INSTABILITY CONSIDERATIONS AT SELF-GRAVITATING CIRCUMPLANETARY RINGLET STRUCTURES

E. WILLERDING and H. J. FAHR

Institut für Astrophysik und Extraterrestrische Forschung der Universität Bonn, Bonn, F.R.G.

(Received 31 October, 1988)

Abstract. The aim of this paper has been to study here specific forms of instabilities in circumstellar and circumplanetary dust ringlets in Keplerian rotation around a central gravitating mass without taking shear flow effects into consideration. Due to the presence of a central mass in the disk, an additional force term appears in the linearized equation of motion. Here we investigate the importance of such a term with respect to the onset of gravitational instabilities in both tangential and radial direction of ring-like substructures in the disk. In addition, we compare the instability tendencies of self-gravitating disks with those of fluid layers where perturbation effects are simply controlled by surface tension. In both cases, the material of the layer is treated as an incompressible inviscid fluid. This assumption, however, as shown from our study of the polytropy of dust gases, was proven to be correct for perturbation wavelengths comparable or larger than the thickness of the layer. From our general dispersion relations for symmetric and anti-symmetric perturbation modes, we can retain for the radial wave propagation the results of Lin and Shu, and Goldreich and Ward in the asymptotic case of an infinitely thin layer without shear flow. However, for the tangential waves we find a different stability criterion showing that the onset of the instability depends on the propagation direction. In the 'finite layer' case, we derive much more general relations showing different instability ranges for 'bending' wave modes and self-excited 'density' wave modes pointing to local and global instability forms in ringlets.

1. Introduction to the Problem

It has been speculated in a series of papers that collective gravitational interaction processes acting in a massive dust layer are the relevant physical processes for the structuration of the circumstellar, or circumplanetary, material into rings, ringlets, gaps, and moons (Goldreich and Lynden-Bell, 1965; Goldreich and Ward, 1973; Goldreich and Tremaine, 1978a, b, 1979; Lin and Bodenheimer, 1981). The study of the gravitational stability of self-gravitating uniform material disks of finite thickness already started two decades ago. One of the most comprehensive works in this respect was already published by Goldreich and Lynden-Bell (1965). It was found by these authors that thermal pressure effects will exert a stabilizing effect on short wavelength perturbations, whereas uniform rotation via Coriolis forces will stabilize long wave perturbations. However, there exists an intermediate range of unstable waves of the order of thickness H of the layer, when the quantity $4\pi G\rho/\Omega^2$ (mass density ρ, revolution period Ω) exceeds a critical value.

These waves, under supercritical conditions, would lead to a fragmentation of the dust layer into discrete mass elements of definite size. The later theory of Goldreich and Ward (1973), based on experiences of the above paper, nowadays is considered as offering the fundamental explanation of the cosmogony of planets and planetesimals in the solar system. The basic assumption of the authors is that during the collapse of a protostellar gas cloud, the dynamically involved dust component at a specific state of the collapse decouples from the gas dynamics and settles down into the equatorial plane of the collapse system. Here a material layer is formed that in first order is stabilized against the central gravitational forces by Keplerian rotation and, after an advanced increase of mass per area, σ, to supercritical values σ_c, becomes subject to self-gravitation. On the basis of a linear dispersion analysis, the properties of density fluctuations in such a material layer are investigated. For supercritical density values, again intermediate wave number ranges can be pointed out where density fluctuations grow unstable. These unstable wave modes, in the view of the authors, are paving the way towards the origin of massive planetesimals with an initial mass of some 10^{18} g.

Since in this analysis the dust layer is treated as being of a homogeneous surface density with no external forces acting in a corotating rest frame with Keplerian rotation period, nothing can be said about the expected mass spectrum of the planetesimals and of the distribution of massive fragments with distance from the collapse centre.

With a slightly different aspect from the above-mentioned papers, the stability of elliptical and ring-like self-gravitating material configurations was investigated by Seboldt (1981) for cylindrical symmetries and a polytropic behaviour of the material. For the stability analysis, Seboldt (1981) takes advantage of methods derived by Schindler *et al.* (1973) in an analogy to plasma physics. Starting from the equlilbrium status of the material configuration, Seboldt applies a specific form of a linear conformal perturbation to this configuration and, using typical variation principle techniques, arrives at an equation of the Euler–Lagrange type from which useful stability criteria can be derived.

Not so much the results as the procedure of this analysis can as well be used for a stability study of planetary rings and disks. This has been done in the work by Seboldt and Schindler (1984) in which a theory of the equilibrium states of dust particle distributions in the Saturnian rings is presented.

Shu *et al.* (1983) address the problem of gravitationally-induced material fluctuations in self-gravitating material configurations again on a different way by studying spiral bending waves (or transversal waves) excited by external moons in the Saturnian ring systems. Especially the two well-observed wave modes due to 5 : 3 resonance with the moon Mimas are studied and are interpreted as a spiral bending wave inside of the Mimas orbit, and as a spiral density wave outside. The work of Shu *et al.* (1983) is, however, unable to explain why bending waves lead to the strongly pronounced condensations observed near the locations of the resonances. A better understanding of structuration processes may be achievable with theoretical considerations worked out by Lin and Bodenheimer (1981) in which the authors follow the idea that elastic and inelastic collisions amongst planetary disk particles may play a major role for material agglomerations and irreversible nonlinear condensations. They show that a collision-dominated particle disk may become unstable against a 'pinch'-instability which is induced by viscous stresses similar to the way how Kelvin–Helmholtz instabilities are excited by supercritical velocity shears. As they propose, for instance, the existence of ringlets around the planet Saturn is to be ascribed to the operation of this instability.

Though the stability of the rotating material disks has already been subject of several investigations in the past, some points in this context have not been taken into account in the calculations carefully enough and thus still seem to merit some ongoing studies. Especially the question of how to correctly describe the forces that are connected with the Keplerian rotation, stabilizing the system to the first-order. Due to the presence of this differential rotation pattern several force terms appear in the linearized equation of motion that up to now did not experience comprehensive and appropriate representations. In this respect one may state that in connection with the first-order Keplerian rotation pattern not only Coriolis forces enter the equations of motion, but also terms describing net gravitational attractions to the centre that are connected with perturbed motions, and terms originating from the convective part of the inertial force term. In the following we will devote some attention to the importance of such terms with respect to gravitational stability of dust-layer configurations.

In addition we will, as a new approach to the problem, study how a rotating material layer behaves under the effect of an artificially introduced surface tension replacing the effect of self-gravitation. It will be interesting in this respect to see whether or not similarities in instability tendencies of self-gravitating and capillary material layers can be found.

2. Stability of a Gravitating, Incompressible Rotating Dust Layer

A circumplanetary dust layer may be locally described as a homogeneous material layer with thickness H. The materal in this layer, though consisting of discrete solid particle elements embedded in a tenuous 'gaseous' environment, may be represented as an incompressible hydrodynamic fluid. Thus for each fluid element of the dust layer, the full set of hydrodynamical equations has to be considered.

In a Cartesian coordinate system X, Y, Z with its origin at O (see Figure 1), the usual form of the hydrodynamic conservation equations holds, with real forces connected with the gradients of the total scalar gravitational potential ϕ and the thermal pressure P. If the absolute space vector is denoted by $\vec{\mathbf{R}}$ and the dust medium in the layer under pressure variations is taken to behave as an incompressible fluid (see Fahr and Willerding, 1988), one thus arrives at the differential equation

$$\frac{\mathrm{d}\vec{\mathbf{U}}}{\mathrm{d}t} = \frac{\partial \vec{\mathbf{U}}}{\partial t} + (\vec{\mathbf{U}} \cdot \nabla)\vec{U} = -\overrightarrow{\mathrm{grad}}\,\phi - \frac{1}{\rho}\,\overrightarrow{\mathrm{grad}}\,P\,, \tag{1}$$

where $\vec{\mathbf{U}}(R, t)$ is the absolute fluid velocity as a function of space and time coordinates.

Since here we aim at the treatment of perturbations U in the velocity field, which are superimposed on the local velocity field of the Keplerian rotation pattern in the disk, it is advised to transcribe Equation (1) into a locally corotating reference system x, y, z with its origin at $\vec{\mathbf{R}} = \vec{\mathbf{R}}_0$ (see Figure 1). To achieve this, we start out from the position vector $\vec{\mathbf{R}} = \vec{\mathbf{R}}_0 + \vec{\mathbf{r}}$ and its time derivatives, which yield the velocities U and u (perturbation velocity) and the corresponding accelerations. The relation between the accelera-

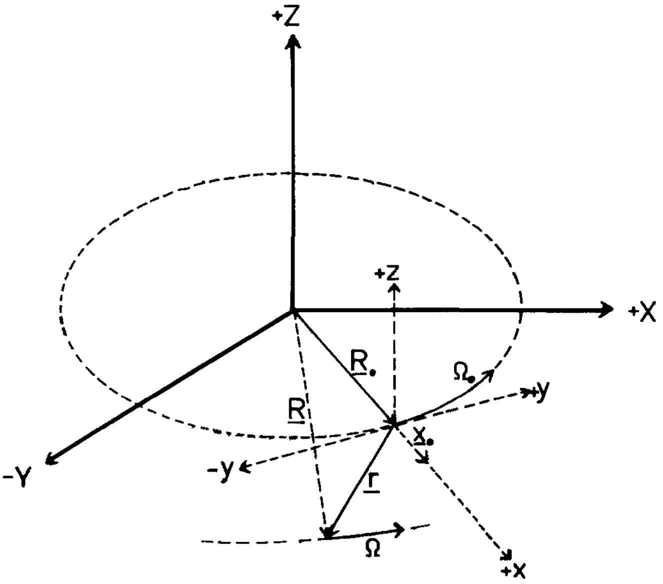

Fig. 1. Schematic diagram showing the geometrical situation in a locally corotating reference system of the Keplerian disk.

tion vector $\mathbf{R}$ in the inertial system X, Y, Z and that r in the corotating reference system x, y, z with its origin at R_0 and rotating with the Keplerian angular velocity $\mathbf{\Omega}_0 = \sqrt{GM/R_0^3}$ is then given by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \ddot{R} = d_0 \times (\mathbf{\Omega}_0 \times R_0) + (\ddot{r} + 2(\mathbf{\Omega}_0 \times \mathbf{r})) + \mathbf{\Omega}_0 \times (\mathbf{\Omega}_0 \times \mathbf{r}), \tag{2}$$

where G is the gravitational constant and M is the central mass. Introducing Equation (2) into Equation (1) we have

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{\Omega}_0 \times (\mathbf{\Omega}_0 \times R_0) + \left(\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) u) + 2(\mathbf{\Omega}_0 \times (\mathbf{\Omega}_0 \times \mathbf{r})\right) =$$

$$= -\frac{GM}{R^3}\, \overline{\mathbf{R}} - \overrightarrow{\mathrm{grad}}\, \delta\phi - \frac{1}{\rho}\, \overrightarrow{\mathrm{grad}} P, \tag{3}$$

where the gravitational force term in Equation (1) has been eparated into two terms describing (a) the centripetal gravitational pull of the central mass M, and (b) the effect of self-gravitation in the dust layer, giving rise to a gravitational potential $\delta\phi$. To remove the absolute space coordinate R from the right- and left-hand sides of the equation, the central gravitational force at $\vec{\mathbf{R}}$ has to be expanded into a Taylor series around the origin

R_0 of the coordinate system x, y, z. This expansion then yields to the first-order in $\{\mathbf{r} \cdot \nabla\}$

$$\frac{GM}{R^3}\vec{\mathbf{R}} = \frac{GM}{R_0^3}\vec{\mathbf{R}}_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left|\{\mathbf{r}\cdot\nabla\}^n \frac{GM}{R^3}\vec{\mathbf{R}}\right|_0 \simeq$$

$$\simeq \frac{GM}{R_0^3}\vec{\mathbf{R}}_0 + \frac{3GM}{R_0^4}\frac{\mathbf{r}\cdot\vec{\mathbf{R}}_0}{R_0}\vec{\mathbf{R}}_0 + \frac{GM}{R_0^3}\vec{\mathbf{r}}. \tag{4}$$

Keeping in mind that the angular velocity $\mathbf{\Omega}_0$ is defined by

$$GM/R_0^2 = R_0\Omega_0^2, \tag{5}$$

and that the vector $\mathbf{\Omega}_0$ is perpendicular to $\vec{\mathbf{R}}_0$, we note that the first term on the left-hand side of Equation (3) cancels the first term of the expansion (4) and thus Equation (3) attains the form

$$\frac{\partial\vec{\mathbf{u}}}{\partial t} + (\mathbf{u}\cdot\nabla) = +\frac{3GM}{R_0^4}\vec{\mathbf{R}}_0 + \frac{GM}{R_0^3}\vec{\mathbf{r}} - 2(\mathbf{\Omega}_0\times\mathbf{u}) - \vec{\mathbf{\Omega}}_0\times(\mathbf{\Omega}_0\times\mathbf{r}) -$$

$$- \overrightarrow{\mathrm{grad}}\,\delta\phi - \frac{1}{\rho}\overrightarrow{\mathrm{grad}}\,P. \tag{6}$$

As can easily be seen in the differential equation (6) for the perturbation velocity u, the second and the fourth terms on the right-hand side cancel each other. Thus one is finally left with the momentum conservation equation of the form

$$\frac{\partial u}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{3GM}{R_0^4}\vec{\mathbf{R}}_0 - 2(\mathbf{\Omega}_0\times\mathbf{u}) - \overrightarrow{\mathrm{grad}}\,\delta\phi - \overrightarrow{\mathrm{grad}}\left(\frac{P}{\rho}\right). \tag{7}$$

The above description is given in the corotating reference system x, y, z with its origin at $\vec{\mathbf{R}}_0$. As is easy to identify, the first terms on the right-hand side of Equation (7) represent the effect of an uncompensated central gravity field in the corotating system and the Coriolis force term, which was also taken into account by Goldreich and Lynden-Bell (1965);

$$\Delta\delta\phi = 0 \quad \text{outside of the layer} \tag{8a}$$

and

$$\Delta\delta\phi = 4\pi G\rho \quad \text{inside the layer}. \tag{8b}$$

When one neglects nonlinear terms in the perturbed quantities (i.e., $0(u^2)$) and effects of the differential Keplerian rotation pattern (because in ringlet structures they can be shown to be of higher order, e.g., Laplacian rings!), one can derive from Equation (7) the following scalar differential equations for the Cartesian velocity components of u:

$$\frac{\partial u_x}{\partial t} = 3\Omega_0^2\xi_x + 2\Omega_0 u_y + \frac{\partial\pi}{\partial x}, \tag{9a}$$

$$\frac{\partial u_y}{\partial t} = -2\Omega_0 u_x + \frac{\partial \pi}{\partial y}, \tag{9b}$$

$$\frac{\partial u_z}{\partial t} = +\frac{\partial \pi}{\partial z}; \tag{9c}$$

where the combined potential $\pi = \delta\phi + P/\rho$ has been introduced.

The equilibrium position of the upper and lower boundaries of the dust layer in z-direction may be defined by $z_0 = \pm H_0/2$. For wave-like perturbations of this boundary, we would thus obtain the description

$$z_0 = \pm H_0/2 + \varepsilon \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \tag{10}$$

where $\mathbf{k}$ and ω are the wave vector and the frequency of the perturbation wave, and where a coplanarity of the wave propagation with the layer surface, yielding $\mathbf{k} = (k_x, k_y, 0)$, can be adopted for obvious reasons. If one now assumes in addition an oscillatory behaviour of the velocity components according to function $\exp(-i\omega t)$ and keeps in mind that due to this representation the following relation (Lagrangian displacement in radial direction) is valid: i.e.,

$$\xi_x = \int_0^t u_x \, \mathrm{d}t = i \frac{u_x}{\omega} e^{-i\omega t}, \tag{11}$$

the system of Equations (9a) through (9c) can be rewritten in the form

$$-u_x\left(i\omega + 3i\frac{\Omega_0^2}{\omega}\right) - 2\Omega_0 u_y = \frac{\partial \pi}{\partial x}, \tag{12a}$$

$$2\Omega_0 u_x - i\omega u_y = \frac{\partial \pi}{\partial y}, \tag{12b}$$

$$-i\omega u_z = \frac{\partial \pi}{\partial z}. \tag{12c}$$

From this system the velocity components $u_{x,y,z}$ can be isolated by the relations

$$u_x = \frac{-i\omega\frac{\partial \pi}{\partial x} + 2\Omega_0\frac{\partial \pi}{\partial y}}{\Omega_0^2 - \omega^2}, \tag{13a}$$

$$u_y = \frac{-i\left(\omega + 3\frac{\Omega_0^2}{\omega}\right)\frac{\partial \pi}{\partial y} - 2\Omega_0\frac{\partial \pi}{\partial x}}{\Omega_0^2 - \omega^2}, \tag{13b}$$

$$u_z = \frac{i}{\omega}\frac{\partial \pi}{\partial z}. \tag{13c}$$

By applying the mass flux continuity equation in the form valid for an incompressible fluid

$$\operatorname{div}(\rho\vec{\mathbf{u}}) = \rho\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0\,, \tag{14}$$

one is able to derive from Equations (13a–c) the following relation for the combined scalar potential π:

$$\omega^2 \frac{\partial^2 \pi}{\partial x^2} + (\omega^2 + 3\Omega_0^2)\frac{\partial^2 \pi}{\partial y^2} + (\omega^2 - \Omega_0^2)\frac{\partial^2 \pi}{\partial z^2} = 0\,. \tag{15}$$

In the case of a nonrotating disk ($\Omega = 0$!), this equation would result in the requirement $\nabla^2 \pi = 0$. However, to solve for the more general case of a rotating disk ($\Omega_0 \neq 0$), we assume for the perturbed potential $\pi(x, y, z)$ the form

$$\pi(x, y, z) = \varepsilon \pi_z(z)\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,. \tag{16}$$

Introduction of Equation (16) in Equation (15), then yields the differential equation for the function π_z

$$\frac{d^2 \pi_z}{dz^2} - k^2\left(\frac{\omega^2 + 3\Omega^2 \sin^2\varphi}{\omega^2 - \Omega^2}\right)\pi_z = 0\,, \tag{17}$$

where it was made use of the fact that the angle of propagation of the surface wave with respect to the x-axis, $\varphi = \measuredangle(\mathbf{x}_0, \mathbf{k})$ is connected with the wave vector components by

$$k_x = k\cos\varphi\,, \qquad k_y = k\sin\varphi\,. \tag{18}$$

With the convention (18) $\varphi = 0$ means radial wave propagation and $\varphi = \pi/2$ means tangential wave propagation in the plane. The solution of Equation (17) is found to be

$$\pi_z(z) = \pi_0 \cosh(\kappa z) \quad \text{(symmetric in } z)\,, \tag{19}$$

$$\pi_z(z) = \pi_0 \sinh(\kappa z) \quad \text{(anti-symmetric in } z)\,; \tag{20}$$

where π_0 is a constant that has to be determined in connection with the boundary conditions and the quantity κ is defined by

$$\kappa = |k|\sqrt{\frac{\omega^2 + 3\Omega_0^2 \sin^2\varphi}{\omega^2 - \Omega_0^2}}\,. \tag{21}$$

As is evident from Equation (21) the quantity κ can also become purely imaginary in which case π_0 had to be complex.

Together with Equations (19) and (20) one then obtains as a solution of Equation (14) the expressions

$$u_z|_{H/2} = \left.\frac{dz}{dt}\right|_{H/2} = \frac{i}{\omega}\pi_0 \kappa \begin{Bmatrix}\cosh(\kappa H/2)\\ \sinh(\kappa H/2)\end{Bmatrix} \varepsilon\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \tag{22}$$

where π_0 contains both the effects of self-gravitation and perturbed hydrostatic pressure. The constant quantity π_0 hereby has to be fixed by use of the boundary conditions of the problem. From Equations (14) and (22) one now derives the relation

$$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{H/2} = -i\omega\varepsilon\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{23}$$

which in combination with Equation (10) then leads to the dispersion relation

$$\omega^2 + \pi_0\kappa \begin{Bmatrix}\cosh(\kappa H/2)\\ \sinh(\kappa H/2)\end{Bmatrix} = 0 \begin{Bmatrix}\text{symmetric}\,,\\ \text{antisymmetric}\,.\end{Bmatrix} \tag{24}$$

This has to be considered as the typical dispersion relation for 'symmetrical' and 'anti-symmetrical' wave modes of a self-gravitating material layer with a local Keplerian rotation period Ω and with no shear flow.

The determination of the constant π_0 has to be carried out on the basis of the definition for π. In the unperturbed state, the hydrostatic pressure in the layer would have to be described by

$$p(z) = (2\pi G\rho^2 + \tfrac{1}{2}\Omega_0^2\rho)\left[\left(\frac{H}{2}\right)^2 - z^2\right], \tag{25}$$

where the first term in the first bracket on the right-hand side is due to the self-gravitation, the second one due to the action of the central force in z-direction connected with the Keplerian rotation. Since at both the unperturbed and the perturbed surface the pressure has to attain a constant, or even a vanishing value, one is then led to the relation

$$p(z) + \delta p = p(H/2) = \text{const.} \tag{26}$$

If we assume no gaseous or dusty material outside the layer, this constant in Equation (26) has in fact to be set equal to zero. Making again use of Equation (10), one can derive for the pressure from Equation (26) the expression

$$\left.\frac{\delta p}{\rho}\right|_{H/2} = (2\pi G\rho H + \tfrac{1}{2}\Omega^2 H)\varepsilon\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,. \tag{27}$$

For the purpose of a determination π_0 in addition to δp one needs an expression for the perturbation in the potential, $\delta\Phi$ caused by the mass redistribution. Contrary to the case of the pressure perturbation, in this case the change of the potential $(\Omega^2 z^2/2)$ caused by the central planetary body does not contribute. For the gravitational potential of the self-gravitating layer, one obtains by use of Equations (9a) and (9b) the following representations for space points inside (i) and outside (o) the material layer, respectively:

$$\Phi_i = -2\pi G\rho z^2 + \varepsilon A \begin{Bmatrix}\sinh(|k|\,z)\\ \cosh(|k|\,z)\end{Bmatrix} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{28}$$

and

$$\Phi_o = \frac{\pi}{2} G\rho H^2 - 2\pi G\rho H z + \varepsilon B\, e^{-|k|\, z}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,; \quad \{z > 0\}\,. \tag{29}$$

In these representations, the two constants A and B have to be fixed such that both the potential and its first derivative are continuous functions at the boundary of the layer z_0 given in Equation (10). The requirement yields the condition

$$A \begin{Bmatrix} \sinh(|k|\, H/2) \\ \cosh(|k|\, H/2) \end{Bmatrix} = B\, e^{-|k|\, H/2}\,. \tag{30}$$

In order also to guarantee the continuous behaviour of the derivative of the potential at the boundary, the following condition has to be fulfilled:

$$A \begin{Bmatrix} \cosh(|k|\, H/2) \\ \sinh(|k|\, H/2) \end{Bmatrix} + B\, e^{-|k|\, H/2} = 4\rho G\rho/|k|\,. \tag{31}$$

From these two conditions, the values for A and B follow from

$$A = (4\pi G\rho/|k|)\, e^{-|k|\, H/2} \tag{32}$$

$$B = (4\pi G\rho/|k|) \begin{Bmatrix} \sinh(|k|\, H/2) \\ \cosh(|k|\, H/2) \end{Bmatrix}. \tag{33}$$

In connection with Equations (32) and (33) from Equation (28) the following result for the boundary value of the potential perturbation $\delta\Phi_{H/2}$ can be obtained:

$$|\delta\Phi|_{H/2} = (4\pi G\rho/|k|)\, e^{-|k|\, H/2} \begin{Bmatrix} \sinh(|k|\, H/2) \\ \cosh(|k|\, H/2) \end{Bmatrix} \varepsilon\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,. \tag{34}$$

The up to now undetermined constant π_0 in Equations (19) and (20) can now be fixed with the help of the relation

$$\pi|_{H/2} = |\delta\Phi|_{H/2} - \frac{\delta p}{\rho}\Big|_{H/2}\,, \tag{35}$$

where the two terms on the right-hand side of Equation can be taken from Equations (34) and (27). Thus one arrives at the following result:

$$\pi_0 \begin{Bmatrix} \sinh(\kappa H/2) \\ \cosh(\kappa H/2) \end{Bmatrix} = (4\pi G\rho/|k|)\, e^{-|k|\, H/2} \begin{Bmatrix} \sinh(|k|\, H/2) \\ \cosh(|k|\, H/2) \end{Bmatrix} - $$

$$- (2\pi G\rho H + \tfrac{1}{2}\Omega^2 H)\,. \tag{36}$$

Introducing Equation (36) in Equation (24), we obtain the two dispersion relations for the symmetric and the antisymmetric modes of a perturbation wave the expression

$$\omega^2 = \{2\pi G\rho(|kH| + e^{-|kH|} - 1) + \Omega^2\,|kH|/2\} \times \\ \times \sqrt{\frac{\omega^2 + 3\Omega^2 \sin^2\varphi}{\omega^2 - \Omega^2}} \coth\left(\sqrt{\left(\frac{\omega^2 + 3\Omega^2 \sin^2\varphi}{\omega^2 - \Omega^2}\right)} \left|\frac{kH}{2}\right|\right) \tag{37}$$

and

$$\omega^2 = \{2\pi G\rho(|kH| - e^{-|kH|} - 1) + \Omega^2\,|kH|/2\} \times \\ \times \sqrt{\frac{\omega^2 + 3\Omega^2 \sin^2\varphi}{\omega^2 - \Omega^2}} \tanh\left(\sqrt{\left(\frac{\omega^2 + 3\Omega^2 \sin^2\varphi}{\omega^2 - \Omega^2}\right)} \left|\frac{kH}{2}\right|\right). \tag{38}$$

With the two above Equations (37) and (38), the general problem of the behaviour of perturbation waves in a rotating material layer is solved. We will now first study these relations in some interesting special cases in order to investigate their physical implications.

3. The Infinitely Thin Layer Approximation

A very interesting special case is the infinitely thin layer. However, this case is only of importance for the study of a self-gravitating material layer if the process $H \to 0$ is carried out such that the amount of gravitating material is kept constant. Otherwise, the effect of self-gravitation necessarily would be lost when going to the limit $H = 0$. In order to take into account this important fact, one usually introduces the surface density of the layer by:

$$\sigma = \rho H\,. \tag{39}$$

We will now consider the behaviour of Equations (37) and (38) for layers with systematically decreasing values of H, however, with a constant value for the surface density σ, i.e., ρ has to be considered as increasing according to $(\sigma/H)_{H\to 0}$. Following this procedure of going towards the limit $H = 0$, one obtains from Equation (37) the relation

$$\omega^2 = \Omega^2 + 2\pi G\sigma\,|k|\,. \tag{40}$$

This simplified form of a dispersion relation represents the well-known relation (see Bertin and Mark, 1980; Shu *et al.*, 1983) for stable transversal bending waves with an arbitrary propagation direction within the layer, i.e., yielding identical results for radially and azimuthally propagating waves. The bending mode character of the waves can be checked with the help of Equation (22) showing the symmetry of u_z with respect to z. In the case of the anti-symmetric mode described by Equation (38) the process $H = 0$ leads to a more complicated relation given by

$$\omega^4 - (\Omega^2 - 2\pi G\sigma\,|k|)\omega^2 + 6\pi G\sigma\Omega^2\,|k|\sin^2\varphi = 0\,. \tag{41}$$

As one may notice, two different dispersion branches for density waves are described by this relation. The character of these waves again with the help of Equation (22) can be identified as longitudinal density waves.

For radially propagating density waves ($\varphi = 0!$), besides the trivial solution $\omega = 0$, one obtains

$$\omega^2 = \Omega^2 - 2\pi G\sigma |k_r| . \tag{42}$$

This is the dispersion relation for purely radial density waves in an infinitely thin layer with Keplerian rotation and self-gravitation. This relation was also obtained earlier by Lin and Shu (1964) and Goldreich and Ward (1973). Stability for such waves only exists if

$$|k_r| < \frac{\Omega^2}{2\pi G\sigma} = k_{r,c} , \tag{43}$$

which will be fulfilled by waves with long wavelengths, whereas the short wavelength perturbations may grow unstable, as already noticed by Goldreich and Ward (1973).

For the case of azimuthally propagating waves (or tangential waves, $\varphi = \pi/2$) one can derive a different stability criterion from Equations (8a) and (8b). This yields

$$k_a = \frac{\Omega^2}{2\pi G\sigma(7 + 4\sqrt{3})} = \frac{k_{r,c}}{(7 + 4\sqrt{3})} . \tag{44}$$

This shows that the onset of perturbation instabilities occurs at different wavelengths depending on the direction into which the wave is propagating. For the ratio of the critical wavelengths of azimuthally and radially propagating waves one obtains

$$(\lambda_{a,c}/\lambda_{r,c}) = (7 + 4\sqrt{3}) \simeq 13.93 . \tag{45}$$

4. The Finite Thickness Layer Approximation

For cases of $H = 0$, the dispersion relations (37) and (38) have to be discussed in their general forms. In a first step towards this generality we will consider cases here in which the quantity $k(H/2) = \chi$ can be considered as small compared to 1. In this case the function $\tanh(\chi)$ in Equation (38) would then lead to the simpler relation

$$\omega^2 = F(kH)\left(\frac{\omega^2 + 3\Omega^2}{\omega^2 - \Omega^2}\right), \tag{46}$$

where the function $F(kH)$ is given by

$$F(kH) = \pi G\rho(|kH| - e^{-|kH|} - 1)|kH| + \frac{\Omega^2 |kH|^2}{4} . \tag{47}$$

The solutions of Equation (46) can be rewritten in the form

$$\omega^2 = \tfrac{1}{2}[(\Omega^2 + F(kH) \pm \sqrt{\Omega^4 + 14\Omega^2 F(kH) + F^2(kH)}] . \tag{48}$$

These solutions immediately show that one of the dispersion branches of Equation (46) for longitudinal perturbation waves in azimuthal direction yields instability of $F(kH) > 0$. If at all, stability can only be guaranteed for both of the existing branches if $F(kH) < 0$. As is then evident from Equation (47) the latter relation can only be valid for $kH \ll 1$. Furthermore, from Equation (48) one can derive that even in the case of negative values for $F(kH)$ no complex frequency values ω appear as long as

$$|F(kH)/\Omega^2| < 7 - 4\sqrt{3}\,. \tag{49}$$

Briefly, the instability analysis may be thus expressed in the following form:

$$\begin{aligned} \operatorname{sgn}(F(kH)) &= +1\,, \quad \text{instability}\,; \\ \operatorname{sgn}(F(kH)) &= -1\,, \quad \text{stability, if Equation (49) is fulfilled}\,. \end{aligned} \tag{50}$$

A similar result as Equation (46) could also be derived from Equation (41) for azimuthal waves ($\phi = \pi/2$), yielding

$$\omega^2 = -2\pi G\sigma k\left(\frac{\omega^2 + 3\Omega^2}{\omega^2 - \Omega^2}\right) = F_0(kH)\left(\frac{\omega^2 + 3\Omega^2}{\omega^2 - \Omega^2}\right). \tag{51}$$

As is evident, $F_0(kH)$ is negative, and whenever it fulfills Equation (49) it proves the stability of infinitely thin layers with respect to azimuthally propagating longitudinal waves.

Furthermore, it is interesting to note that also the dispersion relation for a rotating and self-gravitating material chain can be brought into a form similar to Equation (48) when writing

$$\omega^2 = -2G\rho S(k)\left(\frac{\omega^2 + 3\Omega^2 - G\rho S(k)}{\omega^2 - \Omega^2 - G\rho S(k)}\right), \tag{52}$$

where in this case the factor $(-2G\rho S(k))$ takes into account the tangential coupling between the chain members treated as material clusterings (see Willerding, 1986). The similarity between the dispersion relation (51) and the one obtained for a rotating chain of dust material clearly shows that our hydrodynamical calculations neglect shear-flow-induced effects. Hence, Equations (51) to (52) are restricted in their validity to ring-like structures.

5. Stability of Fluid Layers with Surface Tension

In the previous sections we have solved to the first-order the stability problem of a self-gravitating incompressible fluid layer with Keplerian rotation, but without shear flow. Under the same conditions we shall now consider for purpose of comparison the stability of a liquid layer held together in its equilibrium state and controlled in its perturbed state by capillary forces. The dynamical equations for this case are still of an identical form as those given in Equations (9a–c), of course, with a different meaning of Π. We have to introduce

$$\Pi_c = -\mathrm{d}P_c/\rho\,, \tag{53}$$

since we want to replace both the effect of self-gravitation and thermal pressure by an artificial capillary force to be derived from a capillary pressure p. At a deformed boundary the following equations must be satisfied

$$P_c + \mathrm{d}P_c = \tau(1/R1 + 1/R2)\,, \tag{54}$$

where τ is the constant of surface tension, and where $R1$ and $R2$ are the two principal radii of curvature, respectively. In the unperturbed state the pressure P_c inside the layer vansihes because the principal radii of curvature of the surface are infinite. According to Equation (10), the boundary of the perturbed layer is given by $z_0 = H_0/2 + \varepsilon \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t))$.

To first order in ε, we can then calculate the reciprocal sum of the principal radii of curvature for the perturbed surface by the expression (see for example Landau and Lifshitz, 1978)

$$1/R1 + 1/R2 = -(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2) + 0(\varepsilon^2)\,. \tag{55}$$

In the present one-dimensional case, formula (55) for the capillary pressure-variation yields

$$\mathrm{d}p_c = \tau k^2 \varepsilon \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t))\,. \tag{56}$$

Now following the straightforward analysis of Section 2 concerning the determination of the constant Π, we now obtain with the aid of Equations (53), (54) and (19), (20) for $z = H/2$ that

$$\Pi_{c_0} = -(\tau/\rho)k^2(\cosh \kappa H/2)^{-1}\,, \tag{57}$$

$$\Pi_{c_0} = -(\tau/\rho)k^2(\sinh \kappa H/2)^{-1}\,, \tag{58}$$

Inserting this expressions for Π_{c_0} in the boundary condition (24), we find (after some rearranging) the final form of the dispersion relations for capillary layers with Keplerian-rotation to be

$$\omega^2 = (\tau/\rho)k^2 \kappa \tanh(\kappa H/2) \quad \text{(antisymmetric)}\,, \tag{59}$$

$$\omega^2 = (\tau/\rho)k^2 \kappa \coth(\kappa H/2) \quad \text{(symmetric)}\,. \tag{60}$$

To discuss the stability properties of the capillary layer, we have to distinguish between the radial and non-radial wave-perturbations. As in the foregoing section, we expand the right-hand side of the dispersion relations (59) and (60) to the first-order in H. Thus, we obtain

$$\omega^2 = \tfrac{1}{2}(\tau/\rho)k^4 H(\omega^2 + 3\Omega^2 \sin^2(\varphi))/(\omega^2 - \Omega^2)\,, \tag{61}$$

$$\omega^2 = 2(\tau/\rho)k^2/H\,. \tag{62}$$

By analogy with the criterion (50), a necessary condition for the stability of the capillary layer is that the factor $F(kH) = \frac{1}{2}(\tau/\rho)k^4H$ in (61) should be negative. As is then evident from relation (61), the longitudinal perturbation waves in azimuthal direction are unstable in a capillary layer for all wave numbers. Only in the singular case of purely radial antisymmetric waves does stability exist in the whole range. The dispersion relations for the capillary layer are displayed in Figure 4.

6. Comparison of the Dispersion Relations and Discussions

In the previous sections we have investigated the stability behaviour of an incompressible fluid layer under the action of both a self-induced gravity and an intrinsic surface tension of the layer material. An important aspect in our stability analysis is the fact that we did not take into account the second inertial force term as described by Equation (6). Due to this procedure our investigations are restricted to hydrodynamic models without shear flows. The question whether or not the shear flow in the Keplerian disk is efficient certainly may find an answer in the radial perturbation wavelengths that are involved. If the amplitude of the velocity field perturbation $u(r, t)$ is small in comparison to $\lambda(du/dr)$, the 'Keplerian shears' may be considered as unimportant (for instance the situation in narrow ringlets). In the following we will discuss on this ground the different results in more detail.

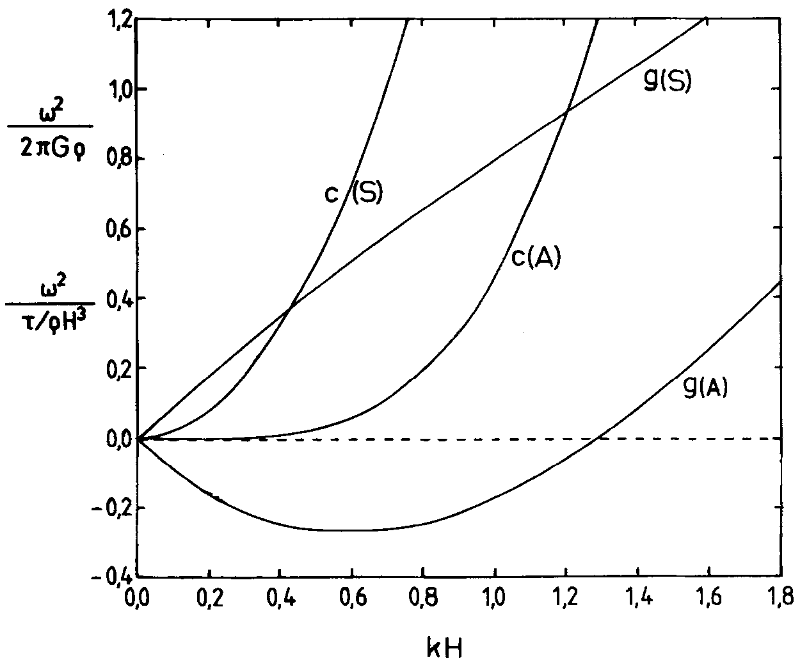

Fig. 2. Dispersion relations for a self-gravitating layer and a capillary layer without rotation. The curv belonging to the case of acting gravity are labeled with g, the curves belonging to capillarity are labeled wi c, respectively.

Figure 2 shows for the case $\Omega = 0$ the dispersion relations given by the expressions (37), (38), (61), and (62), respectively. Since stable oscillations occur only above the dashed line $\omega^2 = 0$, we see that, when rotation is not present, the capillary layer is stable in all dispersion branches ($c(A)$ and $c(S)$) and in the whole range of wave numbers. A different situation was found in the case of a self-gravitating layer. As one can see in Figure 2, the symmetrical wave mode ($g(S)$) shows no instabilities for wave numbers $kH \geq 0$, but, contrary to the capillary case, the antisymmetric wave mode ($g(A)$) is stable only in the range $kH \geq 1.27846$. Therefore, we establish that, for wave perturbations with wavelengths exceeding a critical limit, the self-gravitating layer is unstable. It is interesting to note that in the unstable region of the dispersion branch $g(A)$ a mode of maximum instability occurs for $kH = 0.60701$. Due to this fact the stability properties of a capillary layer and a self-gravitating layer show no similarities as in the case of certain cylindrical systems (Chandrasekhar, 1961).

Figure 3 shows the function $F(kH)/\Omega^2$ for different values of the parameter $\alpha = 4\pi G\rho/\Omega^2$, as given by the expression (47). According to Equation (50), stability can only be guaranteed for the anti-symmetrical dispersion branches ($g(A)$, $\varphi = \pi/2$), if $F(kH)/\Omega^2 \leq 0$. In this region we have stability unless the parameter α exceeds the critical

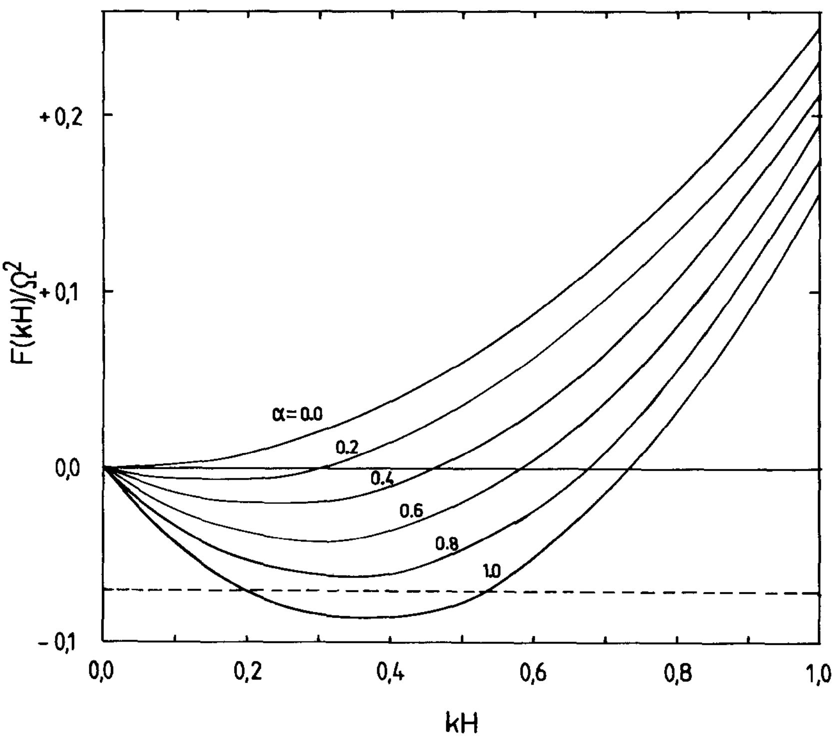

Fig. 3. The function $F(kH)/\Omega^2$ versus wave number (kH) as described in the text (Equation (47)). Different curves are for various values of $\alpha = 4\pi G\rho/\Omega^2$. The dashed line indicates the critical value $F_c(kH)/\Omega^2 = -0.071797$ (see Equation (49)), where complex solutions for ω have to be expected.

value $\alpha_c = 0.8636$ (see dashed line of Figure 3). Beyond this value, solutions of the characteristic Equation (46) are complex, meaning a non-local instability behavior for tangential wave propagations in self-gravitating ringlets. In regions where the function $F(kH)/\Omega^2$ gives values greater than zero, we derive from Equation (46) purely imaginary frequency values. Consequently, we can speak in this cases of a local instability which occurs for short wavelengths depending on the parameter $\alpha = 4\pi G\rho/\Omega^2$ (see Figure 3). To see these results more clearly, we have displayed the different dispersion branches (37) and (38) in case of the finite thickness layer approximation for the value $\alpha = 0.8$ in Figure 4. As mentioned above, the non-local instability does not arise because the value $\alpha = 0.8$ lies below the critical value $\alpha_c = 0.8636$. Nevertheless the local instability occurs in the short wavelength range as can be seen in Figure 4. For the case of self-gravitation, it can be said that rotation has the important effect to stabilize the anti-symmetrical wave mode in the long wavelength range.

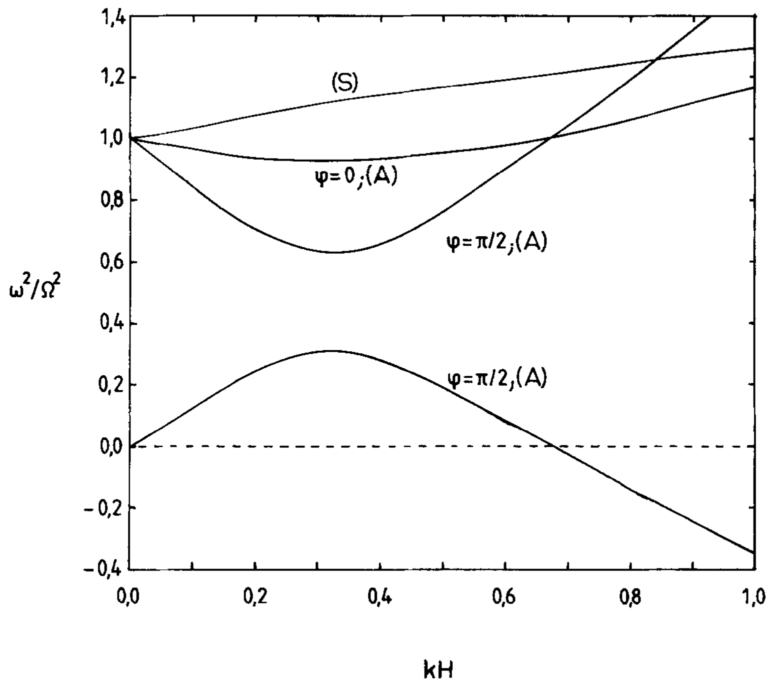

Fig. 4. Dispersion relations for perturbation waves in radial ($\varphi = 0$) and tangential ($\varphi = \pi/2$) directions in a self-gravitating Keplerian disk in the case of finite thicknesses H of the layer without shear flow ($4\pi G\rho/\Omega^2 = 0.8$).

Figure 5 shows the dispersion branches of a capillary layer with rotation given by the relations (61) and (62) assuming $\tau/(2\rho H^3\Omega^2) = 0.1$. Paradoxically, as can be seen, the capillary layer with rotation and an uncompensated central gravity field is completely unstable, because the factor $F(kH) = \frac{1}{2}(\tau/\rho)k^4H$ is positive. So we can repeat our

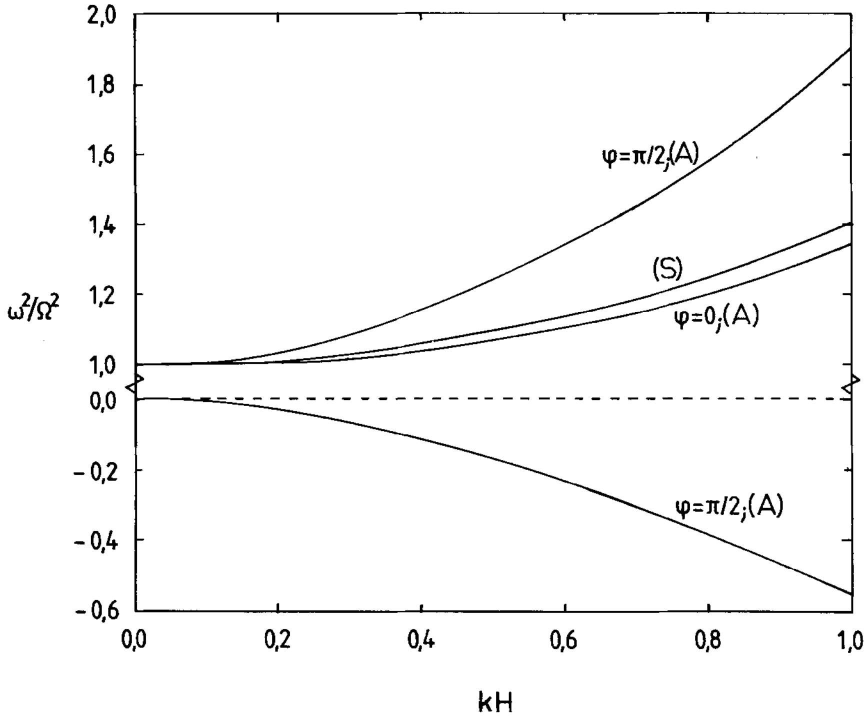

Fig. 5. Dispersion relations for perturbation waves in radial ($\varphi = 0$) and tangential ($\varphi = \pi/2$) directions for a 'capillary disk' without shear flow $\tau/(2\rho H^3\Omega^2) = 0.1$).

statement that, at least for the stability behaviour, similarities between self-gravitating and capillary ringlike structures do not exist.

7. Concluding Remarks

In the present paper, we have calculated linear dispersion relations for self-gravitating, incompressible ringlet structures with a finite thickness and an appropriate planar extent, taking into account additional force terms due to rotation and tidal acceleration. In order to derive results in an analytical form we made the following approximations:

(i) Transformation of the relevant dynamical equations of a homogeneous material layer with thickness H in a local frame of reference, rotating with Keplerian velocity around a central mass M.

(ii) Linearization of the incompressible, inviscid fluid equations for small departures from equilibrium.

(iii) Neglect of the convective part of the material time derivative operator in the hydrodynamical equations, i.e., of the differential rotation pattern of the layer. However, the tidal force term was adequately taken into account.

With these simplifications, we came to our main result, which is represented by two transcendental dispersion relations (Equations (37) and (38), respectively). These equations have been interpreted for two approximative cases, namely the 'infinitely thin layer' and the 'finite layer' approximation. However, it should be mentioned that in addition to these lowest modes (or lowest dispersion branches) described in terms of the finite thickness layer approximation in Section 4 and illustrated in Figure 4, there exists an infinity of higher order modes which can be derived from Equations (37) and (38) by accurately observing the quadrants of the functions $\tanh(\kappa H/2)$ and $\coth(\kappa H/2)$ (κ may be complex or purely imaginary). We have not considered these modes here because the principal results already follow from the lowest dispersion branches.

Reviewing synoptically our results on the properties of self-gravitating dust layers, we may start out from the statement that the tendencies of waves to grow unstable strongly depend on the local Keplerian rotation period. Our results presented for the case $\Omega = 0$ may be taken as characteristic for the outermost fringes of circumplanetary disks. Here it is pointed out that although bending waves are generally revealed as stable, specific types of density waves grow unstable at long wavelengths (small value of kH). For tangential waves of this type it can be seen that $\Omega \to 0$ also implies $\alpha = 4\pi G\rho/\Omega^2 \to \infty$ and that critical values of $\alpha \geq \alpha_c$ lead to the case when the function $F(kH)/\Omega^2$ becomes smaller than $[-(7 - 4\sqrt{3})]$ (see Figure 3) such that global instabilities may develop. For instance, tangential waves propagating prograde may superimpose with waves propagating retrograde and thus may form standing waves on a dust ring, which, due to long-periodic increases in amplitude, may lead to a fragmentation or conglomeration of the diffuse material into discrete masses distributed along this ring.

The opposite case of large values for Ω, which need to take the rotation into account, will be important, especially at the innermost fringes of a circumplanetary disk. Here again we find that bending waves ($g(S)$) remain stable over all wavelengths; however, in contrast to the case $\Omega = 0$, we now obtain instabilities for the density waves ($g(A)$) at small wavelengths (i.e., large values of kH), whereas the long wavelengths are now stable. In connection with the existence of infinitely higher order modes mentioned above, we can expect for all density values complicated conglomeration processes in the short wavelength range. Since for a finite value Ω an increase of α is caused by increasing values of ρ, one can also obtain here the case of complex solutions for ω.

This could also give rise to a global fragmentation of diffuse matter in these inner regions of a disk into a discrete mass distribution. For a cosmogonically relevant circumplanetary or circumstellar dust layer, one should in addition keep in mind that both H and ρ will be functions of a central distance R which definitely determines the absolute values for the unstable wave numbers and imaginary frequency parts. Starting from an available cosmogonical model for a young dust accretion disk, we will look into this problem more quantitatively in a forthcoming paper. In connection with these studies, we can also investigate the compatibility of wavelengths belonging to complex frequency solutions with the local circumference line. We hope to be able to localize regions where, due to complex solutions of the dispersion relation for tangential perturbation waves, only two or three discrete mass condensations are to be expected. In connection with the shear in the rotation pattern, these mass concentrations give rise to the development of self-excited spiral density wave modes. The problem of shear flow on the radial stability of dust layers ('pinch' instability due to viscosity effects) will be a subject for us to study further.

Acknowledgement

We gratefully acknowledge the financial support of this work by the Bundesministerium für Forschung und Technologie des Landes Nordrhein-Westfalen under Grant No. 101-003-86.

References

Bertin, G. and Mark, W. K.: 1980, *Astron. Astrophys.* **88**, 289.
Chandrasekhar, S.: 1961, *Hydrodynamic and Hydromagnetic Stability*, Dover Publications, Inc., New York.
Fahr, H. J. and Willerding, E.: 1989, The Polytropy of Dust Gases', Preprint Univ. Bonn.
Goldreich, P. and Lynden-Bell, D.: 1965, *Monthly Notices Roy. Astron. Soc.* **130**, 97.
Goldreich, P. and Tremaine, S.: 1978a, *Icarus* **34**, 227.
Goldreich, P. and Tremaine, S.: 1978b, *Icarus* **34**, 240.
Goldreich, P. and Tremaine, S.: 1979, *Astrophys. J.* **222**, 850.
Goldreich, P. and Ward, W. R.: 1973, *Astrophys. J.* **183**, 1051.
Landau, L. D., Lifshitz, E. M.: 1978, *Hydrodynamik*, Akademie Verlag, Berlin, pp. 266-271.
Lin, D. N. C. and Bodenheimer, P.: 1981, *Astrophys. J.* **248**, L83.
Schindler, K., Pfirsch, D., and Wobig, H.: 1973, *Plasma Phys.* **15**, 1065.
Seboldt, W.: 1981, *Astrophys. J.* **243**, 1031.
Seboldt, W. and Schindler, K.: 1984, *Astron. Astrophys.* **138**, 290.
Shu, F. H., Cuzzi, J. N., and Lissauer, J. J.: 1983, *Icarus* **53**, 185.
Stewart, G. R., Lin, D. N. C., and Bodenheimer, P.: 1984, in R. Greenberg and A. Brahic (eds.), *Planetary Rings*, Space Science Series, pp. 447-513.
Willerding, E.: 1986, *Astron. Astrophys.* **161**, 403.