been seriously utilized in constructing special perturbation techniques for satellite prediction.

The aim of the present paper is threefold. First, to establish the connections between orbit dynamics and rigid body dynamics and these are the subjects of Sections 2 and 3. Second, to derive general equations of motion in terms of the Eulerian parameters (Section 4), these equations include perturbations which can

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arise from a potential and perturbations which cannot be derived from a potential, also they are valid for any type of orbital motion (elliptic, parabolic or hyperbolic). Third, to construct a general, recursive and stable computational algorithm for the initial value problem of the Eulerian Parameters for satellite motions in the Earth's gravitational field with axial symmetry (Section 5). Applications of the algorithm for short and long term predictions are illustrated by numerical examples of some typical ballistic missiles and Explorer 28 satellite.

2. Rotation of a Rigid Body in Eulerian Parameters

2.1. Eulerian Angles and Eulerian Parameters

When defining the orientation of a body with respect to a reference frame a series of pure rotations is used, and this results in an orthogonal transformation. The associated rotations are the *Eulerian angles*, and they uniquely determine orientation of the body. Start by assuming both the reference X_1, X_2, X_3 and body-fixed x_1, x_2, x_3 frames coincide. One convenient sequence of rotations can be listed as:

1. Rotation about X_3 axis through angle ψ to produce α', β', γ', axes.
2. Rotation about α' axis through angle ϑ to produce α'', β'', γ'', axes.
3. Rotation about γ'' axis through angle ϕ to produce x_1, x_2, x_3 axes.

Each rotation is characterized as an orthogonal transformation. The α'' axis which is known as the *line of nodes* is the intersection of X_1, X_2 and x_1, x_2 planes. Combining this sequence of rotations we get for the transformation from $\mathbf{X}$ to $\mathbf{x}$ the equation

$$\mathbf{x} = \mathbf{c}\mathbf{X}\,, \tag{2.1}$$

where

$$\mathbf{x}^T = [x_1, x_2, x_3];\ \mathbf{X}^T = [X_1, X_2, X_3];\ \mathbf{C} = [C_{ij}];\ i, j = 1, 2, 3\,, \tag{2.2}$$

$$C_{11} = \cos\phi\cos\psi - \sin\phi\cos\vartheta\sin\psi\,, \tag{2.3.1}$$

$$C_{12} = \cos\phi\sin\psi + \sin\phi\cos\vartheta\cos\psi\,, \tag{2.3.2}$$

$$C_{13} = \sin\phi\sin\vartheta\,, \tag{2.3.3}$$

$$C_{21} = -\sin\phi\cos\psi - \cos\phi\cos\vartheta\sin\psi\,, \tag{2.3.4}$$

$$C_{22} = -\sin\phi\sin\psi + \cos\phi\cos\vartheta\cos\psi\,, \tag{2.3.5}$$

$$C_{23} = \cos\phi\sin\vartheta\,, \tag{2.3.6}$$

$$C_{31} = \sin\vartheta\sin\psi\,, \tag{2.3.7}$$

$$C_{32} = -\sin\vartheta\cos\psi\,, \tag{2.3.8}$$

$$C_{33} = \cos\vartheta\,. \tag{2.3.9}$$

The amplitudes of the Eulerian angles ψ, ϑ and ϕ satisfy the conditions

$$0 \leq \psi < 2\pi ; \qquad 0 \leq \vartheta \leq \pi ; \qquad 0 \leq \phi < 2\pi . \tag{2.4}$$

According to this definition, we can express the *Eulerian Parameters* (u_1, u_2, u_3, u_4) by means of the equalities

$$u_1 = \sin\frac{\vartheta}{2}\cos\left(\frac{\psi-\phi}{2}\right); \quad u_2 = \sin\frac{\vartheta}{2}\sin\left(\frac{\psi-\phi}{2}\right), \tag{2.5.1}$$

$$u_3 = \cos\frac{\vartheta}{2}\sin\left(\frac{\psi+\phi}{2}\right); \quad u_4 = \cos\frac{\vartheta}{2}\cos\left(\frac{\psi+\phi}{2}\right). \tag{2.5.2}$$

From these equations it is clear that

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1. \tag{2.6}$$

Eulerian parameters define two *quaternions*, $u = (u_1, u_2, u_3, u_4)$ and its diametric opposite $-u = (-u_1, -u_2, -u_3, -u_4)$, hence according to the real algebra of quaternions (cf., e.g., Porteous, 1969) u and its *conjugate* $\bar{u}$ are written as linear combinations

$$u = u_1 + iu_2 + ju_3 + ku_4 ,$$
$$\bar{u} = u_1 - iu_2 - ju_2 - ku_4 ,$$

Where $\{1, i, j, k\}$ is the standard basis of the Euclidean space R^4 with the basic rules $i^2 = j^2 = k^2 = ijk = -1$. Since a symplectic inner product of two quaternions q and p (say) is defined by $\langle q, p\rangle = q\bar{p}$, consequently, the *norms* of the quaternions u and $-u$ are given in accordance with Equation (2.6) as

$$\langle u, \bar{u}\rangle = u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 ,$$
$$\langle -u, -\bar{u}\rangle = u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 .$$

That is, u and $-u$ are unit quaternions, which means that *every rotation is represented on the unit sphere S^3 in the Euclidean space R^4 by two points at the extremity of a diameter*.

Some important relations between Euler's angles and Euler's parameters are given in what follows.

From Equations (2.3) and (2.5) we get

$$C_{11} = u_1^2 - u_2^2 - u_3^2 + u_4^2 ; \quad C_{12} = 2(u_1u_2 + u_3u_4) , \tag{2.7.1}$$

$$C_{13} = 2(u_1u_3 - u_2u_4); \quad C_{21} = 2(u_1u_2 - u_3u_4) , \tag{2.7.2}$$

$$C_{22} = -u_1^2 + u_2^2 - u_3^2 + u_4^2 ; \quad C_{23} = 2(u_2u_3 + u_1u_4) , \tag{2.7.3}$$

$$C_{31} = 2(u_1u_3 + u_2u_4) ; \quad C_{32} = 2(u_2u_3 - u_1u_4) , \tag{2.7.4}$$

$$C_{33} = -u_1^2 - u_2^2 + u_3^2 + u_4^2 , \tag{2.7.5}$$

where

$$\sum_{j=1}^{3} C_{1j} \cdot C_{ij} = \sum_{j=1}^{3} C_{j1} \cdot C_{ji} = 0 ; \quad i = 2, 3 , \tag{2.8.1}$$

$$\sum_{j=1}^{3} C_{kj}^2 = \sum_{j=1}^{3} C_{jk}^2 = 1; \quad k = 1, 2, 3 , \tag{2.8.2}$$

$$\begin{aligned} C_{31} &= C_{12}C_{23} - C_{13}C_{22} , \\ C_{32} &= C_{13}C_{21} - C_{11}C_{23} , \\ C_{33} &= C_{11}C_{22} - C_{12}C_{21} . \end{aligned} \tag{2.8.3}$$

From Equations (2.5), the inverse transformation from the Eulerian Parameters (u_1, u_2, u_3, u_4) to the Eulerian angles (ψ, ϑ, ϕ) may be written in the form

$$\psi = \tan^{-1}\left\{\frac{u_1u_3 + u_4u_2}{u_1u_4 - u_2u_3}\right\} , \tag{2.9.1}$$

$$\vartheta = 2\tan^{-1}\left\{\sqrt{\frac{a}{b}}\right\} , \tag{2.9.2}$$

$$\phi = \tan^{-1}\left\{\frac{u_1u_3 - u_2u_4}{u_1u_4 + u_2u_3}\right\} , \tag{2.9.3}$$

where

$$a = u_1^2 + u_2^2 = \sin^2\frac{\vartheta}{2}; \quad b = u_3^2 + u_4^2 = \cos^2\frac{\vartheta}{2} . \tag{2.9.4}$$

However, ambiguities in the Eulerian angles will be removed by using the following relations

$$\sin\psi = (u_1u_3 + u_2u_4)/\sqrt{ab}; \ \cos\psi = (u_1u_4 - u_2u_3)/\sqrt{ab} , \tag{2.10.1}$$

$$\sin\vartheta = 2\sqrt{ab}; \quad \cos\vartheta = b - a , \tag{2.10.2}$$

$$\sin\phi = (u_1u_3 - u_2u_4)/\sqrt{ab}; \cos\phi = (u_1u_4 + u_2u_3)/\sqrt{ab} . \tag{2.10.3}$$

Also from Equations (2.5) we deduce that

$$u_1 = (u_3\sin\psi + u_4\cos\psi)\tan\frac{\vartheta}{2} = (u_3\sin\phi + u_4\cos\phi)\tan\frac{\vartheta}{2} , \tag{2.11.1}$$

$$u_2 = (-u_3\cos\psi + u_4\sin\psi)\tan\frac{\vartheta}{2} = (u_3\cos\phi - u_4\sin\phi)\tan\frac{\vartheta}{2} , \tag{2.11.2}$$

$$u_3 = (u_1\sin\psi - u_2\cos\psi)\cot\frac{\vartheta}{2} = (u_1\sin\phi + u_2\cos\phi)\cot\frac{\theta}{2} , \tag{2.11.3}$$

$$u_4 = (u_1\cos\psi + u_2\sin\psi)\cot\frac{\vartheta}{2} = (u_1\cos\phi - u_2\sin\phi)\cot\frac{\vartheta}{2} . \tag{2.11.4}$$

2.2. Angular velocity components

Frequently in the dynamics of rigid body one needs to express the components $(\omega_1, \omega_2, \omega_3)$ of the angular velocity vector $\boldsymbol{\omega}$ about the body axes x_1, x_2, x_3 in terms of the Eulerian angles.

However in the present study we need the expressions of these components in terms of the Eulerian Parameters (u_1, u_2, u_3, u_4), to find so the following analyses are devoted.

2.2.1. *Expressions of ω_1, ω_2, ω_3 in terms of the Eulerian Angles*

Referring to the axes of Section 2.1, resolve the angular velocity $\dot{\psi}$ along γ'' and β'' axes so that the orthogonal components of $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ are $\dot{\vartheta}$ along α'', $\dot{\psi}\sin\vartheta$ along β'', $\dot{\phi}+\dot{\psi}\cos\vartheta$ along γ''. Next resolve the components along the α'' and β'' axes to the x_1, x_2 direction, the result being

$$\omega_1 = \dot{\psi}\sin\vartheta\sin\phi + \dot{\vartheta}\cos\phi\,, \tag{2.12.1}$$

$$\omega_2 = \dot{\psi}\sin\vartheta\cos\phi - \dot{\vartheta}\sin\phi\,, \tag{2.12.2}$$

$$\omega_3 = \dot{\phi} + \dot{\psi}\cos\vartheta\,. \tag{2.12.3}$$

Of course, the Eulerian rates can be expressed in terms of ω_1, ω_2, ω_3. In order to avoid coupling of the rates, only normal components of $\dot{\psi}$, $\dot{\vartheta}$, and $\dot{\phi}$ can be used, because the Eulerian rates are not orthogonal. The three appropriate components are:

(a) $\dot{\psi}\sin\vartheta$ which is normal to $\dot{\vartheta}$ and $\dot{\phi}$,
(b) $\dot{\phi}\sin\vartheta$ which is normal to $\dot{\psi}$ and $\dot{\vartheta}$,
(c) $\dot{\vartheta}$ which is already normal to $\dot{\psi}$ and $\dot{\phi}$.

The transformation are then easily obtained as,

$$\dot{\psi} = \operatorname{cosec}\vartheta(\omega_1\sin\phi + \omega_2\cos\phi)\,, \tag{2.13.1}$$

$$\dot{\phi} = \omega_3 - \cot\vartheta(\omega_1\sin\phi + \omega_2\cos\phi)\,, \tag{2.13.2}$$

$$\dot{\vartheta} = \omega_1\cos\phi - \omega_2\sin\phi\,. \tag{2.13.3}$$

2.2.2. *Partial Derivatives of the Eulerian Angles with respect to the Eulerian Parameters*

From Equations (2.5) and (2.9) we deduce that

$$\frac{\partial\psi}{\partial u_1} = \frac{-u_2}{a} = -\sin\tfrac{1}{2}(\psi-\phi)/\sin\frac{\vartheta}{2}\,, \tag{2.14.1}$$

$$\frac{\partial\psi}{\partial u_2} = \frac{u_1}{a} = \cos\tfrac{1}{2}(\psi-\phi)/\sin\frac{\vartheta}{2}\,, \tag{2.14.2}$$

$$\frac{\partial\psi}{\partial u_3} = \frac{u_4}{b} = \cos\tfrac{1}{2}(\psi+\phi)/\cos\frac{\vartheta}{2}\,, \tag{2.14.3}$$

$$\frac{\partial \psi}{\partial u_4} = \frac{-u_3}{b} = -\sin \tfrac{1}{2}(\psi + \phi)/\cos \frac{\vartheta}{2}, \tag{2.14.4}$$

$$\frac{\partial \vartheta}{\partial u_1} = 2u_1 \sqrt{\frac{b}{a}} = 2 \cos \tfrac{1}{2}(\psi - \phi) \cos \frac{\vartheta}{2}, \tag{2.15.1}$$

$$\frac{\partial \vartheta}{\partial u_2} = 2u_2 \sqrt{\frac{b}{a}} = 2 \sin \tfrac{1}{2}(\psi - \phi) \cos \frac{\vartheta}{2}, \tag{2.15.2}$$

$$\frac{\partial \vartheta}{\partial u_3} = -2u_3 \sqrt{\frac{a}{b}} = -2 \sin \tfrac{1}{2}(\psi + \phi) \sin \frac{\vartheta}{2}, \tag{2.15.3}$$

$$\frac{\partial \vartheta}{\partial u_4} = -2u_4 \sqrt{\frac{a}{b}} = -2 \cos \tfrac{1}{2}(\psi + \phi) \sin \frac{\vartheta}{2}, \tag{2.15.4}$$

$$\frac{\partial \phi}{\partial u_1} = \frac{u_2}{a} = -\frac{\partial \psi}{\partial u_1} = \sin \tfrac{1}{2}(\psi - \phi)/\sin \frac{\vartheta}{2}, \tag{2.16.1}$$

$$\frac{\partial \phi}{\partial u_2} = -\frac{u_1}{a} = -\frac{\partial \psi}{\partial u_2} = -\cos \tfrac{1}{2}(\psi - \phi)/\sin \frac{\vartheta}{2}, \tag{2.16.2}$$

$$\frac{\partial \phi}{\partial u_3} = \frac{u_4}{b} = \frac{\partial \psi}{\partial u_3} = \cos \tfrac{1}{2}(\psi + \phi)/\cos \frac{\vartheta}{2}, \tag{2.16.3}$$

$$\frac{\partial \phi}{\partial u_4} = \frac{-u_3}{b} = \frac{\partial \psi}{\partial u_4} = -\sin \tfrac{1}{2}(\psi + \phi)/\cos \frac{\vartheta}{2}. \tag{2.16.4}$$

2.2.3. *Eulerian Rates in terms of the Eulerian Parameters and their Rates*

Since any of the Eulerian angles can be expressed as a function of the Eulerian Parameters, then

$$\dot{\psi} = \sum_{j=1}^{4} \frac{\partial \psi}{\partial u_j} \dot{u}_j ; \quad \dot{\vartheta} = \sum_{j=1}^{4} \frac{\partial \vartheta}{\partial u_j} \dot{u}_j ; \quad \dot{\phi} = \sum_{j=1}^{4} \frac{\partial \phi}{\partial u_j} \dot{u}_j .$$

Using Equations (2.14), (2.15) and (2.16) into the above expressions we get

$$\dot{\psi} = (-u_2\dot{u}_1 + u_1\dot{u}_2)/a + (u_4\dot{u}_3 - u_3\dot{u}_4)/b , \tag{2.17.1}$$

$$\dot{\vartheta} = 2\left[(u_1\dot{u}_1 + u_2\dot{u}_2)\sqrt{\frac{b}{a}} - (u_4\dot{u}_4 + u_3\dot{u}_3)\sqrt{\frac{a}{b}}\right], \tag{2.17.2}$$

$$\dot{\phi} = (u_2\dot{u}_1 - u_1\dot{u}_2)/a + (u_4\dot{u}_3 - u_3\dot{u}_4)/b. \tag{2.17.3}$$

2.2.4. *Expressions for* ω_1, ω_2, ω_3 *in terms of the Eulerian Parameters and their Rates*

Using Equations (2.17) and Equations (2.11) into Equations (2.12) we get

$$\omega_1 = 2(-u_1\dot{u}_4 + u_4\dot{u}_1 + u_3\dot{u}_2 - u_2\dot{u}_3) , \tag{2.18}$$

$$\omega_2 = 2(-u_2\dot{u}_4 - u_3\dot{u}_1 + u_4\dot{u}_2 + u_1\dot{u}_3) , \tag{2.19}$$

$$\omega_3 = 2(-u_3\dot{u}_4 + u_2\dot{u}_1 - u_1\dot{u}_2 + u_4\dot{u}_3) . \tag{2.20}$$

These equations are what we required to set up for the present subsection.

3. Motion of the Orbital Frame

A most interesting connection between orbital dynamics and rigid body dynamics could be established if we consider the orbit normal $\boldsymbol{\zeta}$, the radius vector $\boldsymbol{\xi}$, and the orthogonal vector $\boldsymbol{\eta} = \boldsymbol{\zeta} \times \boldsymbol{\xi}$ as a rigid body. Since this triad is a rotating coordinate system, its motion can be investigated by applying well-known methods of rigid body dynamics. The present section is devoted to establish the basic formulations for the connection between orbital and rigid body dynamics.

3.1. Orbital frame

The unit vector $\boldsymbol{\xi}$ is defined as a vector which always points at the body under consideration (hereafter we shall consider such body as a given artificial satellite). The unit vector $\boldsymbol{\eta}$ is advanced to $\boldsymbol{\xi}$ in the sense of increasing true anomaly f by a right angle in the plane of instantaneous motion. Finally, the unit vector $\boldsymbol{\zeta}$ completes the orthogonal set and is always directed along the angular momentum vector H. The rotating triad $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ will be called the *orbital frame.*

Now, the first step for the connection between orbital and rigid body dynamics is to find the relations between the unit vectors $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$, the position and velocity vectors $(\mathbf{x}, \dot{\mathbf{x}})$ in the inertial frame (in which the orbital motion is described).

3.2. Relations between $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ and $(\mathbf{x}, \dot{\mathbf{x}})$

The unit vector $\boldsymbol{\xi}$ is related to the position vector $\mathbf{x}$ by

$$\boldsymbol{\xi} = \frac{\mathbf{x}}{r} , \tag{3.1}$$

where $r = |\mathbf{x}|$. Since in the pure Keplerian motion ϕ is a part from an additive constant the true anomaly f then

$$\boldsymbol{\eta} = \frac{r\dot{\mathbf{x}} - \dot{x}\mathbf{x}}{r^2\dot{\phi}} = \frac{r\dot{\mathbf{x}} - \dot{x}\mathbf{x}}{\sqrt{\mu p}} , \tag{3.2}$$

where p is the semi-latus rectum, μ in the present study (artificial satellite motions) is the Earth's gravitational constant.

Finally, since the angular momentum $\mathbf{H}$ is given as

$$\mathbf{H} = H\boldsymbol{\zeta} = \sqrt{\mu p}\,\boldsymbol{\zeta} = \mathbf{x} \times \dot{\mathbf{x}}\,,$$

then

$$\boldsymbol{\zeta} = \frac{1}{\sqrt{\mu p}}(\mathbf{x} \times \dot{\mathbf{x}}). \tag{3.3}$$

Since p vanishes on collision orbits, Equation (3.2) and (3.3) then become meaningless. We therefore exclude collision, hence $p > 0$. This is no restriction in satellite applications since the central mass has finite dimensions.

3.3. Relations between $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ and (u_1, u_2, u_3, u_4)

The next step in our analyses is to find the relations between the unit vectors $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ and the Eulerian Parameters. To obtain so, let us define the following vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} u_2 \\ -u_1 \\ u_4 \\ -u_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} u_3 \\ -u_4 \\ -u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} u_4 \\ u_3 \\ -u_2 \\ -u_1 \end{bmatrix}. \tag{3.4}$$

Clearly, these vectors are mutually orthogonal. Since the unit vectors $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ is related to the unit vectors $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ of the inertial frame by the linear system.

$$\begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \\ \boldsymbol{\zeta} \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix}, \tag{3.5}$$

then by using Equations (2.7) into Equations (3.5) we get by taking account of definitions (3.4) that

$$\boldsymbol{\xi} = \boldsymbol{\Lambda}(\mathbf{u})\mathbf{u}\,, \tag{3.6.1}$$

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}(\mathbf{u})\mathbf{v}\,, \tag{3.6.2}$$

$$\boldsymbol{\zeta} = \boldsymbol{\Lambda}(\mathbf{u})\mathbf{w}\,, \tag{3.6.3}$$

$$\boldsymbol{\Lambda}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & u_4 & u_3 \\ u_3 & -u_4 & u_1 & -u_2 \end{bmatrix}. \tag{3.7}$$

Note that

$$\boldsymbol{\Lambda}(\mathbf{L})\mathbf{M} = \boldsymbol{\Lambda}(\mathbf{M})\mathbf{L}\,, \tag{3.8.1}$$

$$\dot{\Lambda}(\mathbf{L}) = \frac{\mathrm{d}}{\mathrm{d}t}\{\Lambda(\mathbf{L})\} = \Lambda(\dot{\mathbf{L}}) , \tag{3.8.2}$$

where **L, M** are column vectors in the four-dimensional space.

The relations between the rates could be obtained from Equations (3.6), (3.8) and we get

$$\dot{\boldsymbol{\xi}} = 2\Lambda(\mathbf{U})\dot{\mathbf{u}} , \tag{3.9.1}$$

$$\dot{\boldsymbol{\eta}} = 2\Lambda(\mathbf{U})\dot{\mathbf{v}} , \tag{3.9.2}$$

$$\dot{\boldsymbol{\zeta}} = 2\Lambda(\mathbf{U})\dot{\mathbf{w}} . \tag{3.9.3}$$

3.4. Relations between $(\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4)$ and $(\omega_1, \omega_2, \omega_3)$

Differentiating Equations (2.5) with respect to the time t we get

$$\dot{u}_1 = \frac{1}{2}\dot{\vartheta}u_1 \cot\frac{\vartheta}{2} - \frac{1}{2}(\dot{\psi} - \dot{\phi})u_2 , \tag{3.10.1}$$

$$\dot{u}_2 = \frac{1}{2}\dot{\vartheta}u_2 \cot\frac{\vartheta}{2} + \frac{1}{2}(\dot{\psi} - \dot{\phi})u_1 , \tag{3.10.2}$$

$$\dot{u}_3 = -\frac{1}{2}\dot{\vartheta}u_3 \tan\frac{\vartheta}{2} + \frac{1}{2}(\dot{\psi} + \dot{\phi})u_4 , \tag{3.10.3}$$

$$\dot{u}_4 = -\frac{1}{2}\dot{\vartheta}u_4 \tan\frac{\vartheta}{2} - \frac{1}{2}(\dot{\psi} + \dot{\phi})u_3 . \tag{3.10.4}$$

Using Equations (2.13) into Equations (3.10) we get by the aid of Equations (2.11) that

$$\dot{u}_1 = \tfrac{1}{2}(\omega_1 u_4 - \omega_2 u_3 + \omega_3 u_2) , \tag{3.11.1}$$

$$\dot{u}_2 = \tfrac{1}{2}(\omega_1 u_3 + \omega_2 u_4 - \omega_3 u_1) , \tag{3.11.2}$$

$$\dot{u}_3 = \tfrac{1}{2}(-\omega_1 u_2 + \omega_2 u_1 + \omega_3 u_4) , \tag{3.11.3}$$

$$\dot{u}_4 = \tfrac{1}{2}(-\omega_1 u_1 - \omega_2 u_2 - \omega_3 u_3) . \tag{3.11.4}$$

These equations can be written in a matrix form as

$$\dot{\mathbf{u}} = \tfrac{1}{2}\mathbf{B}(\boldsymbol{\omega})\mathbf{u} , \tag{3.12.1}$$

where

$$\mathbf{B}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} . \tag{3.12.2}$$

Note that the matrix **B** is skew symmetric.

Referring to the column vectors defined by Equation (3.4), Equations (3.11) could be written in another form as

$$\dot{\mathbf{u}} = \tfrac{1}{2}[\omega_1 \mathbf{q} - \omega_2 \mathbf{w} + \omega_3 \mathbf{v}] . \tag{3.13.1}$$

It follows also that

$$\dot{\mathbf{v}} = \tfrac{1}{2}[\omega_1 \mathbf{w} + \omega_2 \mathbf{q} - \omega_3 \mathbf{u}] , \tag{3.13.2}$$

$$\dot{\mathbf{w}} = \tfrac{1}{2}[-\omega_1 \mathbf{v} + \omega_2 \mathbf{u} + \omega_3 \mathbf{q}] . \tag{3.13.3}$$

3.5. Relations between $(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\zeta}})$ and $(\omega_1, \omega_2, \omega_3)$

Using Equations (3.13) into Equations (3.9), then using Equations (3.6) in the resulting Equations and the fact that

$$\boldsymbol{\Lambda}(\mathbf{u})\mathbf{q} = \mathbf{0} ,$$

we get

$$\dot{\boldsymbol{\xi}} = \omega_2 \boldsymbol{\eta} - \omega_2 \boldsymbol{\zeta} , \tag{3.14.1}$$

$$\dot{\boldsymbol{\eta}} = \omega_1 \boldsymbol{\zeta} - \omega_3 \boldsymbol{\xi} , \tag{3.14.2}$$

$$\dot{\boldsymbol{\zeta}} = \omega_2 \boldsymbol{\xi} - \omega_1 \boldsymbol{\eta} . \tag{3.14.3}$$

By these relations we come to the end of the present section after establishing the basic formulations for the connection between orbital and rigid body dynamics.

4. Satellite Motions in Terms of the Eulerian Parameters

In this section, the equations of motion in terms of the Eulerian Parameters will be derived.

4.1. Fundamental equations in rectangular variables

The sixth-order system of differential equations in rectangular variables (x_1, x_2, x_3) describing the rate of change of the position and velocity for a satellite orbiting the Earth is given by

$$\ddot{\mathbf{x}} + \mu \frac{\mathbf{x}}{r^3} = \mathbf{P} = \mathbf{P}^* - \frac{\partial V}{\partial \mathbf{x}} , \tag{4.1}$$

where V is the perturbed time-independent potential, and $\mathbf{P}^*$ is the resultant of all nonconservative perturbing forces and forces derivable from a time-dependent potential, the other variables are defined previously.

The coordinate system is inertially fixed with $x_1 x_2$ plane corresponding to the Earth's equatorial plane. Associated with Equation (4.1), the energy equations and the laws of energy defined, respectively as,

$$h_k = \frac{\mu}{r} - \frac{1}{2}\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle , \tag{4.2.1}$$

$$h = h_k - V \tag{4.2.2}$$

and

$$\dot{h}_k = \left\langle \frac{\partial V}{\partial \mathbf{x}}, \dot{\mathbf{x}} \right\rangle - \langle \mathbf{P}^*, \dot{\mathbf{x}} \rangle\,, \tag{4.3.1}$$

$$\dot{h} = -\frac{\partial V}{\partial t} - \langle \mathbf{P}^*, \dot{\mathbf{x}} \rangle\,, \tag{4.3.2}$$

where $(-h_k)$ is the Keplerian energy, $(-h)$ the total energy and $\langle \mathbf{a}, \mathbf{b} \rangle$ is used to denote the scalar product of two vectors **a** and **b**. By means of Equations (4.2) the decision on the type of the orbit could be made, if $h > 0$ it is elliptic, if $h < 0$ hyperbolic and if $h = 0$ parabolic. This decision is very important, since the type of the orbit is occasionally changed by perturbing forces acting during a finite interval of time.

4.2. Equations of Motion in Terms of the Eulerian Parameters

Differentiating Equation (3.1) with respect to t, then using Equation (3.2) we get

$$\dot{\boldsymbol{\xi}} = \frac{\sqrt{\mu p}}{r^2} \boldsymbol{\eta}\,, \tag{4.4}$$

using

$$\mathbf{x} = r\boldsymbol{\xi} \tag{4.5}$$

and

$$\dot{\mathbf{x}} = r\dot{\boldsymbol{\xi}} + \dot{r}\boldsymbol{\xi} \tag{4.6}$$

into Equation (3.3) we obtain

$$\boldsymbol{\zeta} = \frac{r^2}{\sqrt{\mu p}} (\boldsymbol{\xi} \times \dot{\boldsymbol{\xi}})\,. \tag{4.7}$$

Using the first derivative of Equation (4.6) into Equation (4.1) we get

$$r\ddot{\boldsymbol{\xi}} + 2\dot{r}\dot{\boldsymbol{\xi}} + \left(\ddot{r} + \frac{\mu}{r^2}\right)\boldsymbol{\xi} = P\,. \tag{4.8}$$

Since

$$r^2 = \langle \mathbf{x}, \mathbf{x} \rangle\,, \tag{4.9}$$

then

$$r\dot{r} = \langle \mathbf{x}, \dot{\mathbf{x}} \rangle\,, \tag{4.10}$$

$$r\ddot{r} + \dot{r}^2 = \langle \mathbf{x}, \ddot{\mathbf{x}} \rangle + \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle\,. \tag{4.11}$$

Taking the scalar product of Equation (4.1) with the vector **x**, then using Equation (4.9) we obtain

$$\langle \mathbf{x}, \ddot{\mathbf{x}} \rangle = \langle \mathbf{P}, \mathbf{x} \rangle - \frac{\mu}{r}. \tag{4.12}$$

Since p is given as

$$p = \frac{r^2}{\mu}\{\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle - \dot{r}^2\}, \tag{4.13}$$

then by using Equations (4.5), (4.12) and (4.13) into Equation (4.11) we get

$$\ddot{r} = \langle \mathbf{P}, \boldsymbol{\xi} \rangle + \frac{\mu p}{r^3} - \frac{\mu}{r^2}. \tag{4.14}$$

From Equation (4.14) into Equation (4.8), and using the fact.

$$\mathbf{P} = \langle \mathbf{P}, \boldsymbol{\xi} \rangle \boldsymbol{\xi} + \langle \mathbf{P}, \boldsymbol{\eta} \rangle \boldsymbol{\eta} + \langle \mathbf{P}, \boldsymbol{\zeta} \rangle \boldsymbol{\zeta}$$

we deduce that

$$r\ddot{\boldsymbol{\xi}} + 2\dot{r}\dot{\boldsymbol{\xi}} = \left(-\frac{p\mu}{r^3}\right)\boldsymbol{\xi} + \langle \mathbf{P}, \boldsymbol{\eta} \rangle \boldsymbol{\eta} + \langle \mathbf{P}, \boldsymbol{\zeta} \rangle \boldsymbol{\zeta}. \tag{4.15}$$

Now differentiating Equation (4.4) with respect to t we get

$$\ddot{\boldsymbol{\xi}} = \frac{\dot{p}}{2p}\frac{\sqrt{\mu p}}{r^2}\boldsymbol{\eta} - \frac{2\dot{r}}{r}\dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\eta}}\frac{\sqrt{\mu p}}{r^2}.$$

Note that, p is considered variable due to the existence of perturbations.

Using Equation (3.14.2) in the last equation we obtain

$$r\ddot{\boldsymbol{\xi}} + 2\dot{r}\dot{\boldsymbol{\xi}} = \left(-\omega_3\frac{\sqrt{\mu p}}{r}\right)\boldsymbol{\xi} + \left(\frac{\dot{p}}{2p}\frac{\sqrt{\mu p}}{r}\right)\boldsymbol{\eta} + \left(\omega_1\frac{\sqrt{\mu p}}{r}\right)\boldsymbol{\xi}. \tag{4.16}$$

Comparing Equations (4.15) and (4.16) we get

$$\omega_1 = \frac{r}{\sqrt{\mu p}}\langle \mathbf{P}, \boldsymbol{\xi} \rangle, \tag{4.17}$$

$$\omega_3 = \frac{\sqrt{\mu p}}{r^2}, \tag{4.18}$$

$$\dot{p} = 2r\sqrt{\frac{p}{\mu}}\langle \mathbf{P}, \boldsymbol{\eta} \rangle = \frac{2r^3}{\mu}\langle \mathbf{P}, \dot{\boldsymbol{\xi}} \rangle. \tag{4.19}$$

The value of ω_2 may be obtained by substituting Equation (3.14.3) into Equation (4.7) and we get

$$\sqrt{\mu p}\,\boldsymbol{\xi} = r^2\{\omega_3\boldsymbol{\xi} + \omega_2\boldsymbol{\eta}\}\,,$$

using Equation (4.18) into the last equation we get for both the perturbed and unperturbed motions that

$$\omega_2 = 0\,.$$

From the above equations and Equations (3.11) we finally have for perturbed motions in terms of the Eulerian Parameters the following equations

$$\dot{u}_1 = \tfrac{1}{2}(\omega_1 u_4 + \omega_3 u_2)\,, \tag{4.20.1}$$

$$\dot{u}_2 = \tfrac{1}{2}(\omega_1 u_3 - \omega_3 u_1)\,, \tag{4.20.2}$$

$$\dot{u}_3 = \tfrac{1}{2}(-\omega_1 u_2 + \omega_3 u_4)\,, \tag{4.20.3}$$

$$\dot{u}_4 = \tfrac{1}{2}(-\omega_1 u_1 - \omega_3 u_3)\,, \tag{4.20.4}$$

$$\ddot{r} = \langle \mathbf{P}, \boldsymbol{\xi}\rangle + \frac{\mu p}{r^3} - \frac{\mu}{r^2}\,, \tag{4.20.5}$$

$$\dot{p} = 2r\sqrt{\frac{p}{\mu}}\,\langle \mathbf{P}, \boldsymbol{\eta}\rangle\,, \tag{4.20.6}$$

$$\omega_1 = \frac{r}{\sqrt{\mu p}}\langle \mathbf{P}, \boldsymbol{\xi}\rangle\,, \tag{4.20.7}$$

$$\omega_3 = \frac{\sqrt{\mu p}}{r^2}\,. \tag{4.20.8}$$

System (4.20) is of the seventh order, general since it includes perturbations which can arise from a potential and perturbations which cannot be derived from a potential, and uniform in the sense that it is valid for all values of the energy (that is, the same equations describe the motion whether it is elliptic, parabolic, or hyperbolic).

It is noted that in the pure Keplerian motion ($\mathbf{P} = V = \mathbf{P}^* = 0$) we have

$$\omega_1 = 0; \qquad \omega_3 = \frac{\sqrt{\mu p}}{r^2} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\,, \tag{4.21}$$

where p = constant, is the semi-latus rectum of osculating orbit. The components of the perturbing forces $\mathbf{P}(\mathbf{P} = \mathbf{P}^* - \partial V/\partial \mathbf{x})$ in terms of the Eulerian Parameters could be obtained from Equations (3.6), (3.8.1) and we get

$$P_\xi = \langle \mathbf{P}, \boldsymbol{\xi}\rangle = \langle \mathbf{P}, \boldsymbol{\Lambda}(\mathbf{u})\mathbf{u}\rangle = \langle \boldsymbol{\Lambda}^T(\mathbf{u})\mathbf{P}, \mathbf{u}\rangle\,,$$

$$P_\eta = \langle \mathbf{P}, \boldsymbol{\eta} \rangle = \langle \mathbf{P}, \boldsymbol{\Lambda}(\mathbf{u})\mathbf{v} \rangle = \langle \boldsymbol{\Lambda}^T(\mathbf{u})\mathbf{P}, \mathbf{v} \rangle ,$$

$$P_\zeta = \langle \mathbf{P}, \boldsymbol{\xi} \rangle = \langle \mathbf{P}, \boldsymbol{\Lambda}(\mathbf{u})\mathbf{w} \rangle = \langle \boldsymbol{\Lambda}^T(\mathbf{u})\mathbf{P}, \mathbf{w} \rangle .$$

Using Equations (3.4) and (3.7) into the above equations we get:

$$P_\xi = P_1 C_{11} + P_2 C_{12} + P_3 C_{13} , \tag{4.22.1}$$

$$P_\eta = P_1 C_{21} + P_2 C_{22} + P_3 C_{23} , \tag{4.22.2}$$

$$P_\zeta = P_1 C_{31} + P_2 C_{32} + P_3 C_{33} , \tag{4.22.3}$$

where

$$P_j = P_j^* - \frac{\partial V}{\partial x_j} ; \qquad j = 1, 2, 3 ,$$

and C's are given in terms of u's by Equations (2.7). In general P_j, $j = 1, 2, 3$ and $\partial V / \partial t$ are functions of $(\mathbf{x}, \dot{\mathbf{x}}, t)$ (see for example Sharaf *et al.*, 1989) and hence they can be expressed in terms of the Eulerian parameters throughout the relations between $\mathbf{x}$, $\dot{\mathbf{x}}$ and these parameters. A typical example of such application will be considered in the following section. Finally it is worth mentioning that, Equation (2.6) and the condition that $\omega_2 = 0$ [Equation (2.19)] could be used as checks for numerical integration accuracies of the Equations (4.20).

5. Satellite Motions in the Earth's Gravitational Field with Axial Symmetry

In this section, the initial value problem of the Eulerian Parameters will be considered in full detail for satellite motions in the Earth's gravitational field with axial symmetry. A general, recursive and stable computational algorithm of this problem will be established for any conic motion and for any number $N \geqslant 2$ of the zonal harmonic coefficients of the Earth's gravitational potential. Applications of the algorithm are considered for the two cases of short and long term predictions. For the short-term prediction, we consider the final state prediction of some typical ballistic missiles in the geopotential model with zonal harmonic terms up to J_{36}, while for the long-term prediction, we consider the perturbed J_2 motion of Explorer 28 over 100 revolutions (about 580 days).

5.1. Expressions of V, $\partial V / \partial \mathbf{x}$ and (P_ξ, P_η, P_ζ)

For the case of axial symmetry we have

$$\mathbf{P}^* = \mathbf{0}, \qquad \frac{\partial V}{\partial t} = 0 , \tag{5.1}$$

$$V = \left(\frac{\mu}{R}\right) \sum_{k=2}^{N} J_k (R/r)^{k+1} P_k(\sin \Gamma) , \tag{5.2}$$

where V is the Earth's gravitational field with axial symmetry, R is the Earth's mean equatorial radius; Γ the latitude of the satellite, J_k, $k = 2(1)N$ are dimensionless numerical coefficients (note that the infinite series of Equation (5.2) is truncated at some positive integer N) and $P_l(Z)$ is the Legendre polynomial in Z of order l defined $\forall Z \in [-1, 1]$ as

$$P_l(Z) = \frac{1}{2^l} \sum_{k=0}^{[l/2]} \frac{(-1)^k (2l-2k)!}{k!(l-k)!(l-2k)!} Z^{l-2k}, \tag{5.3}$$

$[q]$ denotes the largest interger $\leq q$.

By the same argument that has been established by Sharaf and Awad (1985) for the economical and stable recurrent computations of V and $\partial V/\partial \mathbf{x}$ in terms of $\mathbf{x}$, we can derive for the corresponding computations of these functions in terms of the Eulerian Parameters the following formulations:

$$V = \left(\frac{\mu}{R}\right) \sum_{k=2}^{N} J_k Q_k, \tag{5.4}$$

$$\frac{\partial V}{\partial x_1} = -\left(\frac{\mu C_{11} \rho^2}{1 - C_{13}^2}\right) G, \tag{5.5}$$

$$\frac{\partial V}{\partial x_2} = -\left(\frac{\mu C_{12} \rho^2}{1 - C_{13}^2}\right) G, \tag{5.6}$$

$$\frac{\partial V}{\partial x_3} = -\mu \rho^2 S, \tag{5.7}$$

where

$$\rho = \frac{1}{r}; \quad G = \sum_{k=2}^{N} I_k F_k; \quad S = \sum_{k=2}^{N} I_k D_k; \quad I_k = (k+1) J_k, \tag{5.8}$$

Q's and D's satisfy economical and stable recurrence formulae of the forms

$$Q_k = Q_0\{C_{13} Q_{k-1} - Q_0 Q_{k-2} + C_{13} Q_{k-1} - (C_{13} Q_{k-1} - Q_0 Q_{k-2})/k\}, \tag{5.9}$$

$$D_k = Q_0\{D_0 D_{k-1} - Q_0 D_{k-2} + D_0 D_{k-1} - (D_0 D_{k-1} - Q_0 D_{k-2})/(k+1)\}, \tag{5.10}$$

$$F_k = Q_0 D_{k-1} - D_0 D_k, \tag{5.11}$$

$$Q_0 = R\rho; \quad Q_1 = Q_0^2 C_{13}; \quad D_0 = C_{13}, \quad D_1 = 0.5 Q_0 (3D_0^2 - 1), \tag{5.12}$$

C's are given in terms of u's by Equations (2.7).

Finally, by using Equations (5.5), (5.6) and (5.7) into Equations (4.22) (note that $P_j = -(\partial V)/(\partial x_j)$; $j = 1, 2, 3$) we get by means of Equations (2.8) that

$$P_\xi = \mu\rho^2 Q_0 \sum_{k=2}^{N} I_k D_{k-1} , \tag{5.13}$$

$$P_\eta = \frac{\mu C_{23}\rho^2}{1 - C_{13}^2} \sum_{k=2}^{N} I_k \{D_k - D_0 Q_0 D_{k-1}\} , \tag{5.14}$$

$$P_\zeta = \frac{\mu C_{33}\rho^2}{1 - C_{13}^2} \sum_{k=2}^{N} \{D_k - D_0 Q_0 D_{k-1}\} . \tag{5.15}$$

5.1.1. *The Special Case of* $N = 2$

Particularization of the above equations for the case $N = 2$ gives for the J_2 gravity perturbed motions the following formulations

$$V = \frac{1}{2} J_2 \mu R^2 \rho^3 (3C_{13}^2 - 1) , \tag{5.16}$$

$$\frac{\partial V}{\partial x_1} = \frac{3}{2} J_2 \mu R^2 \rho^4 C_{11} (1 - 5C_{13}^2) , \tag{5.17}$$

$$\frac{\partial V}{\partial x_2} = \frac{3}{2} J_2 \mu R^2 \rho^4 C_{12} (1 - 5C_{13}^2) , \tag{5.18}$$

$$\frac{\partial V}{\partial x_3} = \frac{3}{2} J_2 \mu R^2 \rho^4 C_{13} (3 - 5C_{13}^2) , \tag{5.19}$$

$$P_\xi = -\frac{3}{2} J_2 \mu R^2 \rho^4 (1 - 3C_{13}^2) , \tag{5.20}$$

$$P_\eta = -3 J_2 \mu R^2 \rho^4 C_{13} C_{23} , \tag{5.21}$$

$$P_\zeta = -3 J_2 \mu R^2 \rho^4 C_{33} C_{13} . \tag{5.22}$$

From these equations it is clear that

$$C_{11} \frac{\partial V}{\partial x_1} + C_{12} \frac{\partial V}{\partial x_2} + C_{13} \frac{\partial V}{\partial x_3} + 3V\rho = 0 . \tag{5.23}$$

5.2. The initial-value problem

In this subsection, we shall develop a general, economical and stable recursive computational algorithm of the initial value problem of the Eulerian Parameters for zonal gravity perturbed orbital motions. The algorithm is general in the sense that it could be used for any type of orbital motion (elliptic, hyperbolic, or parabolic) and for any number $N \geqslant 2$ of the zonal harmonic coefficients of the Earth's gravitational potential. Its recursive, economical and stability are due to the usage of the fundamental equations of Section 5.1. With the substitutions

$$u_5 = r; \quad u_6 = \dot{r}; \quad u_7 = \sqrt{\mu p} , \tag{5.24}$$

Equations (4.20) reduced by means of Equations (5.13), (5.14) and (5.15) to the system

$$\dot{u}_1 = \tfrac{1}{2}(\omega_1 u_4 + \omega_3 u_2)\,, \tag{5.25.1}$$

$$\dot{u}_2 = \tfrac{1}{2}(\omega_1 u_3 - \omega_3 u_1)\,, \tag{5.25.2}$$

$$\dot{u}_3 = \tfrac{1}{2}(-\omega_1 u_2 + \omega_3 u_4)\,, \tag{5.25.3}$$

$$\dot{u}_4 = \tfrac{1}{2}(-\omega_1 u_1 - \omega_3 u_3)\,, \tag{5.25.4}$$

$$\dot{u}_5 = u_6 \tag{5.25.5}$$

$$\dot{u}_6 = u_5^{-2}\left\{\mu Q_0 \sum_{k=2}^{N} I_k D_{k-1} + u_7^2 u_5^{-1} - \mu\right\}, \tag{5.25.6}$$

$$\dot{u}_7 = \frac{\mu C_{23}}{u_5(1 - C_{13}^2)} \sum_{k=2}^{N} I_k\{D_k - D_0 Q_0 D_{k-1}\}\,, \tag{5.25.7}$$

where

$$\omega_1 = \frac{\mu C_{33}}{u_5 u_7 (1 - C_{13}^2)} \sum_{k=2}^{N} I_k\{D_k - D_0 Q_0 D_{k-1}\}\,, \tag{5.26.1}$$

$$\omega_3 = u_7 u_5^{-2}\,. \tag{5.26.2}$$

The initial conditions of the above differential system could be computed from the initial position $\mathbf{x}_0 = \mathbf{x}(t_0)$ and velocity $\dot{\mathbf{x}}_0 = \dot{\mathbf{x}}(t_0)$ at the epoch t_0 throughout the following two main steps.

STEP 1: To compute the initial value of V $(= V_0)$.

1. Compute

$$r_0 = (x_{01}^2 + x_{02}^2 + x_{03}^2)^{1/2}$$

$$Q = R/r_0$$

$$E = x_{03}/r_0$$

$$H_1 = Q_0^2 E$$

2. Set

$$S = 0$$

$$H_0 = Q_0$$

$$H_2 = H_1$$

3. For all $k = 2(1)N$, compute

$$A = Q_0 H_0$$

$$G = EH_2$$

$$B = G - A$$

$$H_2 = Q_0(B + G - B/k)$$

$$H_0 \leftarrow H_1$$

$$H_1 \leftarrow H_2$$

$$S \leftarrow S + J_k H_2$$

4. Compute V_0 from

$$V_0 = (\mu/R) * S$$

STEP 2. To compute the initial values of u_j, $j = 1(1)8$.
5. $u_5 = r_0$.

6. Compute

$$u_6 = (x_{01}\dot{x}_{01} + x_{02}\dot{x}_{02} + x_{03}\dot{x}_{03})/u_5$$

$$u_7 = u_5(\dot{x}_{01}^2 + \dot{x}_{02}^2 + \dot{x}_{03}^2 - u_6^2 + 2V_0)^{1/2}$$

$$C_{11} = x_{01}/u_5$$

$$C_{12} = x_{02}/u_5$$

$$C_{13} = x_{03}/u_5$$

7. For all $i = 1, 2, 3$, compute

$$C_{2i} = (u_5\dot{x}_{0i} - u_6 x_{0i})/u_7$$

8. Compute

$$C_{31} = C_{12}C_{23} - C_{13}C_{22}$$

$$C_{32} = C_{13}C_{21} - C_{11}C_{23}$$

$$C_{33} = C_{11}C_{22} - C_{12}C_{21}$$

9. Compute

$$u_4 = (1 + C_{11} + C_{22} + C_{33})^{1/2}/2$$

$$u_1 = 0.25(C_{23} - C_{32})/u_4$$

$$u_2 = 0.25(C_{31} - C_{13})/u_4$$

$$u_3 = 0.25(C_{12} - C_{21})/u_4$$

10. Set

$$u_8 = t_0 .$$

The position and velocity in Cartesian space could be computed at any time $t \neq t_0$ from u's and their derivatives as follows

$$r = u_5$$

$$x_1 = r(u_1^2 - u_2^2 - u_3^2 + u_4^2)$$

$$x_2 = 2r(u_1 u_2 + u_3 u_4) ,$$

$$x_3 = 2r(u_1 u_3 - u_2 u_4) ,$$

$$\dot{x}_1 = 2r(u_1 \dot{u}_1 - u_2 \dot{u}_2 - u_3 \dot{u}_3 + u_4 \dot{u}_4) + x_1 u_6 / r ,$$

$$\dot{x}_2 = 2r(u_2 \dot{u}_1 + u_1 \dot{u}_2 + u_4 \dot{u}_3 + u_3 \dot{u}_4) + x_2 u_6 / r ,$$

$$\dot{x}_3 = 2r(u_3 \dot{u}_1 + u_1 \dot{u}_3 - u_4 \dot{u}_2 - u_2 \dot{u}_4) + x_3 u_6 / r .$$

The accuracy of the computed values during the numerical integration could be checked by the conditions

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 , \tag{5.27}$$

$$\omega_2 = u_2 \dot{u}_4 + u_3 \dot{u}_1 - u_4 \dot{u}_2 - u_1 \dot{u}_3 = 0 . \tag{5.28}$$

In addition to these two general conditions, the present problem provides a third one which is the constancy of the total energy (since the potential with axial symmetry is conservative), that is

$$\Delta h = h(t) - h(0) = 0 , \tag{5.29}$$

where $h(t)$ and $h(0)$ are the values of the total energy h at any time t and at the initial epoch $t = 0$, respectively.

5.3. Numerical Applications

5.3.1. *Test Cases*

For the purposes of the numerical applications of our algorithm we consider two types of test cases, the first type for short-term predications, while the second type for long-term predications. For the first type, we consider four fractional orbit cases typical of ballistic missiles all with the same initial time $t_0 = 0$, while the other initial values are listed for each case in the first column of Table I to IV. In each of these columns, the type of the orbit, the initial values of $\mathbf{x}$, $\dot{\mathbf{x}}$ and E_j; $j = 1, 2, \ldots, 7$ are given, where

$$E_1 \equiv \begin{cases} a \equiv \text{(semi-major axis for elliptic or hyperbolic orbit)}, \\ q \equiv \text{(Pericentre distance for parabolic orbit)}; \end{cases}$$

$E_2 \equiv n$ (mean motion); $E_3 = e$ (eccentricity); $E_4 = i$ (Orbital inclination); $E_5 \equiv \Omega$ (longitude of the ascending node); $E_6 = \pi$ (argument of

TABLE I

Initial and final states with zonal harmonics up to J_{36} for the test case No. 1

Initial	Final	Accuracy checks
Elliptic orbit	Elliptic orbit	CHECK1 = 0.1000000000D + 01
t_0 = 0.000000000D + 00	TF = 0.1800000900D + 04	CHECK2 = 0.9859992902D − 21
x_1 = 0.6478000000D + 04	x_1 = 0.1097091933D + 05	CHECK3 = 0.2629760517D − 11
x_2 = 0.0000000000D + 00	x_2 = 0.1435479581D + 04	
x_3 = 0.0000000000D + 00	x_3 = 0.4304935816D + 04	
$\dot{x}_1$ = 0.7000000000D + 01	$\dot{x}_1$ = −0.4446857760D + 00	
$\dot{x}_2$ = 0.1000000000D + 01	$\dot{x}_2$ = 0.5322856247D + 00	
$\dot{x}_3$ = 0.3000000000D + 01	$\dot{x}_3$ = 0.1595330525D + 01	
E_1 = 0.6222020413D + 04	E_1 = 0.6216360170D + 04	
E_2 = 0.1286387868D − 02	E_2 = 0.1288145229D − 02	
E_3 = 0.9114797593D + 00	E_3 = 0.9115450165D + 00	
E_4 = 0.1249045772D + 01	E_4 = 0.1248776742D + 01	
E_5 = 0.0000000000D + 00	E_5 = 0.6283113665D + 01	
E_6 = 0.3547320780D + 01	E_6 = 0.3546885311D + 01	
E_7 = 0.7053972527D + 00	E_7 = 0.3025566073D + 01	

TABLE II

Initial and final states with zonal harmonics up to J_{36} for the test case No. 2

Initial	Final	Accuracy checks
Elliptic orbit	Elliptic orbit	CHECK1 = 0.1000000000D + 01
t_0 = 0.0000000000D + 00	TF = 0.190056000D + 04	CHECK2 = 0.1609680237D − 17
x_1 = −0.7662907520D + 03	x_1 = 0.1838084668D + 03	CHECK3 = −0.3442699195D − 13
x_2 = 0.9227347132D + 03	x_2 = 0.4838898617D + 04	
x_3 = −0.5725639475D + 04	x_3 = 0.5196034664D + 04	
$\dot{x}_1$ = −0.6667070998D + 00	$\dot{x}_1$ = 0.1030753722D + 01	
$\dot{x}_2$ = 0.8523790555D + 01	$\dot{x}_2$ = −5053060649D + 01	
$\dot{x}_3$ = 0.8253522788D + 00	$\dot{x}_3$ = 0.4825394487D + 01	
E_1 = 0.6378135000D + 04	E_1 = 0.6391944396D + 04	
E_2 = 0.1239448602D − 02	E_2 = 0.1235434146D − 02	
E_3 = 0.1100000000 + 00	E_3 = 0.1123414668D + 00	
E_4 = 0.1689129650D + 01	E_4 = 0.1689032460D + 01	
E_5 = 0.1660331717D + 00	E_5 = 0.1660642310D + 01	
E_6 = 0.4087910174D + 01	E_6 = 0.4098046685D + 01	
E_7 = 0.6459794785D + 00	E_7 = 0.2981988290D + 01	

perigree); $E_7 \equiv M$ (mean anomaly). The adopted units of the time, distances and angles are respectively second, kilometer and radian.

For the second type of the test cases, we consider a typical highly eccentric satellite orbit – Explorer 28. The satellite was launched on May 29, 1965. Orbital period about 5.8 days, the initial values of $\mathbf{x}$ and $\dot{\mathbf{x}}$ (Lowrey, 1972) are,

$$x_1 = 6099.5844, \quad x_2 = 602.05128, \quad x_3 = 2409.1608$$

$$\dot{x}_1 = 1.1047527, \quad \dot{x}_2 = 9.8556127, \quad \dot{x}_3 = -4.4520836$$

TABLE III

Initial and final states with zonal harmonics up to J_{36} for the test case No. 3

Initial	Final	Accuracy checks
Parabolic orbit	Parabolic orbit	CHECK1 = 0.1000000000D + 01
t_0 = 0.0000000000D + 00	TF = 0.1946250000D + 04	CHECK2 = 0.2394687873D − 20
x_1 = 0.9592151798D + 04	x_1 = −0.1835330831D + 05	CHECK3 = −0.3049141110D − 08
x_2 = 0.4539210547D + 04	x_2 = 0.1095613853D + 05	
x_3 = −0.2198098325D + 04	x_3 = 0.6018407546D + 04	
$\dot{x}_1$ = −0.6217477283D + 01	$\dot{x}_1$ = −0.3465737904D + 01	
$\dot{x}_2$ = 0.4184991210D + 01	$\dot{x}_2$ = 0.2718405654D + 01	
$\dot{x}_3$ = 0.4170160618D + 01	$\dot{x}_3$ = 0.4060604939D + 01	
E_1 = 0.4783601250D + 04	E_1 = 0.36720184021 + 08	
E_2 = 0.6313483983D + 03	E_2 = 0.2837342804D − 08	
E_3 = 0.1000000000 + 01	E_3 = 0.9998697206D + 00	
E_4 = 0.1765051473D + 01	E_4 = 0.1765052094D + 01	
E_5 = 0.2658834582D + 01	E_5 = 0.2658836257D + 01	
E_6 = 0.4386710542D + 01	E_6 = 0.4386673859D + 01	
E_7 = 0.2447259744D + 01	E_7 = 0.8885995428D − 05	

TABLE IV

Initial and final states with zonal harmonics up to J_{36} for the test case No. 4

Initial	Final	Accuracy checks
Hyperbolic orbit	Hyperbolic orbit	CHECK1 = 0.1000000000D + 01
t_0 = 0.0000000000D + 00	TF = 0.9753400000D + 03	CHECK2 = 0.2547716287D − 20
x_1 = −0.7515331845D + 03	x_1 = −0.2063161960D + 04	CHECK3 = −0.1671315173D − 13
x_2 = −0.1719532694D + 05	x_2 = −0.9384960103D + 04	
x_3 = −0.1922853605D + 05	x_3 = −0.2434255191D + 05	
$\dot{x}_1$ = −0.1358441628D + 01	$\dot{x}_1$ = −0.1326442061D + 01	
$\dot{x}_2$ = 0.7840211728D + 01	$\dot{x}_2$ = 0.8144014726D + 01	
$\dot{x}_3$ = −0.5483792681D + 01	$\dot{x}_3$ = −0.4987127484D + 01	
E_1 = −0.6378135000D + 04	E_1 = −0.6378225176D + 04	
E_2 = 0.1239448602D − 02	E_2 = 0.1239422317D − 02	
E_3 = 0.5014684674D + 01	E_3 = 0.5014625734D + 01	
E_4 = 0.1689129650D + 01	E_4 = 0.1689130265D + 01	
E_5 = 0.1660331717D + 01	E_5 = 0.1660332765D + 01	
E_6 = 0.4126772960D + 01	E_6 = 0.4126764512D + 01	
E_7 = 0.5832833793D + 01	E_7 = 0.7585445186D + 00	

5.3.2. *The adopted Physical Constants*

$$\mu = 398600.8 \text{ km}^3 \text{ s}^{-2}; \; R = 6378.135 \text{ km}.$$

The numerical values of the Earth's zonal harmonic coefficients J_k; $k = 2, 3, \ldots, 36$ are taken from Hough (1981).

5.3.3. *Numerical Results*

The previous equations of the present section were programmed and applied with fixed step size, fourth-order Runge–Kutta-Gill method, together with the basic

TABLE V

J_2 Perturbed values of **x** and $\dot{\mathbf{x}}$ at every 10 revolutions for Explorer 28 Satellite

I	x_1	x_2	x_3	$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$
1	−0.785715899D + 04	−0.162602171D + 05	+0.466947794D + 04	0.528938994D + 01	0.337992463D + 01	0.653208872D + 00
11	−0.901332678D + 05	−0.410523054D + 05	−0.181030630D + 05	0.208560153D + 01	0.290333728D + 00	0.755908799D + 00
21	−0.137501788D + 06	−0.459617512D + 05	−0.342433199D + 05	0.144694423D + 01	0.512982849D − 01	0.585092597D + 00
31	−0.172186616D + 06	−0.470249931D + 05	−0.460543421D + 05	0.108876702D + 01	−0.479175548D − 01	0.473724510D + 00
41	−0.198810708D + 06	−0.462790058D + 05	−0.548061009D + 05	0.834133171D + 00	−0.104859390D + 00	0.390555766D + 00
51	−0.219361744D + 06	−0.444306468D + 05	−0.611850440D + 05	0.630804768D + 00	−0.143247217D + 00	0.323745984D + 00
61	−0.234877285D + 06	−0.417984222D + 05	−0.656296566D + 05	0.456327310D + 00	−0.171903647D + 00	0.267572530D + 00
71	−0.245948749D + 06	−0.385440762D + 05	−0.684496879D + 05	0.298770122D + 00	−0.194895096D + 00	0.218789110D + 00
81	−0.252912868D + 06	−0.347542287D + 05	−0.698786121D + 05	0.150705743D + 00	−0.214351930D + 00	0.175312613D + 00
91	−0.255936947D + 06	−0.304756372D + 05	−0.700996980D + 05	0.677926549D − 02	−0.231476625D + 00	0.135641600D + 00
101	−0.255057015D + 06	−0.257328321D + 05	−0.692594621D + 05	−0.137514797D + 00	−0.246959168D + 00	0.985430284D − 01

TABLE VI

J_2 perturbed values of the orbital osculating elements at every 10 revolutions for explorer 28 Satellite

I	E_1	E_2	E_3	E_4	E_5	E_6	E_7
1	0.13695981D + 06	0.12456022D − 04	0.95193959D + 00	0.59038004D + 00	0.38661207D + 01	0.23702427D + 01	0.62495202D + 01
11	0.13685725D + 06	0.12470026D − 04	0.95190180D + 00	0.59039315D + 00	0.38452381D + 01	0.24010689D + 01	0.59082332D + 01
21	0.13685673D + 06	0.12470098D − 04	0.95190194D + 00	0.59038817D + 00	0.38243418D + 01	0.24318972D + 01	0.55669864D + 01
31	0.13685662D + 06	0.12470113D − 04	0.95190216D + 00	0.59038419D + 00	0.38034445D + 01	0.24627256D + 01	0.52257443D + 01
41	0.13685658D + 06	0.12470118D − 04	0.95190239D + 00	0.59038057D + 00	0.37825465D + 01	0.24935542D + 01	0.48845063D + 01
51	0.13685655D + 06	0.12470122D − 04	0.95190261D + 00	0.59037715D + 00	0.37616480D + 01	0.25243829D + 01	0.45432719D + 01
61	0.13685654D + 06	0.12470124D − 04	0.95190282D + 00	0.59037388D + 00	0.37407491D + 01	0.25552117D + 01	0.42020413D + 01
71	0.13685653D + 06	0.12470125D − 04	0.95190302D + 00	0.59037073D + 00	0.37198497D + 01	0.25860408D + 01	0.38608144D + 01
81	0.13685652D + 06	0.12470126D − 04	0.95190322D + 00	0.59036770D + 00	0.36989499D + 01	0.26168699D + 01	0.35195911D + 01
91	0.13685651D + 06	0.12470127D − 04	0.95190341D + 00	0.59036479D + 00	0.36780498D + 01	0.26476991D + 01	0.31783715D + 01
101	0.13685651D + 06	0.12470128D − 04	0.95190359D + 00	0.59036199D + 00	0.36571493D + 01	0.26785284D + 01	0.28371555D + 01

equations (see for example, Escobal, 1965) for converting $\mathbf{x}$, $\dot{\mathbf{x}}$ to the orbital elements of any conic motion. Conditions (5.27), (5.28) and (5.29) are used for checking the accuracies of numerical integration. Although the program is developed to include up to any number of Earth's zonal harmonic terms J_n, however, the numerical computations are done with terms up to J_{36}. The output of the program was arranged for each case study in the second and the third columns of Tables I to IV, where CHECK 1, 2, 3 correspond respectively to the conditions (5.27), (5.28) and (5.29). Tables V and VI represent the applications of our algorithm for long-term predication of the J_2 perturbed motion of Explorer 28 Satellite during 100 revolutions given at every 10 revolutions. The accuracy of the computed values is also checked by the conditions (5.27) to (5.29). The first condition is exactly satisfied, while the second and the third conditions are satisfied respectively up to 10^{-17} and 10^{-8} at least.

In concluding this paper, the connections between orbit dynamics and rigid body dynamics are developed throughout the Eulerian redundant Parameters and utilized to establish special perturbation technique for the initial value problem for any conic motion of artificial satellites. A motion prediction algorithm using the Eulerian Parameters has been developed for the motions in the Earth's gravitational field with axial symmetry. The algorithm is of recursive nature, and moreover could be applied for any conic motion whatever the number of zonal harmonic coefficients may be.

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