AN ANALYTICAL MODEL OF THE COMET'S COLLISION WITH JUPITER

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Abstract. An important feature observed in the wake of the Jupiter-comet clash was the appearance of the ring structure axisymmetrically positioned around the center of the impact. The persistent expansion of the dark rings and its speed indicated an outward propagating gravity wave (Benka, 1995). We employ an analytical model of constant density, uniform finite depth and inviscid fluid layer to investigate the wave motion produced by the impact of Comet Shoemaker-Levy 9 on the Jovian atmosphere. The relevant thermal effects are neglected and an explosion resulting from the collision is then described by an initial impulsive pressure at the surface of the Jovian atmosphere. Under the assumption that all the kinetic energy of a comet fragment is completely converted into the energy of wave motions in the Jovian atmosphere, an analytical formula describing the relationship between the resulting wave motion in the atmosphere and the parameters of a comet fragment (the radius, density and speed) is derived. Results from the present simple analytical model give a qualitative agreement with observations regarding the distance and speed of the waves.

Key words: Comets, Collision, Jupiter

1. Introduction

The cosmic collision between Comet Shoemaker-Levy 9 and Jupiter provides perhaps the unique opportunity to study cosmic impacts in the solar system, which may have taken place frequently in its evolution. A familiar feature as a consequence of such cosmic impacts is perhaps the ring structure on the Moon's surface. It was believed that the collisions between the Moon and small bodies like asteroids may produce outward propagating waves due to the fluidized lunar crust and subsequently created the ring structure as the waves were frozen at a latter time (Van Dorn, 1968). Noting the difference between the Moon's solid surface and the Jovian fluid atmosphere, it is remarkable that the similar ring structure around the center of the impact was observed in the wake of the Jupiter-comet clash (see, for example, Benka, 1995). The persistent expansion of the dark rings was seen for several hours with the speed of order $1000 m/s$. It was suggested that the ring structure represents an outward propagating gravity waves (Benka, 1995). The group speed of a gravity wave is given by

$$V_g = f\sqrt{Hg}$$

where H is the depth of the atmosphere layer and $0 < f \leq 1$. Note that $f - 1$ corresponds to the limit of an infinitely long wavelength. In the atmosphere of

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Jupiter, the thickness of the outer layer is $H = O(1000)$ km with g = 23 m/s^2, giving the correct order of magnitude for wave speed. The primary aim of this paper is at providing some insight into the basic fluid dynamic process of the gravity waves generated by the comet-Jupiter collision through the examination of a relatively simple analytical model.

A complete understanding of the cosmic collision is certainly difficult, involving complicated thermal, fluid dynamical, chemical and nonlinear processes. In this paper, we attempt to understand the some fundamental fluid dynamical processes after the comet collision. It is of importance to note that a realistic fluid dynamical modelling for the comet collision can be extremely complex. Spherical geometry, nonlinearity and inhomogeneous density of the Jovian atmosphere may have some influences. Since the spots caused by the impact were observed for a few days, rotational effects may also play a role in the dynamical processes. To include all these effects in a realistic model, a numerical approach must be adopted, which have been discussed (see, for example, the special June issue of *Geophysical Research Letter*, 1994). However, the fragments of the comet are quite small, estimated from less than half a kilometer to almost 5 kilometers. The radius of Jupiter is about 7×10^4 kilometers while the Jovian gaseous atmosphere above the liquid hydrogen is only about 1000 kilometers thick. It is also likely that the amplitude of wave motions in the Jovian atmosphere generated by the collision is likely to be much smaller compared to the depth of the atmosphere. In addition, the value of viscosity of the Jovian atmosphere is of secondary importance for this short-time-scale phenomenon. It is thus expected that a linear inviscid theory in a uniform finite fluid layer may give a reasonable description of the collision with regard to the dynamical aspect of the event.

In this paper, we presents an analytical model describing the amplitude and time-dependent profile of wave motions generated by the comet impact in the Jovian atmosphere in a non-rotating fluid layer of uniform depth, inviscid and constant density. The relevant thermal effects are not included in this simple fluid dynamical model, an explosion caused by the collision is described by an initial impulsive pressure prescribed at the surface of Jupiter. Upon assuming that all the kinetic energy of a comet fragment is completely converted into the energy of wave motions in the Jovian atmosphere, an analytical formula describing the relationship between the resulting wave motions after the collision and the parameters of a comet fragment (the radius, density and speed of the collision) is obtained. The results from this simple analytical model apparently show a qualitative agreement with observations in some important aspects.

This paper is arranged as follows. Section 2 presents a model of the comet impact. This is followed by a discussion of solutions of the model and comparison with observations in section 3. The paper closes in section 4 with some remarks.

2. Description of the model

On the basis of the thin atmospheric layer of Jupiter and the small size of a comet fragment, we assume that the effect of spherical curvature is small so that the Jovian atmosphere can be modeled by a fluid layer of uniform finite depth H. We also assume that the fragment of Comet Shoemaker-Levy 9 has spherical symmetry with the radius R and uniform density ρ_0. The relative speed at the time of the impact on the Jovian surface is denoted by U. An initial impulsive pressure on the Jovian surface at $z = 0$ generated by the comet impact action produces wave motions propagating outward from the origin $r = 0$ of the impact, where cylindrical polar coordinates (r, ϕ, z) are used. The form of the initial impulse caused by the comet collision is assumed to be

$$I(r) = I_0 e^{-(r/R)^2}, \quad z = 0, \tag{2.1}$$

which describes the exponentially diminishing effect of the impact from the center with a characteristic length-scale of the size of the fragment. Additionally, we assume that the resulting wave motions are axisymmetric, irrotational and incompressible, which lead to

$$\mathbf{V} = -\nabla\Phi, \tag{2.2}$$

where $\mathbf{V}$ is the velocity and Φ is the velocity potential satisfying

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Phi}{\partial r}) + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{2.3}$$

The effect of compressibility of the fluid has been neglected since we are only interested in gravity waves rather than acoustic waves. At the interface $z = -H$ between the atmosphere and the internal liquid hydrogen, we may impose

$$\hat{z} \cdot \mathbf{V} = 0, \tag{2.4}$$

where $\hat{z}$ is a vertical unit vector. The vertical fluid motion is described by the $z-$component of momentum equation

$$-\rho\frac{\partial^2\Phi}{\partial t\partial z} + \frac{\partial P}{\partial z} + \rho g = 0, \tag{2.5}$$

where ρ is the average density of the Jovian atmosphere and P is the pressure field. Integrating (2.5) once and noting that

$$\frac{\partial\Phi}{\partial z} + \frac{\partial\eta}{\partial t} = 0$$

at the free surface, where η is the amplitude (elevation) of a wave, we obtain the boundary condition at $z = 0$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0. \tag{2.6}$$

Here we have assumed the wave amplitude is sufficiently small so that the nonlinear effects can be neglected. We may write two initial conditions, a still atmosphere, $\eta = 0$, before the impact and an impulsive pressure, $I(r)$, at the impact $t = 0$, in terms of the velocity potential Φ

$$\frac{\partial \Phi}{\partial t} = 0, \quad \Phi(r, 0, 0) = I(r)/\rho = I_0 e^{-(r/R)^2}/\rho. \tag{2.7a,b}$$

Under the assumptions that the wave motions are produced originally from the rest and that all the kinetic energy of a comet fragment is converted into the wave motions in the Jovian atmosphere, we are able to relate the value of the parameter I_0 to the observational parameters of the comet fragment (see section 3 for detail)

$$I_0^2 = \frac{8}{3}R^2\rho_0\rho U^2/K \tag{2.8}$$

with

$$K = \int_0^\infty s^2 e^{-s^2/2}\tanh(sH/R)ds. \tag{2.9}$$

Our problem is thus defined by the governing Equation (2.3) subject to the boundary conditions (2.4) and (2.6) and initial conditions (2.7).

3. Solution and discussion

To solve Equation (2.3) subject to the relevant boundary and initial conditions, we introduce the zero-order Hankel transform (Lamb, 1932; Whitham, 1974)

$$\Phi^*(\xi, z, t) = \int_0^\infty r\Phi(r, z, t)J_0(\xi r)dr \tag{3.1}$$

of the velocity potential Φ, where J_0 is the zero-order Bessel function. Upon multiplying Equation (2.3) throughout by $rJ_0(\xi r)$ and integrating with respect to r, we find that Φ^* satisfies

$$\frac{\partial^2 \Phi^*}{\partial z^2} - \xi^2\Phi^* = 0, \tag{3.2}$$

where the assumption that there are no influences of the impact at $r \to \infty$ is made. The boundary condition for Φ^* at the bottom of the fluid layer $z = -H$ is then

$$\frac{\partial \Phi^*}{\partial z} = 0, \tag{3.3}$$

while the boundary condition on the free surface may be expressed as

$$\frac{\partial^2 \Phi^*}{\partial t^2} + g\frac{\partial \Phi^*}{\partial z} = 0. \tag{3.4}$$

It is straightforward to show that the solution of Equation (3.2) satisfying Equations (3.3) and (3.4) is

$$\Phi^*(\xi, z, t) = [(F(\xi)/\xi)\cos\omega t + (G(\xi)/\xi)\sin\omega t]\frac{\cosh\xi(H+z)}{\rho\cosh(\xi H)}, \tag{3.5}$$

where the dispersion relation between ω and ξ is given by

$$\omega^2 = g\xi\tanh(\xi H). \tag{3.6}$$

It follows that the velocity potential $\Phi(r, z, t)$ can be expressed as

$$\Phi(r, z, t) = \int_0^\infty [\frac{F(\xi)}{\xi}\cos\omega t + \frac{G(\xi)}{\xi}\sin\omega t]\frac{\cosh[\xi(H+z)]}{\rho\cosh(\xi H)}\xi J_0(\xi r)d\xi. \tag{3.7}$$

Using the initial condition (2.7a), it can be shown that

$$G(\xi) = 0,$$

while the initial condition (2.7b) associated with the impulsive pressure at the time of the impact on the surface $z = 0$ (2.6b) gives rise to

$$I(r) = I_0 e^{-(r/R)^2} = \int_0^\infty \frac{F(\xi)}{\xi} J_0(\xi r)\xi d\xi, \tag{3.8}$$

with its inverse transform given by

$$I^*(\xi) = \frac{F(\xi)}{\xi} = \int_0^\infty I_0 e^{-(r/R)^2} J_0(\xi r) r dr = \frac{I_0 R^2}{2} e^{-\xi^2 R^2/4}. \tag{3.9}$$

Consequently, we may write the velocity potential Φ and the elevation η of the corresponding wave as

$$\Phi(r, z, t) = \frac{I_0 R^2}{2}\int_0^\infty \frac{\cosh[\xi(H+z)]}{\rho\cosh(\xi H)} J_0(\xi r)\cos(\omega t) e^{-\xi^2 R^2/4}\xi d\xi, \tag{3.10}$$

and

$$\eta(r,t) = \frac{-I_0 R^2}{2}\int_0^\infty \frac{\omega\sin(\omega t)}{\rho g} J_0(\xi r) e^{-\xi^2 R^2/4}\xi d\xi. \tag{3.11}$$

We will focus on the profile of the free surface $\eta(r,t)$ as it represents an observable quantity.

To make further progress of the problem, we must connect the unknown parameter I_0 in Equations (3.10–3.11) with the observable variables of a comet. Since the viscosity of the atmosphere is small and the time-scale of the waves is much shorter compared to the diffusion time-scale, it appears appropriate to assume that the energy of a fragment is completely converted into the kinetic energy of the waves. The total kinetic energy of the wave motions resulting from an impact is

$$E_w = \frac{1}{2}\int_V \rho|\nabla\Phi|^2 dV = \frac{\pi I_0^2 R^4}{4\rho}\int_0^\infty \xi^2 e^{-R^2\xi^2/2}\tanh(\xi H)d\xi. \tag{3.12}$$

Let $s = \xi R$, we then have

$$E_w = \frac{\pi I_0^2 R K}{4\rho},$$

where K is defined by Equation (2.9). Equating E_w to the kinetic energy of a spherical fragment, we obtain

$$F(\xi)/\xi = I^*(\xi) = R^3 U\sqrt{\frac{2\rho_0\rho}{3K}} e^{-R^2\xi^2/4}. \tag{3.13}$$

Making use of Equation (3.11), we obtain for the elevation of the free surface

$$\eta(r,t) = -\frac{UR^3}{g}\sqrt{\frac{2\rho_0}{3K\rho}}\int_0^\infty e^{-R^2\xi^2/4}\omega\sin(\omega t)J_0(\xi r)\xi d\xi. \tag{3.14}$$

The exact evaluation of the above integral is quite complicated even for the simplest model of the initial distribution $I(r)$. However, sufficiently away from the center of the impact, $\xi r \gg 1$, an asymptotical expression of the integral for large $r\xi$ can be derived by using the principle of the stationary phase, first proposed by Kelvin in 1887 (see Whitham, 1974, for detail). First we note that the Bessel function $J_0(r\xi)$ for $r\xi \gg 1$ is approximately given by

$$J_0(\xi r) = \sqrt{\frac{2}{\pi\xi r}}\cos(\xi r - \pi/4).$$

Equation (3.14) then becomes

$$\eta(r,t) = \frac{-UR^3}{g}\sqrt{\frac{2\rho_0}{3\pi K\rho}}\int_0^\infty \omega(1+i)\sqrt{\frac{\xi}{r}} \times \exp\{-(R\xi/2)^2 + i(\xi r - \omega t)\}d\xi, \tag{3.15}$$

where the real part of the expression represents required solutions. The Kelvin's approximation is based on the fact that the major contribution arises from the region of the integration where the phase of oscillation is stationary. For the present case, the stationary point is given by

$$t\frac{d\omega}{d\xi} - r = 0. \tag{3.16}$$

Under this approximation, the elevation of the free surface produced by the collision can be expressed as

$$\eta(r,t) = R^3U\omega(\xi^*)\sqrt{\frac{\rho_0}{6g^2\rho K}}exp[-(R\xi^*/2)^2] \times\sqrt{\frac{4\xi^*}{|\omega''(\xi^*)|rt}}\sin[\xi^* r - \omega(\xi^*)t], \tag{3.17}$$

where ξ^* represents the solution of Equation (3.16) for given r and t. Note that the relation $\omega(\xi)$ is given by Equation (3.6). Combining the relevant equations, we can obtain a formula describing the wave elevation of the Jovian surface after the collision

$$\eta(r,t) = \frac{R^3U}{g^{1/2}H^{3/2}r}\sqrt{\frac{2\rho_0}{3\rho K}}\exp[-(R\xi^*/2)^2]F(H\xi^*) \times \sin[\xi^* r - \omega(\xi^*)t] \tag{3.18}$$

where

$$F(x) = \sqrt{\frac{2x^3\tanh x[\sinh(2x) + 2x]\sinh(2x)}{4x[2x\cosh(2x) - x - \sinh(2x)] + \sinh^2(2x)}}. \tag{3.19}$$

The relation between the r, t and ξ^* is defined by

$$2r = t\sqrt{gH}\{\sqrt{\tanh(\xi^* H)/(\xi^* H)} + \sqrt{\xi^* H/[\cosh^3(\xi^* H)\sinh(\xi^* H)]}\}. \tag{3.20}$$

For any given values of radius R, speed U and density ρ_0 of a comet fragment, the resulting wave motions of the Jovian atmosphere after the impact can be readily calculated using Equations (3.18–3.20).

In our evaluation of Equations (3.18–20), we have chosen that H = 1000 km ρ = 500 kg/m^3 and g = 23 m/s^2 for the Jovian atmosphere, though we have used different sizes (R) of a fragment of the comet. Figure 1(a) shows the elevation η as a function of the distance r from the center of the impact on the Jovian surface obtained for R = 3 km, ρ_0 = 500 kg/m^3 and U = 55 km/s at one hour after the impact. The waves produced by the collision reach to about 500 km from the origin of the collision with the amplitude as high as 10 km. At t = 12 hours, the waves propagate to about 4×10^3 km with the wave amplitude about 300 m; at t = 24 hours, our calculation shows that the waves reach the distance slightly more than 10^4 km. Note that a similar plot for small size of the fragment (R = 1 km) at t = 1 hour is shown in Figure 1(b), indicating a much smaller amplitude of the wave elevation.

Our results calculated from Equations (3.18–3.20) appears to be in a qualitative agreement with the rough estimation of observation of the event (for example, Kerr, 1994). It was reported that a black spot with the radius of about 10^4 km was observed a day after the impact of fragment G. It is, though, not our intention here to present an accurate comparison between our simple model and observations. Instead, it is of hope that a simple analytical formula like Equations (3.18–3.20) may be used to understand the basic fluid dynamical processes in connection with the comet collision when the detailed observations become available.

4. Concluding Remarks

We have presented a simple analytical model describing the wave motions after the collision of Comet Shoemaker-Levy 9 on the Jovian atmosphere. With the estimated size, density and speed of a comet fragment, we can use Equations (3.18–20) to produce the profile of the Jovian atmosphere surface resulting from an impact at a given time or a given distance. Our result based on this model appears to be qualitatively in agreement with observations regarding the distance and speed of the waves. With the fully analytical expressions (3.18–3.20), observable quantities like the speed of wave can be readily estimated, providing a complementary alternative to the full numerical solutions of a more complicate, realistic model. Our formulae can also be used to estimate the ring structure on the Moon's surface. In that case, we replace R with the radius of an impact body. Both the height of the rings and the space between the rings may be estimated on the basis of Equations (3.17–3.20). But it remains unknown concerning when and how a propagating wave is frozen.

It is evident that there are several simplifications in our model which need to be improved. One of them is the assumption of constant density for the Jovian

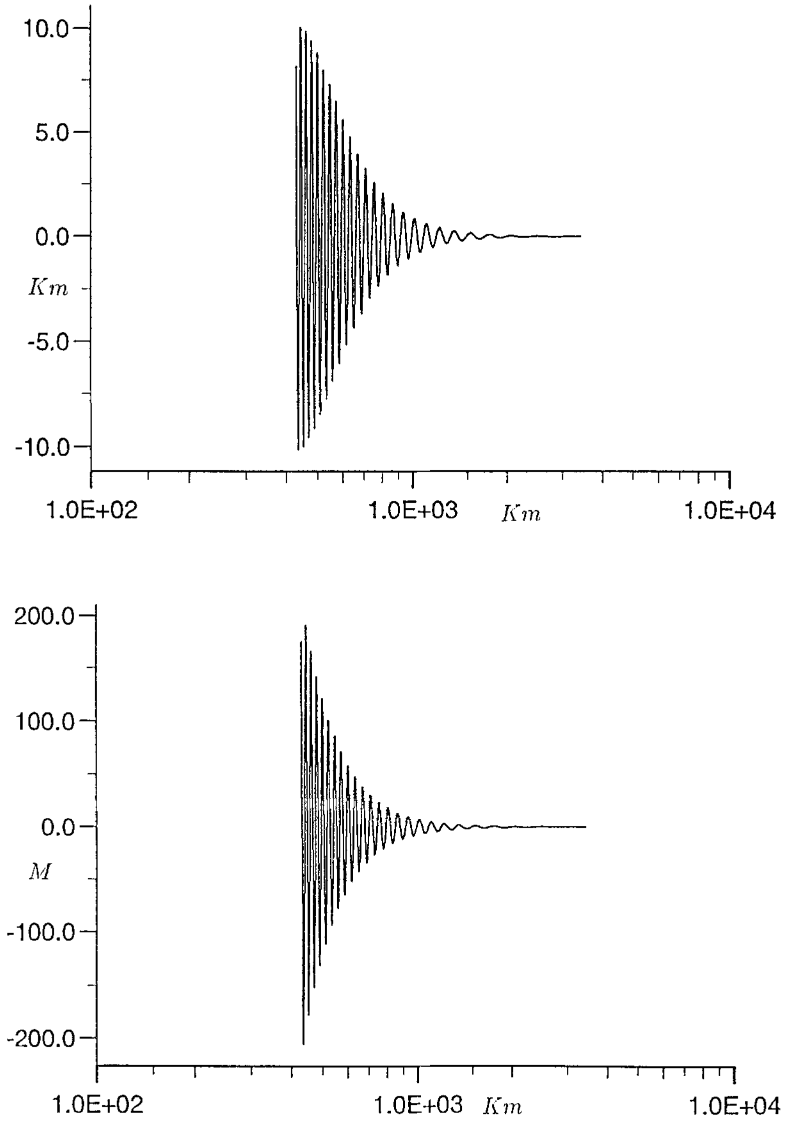

Fig. 1. The elevation of the surface η as a function of the distance r at one hour after the impact with (a) $R = 5000$ m and (b) $R = 3000$ m.

atmosphere, which is obviously oversimplified. Stratification can be of significance and may have important effects, especially, on the depth which can be stirred by an impact. A more realistic model therefore should include a density distribution $\rho = \rho(z)$. Furthermore, the compressibility of the Jovian atmosphere and the possibility of forming nonlinear shock-waves may also be of importance.

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