THE LIMITS OF POSSIBLE VALUES OF INCLINATION OF MERCURY'S EQUATOR

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Abstract. The method of estimation of the limits, containing the equator inclination of a celestial body, had been developed. In this method it is necessary to know the orbital elements and the mass of a celestial body. Another condition is that the axial rotation of a body should be in the resonance with its orbital motion. It has been found that the equator inclinations should have the values between $1^\circ.7$ and $2^\circ.6$ for Mercury and between $1^\circ.0$ and $1^\circ.8$ for the Moon. It also has been found that largest harmonics in Mercury's physical libration are the harmonics $\sin(\phi - 3g)$, $\cos(\phi - 3g)$, $\sin g$ and $\sin 2\omega$.

1. Introduction

At this time there is still no proper determination of Mercury's equator inclination to the ecliptic. Schiaparelli found that the period of rotation of Mercury is equal to the orbit period and he also supposed that Mercury's equator plane coincide with its orbital plane. As it was found later, during identification of the spot's on Mercury's surface using Schiaparelli's maps and photo of Dollfus (1953), disagreement in the spot's latitudes exist. Then Sharonov (1958) pointed out that these disagreements may be resolved if it is suggested that Mercury's equator plane lie in the ecliptic plane.

Radar range observations in 1965 by Pettengill, Dyce and Shapiro (1965, 1967) gave more accurate information about Mercury's rotation. According to these measurements, the period of the axial rotation is about 59 ± 3 days, contrary to the early determinations (88 days). Colombo (1965) was the first who found that this new value of the rotation period is very close to the 3:2 resonance with the orbital period. This fact had induced few researches to study theoretical basis of the Cassini's laws. So, Colombo (1966) and Peale (1969) generalized the second and the third Cassini's laws.

The most complete theoretical study of a resonances in the rotations of a celestial bodies had been made by Beletskii (1975, 1977). He not only explained Cassini's laws but also made their generalizations. As for the 3:2 spin-orbital resonance in Mercury motion, he pointed out that:

1. such resonance cannot take place on the circle orbit, the orbit must be elliptic;
2. rotation must take place round the smallest axis of the inertia ellipsoid;
3. the greatest axis of the inertia ellipsoid is in the direction to the orbital semi-major axis when passing perihelion;

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4. the spin axis, the normal to the orbital plane and the perpendicular to the ecliptic must lie in one plane, perpendicular to the line of nodes. There are four stable positions of Mercury's spin axis. One of them is placed on the meridian $F = (l - \Omega) = 90°$, where l is the mean ecliptic longitude of Mercury and Ω is the longitude of ascending node of the orbit on the ecliptic. Another three stable positions are on the meridian $F = 270°$, two in the region $\rho_0 < \pi/2$ and one in $\rho_0 < \pi/2$, where ρ_0 is the obliquity of the equator's axis to the orbital axis. Most probable position for the stable point is the last one. According to the radar ranging observations ρ_0 had been estimated as $\rho_0 < 25°$, but according to Beletskii (1975) it satisfies to $\rho_0 \leqslant i = 7°$, where i is the orbital inclination. In all three cases ascending node of the equator must coincide with the orbit ascending node.

In this article the limits of possible values of the equator inclination to the ecliptic and the main harmonics in physical libration of Mercury had been estimated.

2. Coordinate Systems

Let the ecliptic coordinate system $OXYZ$ be a basic coordinate system for the further investigations. The origin of this system O is placed at the center of mass of Mercury and the OX axis is in the direction to the ascending node of the orbit on the ecliptic. Let us also introduce "accompanying"-coordinate system $OX'Y'Z'$, which rotate uniformly around the ecliptic axis OZ with the angular rate

$$\rho = \dot{\omega} + 3/2\dot{g}\,. \tag{1}$$

The position of the $OX'Y'Z'$ system relative to the $OXYZ$ is determined by the angle

$$\phi = \omega + 3/2g\,, \tag{2}$$

where ω is the argument of perihelion, g the mean anomaly of Mercury. Let us consider Mercury as a rigid body and let us take the coordinate system $Oxyz$, which is fixed in the body and coincides with the principal inertia axes. So, x axis coincides with the greatest axis and is in the direction to the Sun in perihelion, mean inertia axis coincides with the y axis and smallest with the z axis. Principal inertia moments relative to these axis are denoted as A, B and C respectively and satisfy to the condition

$$A < B < C\,. \tag{3}$$

The position of the xyz frame relative to the $X'Y'Z'$ triad are determined by the

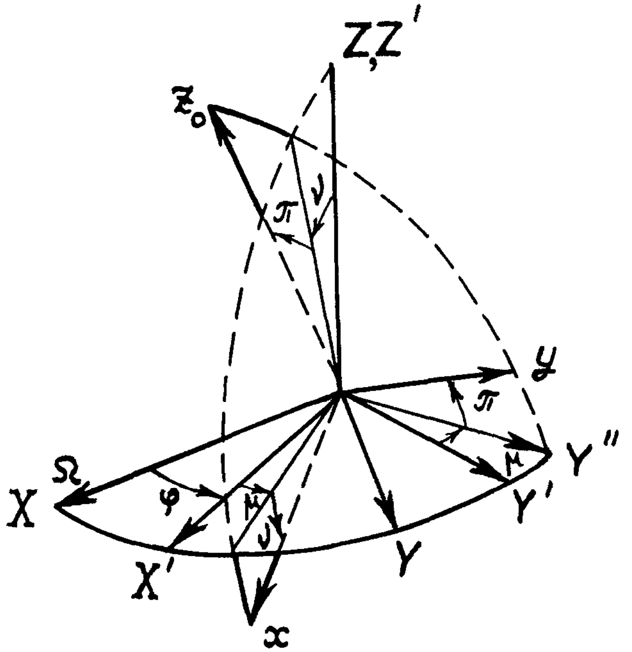

Fig. 1. Coordinate systems: (X, Y, Z)-ecliptic; (X', Y', Z')-uniformly rotating with angular rate $= \dot{\rho}$; (x, y, z)-coinciding with the principal inertia axis. The transformation $(X', Y', Z') \rightarrow (x, y, z)$ is carrying out by the positive turns on the angles μ, ν, π.

rotation angles: μ – around the Z' axis, ν – around the Y'' axis and π – around the x axis (Figure 1).

Angular variables, μ, ν, π are the components of Mercury's physical libration. Because the values of these variables are the small quantities we can use for the further studying only linear part of the differential equation. The coordinate system transformation $(X'Y'Z') \rightarrow (x, y, z)$ are represented here by the simplified formula:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \mu & -\nu \\ -\mu & 1 & \pi \\ \nu & -\pi & 1 \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} .$$

3. Transformation of Euler's Equations

Let (p, q, r) be the projections of the angular velocity vector of Mercury along to the axis of the rotating frame (x, y, z) and are determined by the kinematic Euler equations

$$\begin{aligned} p &= -(\rho + \dot{\mu}) \sin \nu + \dot{\pi} , \\ q &= (\rho + \dot{\mu}) \cos \nu \sin \pi + \dot{\nu} \cos \pi , \\ r &= (\rho + \dot{\mu}) \cos \nu \cos \pi - \dot{\nu} \sin \pi . \end{aligned} \tag{5}$$

By use of the leading terms in the force function, describing interaction in Mercury–Sun gravitational field in the form

$$U = -\left(\frac{3}{2}\frac{fM_\odot}{a^3}\right)\left(\frac{R}{a}\right)^{-3}(Au_1^2 + Bu_2^2 + Cu_3^2)\,, \tag{6}$$

where f is the gravitational constant; $M_\odot$ is the mass of the Sun; a is the mean distance; R is Mercury's radius vector; u_1, u_2, u_3 is the direction cosines of Mercury's radius vector relative to the principal inertia axes and coefficient Q according to the third Kepler law

$$Q = \frac{3}{2}\frac{fM_\odot}{a^3} = \frac{3}{2}\frac{n^2}{1 + (M☿/M_\odot)}\,, \tag{7}$$

with n Mercury's mean motion and $M☿$ the mass of Mercury. Euler's dynamical equations then are

$$\begin{aligned}
\dot{p} + \alpha qr &= \alpha Q\left(\frac{R}{a}\right)^{-3}(2u_2u_3)\,,\\
\dot{q} - \beta rp &= -\beta Q\left(\frac{R}{a}\right)^{-3}(2u_1u_3)\,,\\
\dot{r} + \gamma pq &= \gamma Q\left(\frac{R}{a}\right)^{-3}(2u_1u_2)\,;
\end{aligned} \tag{8}$$

where

$$\alpha = \frac{C-B}{A}; \qquad \beta = \frac{C-A}{B}; \qquad \gamma = \frac{B-A}{C}\,, \tag{9}$$

which we call "dynamical flattening".

For our further estimations we shall calculate direction cosines u_1, u_2, u_3 using unperturbed keplerian Mercury orbit. Direction cosines l', m', n' of Mercury's radius-vector in $X'Y'Z'$ frame are

$$\begin{pmatrix} l' \\ m' \\ n' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\omega + v) \\ \sin(\omega + v)\cos i \\ \sin(\omega + v)\sin i \end{pmatrix}.$$

and direction cosines relative to the rotating xyz frame are

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & \mu & -\nu \\ -\mu & 1 & \pi \\ \nu & -\pi & 1 \end{pmatrix}\begin{pmatrix} l' \\ m' \\ n' \end{pmatrix}.$$

Here v is the true anomaly and i, Mercury's orbital inclination to the ecliptic. Using (11) we can produce the following multiplications:

$$
\begin{aligned}
u_1u_2 &= l'm' - \mu(l'^2 - m'^2) + \nu(m'n') + \pi(l'n')\,, \\
u_1u_3 &= l'n' + \mu(m'n') + \nu(l'^2 - n'^2) - \pi(l'm')\,, \\
u_2u_2 &= m'n' - \mu(l'n') + \nu(l'm') - \pi(m'^2 - n'^2)\,.
\end{aligned}
\tag{12}
$$

The functions in a right sides of these multiplications are determined from (10). They have the following trigonometric form:

$$
\begin{aligned}
l'm' &= -\frac{1}{4}\sin^2 i \sin 2\phi - \frac{1}{2}\cos^4\frac{i}{2}\sin(3g - 2v) - \\
&\quad -\frac{1}{2}\sin^4\frac{i}{2}\sin(4\phi - 3g + 2v)\,, \\
l'n' &= \frac{1}{4}\sin 2i \sin\phi - \frac{1}{2}\sin i\cos^2\frac{i}{2}\sin(\phi - 3g + 2v) + \\
&\quad +\frac{1}{2}\sin i\sin^2\frac{i}{2}\sin(3\phi - 3g + 2v)\,, \\
m'n' &= \frac{1}{4}\sin 2i\cos\phi - \frac{1}{2}\sin i\cos^2\frac{i}{2}\cos(\phi - 3g + 2v) + \\
&\quad +\frac{1}{2}\sin i\sin^2\frac{i}{2}\cos(3\phi - 3g + 2v)\,, \\
l'^2 - m'^2 &= -\frac{1}{2}\sin^2 i\cos 2\phi + \cos^4\frac{i}{2}\cos(3g - 2v) + \\
&\quad +\sin^4\frac{i}{2}\cos(4\phi - 3g + 2v)\,, \\
l'^2 - n'^2 &= \left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right) + \frac{1}{4}\sin^2 i\cos 2\phi + \\
&\quad +\frac{3}{4}\sin^2\cos(2\phi - 3g + 2v) + \frac{1}{2}\cos^4\frac{i}{2}\cos(3g - 2v) + \\
&\quad +\frac{1}{2}\sin^4\frac{i}{2}\cos(4\phi - 3g + 2v)\,, \\
m'^2 - n'^2 &= \left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right) - \frac{1}{4}\sin^2 i\cos 2\phi +
\end{aligned}
\tag{13}
$$

$$+\frac{3}{4}\sin^2\cos(2\phi-3g+2v)-\frac{1}{2}\cos^4\frac{i}{2}\cos(3g-2v)-$$

$$-\frac{1}{2}\sin^4\frac{i}{2}\cos(4\phi-3g+2v)\,;$$

where 2ω in arguments of trigonometric functions had been substituted by the $2\omega=2\phi-3g$.

Let us also introduce the following functions:

$$S_1=2\left(\frac{R}{a}\right)^{-3} l'm',\qquad S_2=2\left(\frac{R}{a}\right)^{-3} l'n',\qquad S_3=2\left(\frac{R}{a}\right)^{-3} m'n',$$

$$S_4=2\left(\frac{R}{a}\right)^{-3}(l'^2-m'^2),\qquad S_5=2\left(\frac{R}{a}\right)^{-3}(l'^2-n'^2),\tag{14}$$

$$S_6=2\left(\frac{R}{a}\right)^{-3}(m'^2-n'^2),$$

which are represented in a form of series with arguments multiplied to the mean anomoly g: i.e.,

$$\left(\frac{R}{a}\right)^{-3}=\sum_{k=0}C_k^{-3,0}\cos kg\,,$$

$$\left(\frac{R}{a}\right)^{-3}{\textstyle\frac{\sin}{\cos}}(x)=\sum_{k=0}\frac{1}{2}C_k^{-3,0}{\textstyle\frac{\sin}{\cos}}(x-kg)+\sum_{k=0}\frac{1}{2}C_k^{-3,0}{\textstyle\frac{\sin}{\cos}}(x+kg)\,,$$

$$\left(\frac{R}{a}\right)^{-3}{\textstyle\frac{\sin}{\cos}}(x\mp v)=\sum_{k=1}\frac{1}{2}(C_k^{-3,1}+S_k^{-3,1}){\textstyle\frac{\sin}{\cos}}(x\mp kg)+\tag{15}$$

$$+\sum_{k=1}\frac{1}{2}(C_k^{-3,1}-S_k^{-3,1}){\textstyle\frac{\sin}{\cos}}(x\pm kg)\,,$$

$$\left(\frac{R}{a}\right)^{-3}{\textstyle\frac{\sin}{\cos}}(x\mp 2v)=\sum_{k=1}\frac{1}{2}(C_k^{-3,2}+S_k^{-3,2}){\textstyle\frac{\sin}{\cos}}(x\mp kg)+$$

$$+\sum_{k=1}\frac{1}{2}(C_k^{-3,2}-S_k^{-3,2}){\textstyle\frac{\sin}{\cos}}(x\mp kg)\,.$$

The coefficients C_k and S_k are the functions of the orbital eccentricity e. Up to

TABLE I

Trigonometrical expansions of S_1, S_4, S_5, S_6

Arg δ_i	S_1(sin) $A_{1i}10^{-3}$	S_4(cos) $A_{4i}10^{-3}$	S_5(cos) $A_{5i}10^{-3}$	S_6(cos) $A_{6i}10^{-3}$
0		648.00	1687.71	391.70
g	−530.87	1244.37	1876.78	−611.95
$2g$	101.51	−101.51	84.53	287.55
$3g$			56.32	56.32
2ϕ	−7.91	7.91	22.47	6.65
$2\phi - g$	−2.40	2.40	22.35	17.54
$2\phi - 2g$		0.71		
$2\phi - 3g$	−0.21	0.21	0.21	−0.21
$2\phi + g$	−2.40	2.40	10.42	5.61
$2\phi + 2g$		0.71		
$2\phi + 3g$	−2.14	0.21	0.21	−0.21

TABLE II

Trigonometric expansions of S_2, S_3

Arg γ_i	S_2(sin), S_3(cos) A_i10^{-3}
ϕ	49.40
$\phi - g$	−69.54
$\phi - 2g$	23.94
$\phi - 3g$	3.49
$\phi + g$	−4.56
$\phi + 2g$	11.52
$\phi + 3g$	3.48

the e^2 they can be taken from the Cayley's (1861) tables. In Tables I and II we represent expansion of the S_k function in to the series with multiple to g arguments. In these tables the coefficient A_{ki} of a harmonics are in the radians. In calculations the values $i = 7°00'18''$ and $e = 0.205632$ had been used.

4. Differential Equations of the Problem in a Linear Approximation

From the suggestion that the components of Mercury's physical libration, μ, ν, π are the small quantities, we can write dynamic Equation (8) as

$$\begin{aligned} &\ddot{\mu} = \gamma Q[S_1 - \mu(A_{40} + \tilde{S}_4) + \nu S_3 + \pi S_2] , \\ &\ddot{\nu} + n(1-\beta)\dot{\pi} + \beta n^2 \nu = -\beta Q[S_2 + \mu S_3 + \nu(A_{50} + \tilde{S}_5) + \pi S_1] , \\ &\ddot{\pi} - n(1-\alpha)\dot{\nu} + \alpha n^2 \pi = \alpha Q[S_3 - \mu S_2 - \nu S_1 - \pi(A_{60} + \tilde{S}_6)] . \end{aligned} \tag{16}$$

If we retain in the right sides of the equations only secular terms, then we can get the following system of a linear differential equations of oscillating motions,

$$\ddot{\mu} + (\gamma Q A_{40})\mu = \gamma Q S_1 ,$$

$$\ddot{\nu} + n(1-\beta)\dot{\pi} + \beta(n^2 + QA_{50})\nu = -\beta QS_2 ,$$
$$\ddot{\pi} - n(1-\alpha)\dot{\nu} + \alpha(n^2 + QA_{60})\pi = \alpha QS_3 , \tag{17}$$

where A_{40}, A_{50}, A_{60} are the coefficients of a secular terms in a series S_4, S_5, S_6.

In (17) the first equation, describing librations in longitude μ, may be separated from two others. It represents the equation of a forced harmonic oscillations

$$\ddot{\mu} + \omega^2\mu = \sum_i \gamma QA_{1i} \sin \delta_i ,$$
$$\omega^2 = \gamma QA_{40} ; \tag{18}$$

and its solution has the following form

$$\mu = B \sin(\omega t + \Theta) + \sum \frac{\gamma QA_{1i}}{\omega^2 - \dot{\delta}_i} \sin \delta_i , \tag{19}$$

where B and Θ are the integration constants.

In our problem for the determination of a bounds of the equator's inclination to the ecliptic, the main interest is presented by two last equations in (17). They present the system with two degrees of freedom and may be written in compact form

$$\ddot{\nu} + \alpha_{11}\dot{\pi} + \alpha_{12}\nu = -\sum \beta QA_i \sin \gamma_i ,$$
$$\ddot{\pi} - a_{21}\dot{\nu} + a_{22}\pi = \sum \alpha QA_i \cos \gamma_i ; \tag{20}$$

where

$$a_{11} = n(1-\beta), \qquad a_{21} = n(1-\alpha) ,$$
$$a_{12} = \beta(n^2 + QA_{50}), \qquad a_{22} = \alpha(n^2 + QA_{60}) , \tag{21}$$

The full solution of (20) is

$$\nu = B_1 \sin(\eta\tau + \Theta_1) + B_2 \sin(\xi t + \Theta_2) + \sum \mathrm{K}_i \sin \gamma_i ,$$
$$\pi = B_1\kappa_1 \cos(\eta t + \Theta_1) + B_2\kappa_2 \cos(\xi t + \Theta_2) + \sum L_i \sin \gamma_i ; \tag{22}$$

where B_1, B_2, Θ_1, Θ_2 are constants of a free librations,

$$\left.\begin{aligned} \eta &= \sqrt{(n^2 + QA_{50})\beta} \\ \xi &= \sqrt{(n^2 + QA_{50})\alpha\beta} \end{aligned}\right\} \quad \text{proper frequencies ,}$$

$$\left.\begin{aligned} \kappa_1 &= \frac{a_{21}\eta}{-a_{22} + \eta^2} \\ \kappa_2 &= \frac{a_{21}\xi}{-a_{22} + \xi^2} \end{aligned}\right\} \quad \text{coefficients of distribution .}$$

We shall consider further only the forced oscillations. The solutions will be a quasi-periodic functions in a simple form

$$\nu = \sum K_i \sin \gamma_i , \qquad \pi = \sum L_i \cos \gamma_i .$$

If we substitute these expressions in (20) and take the coefficients attached to the identical trigonometrical functions, then two equations with two unknowns K_i and L_i can be written for each ith harmonic: i.e.,

$$-(\dot{\gamma}_i^2 - a_{12})K_i - \alpha_{11}\dot{\gamma}_i L_i = -\beta Q A_i ,$$

$$-a_{21}\dot{\gamma}_i K_i - (\dot{\eta}_i^2 - a_{22})L_i = \alpha Q A_i ,$$

Solving this we get

$$K_i = \frac{QA_i[\beta(\dot{\gamma}_i^2 - a_{22}) + \alpha a_{11}\dot{\gamma}_i]}{\dot{\gamma}_i^4 - (a_{12} + a_{22} + \alpha_{11}a_{21})\dot{\gamma}_i^2 + a_{12}a_{22}} ,$$
$$L_i = \frac{QA_i[\alpha(\dot{\gamma}_i^2 - a_{12}) + \beta a_{21}\dot{\gamma}_i]}{\dot{\gamma}_i^4 - (a_{12} + \alpha_{22} + a_{11}a_{21})\dot{\gamma}_i^2 + a_{12}a_{22}} . \tag{23}$$

Thus, the problem of determination of a forced oscillations in ν and π in a linear approximation may be solved quite easily.

5. Determination of the ν, π and μ Component of Mercury's Physical Libration

At the moment we know well from the observations the orbital inclination i, orbital eccentricity e, mean daily motion n, mass ratio $M☿/M_{\odot}$, and mean daily angular rate ρ. But still there is no proper data about Mercury's moments of inertia.

Nevertheless we are in the position to determine their bounds, containing possible values of physical libration of Mercury.

Beletskii (1975) had proved, that stable resonance (3:2) rotation of Mercury is taking place around the minimum inertia axis if

$$A < B < C . \tag{24}$$

This means that the physical libration parameter

$$f = \frac{\alpha}{\beta} = \frac{B(C - B)}{A(C - A]} \tag{25}$$

lies inside the interval

$$0 \leq f < 1 . \tag{26}$$

The β value is a characteristic of a inertia ellipsoid flattening in the xz plane. We believe that the most probable value of β lies between 10^{-3} and 10^{-6}. From this suggestion we calculated the components of the physical libration of Mercury for different values of f and β to obtain,

$$f = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$$

$$\beta = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6},$$

setting

$$i = 7°00'18'', \quad e = 0.205632, \quad n = 0.0714248.$$
$$M_{\odot}/M_{☿} = 6000000, \quad \rho = 0.1071374,$$

The result of calculations of a main harmonic in ν and π, using (23), are presented in Table III. The forced librations in μ had been calculated using (19). In both cases the calculations were carried out for the same values of α and β. The value of γ has been found as

$$\gamma = \frac{\beta - \alpha}{1 - \alpha\beta}. \tag{27}$$

6. Discussion

If Mercury is rotating strictly in accordance with Cassini's laws, then the coordinates ν and π must be the functions of only one argument – namely ϕ. The mutual location of the ecliptic, equator and orbit of Mercury in this case are shown on Figure 2. In case of pure Cassini's motion the pole of the ecliptic, pole of the equator and of the orbit (Z, z and P respectively) must be in the plane which is perpendicular to the node line. Coordinate μ in that case must be zero, but components ν and π should have the different signs. If the ascending nodes of the orbit and of the equator are coincided, then ν will have negative sign and π-positive (Mercury case). And vice versa, if the decending node of the equator will coincide with ascending node of the orbit, then ν win be positive and π-negative (Moon case).

From spherical triangle ZNz it follows that

$$\tan(-\nu) = \tan I \sin\phi,$$
$$\sin\pi = \sin I \cos\phi; \tag{28}$$

where I is the inclination of the equator to the ecliptic*.

So, the mean rotation satisfying to the Cassini's laws may be presented with the enough precision by the functions.

* Let us note, that negative turn on the ν angle round the $Y'O$ line is correspond to the positive turn of I around the node line ΩO.

TABLE III

Mercury's physical libration components for the values of f (0.0–0.8) and β ($10^{-3} - 10^{-6}$)

$f = 0.0$

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−6037″.44	−6037″.33	−6037″.33	−6037″.33	6037″.33	6037″.33	6037″.33	6037″.33
$\phi - g$	1.08	0.11	0.01		−3.23	−0.32	−0.03	
$\phi - 2g$	−0.37	−0.04			−1.11	−1.11	−0.01	
$\phi - 3g$	−421.17	−381.55	−196.60	−33.62	−421.17	−381.55	−196.60	−33.62
$\phi + g$	−0.26	−0.03			0.16	0.02		
$\phi + 2g$	0.30	0.03			−0.13	−0.01		
$\phi + 3g$	0.05	0.01			−0.02			

$I = 1°.667$

$f = 0.0$

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$ μ	$\beta = 10^{-6}$
g	164″.42	16″.43	1″.64	0.16
$2g$	−7.85	−0.79	−0.08	−0.01
2ϕ	0.27	0.03		
$2\phi - g$	0.19	0.02		
$2\phi - 2g$	0.22	0.22		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18
$2\phi + g$	0.05			

$f = 0.1$

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−6490″.04	−6490″.38	−6490″.42	−6490″.42	6489″.53	6490″.33	6490″.41	6490″.42
$\phi - g$	1.40	0.14	0.01		−3.33	−0.33	−0.03	
$\phi - 2g$	−0.26	−0.03				−1.07	−0.11	−0.01
$\phi - 3g$	−370.60	−336.41	−175.06	−30.20	−370.66	−336.41	−175.07	−30.20
$\phi + g$	−0.27	−0.03			0.18	0.02		
$\phi + 2g$	0.31	0.03			−0.16	−0.02		
$\phi + 3g$	0.06	0.01			−0.02			

$I = 1°.802$

$f = 0.1$

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$ μ	$\beta = 10^{-6}$
g	147″.96	14″.78	1″.48	0.15
$2g$	−7.07	−0.71	−0.07	−0.01
2ϕ	0.24	0.02		
$2\phi - g$	0.17	0.02		
$2\phi - 2g$	0.20	0.02		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18
$2\phi + g$	0.04			

TABLE III. Continued

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
	$f = 0.2$							
	ν sin				π cos			
ϕ	−6922″79	−6923″36	−6923″41	−6923″42	6921″79	6923″26	6923″40	6923″42
$\phi - g$	1.72	0.17	0.02		−3.44	−0.34	−0.03	
$\phi - 2g$	−0.15	−0.01			−1.04	−0.10	−0.01	
$\phi - 3g$	−322.23	−293.06	−153.98	−26.80	−322.34	−293.08	−153.98	−26.80
$\phi + g$	−0.29	−0.03			0.21	0.02		
$\phi + 2g$	0.33	0.03			−0.19	−0.02		
$\phi + 3g$	0.06	0.01			−0.03			

$I = 1°922$

$f = 0.2$

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
	μ			
g	131″51	13″14	1″31	0.13
$2g$	−6.28	−0.63	−0.06	−0.01
2ϕ	0.22	0.02		
$2\phi - g$	0.15	0.01		
$2\phi - 2g$	0.18	0.02		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18
$2\phi + g$	0.04			

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
	$f = 0.3$							
	ν sin				π cos			
ϕ	−7336″87	−7337″55	−7337″62	−7337″62	7335″41	7337″40	7337″60	7337″62
$\phi - g$	2.04	0.20	0.02		−3.54	−0.35	−0.04	
$\phi - 2g$	−0.04				−1.00	−0.10	−0.01	
$\phi - 3g$	−275.92	−251.42	−133.34	−23.41	−276.09	−251.43	−133.34	−23.41
$\phi + g$	−0.31	−0.03			0.23	0.02		
$\phi + 2g$	0.34	0.03			−0.22	−0.02		
$\phi + 3g$	0.06	0.01			−0.03			

$I = 2°307$

$f = 0.3$

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
	μ			
g	115″06	11″50	1″15	0.11
$2g$	−5.50	−0.55	−0.05	−0.01
2ϕ	0.19	0.02		
$2\phi - g$	0.13	0.01		
$2\phi - 2g$	0.15	0.02		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18
$2\phi + g$	0.03			

TABLE III. Continued

Arg	$f = 0.4$ $\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−7733.″47	−7735.″16	−7734.″23	−7734.″24	7731.″56	7733.″97	7734.″21	7734.″23
$\phi - g$	2.36	0.24	0.02		−3.65	−0.37	−0.04	
$\phi - 2g$	0.08	0.01			−0.97	−0.10	−0.01	
$\phi - 3g$	−231.55	−211.37	−113.11	−20.03	−231.76	−211.39	−113.12	−20.03
$\phi + g$	−0.32	−0.03			0.26	0.03		
$\phi + 2g$	0.35	0.04			−0.25	−0.02		
$\phi + 3g$	0.06	0.01			−0.04			

$I = 2.°147$

Arg sin	$f = 0.4$ $\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$ μ	$\beta = 10^{-6}$
g	98.″61	9.″86	0.99	0.10
$2g$	−4.71	−0.47	−0.05	
2ϕ	0.16	0.02		
$2\phi - g$	0.11	0.01		
$2\phi - 2g$	0.13	0.01		
$2\phi - 3g$	−68.18	−68.18	−68.16	−68.18
$2\phi + g$	0.03			

Arg	$f = 0.5$ $\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−8113.″67	−8114.″29	−8114.″35	−8114.″35	8111.″33	8114.″05	8114.″32	8114.″35
$\phi - g$	2.68	0.27	0.03		−3.75	−0.38	−0.04	
$\phi - 2g$	0.19	0.02			−0.93	−0.09	−0.01	
$\phi - 3g$	−188.99	−172.82	−93.30	−16.66	−189.26	−172.85	−93.31	−16.66
$\phi + g$	−0.34	−0.03			0.29	0.03		
$\phi + 2g$	0.37	0.04			−0.28	−0.03		
$\phi + 3g$	0.06	0.01			−0.04			

$I = 2.°253$

Arg sin	$f = 0.5$ $\beta = 10^{-3}$	$\beta = 10^{-4}$	$\beta = 10^{-5}$ μ	$\beta = 10^{-6}$
g	82.″17	8.″21	0.82	0.08
$2g$	−3.93	−0.39	−0.04	
2ϕ	0.14	0.01		
$2\phi - g$	0.09	0.01		
$2\phi - 2g$	0.11	0.01		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18
$2\phi + g$	0.02			

TABLE III. Continued

f = 0.6

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−8478″.46	−8478″.93	−8478″.98	−8478″.98	8475″.71	8478″.66	8478″.95	8478″.98
$\phi - g$	3.00	0.30	0.03		−3.86	−0.39	−0.04	
$\phi - 2g$	0.30	0.03			−0.89	−0.09	−0.01	
$\phi - 3g$	−148.14	−135.70	−73.89	−13.30	−148.45	−135.73	−73.89	−13.30
$\phi + g$	−0.35	−0.04			0.31	0.03		
$\phi + 2g$	0.38	0.04			−0.31	−0.03		
$\phi + 3g$	0.06	0.01			−0.05	−0.01		

$I = 2°.354$

f = 0.6

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$ μ	$\beta = 10^{-5}$	$\beta = 10^{-6}$
g	65″.73	6″.57	0.66	0.07
$2g$	−3.14	−0.31	−0.03	
2ϕ	0.11	0.01		
$2\phi - g$	0.07	0.01		
$2\phi - 2g$	0.07	0.01		
$2\phi - 3g$	0.09	−68.18	−68.18	−68.18
$2\phi + g$	−68.18			
	0.02			

f = 0.7

Arg	$\beta = 10^{-3}$	$\beta = 10^{-4}$ ν sin	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-3}$	$\beta = 10^{-4}$ π cos	$\beta = 10^{-5}$	$\beta = 10^{-6}$
ϕ	−8828″.77	−8829″.02	−8829″.05	−88.29″.05	8825″.63	8828″.71	8829″.02	8829″.05
$\phi - g$	3.32	0.33	0.03		−3.96	−0.40	−0.04	
$\phi - 2g$	0.41	0.04			−0.86	−0.09	−0.01	
$\phi - 3g$	−108.90	−99.93	−54.87	−9.96	−109.25	−99.96	−54.87	−9.96
$\phi + g$	−0.37	−0.04			0.34	0.03		
$\phi + 2g$	0.39	0.04			−0.34	−0.03		
$\phi + 3g$	0.07	0.01			−0.06	−0.01		

$I = 2°.451$

f = 0.7

Arg sin	$\beta = 10^{-3}$	$\beta = 10^{-4}$ μ	$\beta = 10^{-5}$	$\beta = 10^{-6}$
g	49″.29	4″.93	0.49	0.05
$2g$	−2.36	−0.24	−0.02	
2ϕ	0.08	0.01		
$2\phi - g$	0.06	0.01		
$2\phi - 2g$	0.07	0.01		
$2\phi - 3g$	−68.18	−68.18	−68.18	−68.18

TABLE III. Continued

	f = 0.8							
Arg	$\beta=10^{-3}$	$\beta=10^{-4}$	$\beta=10^{-5}$	$\beta=10^{-6}$	$\beta=10^{-3}$	$\beta=10^{-4}$	$\beta=10^{-5}$	$\beta=10^{-6}$
		ν sin				π cos		
ϕ	−9165″.44	−9165″.42	−9165″.42	−9165″.42	9161″.92	9165″.07	9165″.38	9165″.41
$\phi-g$	3.65	0.37	0.04		−4.07	−0.41	−0.04	
$\phi-2g$	0.52	0.05	0.01		−0.82	−0.08	−0.01	
$\phi-3g$	−71.17	−65.43	−36.22	−6.63	−71.57	−65.47	−36.22	−6.63
$\phi+g$	−0.38	−0.04			0.36	0.04		
$\phi+2g$	0.40	0.04			−0.37	−0.04		
$\phi+3g$	0.07	0.01			−0.06	−0.01		
$I = 2°.544$								

	f = 0.8			
Arg	$\beta=10^{-3}$	$\beta=10^{-4}$	$\beta=10^{-5}$	$\beta=10^{-6}$
sin		μ		
g	32″.86	3″.29	0.33	0.03
$2g$	−1.57	−0.16	−0.02	
2ϕ	0.05	0.01		
$2\phi-g$	0.04			
$2\phi-2g$	0.04			
$2\phi-3g$	−68.18	−68.18	−68.18	−68.19

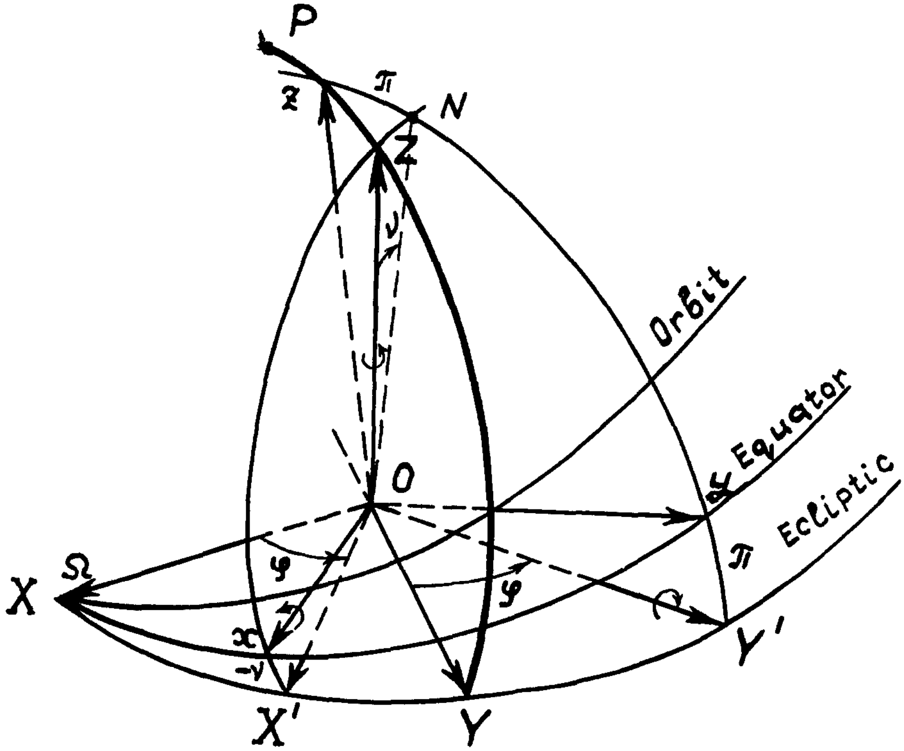

Fig. 2. The position of the (x, y, z) triad, which is fixed in Mercury, in accordance with Cassini's laws.

TABLE IV

Inclinations of the equator to the ecliptic $K_\phi = -I$ as a function of f

f	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mercury	$-1\overset{\circ}{.}68$	$-1\overset{\circ}{.}80$	$-1\overset{\circ}{.}92$	$-2\overset{\circ}{.}04$	-2.15	$-2\overset{\circ}{.}2.25$	$-2\overset{\circ}{.}36$	$-2\overset{\circ}{.}45$	$-2\overset{\circ}{.}54$	$-2\overset{\circ}{.}64$
Moon	$+1\overset{\circ}{.}00$	$+1\overset{\circ}{.}08$	$+1\overset{\circ}{.}17$	$+1\overset{\circ}{.}27$	$+1\overset{\circ}{.}38$	$+1\overset{\circ}{.}46$	$+1\overset{\circ}{.}56$	$+1\overset{\circ}{.}66$	$+1\overset{\circ}{.}76$	$+1\overset{\circ}{.}85$

$$\begin{aligned} -\nu &= \tan I \sin \phi\,, \\ \pi &= \sin I \cos \phi\,. \end{aligned} \tag{29}$$

During discussion of results we must emphasize that in the solutions for ν and the main harmonics are the $K_\phi \sin \phi$ and $L_\phi \cos \phi$, which describe, as was mentioned above, the mean rotation of Mercury. Other harmonics in solution describes oscillations relative to the mean motion*.

Analyzing the formulas in (25) or the Table III, we found the main property of the resonance rotations. That is the inclination I mainly depends on f and slightly on α. Approximate formula, without taking into consideration terms containing α, is written:

$$K_\phi = -I = \frac{A_\phi(1+f)}{2(A_{50} + A_{60}f)}\,. \tag{30}$$

In Table IV there are shown the values of inclination of Mercury (in degrees) as a function of f.

Thus we can see, that in Mercury's resonance rotation phenomenon, the only pairs of α and β may exist, which are satisfied to the condition ($0 \leqslant f < 1$). From this fact we can claim that the value of inclination I is limited by the $1\overset{\circ}{.}7$ and $2\overset{\circ}{.}6$ approximately. In case of $f = 0$ ($C = B$) the inertia ellipsoid of Mercury must be axially symmetric and inclination of the equator would have the smallest value. The value of $f = 1$ ($B = A$) corresponds to the oblate Mercury's inertia ellipsoid for which the condition (24) is not more valid. It is necessary to note that Mercury's equator cannot coincide with the orbital plane, since the parameter f for $I = +7°$ will exceed the permissible limit ($f = 101$). Equator of Mercury cannot likewise coincide with the ecliptic plane since it corresponds to the negative value of f.

The negative value of inclination $I = K_\phi$ automatically corresponds to our values of parameters A_ϕ, A_{50} and A_{56}. And this means that the ascending nodes of the equator and of the orbit coincide.

In order to prove reliability of our conclusions we have made the parallel computations for the case of resonance rotation of the Moon. Table IV shows the results of these computations. As it follows from this table, the inclination of the Moon's equator lies between $+1\overset{\circ}{.}00$ and $+1\overset{\circ}{.}88$. The accepted value in the

* It is necessary to note that the coefficients K_ϕ and L_ϕ have been obtained only in linear approximation. It is possible to calculate them with better accuracy, solving nonlinear differential equations, for example by the method of Eckhardt (1981). But in our problem it is enough to put $\nu_\phi = K_\phi = I$.

"Astronomical Almanac" for inclination is equal to $I = 1^\circ.542$, which roughly corresponds to $f = 0.6$. According to the values of A_ϕ, A_{50}, A_{56} for the Moon the sign of K_ϕ is positive. It means that descending node of the equator coincides with the ascending node of orbit.

Now we can make the suggestion that if Mercury – as well as the Moon – has the parameter f close to $f = 0.6$, then the probable value of its inclination of equator to the ecliptic lies near the value of $I = 2^\circ.3$.

From all presented above it follows that in case of stable rotation of a body under the second and the third generalized Cassini's laws and if the condition $A < B < C$ is satisfied, then the equator plane can not coincide with the orbital plane or with the plane of the ecliptic.

The variations of Mercury's equator relative to mean position are generally small. So, in components ν and π the amplitudes of all harmonics are less than one arc of second, excluding only harmonics $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\phi - 3g)$, which have the amplitudes, for different values of f and β, from a few hundreds to a few tens of seconds. The frequency of these harmonics, with the opposite sign, is very close to the mean angular rate of the Mercury. The signs of harmonics both in ν and in π are keeps negative.

In longitude μ variations, the only $\sin g$ harmonic amplitude may be up to several tens of arc second (for $\beta = 10^{-3}$) and it depends more on β than on f. Especially it is necessary to note the $\sin(2\phi - 3g) = \sin 2\omega$ harmonic. The low variation in the argument of the perihelion latitude ω causes almost the table value of amplitude $A_{2\omega} = -68''$. The rest of a harmonics has the very small amplitudes.

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