URANUS AND NEPTUNE INTERIOR MODELS

C. Z. ZHANG
Department of Astronomy, Nanjing University, Nanjing 210093, China

(Received 16 April, 1996; Accepted 19 December, 1996)

Abstract. This paper is concerned with the interior structure of Uranus and Neptune. Our approach is three-fold. First, a set of three-layer models for both Uranus and Neptune are constructed using a method similar to that used in the study of the terrestrial planets. The variations of the mass density $\rho(s)$ and flattening $e(s)$ with fractional mean radius s for two representative models of Uranus and Neptune are calculated. The results are tabulated. A comparison of these models shows that these two planets are probably very similar to each other in their basic dynamical features. Such similarity is very seldom seen in our solar system. Secondly, we check the conformance between the theoretical results and observational data for the two planets. And thirdly, the 6th degree Stokes zonal parameters for Uranus and for Neptune are predicted, based on the interior models put forward in this paper.

Key words: Uranus, Neptune, comparative study.

1. Introduction

One of the consequences of the many achievements of space probes in our solar system, has been a great improvement in our knowledge of the interior structure of the giant planets (see, e.g. Gudkova et al., 1988a,b; Hubbard and Marley, 1989; Hubbard et al., 1991). Not long ago, Zharkov and Gudkova (1991) reviewed the advancement in this field. The main purpose of the present paper is to make a comparative study between the giant planets (Uranus and Neptune) and the terrestrial planets.

It is well known that the dynamical figures of terrestrial planets, excepting the Earth, deviate from those of hydrostatic equilibrium. This fact demonstrates that the terrestrial planets have undergone tidal evolution (cf. Zhang, 1994a; Zhang and Xia, 1994). In contrast, at present Uranus and Neptune are both in strict hydrostatic equilibrium. Now according to the well-known Radau–Darwin formula (cf. Burša, 1984) we have

$$\frac{C}{Ma^2} = \frac{2}{3} - \frac{4}{15}\left(\frac{5}{2}\frac{q}{\alpha} - 1\right)^{1/2}, \tag{1}$$

where α is the hydrostatic flattening of external equipotential surface of the celestial body; and q is the dimensionless rotation parameter. They are related to J_2, GM, ω and a as follows:

$$\begin{aligned}\alpha = & -\frac{3}{2}J_2 + \frac{1}{2}q - \frac{11}{56}q^2 + \frac{3}{14}J_2 q + \frac{9}{8}J_2^2 \\ & + \frac{9}{98}q^3 - \frac{93}{784}J_2 q^2 + \frac{9}{98}J_2^2 q - \frac{27}{16}J_2^3,\end{aligned} \tag{2}$$

Earth, Moon and Planets **75:** 17–24, 1997.
© 1997 *Kluwer Academic Publishers. Printed in the Netherlands.*

$$q = \frac{\omega^2 a^3}{GM}. \tag{3}$$

Here, GM, a, ω are the planetocentric gravitational constant, the equatorial mean radius, and the angular velocity of rotation, which is assumed to be uniform throughout the planet. Also J_2 is the second zonal Stokes gravitational coefficient and C is the polar principal moment of inertia. Amongst these geodetic parameters for a planet, GM, a, ω, and J_2 are defined as primary parameters, but C/Ma^2, α and q are derived ones given by Equations (1), (2) and (3).

As the above primary parameters of a planet are observationally determined, we are able to calculate the mean density $\bar{\rho}$ and mean moment-of-inertia ratio I/MR^2($\approx C/Ma^2$; R is the mean radius of the body). Next regarding $\bar{\rho}$ and I/MR^2 as two constrained quantities and using the state equations of the constitutive material for the planet, we can then construct models of the internal structure of each body.

Previously we have constructed some interior models of terrestrial planets by solving the Emden equation (see Zhang, 1994a,b; Zhang and Zhang, 1995). The primary objective of the present paper, therefore, is to create a corresponding set of interior models of Uranus and Neptune. By comparing these models with those of the terrestrial planets, we can check whether this procedure is applicable to the giant outer planets as well.

2. Observational Data and Estimation of the Density Distribution in the Interior

2.1. Geodetic Parameters of Uranus and Neptune

Recently, the important geodetic parameters of giant planets have become available with high accuracy (cf. Thomas, 1991, Zharkov and Gudkova, 1991; Burša, 1992). The geodetic parameters of Uranus and Neptune adopted in these studies are collected in Table I.

The observed values of $\bar{\rho}$ and C/Ma^2 given in Table I show that Uranus and Neptune both possess a central core of "rock", an intermediate "ice" envelope, and an outer atmosphere composed mainly of hydrogen and helium. This data enable us to estimate the internal density profile of each planet by means of simple two-layer (core and envelope) model.

2.2. Estimation of Two-Layer Models

In order to construct reasonable two-layer models, we assume that the interior density distribution can be written as (cf. Zhang and Shen, 1988)

$$\left.\begin{array}{ll} \rho(s) = \rho_c = \text{const.}, & 0 \le s \le s_c, \\ \rho(s) = \rho_m - ks^2, & s_c \le s \le 1. \end{array}\right\} \tag{4}$$

Table I
Geodetic parameters of Uranus and Neptune

Parameter	Uranus	Neptune
Primary Parameters		
GM ($km^3\ s^{-2}$)	5793947	6835096
a (km)	25559	24764
$-J_2$ (10^{-6})	3343.43	3411
J_4 (10^{-6})	28.85	26
ω (rad s^{-1})	1.01237×10^{-4}	1.08338×10^{-4}
Derived Parameters		
R (km)	25270	24622
$\bar{\rho}$ (g cm^{-3})	1.2846	1.638
C/Ma^2	0.226	0.237
α	0.01961	0.01802
q	0.029535	0.026078

Table II
Estimation of the two-layer models

	ρ_c (g cm^{-3})	s_c	ρ_m (g cm^{-3})	k	ρ (1) (g cm^{-3})
Uranus	6.00	0.45–0.50	2.38–1.16	2.41–0.83	< 0.37
Neptune	7.00	0.45–0.50	3.14–1.81	3.15–1.42	< 0.40

Here ρ_c is the density of the core, ρ_m and k stand for two positive constants, $s = r/R$ is the fractional radius, and $s_c = r_c/R$ is the fractional radius of the core. It now easily follows that $\bar{\rho}$ and I/MR^2 have the form

$$\left.\begin{aligned} \bar{\rho} &= \rho_m(1 - s_c^3) - \frac{3k}{5}(1 - s_c^5) + \rho_c s_c^3. \\ \frac{5}{2}\bar{\rho}\frac{I}{MR^2} &= \rho_m(1 - s_c^5) - \frac{5k}{7}(1 - s_c^7) + r_c s_c^5. \end{aligned}\right\} \tag{5}$$

When the value of ρ_c is given, Equation (5) can be used for estimating a reasonable range of the values of ρ_m and k. The results so calculated are shown in Table II.

Now using the results listed in Table II, we are in a position to create the interior models of Uranus and Neptune by solving the Emden equation.

3. Models of Uranus and Neptune

In order to simplify the process of solving for the hydrostatic equilibrium, let us assume that the bulk modulus $K(r)$ and pressure $p(r)$ satisfy a linear relation-

Table III

Values, K_0 and b used in the construction of planetary models

Zone	K_0 (Mbar)	b
Core	2.50	3.50
Envelope	1.02	2.52

ship (Bullen's compressibility-pressure hypothesis) in the interior of Uranus and Neptune. We write

$$K(r) = K_0 + bp(r), \tag{6}$$

where both K_a and b stand for two positive constants. We may refer to the Earth and lunar models mentioned earlier and adopt the same values of K_a and b for study. These values are listed in Table III.

Starting with the equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{4\pi G\rho(r)}{r^2}\int_0^r r^2\rho(r)\mathrm{d}r, \tag{7}$$

in the form, we can readily derive the dimensional Emden equation (cf. Zhang and Zhang, 1995) viz.

$$\frac{1}{r^2\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{b-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) + \frac{4\pi G}{K_0}\rho_u^b = 0, \tag{8}$$

where ρ_u denotes the value of density under zero pressure. By choosing appropriate boundary conditions Equation (8) can be solved numerically for the core and the envelope, respectively. When the profile of the density $\rho(r)$ is obtained, other rheological parameters ($p(r)$, $K(r)$, etc.) can be derived from known relations. As regards the external molecular envelope composed of hydrogen and helium, we assume that the density distribution follows a linear relationship, in accordance with Zharkov and Gudkova (1991).

Using the above procedure we have constructed a group of internal structure models for Uranus and for Neptune. The results for two different parametric models are listed for each body in Tables IV and V. The run of $\rho(s)$ with fractional radius s is expressed with simple fitting polynomials in s.

Once the density distribution $\rho(r)$ is known, the profile of the flattening $e(r)$ in the planetary interior can be obtained by numerically solving the Clairaut equation

$$\frac{\mathrm{d}^2e(r)}{\mathrm{d}r^2} + \frac{\rho(r)}{\sigma(r)}\frac{6}{r}\frac{\mathrm{d}e(r)}{\mathrm{d}r} + \frac{6}{r^2}\left[\frac{\rho(r)}{\sigma(r)} - 1\right]e(r) = 0, \tag{9}$$

Table IV
Two parametric models of Uranus

Region	Radius (km)	ρ (s) (g cm^{-3})	e (s)
Uranus 96-01			
Core	0.0–6552.2	6.0000	1.0881e-2
		−0.0264*s	+2.0677e-3*s^2
		−6.8914*s^2	+1.2560e-3*s^3
		−2.4480*s^3	
Envelope	6552.2–18980.5	4.5528	8.2577e-3
		−1.3969*s	+1.2883e-2*s
		−2.1653*s^2	−1.1190e-2*s^2
		−1.4476*s^3	+6.4819e-3*s^3
External layer	18980.5–25270.0	1.5824815	−1.65020e-3
		−1.5822125*s	+2.13677e-2*s
Uranus 96-02			
Core	0.0–7280.2	5.0000	1.2084e-2
		−0.0166*s	+1.7439e-3*s^2
		−4.3220*s^2	+8.3877e-4*s^3
		−1.2532*s^3	
Envelope	7280.2–19110.9	4.5634	1.0775e-2
		−1.4578*s	+5.7577e-3*s
		−2.0525*s^2	−3.8902e-3*s^2
		−1.5150*s^3	+4.2229e-3*s^3
External layer	19110.9–25270.0	1.5824815	−1.20241e-3
		−1.5822125*	+2.10930e-2*s

Here $\sigma(r) = 3/r^3 \int_0^r \rho(r) r^2 \mathrm{d}r$ is the mean density of the material inside the spheroidal surface of radius r. The expressions of $e(r)$ for all four representative models are also presented in Tables IV and V, again using simplified polynomial fits with fractional radius s as the independent variable.

4. Discussion

In this paper, we have not considered the influence of temperature in the model calculations. It is also not certain how suitable Bullen's relationship is in modelling the interior of the giant planets. For these reasons, the results given above are likely to deviate from the actual state of the planets. To check on the validity of our calculations, let us compare the theoretical results and observational data from two different aspects.

First, the calculated results show that the uncompressed density in the models of Uranus is about equal to 2.80 for the "rock" core and about 1.45 g cm^{-3} for

Table V
Two parametric models of Neptune

Region	Radius (km)	ρ (s) (g cm^{-3})	e (s)
Neptune 96-01			
Core	0.0–6766.9	7.0000	1.0573e-2
		−0.0323*s	−1.6750e-3*s
		−7.4678*s^2	+9.7668e-3*s^2
		−2.6527*s^3	−1.1202e-2*s^3
Envelope	6766.9–19353.6	5.6473	8.7881e-3
		−2.8279*s	+6.4904e-3*s
		−0.7840*s^2	
		−2.5113*s^3	
External layer	19353.6–24622.0	2.141845	−8.62282e-4
		−2.141576*s	+1.88564e-2*s
Neptune 96-02			
Core	0.0–10786.8	5.5000	1.1785e-2
		−0.0598*s	+8.2079e-5*s
		−3.7071*s^2	+1.0388e-3*s^2
		−1.7772*s^3	+1.2564e-3*s^3
Envelope	10786.8–19307.3	5.0848	1.1588e-2
		−5.4398*s^2	+1.2573e-3*s^2
			+3.3194e-3*s^3
External layer	19307.3–24622.0	2.141845	−7.28743e-4
		−2.141576*s	+1.87338e-2*s

"ice" envelope respectively. For the Neptune models, the corresponding values are about equal to 2.85 and to 1.52 g cm^{-3} respectively. The "rock" cores of two planets are quite small, with masses about equal to $(1.0 \sim 1.3)M_\oplus$ for Uranus and $(2.7 \sim 5.4)M_\oplus$ for Neptune respectively. These results are consistent with our knowledge of chemical composition of the material inside Uranus and Neptune, though the mass of rock is smaller than expected, especially in the case of Uranus. The same shortfall has also been found by Marley and Gomez (1995). These authors point out that rock may masquerade as ice, if mixed with gas. Further improvement in modelling the interiors of Uranus and Neptune must, therefore, await improvements in our knowledge of the equation of state of rock-ice-gas mixtures.

Next, let us compute the hydrostatic values of the Stokes zonal parameters, $J_2^{(0)}$, $J_4^{(0)}$, etc., and compare these with the observational values, J_2 and J_4 (see Table I). This will provide another check as to whether the results of the models are consistent with observational data.

Table VI
Estimated results for the models of Uranus and Neptune

	Uranus		Neptune	
	U96-01	U96-02	N96-01	N96-02
$-J_2^{(0)}$ (10^{-3})	3.2559	3.3693	3.2801	3.2972
$J_4^{(0)}$ (10^{-5})	2.53	2.55	2.79	2.79
$-J_6^{(0)}$ (10^{-7})	4.8	4.9	3.5	3.5
$e(1)$ (10^{-2})	1.97175	1.98906	1.79941	1.80051
$k_{s,2}$	0.331	0.342	0.377	0.379

Let $e_s = e(1)$ denote flattening on the planetary surface (equipotential surface) and $k_{s,n}$, n = 4, 6 are nth-degree secular Love numbers. Then retaining terms up to e_s^2 in smallness, we can get approximately (cf. Burša, 1992)

$$\left.\begin{aligned} J_4^{(0)} &= \frac{8}{35} k_{s,4} q e_s \left(\frac{a}{a_e}\right)^{-7}, \\ J_6^{(0)} &= -\frac{32}{231} k_{s,6} q e_s^2 \left(\frac{a}{a_e}\right)^{-9} \end{aligned}\right\} . \tag{10}$$

Since $k_{s,2} = -3J_2^{(0)}/q$, and adopting $k_{s,4} = 0.19$ (for Uranus), 0.26 (for Neptune) (cf. Burša, 1992) and $k_{s,6} = \bar{k}_s = 0.30$, the values of $J_4^{(0)}$ and $J_6^{(0)}$ for the four models can now be estimated using Equation (10). The calculated results are presented in Table VI. The values of $J_2^{(0)}$ $(= (A - C)/Ma^2)$ given in Table VI are obtained from the values of C and A given by numerical integration (cf. Zhang, 1994a). The tabulated data show that the relative errors between the values of $J_2^{(0)}$ and J_2 do not exceed 2.6% (for Uranus) and 3.8% (for Neptune). On the other hand, we hope that a new epoch of space explorations will provide an observational check on the predicted value of J_6 for Uranus and for Neptune given in this table.

5. Conclusions

From the preceding discussions in this paper we come to the following conclusions:

(1) A set of three-layer parametric models both for Uranus and Neptune can be obtained by means of the procedure used in the study of the terrestrial planets.

(2) A comparison between the calculated models of Uranus and Neptune, show that these two planets are similar to each other in their dynamical features. This is probably the only example of its kind in our solar system. Nevertheless, the very different obliquity of the rotation axis of the two celestial bodies does provide a very valuable discriminant between process of planetary formation. This fact must be explored in further study (cf. Prentice, 1986).

(3) The calculated results in this paper indicate that the predicted values of $J_6^{(0)}$ for Uranus and Neptune are about -5×10^{-7} and -3×10^{-7}, respectively. These values are in line with the results estimated by Burša (1992).

Acknowledgements

The author is grateful to Professor Baoren Chen and a referee for helping to improve the English. This work was supported by the National Natural Science Foundation of China.

References

Burša, M.: 1984, *Earth, Moon, and Planets* **31**, 135–140.
Burša, M.: 1992, *Earth, Moon, and Planets* **59**, 239–244.
Gudkova, T. V., Zharkov, V. N., and Leont'ev, V. V.: 1988a, *Astron. Vestn.* **22**, 23–40 (In Russian).
Gudkova, T. V., Zharkov, V. N., and Leont'ev, V. V.: 1988b, *Astron. Vestn.* **22**, 252–261 (In Russian).
Hubbard, W. B. and Marley, M. S.: 1989, *Icarus* **78**, 102–118.
Hubbard, W. B. and five others: 1991, *Science* **253**, 640–651.
Marley, M. S. and Gomez, P.: 1995, *J. Geophys. Res.* **100**, 23349–23353.
Prentice, A. J. R.: 1986, *Phys. Lett. A* **114**, 211.
Thomas, P. C.: 1991, *Rev. of Geophys.*, Supplement 182–187.
Zhang, C.-Z.: 1994a, *Earth, Moon, and Planets* **64**, 31–38.
Zhang, C.-Z.: 1994b, *Earth, Moon, and Planets* **64**, 117–124.
Zhang, C.-Z. and Shen, M.: 1988, *Earth, Moon, and Planets* **42**, 179–184.
Zhang, C.-Z. and Xia, Y. F.: 1994, *Earth, Moon, and Planets* **65**, 269–276.
Zhang, C.-Z. and Zhang, K.: 1995, *Earth, Moon, and Planets* **69**, 237–247.
Zharkov, V. N. and Gudkova, T. V.: 1991, *Ann. Geophysicae* **9**, 357–366.