THE RAYLEIGH-TAYLOR INSTABILITY IN A SELF-GRAVITATING TWO-LAYER FLUID SPHERE

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Abstract. The Rayleigh-Taylor instability is studied in a self-gravitating two-layer fluid sphere: an inner sphere and an outer layer. The density and the viscosity are assumed to be constant in each region. Analytic expressions of the dispersion relations are obtained in inviscid and viscid cases. This examination aims at the investigation of the Earth's core formation. The fluid sphere corresponds to the proto-Earth in the accretion stage. The instability is examined without rotation of the fluid sphere, while the proto-Earth is rotating. However, it is shown that the Coriolis force does not influence the conclusion in the Earth's core formation problem. The main properties of the instability are as follows: For $l = 1$ (where l is the subscript of a spherical harmonic Y_l^m), the growth rate σ is determined mainly by deformation of the outer layer, while, for $l \geqslant 2$, by deformation of the more viscous region of the inner sphere and the outer layer. The time scale of the instability is governed by the free-fall time in the case of weak viscosity, and by the viscous-diffusion time in the case of appreciable viscosity. These results are applied to the Earth's core formation problem in another paper (Ida *et al.*, 1987), where we concluded that the Earth's core has formed through the translational mode of the Rayleigh-Taylor instability with the time-scale of 10 h.

1. Introduction

It has been proposed by several authors (e.g. Elsasser, 1963; Stevenson, 1981) that the Earth's core formed through the Rayleigh-Taylor instability. According to Elsasser, the wavelength of instability is as large as the size of proto-Earth; while Stevenson advocated the view that the translational mode occurs. In the investigation of the Earth's core formation, the instability must be considered in the self-gravitating fluid sphere. Since the instability has not been studied in self-gravitating fluid sphere, the authors of the previous studies examined the Earth's core formation using the results in the plane-parallel fluid with constant-gravity (e.g., Chandrasekhar, 1961; Ramberg, 1968), though one can validly use the constant-gravity plane-parallel approximation only when the wavelength of the instability is sufficiently small and cannot treat translational mode with this approximation. In the present paper, we will strictly study the Rayleigh-Taylor instability in a self-gravitating two-layer fluid sphere (inviscid and viscous) assuming that the density and the viscosity are constant in each region. The results are used in the investigation of the Earth's core formation in another paper (Ida *et al.*, 1987) which will be hereafter referred to as Paper II. The recent theories of planetary formation leads to the proto-Earth with the three-layer structure, the innermost undifferentiated solid core, the intermediate metal-metal layer, and the outermost silicate-melt layer.

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This structure is gravitationally unstable, since the density of the intermediate metal-melt layer is larger than that of the innermost undifferentiated solid core. The two-layer fluid sphere studied in the present paper corresponds to the inner two regions of the proto-Earth. In the Earth's core formation problem, it is sufficient to consider the inner two region (Paper II). The instability is examined without rotation in the present paper, while the proto-Earth is rotating and the Coriolis force would influence the fluid motion. However, as shown in Section 5, it is expected that the conclusion based on the results without rotation is still valid even if the Coriolis force is considered, as long as we are concerned with the instability of Earth's core formation.

2. Basic Equations

We will consider a self-gravitating two-layer fluid (an inner sphere and an outer layer); these regions contain incompressible fluid. We assume that the inner sphere and the outer layer have the densities ρ_1 and ρ_2 and the viscosities μ_1 and μ_2, respectively, which are constant in each region. Let R_1 and R_2 be the radius of the interface between the inner sphere and the outer layer and that of the outer surface, respectively.

We perturb the initial hydrostatic state, deforming the interface at $r = R_1$ slightly as $r = R_1 + \delta r$. We denote the corresponding increments in the gravitational potential, the pressure, and the velocity field by $\delta\Phi$, δP, and $\mathbf{u}$, respectively. These increments are infinitesimal since we assume that δr is small enough compared to R_1. If we neglect the higher-order terms in these quantities, the equations governing the perturbed state of the inner sphere and the outer layer become

$$\begin{cases} \partial\mathbf{u}/\partial t = -\mathrm{grad}\,\phi + \nu\nabla^2\mathbf{u}, & (1)\\ \mathrm{div}\,\mathbf{u} = 0, & (2)\\ \nabla^2\delta\Phi = 4\pi G\delta\rho, & (3)\end{cases}$$

where $\phi \equiv \delta P/\rho + \delta\Phi$ and $\nu \equiv \mu/\rho$, and $\delta\rho$ is the Eulerian change of the density due to the boundary deformations at $r = R_1$ and R_2: i.e.,

$$\delta\rho = (\rho_1 - \rho_2)\,\delta r\delta_D(r - R_1) + \rho_2\,\delta r'\delta_D(r - R_2),$$

where δ_D is the Dirac's δ-function and $\delta r'$ the deformation of the surface at $r = R_2$ corresponding to δr. Equations (1), (2), and (3) are the equation of motion, the continuity equation (note that the fluid is assumed to be incompressible), and the Poisson's equation, respectively.

Since the configuration of unperturbed state is spherically-symmetric, the solution can be expanded into a series of normal modes; and these can be specified uniquely by a shape of deformation of the interface at $r = R_1$, given by

$$\begin{cases} \delta r = \varepsilon Y_l^m(\theta, \phi), & (4)\\ \varepsilon = \varepsilon_0 e^{\sigma t}, & (5)\end{cases}$$

where we used the spherical coordinate (r, θ, ϕ), and ε_0 is an infinitesimal constant, σ the growth rate of instability to be determined, and Y_l^m a spherical harmonic. Note that the perturbation is unstable if the real part of σ is positive.

In order to solve the solution to the perturbation equations, we must impose the boundary conditions at $r = R_1$ and R_2. As to the condition at $r = R_2$, we have two cases: i.e., a rigid surface and a free surface. The boundary conditions in the viscous case are as follows:

(1) The radial velocity u_r is continuous and equal to $\partial(\delta r)/\partial t$ at $r = R_1 + \delta r$.

(2) The tangential velocities u_θ and u_ϕ are continuous at $r = R_1 + \delta r$.

(3) The total normal stress (including the hydropressure P) P_{rr} is continuous at $r = R_1 + \delta r$.

(4) The tangential stresses $P_{r\theta}$ and $P_{r\phi}$ are continuous at $r = R_1 + \delta r$.

(5-r) In the rigid-surface condition ($\delta r' = 0$), u_r, u_θ, and $u_\phi = 0$ at $r = R_2$.

(5-f) In the free-surface condition P_{rr}, $P_{r\theta}$, and $P_{r\phi} = 0$ at $r = R_2 + \delta r'$.

In the above, the components of the total stress tensor P_{rr}, $P_{r\theta}$, and $P_{r\phi}$ are given by

$$\begin{cases} P_{rr} = -(P + \delta P) + 2\mu\, \partial u_r/\partial r, & (6) \\ P_{r\theta} = \mu(\partial u_r/\partial\theta + r\, \partial u_\theta/\partial r - u_\theta)/r, & (7) \\ P_{r\phi} = \mu(r\, \partial u_\phi/\partial r + \partial u_r/\sin\theta\, \partial\phi - u_\phi)/r. & (8) \end{cases}$$

On the other hand, the boundary conditions in the inviscid case ($\nu = 0$) are written as follows:

(1′) The radial velocity u_r is continuous and equal to $\partial(\delta r)/\partial t$ at $r = R_1 + \delta r$.

(2′) The pressure $P + \delta P$ is continuous at $r = R_1 + \delta r$.

(3′-r) On the rigid-surface condition, $u_r = 0$ at $r = R_2$.

(3′-f) On the free-surface condition, $P = 0$ and $u_r = \partial(\delta r')/\partial t$ at $r = R_2 + \delta r'$.

In the next section, we will obtain the analytic expressions of dispersion relations, solving algebraically Equations (1) to (3) on the above boundary conditions.

3. Dispersion Relations

3.1. Inviscid case

We will first study the inviscid case, which is helpful to understand the properties of the instability in the viscid case. The dispersion relations in the inviscid case are as follows (their derivations are shown in Appendix A2): in the rigid-surface case,

$$\sigma^2 = \frac{4\pi G}{3}(\rho_2 - \rho_1)\frac{l(l+1)}{2l+1}\frac{2(l-1)\rho_1 + 3\rho_2}{(l+1)(\rho_1 - \rho_2) + (2l+1)\rho_2 D_{inv}}, \quad (9)$$

where $\chi = R_2/R_1$ and $D_{inv} = (1 - \chi^{-(2l+1)})^{-1}$. We should notice that σ^2 is proportional to the density difference $(\rho_2 - \rho_1)$. This mode will be called the R mode hereafter (where R represents the *rigid* surface), and σ of the R mode given by Equation (9) will be denoted by σ_R.

In the free-surface case,

$$\sigma^2 = -\frac{4\pi G}{3}\frac{l(l+1)}{2l+1}\frac{K_1 \pm (K_1^2 - 4K_0K_2)^{1/2}}{2K_2}, \tag{10}$$

where

$$K_2 \equiv (l+1)\{(2l+1)\rho_2 - (l+1)k_1(\rho_2 - \rho_1)\},$$

$$K_1 \equiv (2l+1)(k_2 - 3\rho_1\chi^{-(2l+1)})\rho_2$$
$$\quad - (l+1)(\rho_2 - \rho_1)\{(2l+1)k_3k_4 + 2k_1(l-1)\rho_1\},$$

$$K_0 \equiv (\rho_1 - \rho_2)k_3k_2\{2(l-1)\rho_1 + 3\rho_2\} - 3(2l+1)\rho_1\rho_2\chi^{-(2l+1)},$$

$$k_1 \equiv 1 + l\chi^{-(2l+1)}/(l+1),$$

$$k_2 \equiv (2l+1)k_4 - 3k_3\rho_2,$$

$$k_3 \equiv 1 - \chi^{-(2l+1)}(= D_{inv}^{-1}),$$

$$k_4 \equiv \rho_2 + \chi^{-3}(\rho_1 - \rho_2).$$

As is seen from Equation (10), there are generally two solutions for σ^2 in the free-surface case: one corresponds to a mode subject to the displacement of the interface at $r = R_1$ (i.e., an internal gravity mode, which will be called the F mode, where F represents the *free* surface); and the other, to a mode subject to the displacement of the surface at $r = R_2$; i.e., a surface-oscillational mode, which will be called the F' mode. If $\rho_1 < \rho_2$, the F mode is the solution with minus sign before the square-root in Equation (10), while the F' mode plus sign. If $\rho_1 > \rho_2$, they are interchangeable. The growth rate σ of the F and F' modes given by Equation (10) will be denoted by σ_F and $\sigma_{F'}$, respectively.

The last term in the right-hand side of Equation (9) is always positive, and in Equation (10) $K_0K_2 > 0$ and $K_1 > 0$ if $\rho_1 > \rho_2$ and $K_0K_2 < 0$ if $\rho_1 < \rho_2$. These imply that:

(i) If $\rho_2 < \rho_1$,

$$\sigma_R^2 < 0,\ \sigma_F^2 < 0, \text{ and } \sigma_{F'}^2 < 0,$$

i.e., all the modes are stable.

(ii) If $\rho_2 > \rho_{1'}$

$$\sigma_R^2 > 0,\ \sigma_F^2 > 0, \text{ and } \sigma_{F'}^2 < 0,$$

i.e., R and the F modes are unstable, while the F' mode stable.

(iii) If $\rho_2 = \rho_1 (= \rho)$,

$$\sigma_R^2 = \sigma_F^2 = 0,$$

and

$$\sigma_{F'}^2 = -\frac{8\pi G\rho}{3}\frac{l(l-1)}{2l+1},$$

i.e., the R and F modes do not exist and the F' mode reduces to what is known as Kelvin mode.

For $l=0$ and 1, Equations (9) and (10) reduce to the simple forms:

(iv) If $l=0$,

$$\sigma_R^2=\sigma_F^2=\sigma_{F'}^2=0.$$

This is because the mode $l=0$ – an expansion or contraction mode – is prohibited owing to the assumption of incompressibility.

(v) If $l=1$,

$$\sigma_R^2=\frac{4\pi G}{3}(\rho_2-\rho_1)\frac{2(1-\chi^{-3})\rho_2}{2\rho_1+\rho_2-2\chi^{-3}(\rho_1-\rho_2)}, \tag{11}$$

$$\sigma_F^2=-\frac{4\pi G}{3}\frac{2K_1}{3K_2},$$

$$=\frac{4\pi G}{3}(\rho_2-\rho_1)\frac{2(1-\chi^{-3})\{\rho_2+\chi^{-3}(\rho_1-\rho_2)\}}{2\rho_1+\rho_2+\chi^{-3}(\rho_1-\rho_2)}. \tag{12}$$

And, lastly,

$$\sigma_{F'}^2=0$$

implies that the translational mode is prohibited in the surface oscillation.

Now we will examine the effects of the surface boundary conditions at $r=R_2$. We first notice that when $\chi(=R_2/R_1)$ tends to unity; i.e., the outer layer is quite thin, σ_R^2 and σ_F^2 for $l=1$ given by Equations (11) and (12) become identical: they both reduce to

$$\sigma^2=\frac{8\pi G}{3}(\rho_2-\rho_1)(\chi-1). \tag{13}$$

We next consider the case of large χ; i.e., the outer layer is quite thick. Equation (10) can be rewritten as

$$\{(l+1)\Sigma^2-k_2\}(K_3\Sigma^2-K_4)+(K_5\Sigma^2-K_6)\chi^{-(2l+1)}=0, \tag{14}$$

where

$$\Sigma^2=\sigma^2\left\{\frac{4\pi G}{3}\frac{l(l+1)}{2l+1}\right\}^{-1}$$

$$K_3\equiv(2l+1)\rho_2-(l+1)k_3(\rho_2-\rho_1),$$

$$K_4\equiv-(\rho_2-\rho_1)k_3\{2(l-1)\rho_1+3\rho_2\},$$

$$K_5\equiv(l+1)(\rho_2-\rho_1)\times$$
$$\times(3k_3\rho_2+(2l+1)\{(2l-1)\rho_1+3\rho_2\}/(l+1)),$$

$$K_6\equiv-3(2l+1)(\rho_2-\rho_1)k_3\rho_1\rho_2,$$

$$k_2\equiv(2l+1)\{\rho_2+\chi^{-3}(\rho_1-\rho_2)\}-3k_3\rho_2,$$

$$k_3\equiv1-\chi^{-(2l+1)}.$$

When $\chi^{-(2l+1)} \ll 1$; i.e., the outer layer is thick enough compared with the wave-length of the instability, Equation (14) has such solutions that $\Sigma^2 \simeq k_2/(l+1)$ and K_4/K_3–i.e.,

$$\sigma^2 \simeq -\frac{8\pi G\rho}{3}\frac{l(l-1)}{2l+1},$$

where $\rho \equiv \rho_2 + (2l+1)\chi^{-3}(\rho_1 - \rho_2)/2(l-1)$, and

$$\sigma^2 \simeq \sigma_R^2.$$

The former is the surface-oscillation mode (a generalized Kelvin mode), and the latter is the internal-gravity mode and strictly reduces to σ_R^2 also in this case. These imply that whether the surface boundary at $r \equiv R_2$ is free or rigid, the growth rates σ of the modes subject to the displacement of the interface at $r = R_1$ are the same when $R_2/R_1 \simeq 1$ for $l = 1$ or when $(R_2/R_1)^{2l+1} \gg 1$ for all l: In these limit cases, the surface boundary conditions are not essential to these modes.

We will examine the effect of curvature of the geometry; we can expect it to become less important with increasing l. When $l \gg 1$, Equations (9) and (10) reduce to, respectively,

$$\begin{aligned}\sigma_R^2 &\simeq \frac{4\pi G}{3}(\rho_2-\rho_1)l\frac{\rho_1}{\rho_1-\rho_2+2\rho_2/(1-e^{-2k\,\Delta R})}\\ &= \frac{4\pi G}{3}\rho_1 R_1\frac{l}{\rho_1-\rho_2+2\rho_2/(1-e^{-2k\,\Delta R})}\\ &= gk\frac{(\rho_2-\rho_1)(1-e^{-2k\,\Delta R})}{\rho_1+\rho_2+e^{-2k\,\Delta R}(\rho_2-\rho_1)},\end{aligned} \tag{15}$$

$$\sigma_{F'}^2 \simeq -4\pi G\rho_1 l/3 = -gk, \tag{16}$$

and

$$\begin{aligned}\sigma_F^2 &\simeq \frac{4\pi G}{3}(\rho_2-\rho_1)l\frac{\rho_1}{\rho_1+\rho_2-(\rho_2-\rho_1)e^{-2k\,\Delta R}}\\ &= gk\frac{(\rho_2-\rho_1)(1-e^{-2k\,\Delta R})}{\rho_1+\rho_2-e^{-2k\,\Delta R}(\rho_2-\rho_1)},\end{aligned} \tag{17}$$

where $g = 4\pi G\rho_1 R_1/3$, $k = l/R_1$, and $\Delta R = R_2 - R_1$, and we used the relation

$$\begin{aligned}\lim_{l\to\infty}(R_2/R_1)^{-(2l+1)} &= \lim_{l\to\infty}(1+\Delta R/R_1)^{-(2l+1)},\\ &= \lim_{l\to\infty}(1+2k\,\Delta R/2l)^{-2l},\\ &= e^{-2k\,\Delta R}.\end{aligned}$$

We restrict ourselves to the case of $(R_2/R_1)^{-3} \simeq 1$ (i.e., the simple case where the approximation of constant gravity is valid). Equations (15), (16), and (17) are, respectively, identical to Equations (B-11), (B-12), and (B-13) obtained in Appendix

B, which are the dispersion relations in the plane-parallel geometry: The plane-parallel approximation is valid in the case of $l \gg 1$, as expected.

3.2. Viscous case

We next consider the viscous case. Also in this case, we may consider two types of the boundary conditions at $r = R_2$; viz., the free surface and the rigid one. As shown in Paper II, however, in the study of the Earth's core formation, it is sufficient to know the results in the rigid-surface case. Hence, we consider only the rigid-surface case.

The dispersion relation in the viscous case is found to be (its derivation is shown in Appendix A3)

$$\sigma^2 = \frac{4\pi G}{3}(\rho_2 - \rho_1)\frac{l(l+1)}{2l+1}\frac{2(l-1)\rho_1 + 3\rho_2}{(l+1)(\rho_1 - \rho_2) + (2l+1)\rho_2 D_v}, \tag{18}$$

where

$$D_v \equiv (1 - D_1)/(1 - \chi^{-(2l+1)} - D_2),$$

$$\chi \equiv R_2/R_1,$$

$$D_1 \equiv 2(l+1)(l-1)M\{Mq_{21}^{-2}2l(l+2)(A_3B_1 - A_1B_3) + $$
$$+ (2l+1)(A_4B_3 - A_3B_4) - (210)B_1 + A_1\langle 210\rangle\}/D_3,$$

$$D_2 \equiv (2l+1)\{M(A_2B_5 - A_5B_2) - q_{21}^2(A_2B_4 - A_4B_2)\}/D_3,$$

$$D_3 \equiv \chi^{l+2}\{M((221)B_5 - A_5\langle 221\rangle) - q_{21}^2((221)B_4 - A_4\langle 221\rangle)\},$$

$$M \equiv \mu_1/\mu_2 - 1,$$

$$q_{ij} \equiv (\sigma/\nu_i)^{1/2}R_j, \text{ where } (ij) = (11), (21), \text{ and } (22),$$

$$(ijn) \equiv (-q_{ij})^n(R_jR_1)^{-3/2}I_{l+n+1/2}(q_{ij}) \text{ (where } n = 0 \text{ and } 1),$$

$$\langle ijn\rangle \equiv q_{ij}^n(R_j/R_1)^{-3/2}K_{l+n+1/2}(q_{ij}),$$

$$A_1 \equiv (211) - (221)\chi^{l+2},$$

$$A_2 \equiv (210) - (220)\chi^{1-l},$$

$$A_3 \equiv (\chi^{1-l} - \chi^{l+2})(221)/(2l+1) + A_2,$$

$$A_4 \equiv -\rho_1(110)A_1/\rho_2(111) + (210)$$

$$A_5 \equiv 2(l+1)(l-1)A_3 + 2A_1,$$

and B_k's ($k = 1, 2, 3, 4$ and 5) are given by replacing all (ijn) in A_k's with $\langle ijn\rangle$, and $I_{l+n+1/2}$, $K_{l+n+1/2}$ are the modified Bessel functions of the first and second kind. Equation (18) has the same form as Equation (9) in the inviscid case except for D_v and D_{inv} in the denominators in their right-hand sides. We should notice that D_1 and D_2 in D_v contain σ, so that Equation (18) is a transcendental equation as to σ.

In the limit case where both μ_1 and μ_2 tend to zero, Equation (18) reduces to Equation (9) in the inviscid case, since D_1 and $D_2 \ll 1$ in this limit. On the other hand, when μ_1 and μ_2 tend to infinity, σ^2 approaches zero since $D_2 \simeq 1 - \chi^{(-2l+1)}$ and hence D_v becomes infinite in this limit. In order to see the general properties of the dispersion relation (18), however, we must numerically solve it as to σ, which will be done in the next section. Here we will only introduce some physical quantities and rewrite Equation (18) for the later use.

The dispersion relation (18) contains three characteristic time scales: the free fall time τ_f, the viscous-diffusion time τ_{D1} in the inner sphere, and that in the outer layer, τ_{D2}. They are defined by

$$\tau_f \equiv \{4\pi G(\rho_2 - \rho_1)/3\}^{1/2}, \tag{19}$$

$$\tau_{D1} \equiv R_1^2/\nu_1, \tag{20}$$

$$\tau_{D2} \equiv R_1(R_2 - R_1)/\nu_2. \tag{21}$$

Instead of τ_{D2}, in some case we will use the alternative form τ'_{D2} defined by

$$\tau'_{D2} \equiv (R_2 - R_1)^2/\nu_2. \tag{22}$$

The different forms of τ_{D1}, τ_{D2}, and τ'_{D2} reflect the differences in geometry of the regions. We will also use the frequencies defined by the inverses of these characteristic times:

$$\sigma_f \equiv \tau_f^{-1}, \tag{23}$$

$$\sigma_{D1} \equiv \tau_{D1}{}^{-1}, \tag{24}$$

$$\sigma_{D2} \equiv \tau_{D2}^{-1}, \tag{25}$$

$$\sigma'_{D2} \equiv \tau'^{-1}_{D2}, \tag{26}$$

Using the above quantities, we can rewrite the dispersion relation (18) in a non-dimensional form as

$$\left(\frac{\sigma}{\sigma_f}\right)^2 = \frac{l(l+1)}{2l+1}\,\frac{2(l-1)+3y}{(l+1)(1-y)+\{(2l+1)y(1-D_1)/(1-\chi^{-(2l+1)}-D_2)\}}, \tag{27}$$

where $y \equiv \rho_2/\rho_1$. The non-dimensional quantities q_{ij}'s contained in D_1 and D_2 are also expressed as

$$q_{11} = (\sigma/\sigma_f)^{1/2}(\sigma_f/\sigma_{D1})^{1/2},$$

$$q_{21} = (\sigma/\sigma_f)^{1/2}(\sigma_f/\sigma_{D2})^{1/2}(\chi - 1)^{-1/2},$$

or

$$q_{21} = (\sigma/\sigma_f)^{1/2}(\sigma_f/\sigma'_{D2})^{1/2}(\chi - 1)^{-1},$$

and

$$q_{22} = \chi q_{21}.$$

The normalized growth rate σ/σ_f is determined as a function of l by specifying four independent non-dimensional parameters out of ρ_2/ρ_1, R_2/R_1, μ_2/μ_1, σ_{D1}/σ_f, σ_{D2}/σ_f, and σ'_{D2}/σ_f. In the next section, we will select ρ_2/ρ_1, R_2/R_1, σ_{D1}/σ_f, and σ_{D2}/σ_f (or σ'_{D2}/σ_f).

4. General Properties

Here we will show the dispersion relations explicitly to study the properties of the instability in detail.

4.1. Inviscid case

First, we will show the inviscid case. The growth rates σ in a typical case (with $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 2.0$) are shown as a function of l in Figure 1 in the rigid-surface condition, and those in the free-surface condition in Figure 2. They are plotted according to Equations (9) and (10), respectively. In either case σ is an increasing function of l – approximately, $\sigma \propto l^{1/2}$. It discloses that the time-scale of

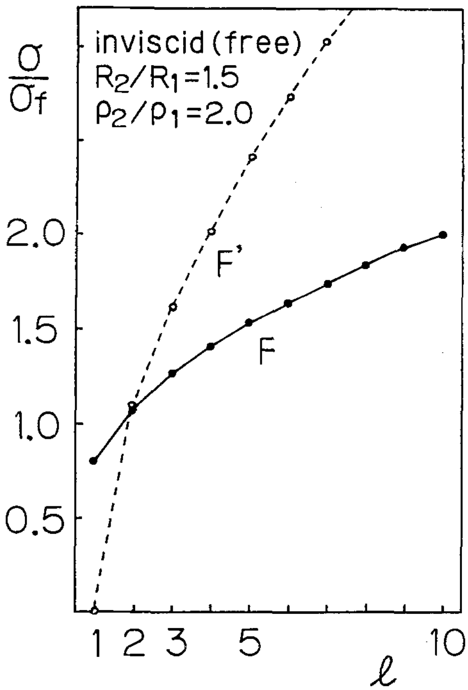

Fig. 1. The dispersion relation in the inviscid case for the rigid surface with $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 2.0$. The growth rate σ is normalized by $\sigma_f \equiv \{4\pi G(\rho_2 - \rho_1)/3\}^{1/2} \approx$ (the free-fall time)$^{-1}$.

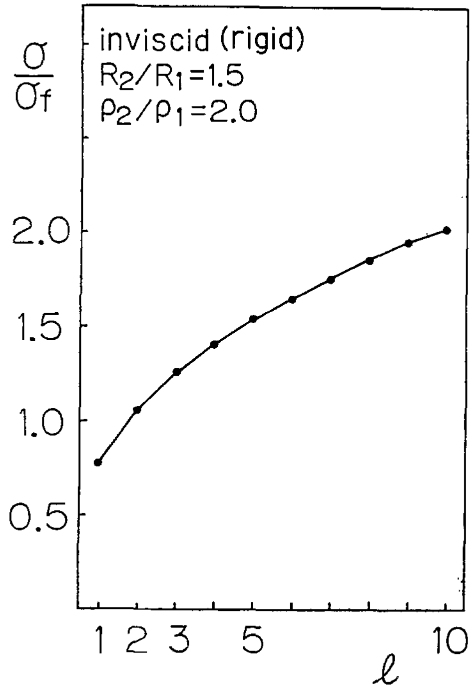

Fig. 2. The dispersion relation in the inviscid case for the free surface with $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 2.0$; the solid curve corresponds to the F mode whereas the dashed curve to the F' mode (Note that $\sigma_{F'}$, is purely imaginary, so this figure shows its absolute value).

instability ($\simeq \sigma^{-1}$) is proportional to the root of the distance ($\propto l^{-1/2}$) between fluid elements to be exchanged, as understood by the following consideration. The time required to exchange the fluid element at $r = R_1 + \delta r$ and $\theta - \theta_0$ for the other at $r = R_1 - \delta r$ and $\theta = \theta_0 + \pi/l$ is approximately given by the time required in the free fall on the slope (see Figure 3),

$$t \simeq 2(\delta r/g')^{1/2} \text{ (i.e., } 2\delta r = -g't^2/2),$$

where g' is the effective gravitational acceleration given by

$$\rho g' \simeq (\rho_2 - \rho_1)g\{2\delta r/(\pi R_1/l)\},$$

in which ρ is the mean density given by $\{\rho_1 + \rho_2)/2$; hence,

$$g' \simeq 16G\rho_1(\rho_2 - \rho_1)\delta r l/3(\rho_1 + \rho_2),$$

where we used the relation $g = 4\pi G\rho_1 R_1/3$. Then we obtain

$$\begin{aligned}\sigma &\simeq t^{-1} \simeq (g'/\delta r)^{1/2}/2 \\ &= \{4G(\rho_2 - \rho_1)/3\}\{l\rho_1/(\rho_1 + \rho_2)\}^{1/2} \propto l^{1/2}.\end{aligned} \tag{28}$$

We should note that Equation (28) is the same as the exact dispersion relation (9) in the case of large l except for the numerical factor π.

In Figures 1 and 2, σ_F in the free-surface condition does not differ so much from σ_R in the rigid-surface one. In the preceding section, it was already shown analytically that σ_F reduces to σ_R when $(R_2/R_1)^{2l+1} \gg 1$ for all l or when $R_2/R_1 \simeq 1$ for $l = 1$. In order to see the difference between σ_F and σ_R, the ratio σ_F/σ_R is plotted as a function of R_2/R_1 in Figure 4. This figure shows the above-mentioned tendency clearly and moreover illustrates that σ_F/σ_R generally increases with decreasing ΔR, and is limited by $(\rho_2/\rho_1)^{1/2}$ when $\Delta R(= R_2 - R_1)$ becomes small, which will be

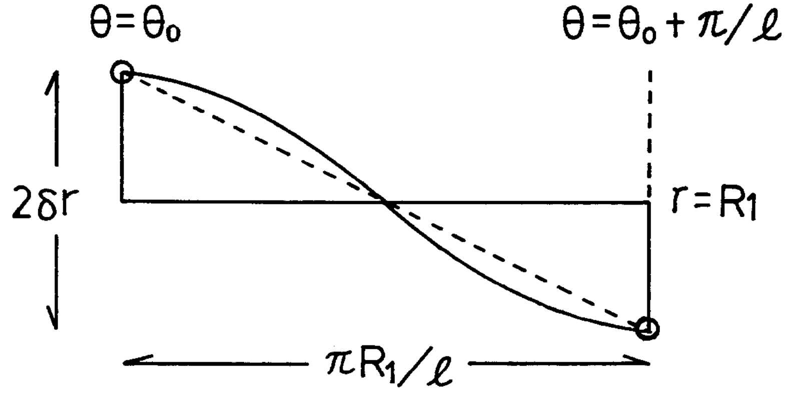

Fig. 3. The schematic illustration for the instability in the inviscid case; the fluid element at $r = R_1 + \delta r$ and $\theta = \theta_0$ and the other at $r = R_1 - \delta r$ and $\theta = \theta + \pi/l$ are interchanged in the free-fall time on the slope.

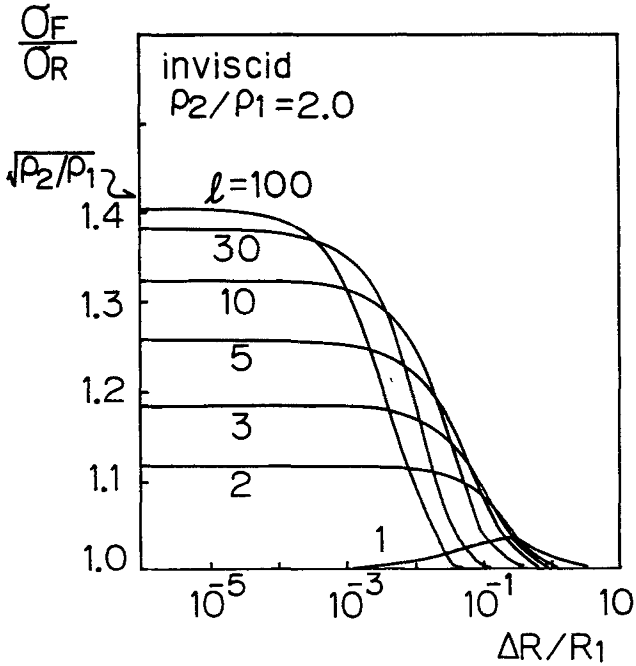

Fig. 4. The ratio σ_F/σ_R as a function of $\Delta R/R_1$ in the inviscid case with $\rho_2/\rho_1 = 2.0$, where $\Delta R = R_2 - R_1$. The ratio is limited by $(\rho_2/\rho_1)^{1/2}$ the limiting value in the plane-parallel case.

important for the discussion in Paper II. These are physically interpreted as follows: Generally, in the inviscid case the difference between σ_F and σ_R corresponds to the difference in δP on the surface at $r = R_2$: On the rigid surface we have non-zero δP, whereas δP is zero on the free surface. When ΔR is large, the deformation at $r = R_1$ can hardly affect the fluid near the surface; i.e., the difference in the surface-boundary conditions is virtually less effective, which is conspicuous when the wavelength ($\propto 1/l$) of perturbation is small compared with ΔR. This is the reason why $\sigma_R/\sigma_R \simeq 1$ when $(R_2/R_1)^{2l+1} \gg 1$. On the other hand, when ΔR is small, the deformation at $r = R_1$ directly affects the motion of the fluid near the surface at $r = R_2$: Then δP at $r = R_2$ on the rigid surface is generally large and increases with the number of nodes of deformation; i.e., l. As mentioned above, σ_F/σ_R increases with the difference between δP at $r = R_2$ on the rigid surface and that on the free surface. Therefore σ_F/σ_R increases with l. When l is large, σ_F/σ_R asymptotically approaches the limiting value $(\rho_2/\rho_1)^{1/2}$ in the plane-parallel geometry (see Equation (B-17) in Appendix B), since the plane-parallel approximation is valid when l is large as shown in the preceding section. We should notice that the mode $l = 1$ exceptionally has no node (i.e., without deformation of the interface and surface) and then the flow in the outer layer tends to be parallel to the boundaries; i.e., the radial component of the flow is very small. In this thin layer case, the radial component of the flow directly corresponds to δP on the rigid surface. Therefore δP

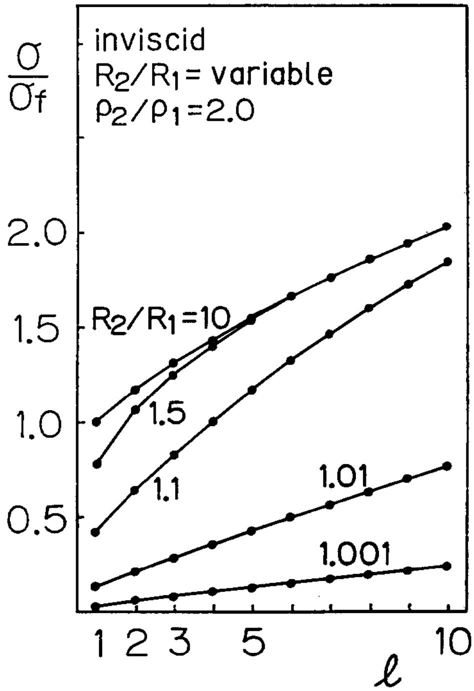

Fig. 5. The dispersion relations in the inviscid case (the rigid surface) with various R_2/R_1.

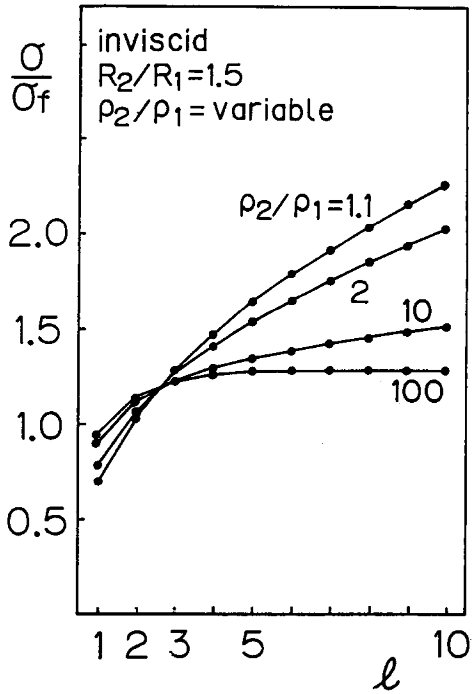

Fig. 6. The dispersion relations in the inviscid case (the rigid surface) with various ρ_2/ρ_1.

is quite small for the mode $l = 1$ even on the rigid surface: $\sigma_F/\sigma_R \simeq 1$ for the mode $l = 1$ when ΔR is small.

The R_2/R_1-dependence of the dispersion relations in the rigid-surface case is shown in Figure 5, and the ρ_2/ρ_1-dependence in Figure 6 (figures which illustrate those dependences in the free-surface case are omitted because σ_F/σ_R is generally not far from unity as mentioned above). Figure 5 shows the instability is largely suppressed when $R_2/R_1 \lesssim 1.1$. This fact will be referred to later. On the other hand, Figure 6 shows that the increase of the ratio ρ_2/ρ_1 flattens the l-dependence of σ.

4.2. Viscous Case

Next we will illustrate the dispersion relation (18) in the viscid case on the rigid-surface condition and examine the properties of the instability. Typical examples (with $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 2.0$) are shown in Figures 7 and 8. In these figures, S_1 and S_2 are non-dimensional parameters which indicate the degree of efficiency of the viscosity and are defined by

$$\begin{cases} S_1 \equiv \sigma_{D1}/\sigma_f = \nu_1/R_1^2\sigma_f, & (29) \\ S_2 \equiv \sigma_{D2}/\sigma_f = \nu_2/R_1\Delta R\sigma_f, & (30) \end{cases}$$

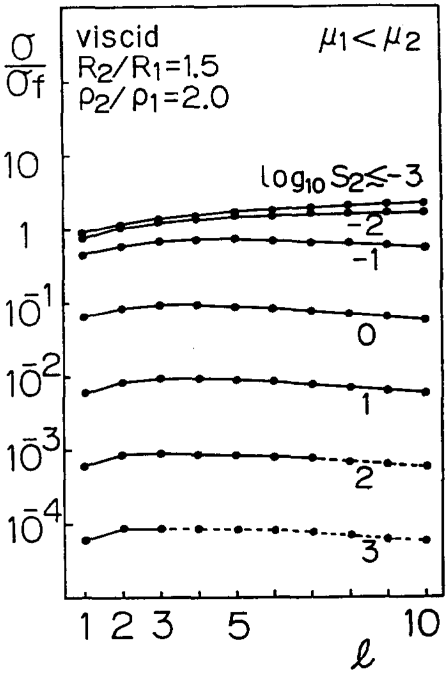

Fig. 7. The dispersion relations in the viscid case (the rigid surface) with $R_2/R_1 = 1.5$, $\rho_2/\rho_1 = 2.0$, and $\mu_1 < \mu_2$: σ is mainly determined by S_2 (see Equation (30) as for S_2).

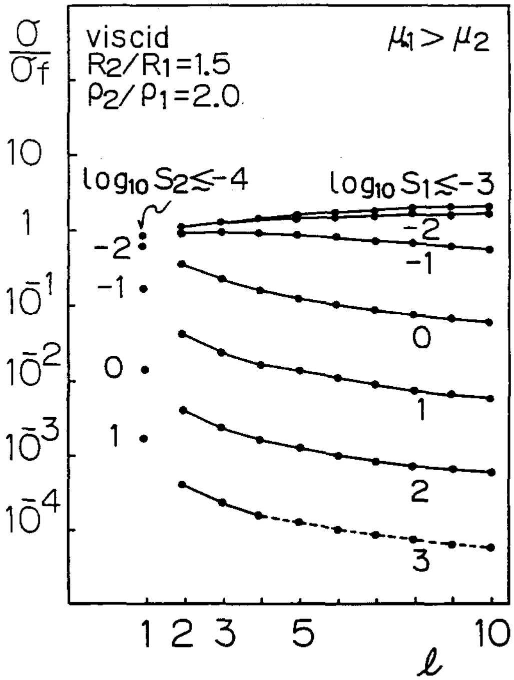

Fig. 8. The dispersion relations in the viscid case (the rigid surface) with $R_2/R_1 = 1.5$, $\rho_2/\rho_1 = 2.0$, and $\mu_1 > \mu^2$; for $l = 1$, σ is mainly determined by S_2, while for $l \geqslant 2$ by S_1 (see Equation (29) as for S_1).

where σ_f, σ_{D1}, and σ_{D2} are given by Equations (23), (24) and (25). It is to be noticed that these two parameters have simple physical meanings: S_1 (or S_2) is the viscous-diffusion time in the inner sphere (or the outer layer) in unit of the free-fall time. Figures 7 and 8 reveal the general properties of the Rayleigh-Taylor instability in this case. First we can see that

(i) For $l = 1$, irrespective of the inequality between μ_1 and μ_2, S_2 mainly determines the growth rate. (31)

(ii) For $l \geqslant 2$, when $\mu_1 > \mu_2$, S_1 determines the rate. (32)

(iii) For $l \geqslant 2$, when $\mu_1 < \mu_2$, S_2 determines the rate. (33)

For $l \geqslant 2$, the instability is governed by a deformation of the regions which has the larger viscosity. For $l = 1$, it is governed by a deformation of the outer layer, since $l = 1$ mode is the translation of the inner sphere without deformation through the outer layer. On one hand, we should notice that when l is large enough the viscous dissipation is inevitably large so long as μ_1 or μ_2 is non-zero, so that σ tends to zero.

Next, we define a non-dimensional parameter S by

$$S \equiv \begin{cases} S_1 \text{ in the case of } \mu_1 > \mu_2 \text{ with } l \geqslant 2, \\ S_2 \text{ in the case of } l = 1 \text{ or the case of } \mu_1 < \mu_2. \end{cases} \tag{34}$$

Then we can see that the time-scale of instability is governed by that in the inviscid case ($\sim$ the free-fall time) in the case of $S \ll 1$:

$$\sigma \simeq \sigma_{\rm inv} \qquad \text{for } S \ll 1, \tag{35}$$

where $\sigma_{\rm inv}$ is the growth rate in the inviscid case given by Equation (9) with the same values of R_2/R_1 and ρ_2/ρ_1. And we can also see that

$$\sigma \simeq (S = 1)/S \ll \sigma_{\rm inv} \quad \text{for} \quad S \gtrsim 1, \tag{36}$$

where $\sigma(S = 1)$ denotes the growth rate for $S = 1$: The instability is governed by the viscous-diffusion time rather than the free-fall time for $S \gtrsim 1$. These properties would be valid also in the free-surface case.

When the outer layer is thin $-\Delta R$ is small, the flow in the outer layer would move parallel to the boundaries. When, say (see Figure. 5) $R_2/R_1 \lesssim 1.1$, a new parameter, given by

$$S_2' \equiv \sigma'_{D2}/\sigma_f = \nu_2/\Delta R^2 \sigma_f, \tag{37}$$

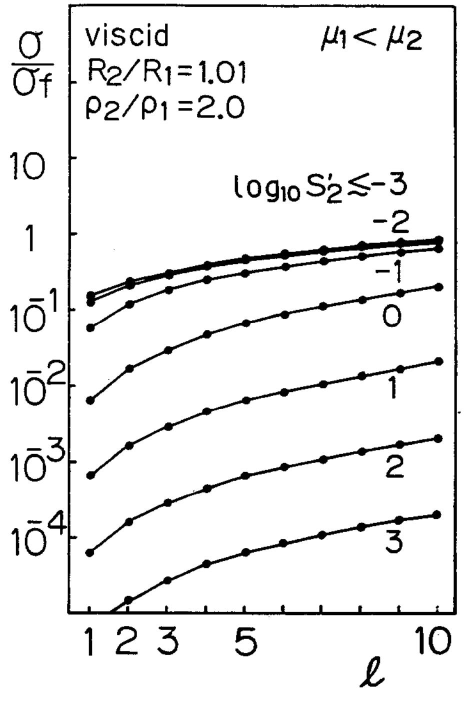

Fig. 9. The dispersion relations in the viscid case (the rigid surface) with $R_2/R_1 = 1.01$, $\rho_2/\rho_1 = 2.0$, and $\mu_1 > \mu_2$; in the thin layer case ($R_2/R_1 \lesssim 1.1$), the growth rate is characterized by S_2' (see Equation (37) as for S_2'). Note that in this case $S_2 = 0.01\ S_2'$.

is more adequate than S_2 as the parameter indicating the degree of efficiency of the viscosity in the outer layer. The parameter S_2' is the square of thickness of the viscous boundary layer in the outer layer in unit of ΔR. The growth rate parameterized by S_2' are illustrated in Figure 9 for $R_2/R_1 = 1.01$ (compare with Figure 7), which shows that it is more suitable to replace all of "S_2" in the statements (31) to (35) by "S_2'" for $R_2/R_1 \lesssim 1.1$.

Finally we will point out that the behaviors of the dispersion relations are not classified in terms of the kinematic viscosity ν but the viscosity μ. In Figures 7 and 8, $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 2.0$, which lead to

$$\mu_2/\mu_1 = 2\nu_2/\nu_1 = S_2/S_1, \tag{38}$$

i.e., a value of μ_2/μ_1 does not so much differ from ν_2/ν_1; only from these figures we cannot see well which quantity of μ and ν classifies the behaviors of the dispersion relations. To clarify it, we examine the case of $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 200$ (see Figure 10). In this case we have

$$\mu_2/\mu_1 = 200\nu_2/\nu_1 = 100S_2/S_1. \tag{39}$$

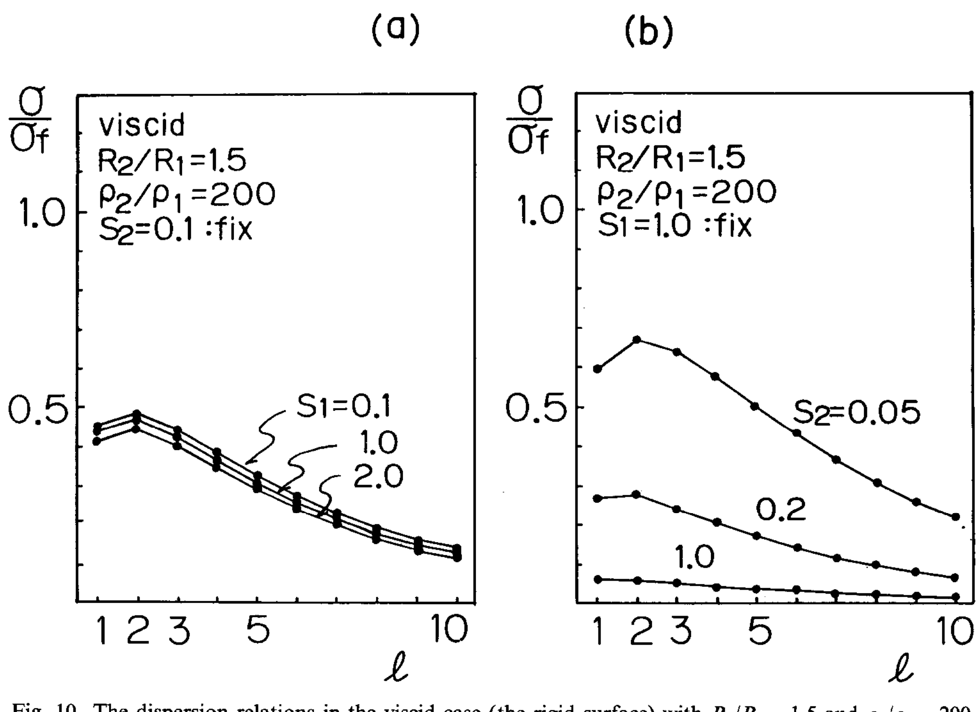

Fig. 10. The dispersion relations in the viscid case (the rigid surface) with $R_2/R_1 = 1.5$ and $\rho_2/\rho_1 = 200$ (i.e., $\mu_2/\mu_1 = 100\, S_2/S_1$ and $\nu_2/\nu_1 = 0.5\, S_2/S_1$). Figure (a) is for fixed S_1 and Figure (b) for fixed S_2. For the values of S_1 and S_2 in these figures, two inequalities $\mu_2 > \mu_1$ and $\nu_1 > \mu_2$ are satisfied at the same time. The dispersion relations are obviously determined by S_2 and almost independent of S_1.

If $0.01 < S_2/S_1 < 2.0$, we have $\mu_2 > \mu_1$ and $\nu_2 < \nu_1$ at the same time. Then if the behaviors are classified by the viscosity μ, σ should be determined by S_2 while if by the kinematic viscosity ν, σ for $l \geqslant 2$ should be determined by S_1. Figure 9 obviously shows that σ is determined by S_2. Hence the behaviour is classified by the viscosity μ.

5. The Validity of the Neglect of the Coriolis Force

So far we have studied the instability in a nonrotating system. However, the proto-Earth was rotating. Hence, the Coriolis force influences the fluid motion. As shown in Paper II, in the Earth's core formation problem the proto-Earth corresponds to the two-layer model with $S_1 \gg 1$ and S_2, $S_2' \ll 1$. In this case, from the results in Section 4, the properties of instability are as follows: the $l = 1$ mode of instability is governed by the free-fall time of the outer layer while the $l \geq 2$ mode by the viscous diffusion time of the inner sphere. Hence the instability occurs through the $l = 1$ mode with the free-fall time (about 10 h, see Paper II). Here we will demonstrate that, in the case with $S_1 \gg 1$ and S_2, $S_2' \ll 1$, The conclusion derived from the results in the nonrotating system remains valid still in the rotating system.

The perturbed Navier-Stokes equations in the rotating (angular velocity $\mathbf{\Omega}$) frame is

$$\mathrm{d}\mathbf{u}/\mathrm{d}t + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla(\delta\Phi + \delta P/\rho) + \nu\nabla^2\mathbf{u}. \tag{40}$$

In the spherical coordinates (r, θ, ϕ),

$$\begin{cases} \mathrm{d}u_r/\mathrm{d}t + 2\Omega_\theta u_\phi = -\partial(\delta\Phi + \delta P/\rho)/\partial r + \nu(\nabla^2 u)_r, & (41) \\ \mathrm{d}u_\theta/\mathrm{d}t - 2\Omega_r u_\phi = -\partial(\delta\Phi + \delta P/\rho)/r\partial\theta + \nu(\nabla^2 u)_\theta, & (42) \\ \mathrm{d}u_\phi/\mathrm{d}t + 2(\Omega_r u_\theta - \Omega_\theta u_r) = -\partial(\delta\Phi + \delta P/\rho)/r \sin\theta\partial\phi + \nu(\nabla^2 u)_\phi, & (43) \end{cases}$$

where $\mathbf{\Omega} = (\Omega_r, \Omega_\theta, \Omega_\phi) = (\Omega\cos\theta, -\Omega\sin\theta, 0)$, that is, the rotating axis is in the direction of $\theta = 0$.

First we will consider the $l = 1$ mode. Since the $l = 1$ mode is governed by the deformation of the outer layer, we will examine the effect of the Coriolis force on the fluid motion in the outer layer. The outer layer in the proto-Earth is thin enough compared with its radius. Since, in the core formation problem, only the long wave mode is of interest (Paper II), we can assume that

$$|u_r| \ll |u_\theta|, |u_\phi|, \tag{44}$$

which enables us to replace the term $(\Omega_r u_\theta - \Omega_\theta u_r)$ in Equation (43) with $\Omega_r u_\theta$. On the other hand, in the radial direction, the gravity is much larger than the Coriolis force: i.e.,

$$\frac{|(\mathbf{\Omega} \times \mathbf{u})_r|}{|\partial(\delta\Phi)/\partial r|} \sim \frac{\Omega\sigma R}{(\delta\rho/\rho)g} \sim \frac{\Omega\sigma R}{g} \sim \frac{3\Omega\sigma R}{4\pi G\rho R} \sim \frac{\Omega}{\sigma_f} \sim 0.07, \tag{45}$$

which enables us to neglect (the radial component of) the Coriolis force in Equation

(41). Therefore, Equations (41), (42), and (43) are approximately reduced to

$$\mathbf{du}/\mathrm{d}t + 2\mathbf{f} \times \mathbf{u} = -\nabla(\delta\Phi + \delta P/\rho) + \nabla^2\mathbf{u}, \tag{46}$$

where $\mathbf{f}$ is the Coriolis parameter defined by $\mathbf{f} = (\Omega \cos \theta, 0, 0)$: The Coriolis force influences the fluid motion only in the $(\theta - \phi)$-plane.

Since $\mathbf{u}$ is solenoidal ($\mathrm{div}\,\mathbf{u} = 0$), $\mathbf{u}$ is decomposed into the poloidal (subscript "P") and toroidal (subscript "T") components

$$\mathbf{u} = \mathbf{u}_P + \mathbf{u}_T, \tag{47}$$

where

$$(\mathbf{u}_T)_r = 0 \text{ and } (\mathrm{rot}\,\mathbf{u}_P)_r = 0. \tag{48}$$

Since the Coriolis parameter $\mathbf{f}$ has only the radial component, $(\mathbf{f} \times \mathbf{u})_r$ vanishes; so the solenoidal component of $\mathbf{f} \times \mathbf{u}$ is toroidal. The gradient of gravitational potential grad $\delta\Phi$ is poloidal, since $\delta\Phi$ satisfies Laplace equation. Thus the poloidal and toroidal components of Equation (46) are

$$\begin{cases} \mathbf{du}_P/\mathrm{d}t = -\nabla(\delta\Phi + \delta P_P/\rho) + (\nu\nabla^2\mathbf{u})_P. & (49) \\ \mathbf{du}_T/\mathrm{d}t + 2\mathbf{f} \times \mathbf{u} = -\nabla(\delta P_T/\rho) + (\nu\nabla^2\mathbf{u})_T. & (50) \end{cases}$$

We must notice that Equation (49) for $\mathbf{u}_P$ is the same form as Equation (1) in the nonrotating system except the viscous dissipation term in the right-hand side where $\mathbf{u}_P$ is viscously coupled with $\mathbf{u}_T$. The toroidal components $\mathbf{u}_T$ is generated by the Coriolis force as shown in Equation (50). Therefore, the poloidal component $\mathbf{u}_P$ generated by the release of gravitational energy is directly connected to the instability, while $\mathbf{u}_T$ generated by $\mathbf{u}_P$ through the Coriolis force is connected to the instability only through the viscous coupling with $\mathbf{u}_P$. The time-scale of the instability in the nonrotating system is about 10 h, which is comparable with or shorter than that of the Coriolis force. In this case, we can reasonably say that, in the non-viscous case when the viscous coupling of $\mathbf{u}_P$ with $\mathbf{u}_T$ is weak, $\mathbf{u}_T$ generated by the Coriolis force hardly affects the instability, while in the considerably viscous case $\mathbf{u}_T$ may depress the instability through the viscous coupling with $\mathbf{u}_P$. Indeed, Chandrasekhar (1961) shows that, in the plane-parallel inviscid case,

$$\sigma_{\mathrm{rot}}^2 = -2\Omega^2 + (4\Omega^4 + \sigma_0^4)^{1/2}, \tag{51}$$

where σ_{rot} is the rate of instability in the rotating system (angular momentum Ω) and σ_0 is that in the nonrotating system; σ_{rot} differs from σ_0 only by a factor as long as σ_0 is comparable with or larger than Ω. As mentioned before, the $l = 1$ mode is nonviscous (governed by the free-fall time of the outer layer). Therefore, we can conclude that the $l = 1$ mode in the rotating system differes only by a factor from that in the nonrotating system: i.e.,

$$\sigma_{\mathrm{rot}}(l = 1) \approx \sigma_0(l = 1) \tag{52}$$

Next we consider $l \geqslant 2$ mode. Since the $l \geqslant 2$ mode is considerably viscous (see Paper II), $\mathbf{u}_T$ (i.e., the Coriolis force) may depress the gravitational instability as mentioned above. So, we reasonably expect that

$$\sigma_{\text{rot}}(l \geqslant 2) \lesssim \sigma_0(l \geqslant 2) \text{ or } \ll \sigma_0(l \geqslant 2). \tag{53}$$

Since $\sigma_0(l = 1) \gg \sigma_0(l \geqslant 2)$ (Paper II), we therefore obtain from Equations (52) and (53) that

$$\sigma_{\text{rot}}(l = 1) \gg \sigma_{\text{rot}}(l \geqslant 2). \tag{54}$$

Thus, still valid in the rotating system is the conclusion derived from the nonrotating system that the instability occurs through the $l = 1$ mode with the time scale about the free-fall time (but the time scale in the rotating system may become larger by a factor than that in the rotating system).

6. Conclusions

We summarize the properties of the Rayleigh-Taylor instability in the self-gravitating two-layer fluid sphere.

The inviscid case:

(1) The growth rate σ is approximately proportional to $l^{1/2}$.

(2) The ratio σ_F/σ_R is nearly of unity when $(R_2/R_1)^{2l+1} \gg 1$ for all l or when $R_2/R_1 \simeq 1$ for $l = 1$.

(3) The ratio σ_F/σ_R increases with decreasing ΔR for $l \geqslant 2$ and is limited by $(\rho_2/\rho_1)^{1/2}$.

(4) When $R_2 R_1 \lesssim 1.1$, the existence of the surface at $r = R_2$ has a large effect to suppress the growth of the instability.

The viscous case:

(1) The degree of efficiency of the viscosity is indicated by the non-dimensional parameters $S_1 = \nu_1/R_1^2\sigma_f$ in the inner sphere, $S_2 = \nu_2(R_1 \Delta R \sigma_f (R_2/R_1 \gtrsim 1.1)$ and $S_2' = \nu_2/\Delta R^2 \sigma_f$ $(R_2/R_1 \lesssim 1.1)$ in the outer layer; they physically mean the viscous-diffusion time in each region in unit of the free-fall time.

(2) For $l = 1$, S_2 mainly determines the growth rate σ, since the mode $l = 1$ is the translation of the inner sphere without deformation through the outer layer. For the mode $l \geqslant 2, S_1$ does when $\mu_1 > \mu_2$ and S_2 does when $\mu_1 < \mu_2$: the instability is governed by deformation of the more viscous region.

(3) The time scale of instability is governed by the free-fall time for $S \ll 1$; i.e.,

$$\sigma \simeq \sigma_{\text{inv}} \quad \text{for } S \ll 1,$$

and governed by the viscous-diffusion time for $S \gtrsim 1$; i.e.,

$$\sigma \simeq \sigma(S = 1)/S \ll \sigma_{\text{inv}} \text{ for } S \gtrsim 1,$$

where $\sigma(S = 1)$ means the growth rate for $S = 1$ and

$$S \equiv \begin{cases} S_1 \text{ in the case of } \mu_1 > \mu_2 \text{ with } l \geqslant 2, \\ S_2 \text{ in the case of } l = 1 \text{ or the case of } \mu_1 < \mu_2. \end{cases}$$

(4) When $R_2/R_1 \lessapprox 1.1$, in the above statements (2) and (3), it is more adequate to replace S_2 by S_2'.

The above results in the viscous case in the rigid-surface condition, as well as the results in the inviscid case, enable us to investigate the initiation of the Earth's core formation, which is done in Paper II; in the proto-Earth, we have

$$\mu_1 \gg \mu_2,$$

$$S_2 \text{ and } S_2' \ll 1,$$

and

$$S_1 \gg 1,$$

which lead to the conclusion that the Earth core formed through the instantaneous ($\sigma \simeq \sigma_{\text{inv}}$) translation ($l = 1$ mode) of the innermost undifferentiated solid core (see Paper II for details). Further, as shown in Section 5, the above conclusion does not change, even if the rotation of the proto-Earth; i.e., the Coriolis force is considered.

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Appendix A. Derivations of Dispersion Relations

A1. Perturbations

In the configuration mentioned in Section 2, the hydrostatic equilibrium state is described in terms of the gravitational potential Φ as follows:

$$\Phi = \begin{cases} 2\pi G\{\rho_1(r^2 - 3R_1^2) + 3\rho_2(R_1^2 - R_2^2)\}/3 & \text{for } r < R_1, \\ 2\pi G\{\rho_2(r^2 - 3R_2^2) + 2(\rho_2 - \rho_1)R_1^3/r\}/3 & \text{for } R_1 < r < R_2, \\ -4\pi G\{\rho_1 R_1^3 - \rho_2(R_1^3 - R_2^3)\}/3r & \text{for } r > R_2. \end{cases} \tag{A-1}$$

The pressure is given by

$$P = \rho\Phi + \text{const.} \tag{A-2}$$

The perturbation of the gravitational potential $\delta\Phi$ is obtained from Equation (3) as

$$\delta\Phi = \begin{cases} 4\pi G\{(\rho_2 - \rho_1)R_1(r/R_1)^l\delta r - \rho_2 R_2(r/R_2)^l\delta r'\}/(2l+1) \\ \qquad\qquad\qquad\qquad\qquad \text{for region I,} \\ 4\pi G\{(\rho_2 - \rho_1)R_1(r/R_1)^{-(l+1)}\delta r - \rho_2 R_2(r/R_2)^l\delta r'\}/(2l+1) \\ \qquad\qquad\qquad\qquad\qquad \text{for region II,} \end{cases} \tag{A-3}$$

where region I denotes the region of $r < R_1 + \delta r$, and region II that of $R_1 + \delta r < r < R_2 + \delta r'$. The divergence of the equation of motion (1) gives

$$\nabla^2\phi = 0,$$

then we can put

$$\phi = \begin{cases} (l+1)Q_0 R_1(r/R_1)^l\delta r & \text{for region I,} \\ \{(l+1)Q_1 R_1(r/R_1)^l - lQ_2 R_1(r/R_1)^{-(l+1)}\}\delta r & \\ & \text{for region II} \end{cases} \tag{A-4}$$

where Q_0, Q_1, and Q_2 are constants to be determined by the boundary conditions. In Equation (A-4) we use the fact that $\delta r' \propto \delta r$ in the linear perturbation theory. From the expressions for ϕ given by Equation (A-4), grad ϕ is found to be a poloidal vector field which is expressed in terms of the following defining scalar function $G(r)$:

$$G(r) = \begin{cases} \varepsilon Q_0(r/R_1)^{l+1}R_1^2 & \text{for region I} \\ \varepsilon\{Q_1(r/R_1)^{l+1} + Q_2(r/R_1)^{-l}\}R_1^2 & \text{for region II,} \end{cases} \tag{A-5}$$

(cf. Chandrasekhar, 1961; Appendix III).

Since div $\mathbf{u} = 0$, the equation of motion (1) is transformed into

$$\partial\mathbf{u}/\partial t = -\text{grad}\,\phi - \nu\,\text{rot}^2\,\mathbf{u}, \tag{A-6}$$

from which $\mathbf{u}$ and $\text{rot}^2\,\mathbf{u}$ are also found to be poloidal vector fields. The defining scalar function of u is denoted by $\varepsilon U(r)$ hereafter. That of $\text{rot}^2\,\mathbf{u}$ is, hence, $\varepsilon\{d^2U/dr^2 - l(l+1)U/r^2\}$. We will solve Equation (A-6) first in the inviscid case ($\nu = 0$) and next in the viscous case to obtain the dispersion relations.

A2. The Inviscid Case

Equating the defining scalar functions of each side of Equation (A-6), we obtain

$$\sigma U = \begin{cases} -(r/R_1)^{l+1}Q_0R_1^2, & \text{for region I;} \\ -\{(r/R_1)^{l+1}Q_1 + (r/R_1)^{-l}Q_2\}R_1^2, & \text{for region II.} \end{cases} \tag{A-7}$$

The boundary conditions are given in Section 2 by (1′), (2′), and ($3' - r$) for rigid surface or ($3' - f$) for free surface. From Equations (A-7) and the relation

$u_r(r) = l(l+1)U(r)\delta r/r^2$, boundary condition (1′) requires that

$$u_r(R_1 + \delta r) = -l(l+1)Q_0\delta r/\sigma = -l(l+1)(Q_1 + Q_2)\delta r/\sigma, \tag{A-8}$$

$$= \sigma\delta r \quad (\because \delta r = \varepsilon e^{\sigma t} Y_l^m). \tag{A-9}$$

From Equations (A-1) to (A-7), the condition (2′) requires that

$$(l+1)\rho_1 Q_0 - (l+1)\rho_2 Q_1 + l\rho_2 Q_2 =$$
$$= -4\pi G\{2(l-1)\rho_1 + 3\rho_2 - 3\rho_2(\delta r'/\delta r)\}/3(2l+1). \tag{A-10}$$

From Equation (A-7), the condition $(3' - r)$ requires that

$$u_r(R_2) = -l(l+1)(Q_1\chi^{l+1} + Q_2\chi^{-l})\delta r/\sigma = 0. \tag{A-11}$$

On the other hand, the condition $(3' - f)$ requires that

$$-4\pi G(\{2(l-1)\rho_2 R_2 - (2l+1)(\rho_2 - \rho_1)R_1^3/R_2^3\}\delta r' +$$
$$+ 3(\rho_2 - \rho_1)\chi^{-(2l+1)}\delta r)/3(2l+1) + \{(l+1)Q_1 - lQ_2\chi^{-(2l+1)}\}\delta r = 0, \tag{A-12}$$

and

$$\sigma\delta r' = -l(l+1)(Q_1\chi^{l+1} + Q_2\chi^{-l})\delta r/\sigma. \tag{A-13}$$

From Equations (A-8) to (A-11), we obtain the dispersion relation for the rigid surface given by Equation (9) in Section 3:

$$\sigma^2 = \frac{4\pi G}{3}(\rho_2 - \rho_1)\frac{l(l+1)}{2l+1}\frac{2(l-1)\rho_1 + 3\rho_2}{(l+1)(\rho_1 - \rho_2) + (2l+1)\rho_2 D_{\text{inv}}}, \tag{A-14}$$

where $\chi = R_2/R_1$ and $D_{\text{inv}} = (1 - \chi^{-(2l+1)})^{-1}$.

From Equations (A-8) to (A-10), (A-12), and (A-13), the dispersion relation for the free surface given by Equation (10) in Section 3: i.e.,

$$\sigma^2 = -\frac{4\pi G}{3}\frac{l(l+1)}{2l+1}\frac{K_1 \pm (K_1^2 - 4K_0K_2)^{1/2}}{2K_2}. \tag{A-15}$$

A3. THE VISCOUS CASE

Equating the defining scalar functions of each side of Equation (A-6), we obtain

$$\sigma U(r) = \begin{cases} -Q_0R_1^2(r/R_1)^{l+1} + \nu_1\{d^2U/dr^2 - l(l+1)U/r^2\} & \text{for region I,} \\ -Q_1R_1^2(r/R_1)^{l+1} + Q_2R_1^2(r/R_1)^{-l} & \\ +\nu_2\{d^2U/dr^2 - l(l+1)U/r^2\} & \text{for region II.} \end{cases} \tag{A-16}$$

We can readily solve Equation (A-16) analytically and obtain

$$U(r) = \begin{cases} F_0r^{1/2}I_{l+1/2}(q_1r) - Q_0R_1^2(r/R_1)^{l+1}/\sigma & \text{for region I,} \\ F_1r^{1/2}I_{l+1/2}(q_2r) + F_2r^{1/2}K_{l+1/2}(q_2r) & \\ -\{Q_1(r/R_1)^{l+1} + Q_2(r/R_1)^{-l}\}R_1^2/\sigma & \text{for region II,} \end{cases} \tag{A-17}$$

where F_0, F_1, and F_2 are constants to be determined by the boundary conditions, $I_{l+1/2}$ and $K_{k+1/2}$ are the modified Bessel functions of the first and the second kind, respectively; and q_1 and q_2 are defined by

$$q_1 = (\sigma/\nu_1)^{1/2} \quad \text{and} \quad q_2 = (\sigma/\nu_2)^{1/2}.$$

In terms of $U(r)$, we obtain three components of the velocity as

$$u_r = l(l+1)U(r)\delta r/r^2 = \\ = \begin{cases} l(l+1)\{F_0 r^{-3/2} I_{l+1/2}(q_1 r) - (r/R_1)^{l-1} Q_0/\sigma\}\delta r & \text{for region I,} \\ l(l+1)[F_1 r^{-3/2} I_{l+1/2}(q_2 r) + F_2 r^{-3/2} K_{l+1/2}(q_2 r) \\ \quad -\{(r/R_1)^{l-1} Q_1 + (r/R_1)^{-(l+1)} Q_2\}/\sigma]\delta r & \text{for region II,} \end{cases} \tag{A-18}$$

$$u_\theta = (\mathrm{d}U/\mathrm{d}r)(\partial\delta r/\partial\theta)/r = \\ = \begin{cases} [F_0\{(l+1) r^{-3/2} I_{l+1/2}(q_1 r) + q_1 r^{-1/2} I_{l+3/2}(q_1 r)\} \\ \quad -(l+1)(r/R_1)^{l-1} Q_0/\sigma]\partial(\delta r)/\partial\theta & \text{for region I,} \\ [F_1\{(l+1) r^{-3/2} I_{l+1/2}(q_2 r) + q_2 r^{-1/2} I_{l+3/2}(q_2 r)\} \\ \quad + F_2\{(l+1) r^{-3/2} K_{l+1/2}(q_2 r) - q_2 r^{-1/2} K_{l+3/2}(q_2 r)\} \\ \quad -\{(l+1)(r/R_1)^{l-1} Q_1 - l(r/R_1)^{-(l+2)} Q_2\}/\sigma]\partial(\delta r)/\partial\theta & \text{for region II,} \end{cases} \tag{A-19}$$

and

$$u_\phi = (\mathrm{d}U/\mathrm{d}r)(\partial\delta r/\partial\phi)/r\sin\theta = \\ = u_\theta(\partial\delta r/\partial\phi)/\{(\partial\delta r/\partial\theta)\sin\theta\}. \tag{A-20}$$

From Equations (A-18) to (A-20), we can also obtain

$$P_{rr} = -(P+\delta P) + 2\mu\partial u_r/\partial r \\ = \begin{cases} -(P+\delta P)_{\rm I} + 2\mu_1 l(l+1)[F_0 q_1^{1/2}\{(l-1) r^{-5/2} I_{l+1/2}(q_1 r) \\ \quad + q_1 r^{-3/2} I_{l+3/2}(q_1 r)\} - (l-1)(r/R_1)^{l-2} Q_0/\sigma R_1]\delta r & \text{for region I,} \\ -(P+\delta P)_{\rm II} + 2\mu_2 l(l+1)[F_1 q_2^{1/2}\{(l-1) r^{-5/2} I_{l+1/2}(q_2 r) + q_2 r^{-3/2} I_{l+3/2}(q_2 r)\} \\ \quad + F_2 q_2^{1/2}\{(l-1)_r^{-5/2} K_{l+1/2}(q_2 r) - q_2 r^{-3/2} K_{l+3/2}(q_2 r)\} \\ \quad -\{(l-1)(r/R_1)^{l-2} Q_1 - (l+2)(r/R_1)^{-(l+3)} Q_2\}/\sigma R_1]\delta r & \text{for region II,} \end{cases} \tag{A-21}$$

$$P_{r\theta} = \mu(\partial u_r/\partial\theta - u_\theta + r\partial u_\theta/\partial r)/r \\ = \begin{cases} \mu_1[F_0\{(2(l+1)(l-1) r^{-5/2} + q_1^2 r^{-1/2}) I_{l+1/2}(q_1 r) - 2q_1 r^{-3/2} I_{l+3/2}(q_1 r)\} \\ \quad -2(l+1)(l-1)(r/R_1)^{l-2} Q_0/\sigma R_1]\partial(\delta r)/\partial\theta & \text{for region I,} \\ \mu_2[F_1(2(l+1)(l-1) r^{-5/2} + q_2^2 r^{-1/2}) I_{l+1/2}(q_2 r) \\ \quad + F_2(2(l+1)(l-1) r^{-5/2} + q_2^2 r^{-1/2}) K_{l+1/2}(q_2 r) \\ \quad -2q_2 r^{-3/2}\{F_1 I_{l+3/2}(q_2 r) - F_2 K_{l+3/2}(q_2 r)\} \\ \quad -2\{(l+1)(l-1)(r/R_1)^{l-2} Q_1 \\ \quad + l(l+2)(r/R_1)^{-(l+3)} Q_2\}/\sigma R_1]\partial(\delta r)/\partial\theta & \text{for region II,} \end{cases} \tag{A-22}$$

The boundary conditions in this viscid case are given in Section 2 by (1), (2), (3), (4), and $(5-r)$. From Equation (A-18), the condition (1) requires that

$$\sigma = l(l+1)(F_0(110) - Q_0/\sigma) = \\ = l(l+1)\{F_1(210) + F_2\langle 210\rangle - (Q_1 + Q_2)/\sigma\}, \tag{A-23}$$

where

$$(ijn) \equiv (-q_{ij})^n (R_j/R_1)^{-3/2} I_{l+n+1/2}(q_{ij}),$$
$$\langle ijn\rangle \equiv q_{ij}^n (R_j/R_1)^{-3/2} K_{l+n+1/2}(q_{ij}),$$
$$q_{ij} \equiv q_i R_j = (\sigma/\nu_i)^{1/2} R_j.$$

From Equations (A-19), (A-20), the condition (2) requires that

$$F_0\{(l+1)(110) - (111)\} - (l+1)Q_0/\sigma = \\ = F_1\{(l+1)(210) - (211)\} + F_2\{(l+1)\langle 210\rangle - \langle 211\rangle\} - \\ - \{(l+1)Q_1 - lQ_2\}/\sigma. \tag{A-24}$$

From Equation (A-21), the condition (3) requires that

$$4\pi G(\rho_1 - \rho_2)\{2(l-1)\rho_1 + 3\rho_2\}/3(2l+1) = \\ = (l+1)(\rho_1 Q_0 - \rho_2 Q_1) + l\rho_2 Q_2 - \\ - 2l(l+1)R_1^{-2}[\mu_1\{F_0(l-1)(110) - F_0(111) - (l-1)Q_0/\sigma\} - \\ - \mu_2\{F_1(l-1)(210) - F_1(211) - (l-1)Q_1/\sigma\} - \\ - \mu_2\{F_2(l-1)\langle 210\rangle - F_2\langle 211\rangle + (l+2)Q_2/\sigma\}]. \tag{A-25}$$

From Equation (A-22), the condition (4) requires that

$$\mu_1 F_0[\{2(l-1)(l+1) + q_{11}^2\}(110) + 2(111)] - 2(l-1)(l+1)\mu_1 Q_0/\sigma = \\ = \mu_2 F_1[\{2(l-1)(l+1) + q_{21}^2\}(210) + 2(211)] - 2(l-1)(l+1)\mu_2 Q_1/\sigma + \\ + \mu_2 F_2[\{2(l-1)(l+1) + q_{21}^2\}\langle 210\rangle + 2\langle 211\rangle] - 2l(l+2)\mu_1 Q_2/\sigma. \tag{A-26}$$

From Equations (A-18) to (A-20), the condition $(5-r)$ requires that

$$F_1(220) + F_2\langle 220\rangle - (Q_1\chi^{l-1} + Q_2\chi^{-(l+2)})/\sigma = 0, \tag{A-27}$$

and

$$F_1\{(l+1)(220) - (221)\} + F_2\{(l+1)\langle 220\rangle - \langle 221\rangle\} \\ - \{(l+1)Q_1\chi^{l-1} - lQ_2\chi^{-(l+2)}\}/\sigma = 0, \tag{A-28}$$

From Equations (A-23) to (A-28), after somewhat lengthy reductions, we obtain the dispersion relation in the viscid case for the rigid surface given by Equation (15) in Section 3; i.e.,

$$\sigma^2 = \frac{4\pi G}{3}(\rho_2 - \rho_1)\frac{l(l+1)}{2l+1}\frac{2(l-1)\rho_1 + 3\rho_2}{(l+1)(\rho_1 - \rho_2) + (2l+1)\rho_2 D_v}. \tag{A-29}$$

Appendix B. The Plane-Parallel Case (Inviscid)

We consider here the Rayleigh-Taylor instability of the incompressible inviscid fluid in the plane-parallel configuration with constant gravity.

A fluid layer of thickness Δz with the constant density ρ_2 is overlaid on a half-infinite fluid with the constant density ρ_1, and their interface is at $z = 0$. The direction of the gravity is vertical, i.e., $\mathbf{g} = (0, 0, -g)$. Then an unperturbed state is described in terms of the pressure: i.e.,

$$P = \begin{cases} -\rho_1 z + P_0 & \text{for} \quad z < 0, \\ -\rho_2 z + P_0 & \text{for} \quad z > 0, \end{cases} \tag{B-1}$$

where P_0 is a constant.

Let the perturbation of the interface δz be

$$\delta z = \varepsilon_0 e^{i\mathbf{kr} + \sigma t}, \tag{B-2}$$

where $\mathbf{k} = (k_x, k_y, 0)$ and $\mathbf{r} = (x, y, z)$. The perturbation equations are

$$\begin{cases} \rho \partial \mathbf{u}/\partial t = -\nabla \delta P - \delta \rho \mathbf{g}, & \text{(B-3)} \\ \operatorname{div} \mathbf{u} = 0, & \text{(B-4)} \end{cases}$$

where δP and $\mathbf{u}(= (u, v, w))$ are, respectively, the corresponding perturbations ($\propto e^{i\mathbf{kr} + \sigma t}$) of the pressure and the velocity field, and $\delta\rho$ the Eulerian change due to the deformation of the interface:

$$\delta\rho = (\rho_1 - \rho_2)\delta_D(z)\delta z + \rho_2 \delta_D(z - \Delta z)\delta z',$$

where $\delta z'$ is the deformation of the surface at $r = \Delta R$ corresponding to δz. From Equations (B-1) to (B-4), we obtain

$$\begin{cases} \rho\sigma u = -ik_x \delta P, & \text{(B-5)} \\ \rho\sigma v = -ik_y \delta P, & \text{(B-6)} \\ \rho\sigma w = -\partial(\delta P)/\partial z - \delta\rho g, & \text{(B-7)} \\ ik_x u + ik_y v + \partial w/\partial z = 0. & \text{(B-8)} \end{cases}$$

From Equations (B-5) to (B-8), we can easily obtain

$$(\partial^2/\partial z^2)\delta P = k^2 \delta P, \tag{B-9}$$

where $k^2 = k_x^2 + k_y^2$. Hence, we have

$$\delta P = \begin{cases} Ae^{kz} & \text{for} \quad z < 0, \\ Be^{kz} + Ce^{-kz} & \text{for} \quad z > 0, \end{cases} \tag{B-10}$$

where A, B, and C are constant. In the region of $z < 0$, there is no term proportional to e^{-kz} because the region of $z < 0$ is semi-infinite.

The boundary conditions are as follows:

(1) The pressure $P + \delta P$ is continuous at $z = \delta z$.

(2) The radial velocity w is continuous and equal to $\partial(\delta r)/\partial t$ at $z = \delta z$.

$(3 - r)$ w (i.e., $\partial(P + \delta P)/\rho \partial z$) is zero at $z = \Delta z$ in the rigid-surface condition.

$(3 - f)$ $P + \delta P$ is zero and $w = \partial(\delta z')/\delta t$ at $z = \Delta z + \delta z'$ in the free-surface condition.

From Equations (B-1) to (B-10), we can readily find the dispersion relation under the boundary conditions, (1) and $(2 - r)$, or (1) and $(2 - f)$, as follows: in the rigid-surface case,

$$\sigma^2 = gk(\rho_2 - \rho_1)(1 - a)/\{\rho_1 + \rho_2 + a(\rho_2 - \rho_1)\}, \tag{B-11}$$

where $a \equiv \exp(-2k\Delta z)$.

For the free-surface case,

$$\sigma^2 = -gk \quad \text{(the surface wave)}, \tag{B-12}$$

or

$$= gk(\rho_2 - \rho_1)(1 - a)/\{\rho_1 + \rho_2 - a(\rho_2 - \rho_1)\} \quad \text{(the internal gravity wave)}. \tag{B-13}$$

If $\Delta z \to \infty$, Equations (B-11) and (B-13) both reduce to the dispersion relation in the case where the overlaid fluid is half-infinite,

$$\sigma^2 = gk(\rho_2 - \rho_1)/(\rho + \rho_2). \tag{B-14}$$

On the other hand, let σ_R be σ given by Equation (B-11) and σ by Equation (B-13), then we have

$$(\sigma_F/\sigma_R)^2 = (1 + b)/(1 - b), \tag{B-15}$$

where $b \equiv a(\rho_2 - \rho_1)/(\rho_1 + \rho_2)$. Note that, if $\rho_2 > \rho_1$, R is always greater than unity, that is, σ_F is always greater than σ_R. Furthermore, we can easily see that

$$(\sigma_F/\sigma_R)^2 \to 1, \quad \text{if} \quad \Delta z \to \infty, \tag{B-16}$$

$$(\sigma_F/\sigma_R)^2 \to \rho_2/\rho_1, \quad \text{if} \quad \Delta z \to 0. \tag{B-17}$$

Equation (B-17) is important for the discussion in Section 4.

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