THE MOTION OF VORTICES WITHIN A ROTATING, FLUID SHELL

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Abstract. We consider a spherical, solid planet surrounded by a thin layer of an incompressible, inviscid fluid. The planet rotates with constant angular velocity.

Within the constraints of the geostrophic approximation of hydrodynamics, we determine the equation that governs the motion of a vortex tube within this rotating ocean. This vorticity equation turns out to be a nonlinear partial differential equation of the third order for the stream function of the motion.

We next examine the existence of particular solutions to the vorticity equation that represent travelling waves of permanent form but decaying at infinity. A particular solution is obtained in terms of I_1 and K_1, the modified Bessel functions of order one.

The question whether these localized vortices that move like solitary waves could even be solitons depends on their behavior during and after collision with each other and has not yet been resolved.

1. Introduction

We consider a spherical, solid planet of radius R whose uppermost layer consists of a thin, liquid shell of constant thickness h, with $h \ll R$. The fluid constituting this ocean is incompressible and inviscid. The planet is rapidly rotating with constant angular velocity ω about an axis fixed with respect to its surface.

We use spherical, polar coordinates with origin O at the center of the planet. We denote by z the radial distance measured from the center, positive upwards, by θ the colatitude, measured from the rotational axis, and by ϕ the longitude, positive in the direction of rotation (East). In the tangent plane to the sphere at a point $P(z, \theta, \phi)$, we use a local, cartesian system with the x-axis tangent to the parallel of latitude, positive toward the East, and the y-axis tangent to the meridian, positive northward. Any point Q on this tangent plane can be identified by its distance r from P and the azimuth angle α that the segment $\overline{PQ}$ makes with the meridian, so that we have $x = r \sin \alpha$, $y = r \cos \alpha$.

We shall make use of the geostrophic approximation to the hydrodynamic equations of motion, which means that we shall limit ourselves to considering motions that correspond to a small Rossby number. Within this context, we are interested in (1) obtaining the equation that governs the motion of a vortex tube within the thin, liquid, rotating shell, and (2) ascertaining whether such equation admits of solutions that represent travelling waves of permanent form that decay at infinity.

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2. Motion of a Vortex Tube within a Thin, Liquid Shell

In the absence of viscosity, the Navier–Stokes equations are

$$\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nabla\Phi - 2\boldsymbol{\omega}\times\mathbf{q}\,, \tag{1}$$

where t is time, ρ the fluid density, p the fluid pressure, Φ the gravitational potential of the planet, ω its rotational velocity, and $\mathbf{q} = (u, v, w)$ is the velocity vector with respect to a coordinate system rotating with the planet, u being the eastward velocity, v the northward, velocity, and w the vertical velocity. The operator $\mathrm{D}/\mathrm{D}t$ is the Lagrangian or molecular derivative defined as

$$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + (\mathbf{q}\cdot\nabla)\,. \tag{2}$$

In considering propagation phenomena within a rotating ocean, we must define a horizontal length scale L and a typical horizontal velocity scale V. In what follows, we shall limit ourselves to motions such that the Rossby number is much smaller than one: i.e.,

$$\epsilon = V/\omega L \ll 1\,. \tag{3}$$

Under such circumstances, we realize that the ratio between the relative acceleration $\mathrm{D}\mathbf{q}/\mathrm{D}t$ of a particle and its Coriolis acceleration $2\boldsymbol{\omega}\times\mathbf{q}$ is of the order

$$\frac{V^2/L}{2\omega V} = \frac{V}{2\omega L} = \frac{1}{2}\epsilon\,. \tag{4}$$

This means that we can neglect the relative acceleration with respect to the Coriolis acceleration; we reach then the geostrophic approximation to the Navier–Stokes equation

$$2\boldsymbol{\omega}\times\mathbf{q} = -\frac{1}{\rho}\nabla p + \nabla\Phi\,, \tag{5}$$

which represents the momentum balance of the liquid column in terms of the Coriolis force, the pressure gradient, and the gravitational force.

In expanding Equation (5), we shall (1) neglect the ratio w/v because the particle trajectories shall be rather flat due to the fact that $h/L \ll 1$, and (2) limit the North–South extent of the motion so that it is legitimate to use the local coordinate system $(P; x, y, z)$ and eventually assign an average value θ_0 for the colatitude within that range. We reach then the system

$$\begin{aligned} \rho f u &= -\partial p/\partial y\,, \\ \rho f v &= \partial p/\partial x\,, \\ \rho g &= -\partial p/\partial z\,; \end{aligned} \tag{6}$$

where g is the acceleration due to gravity, and

$$f = 2\omega \cos \theta \tag{7}$$

is the so-called Coriolis parameter; see, e.g., Batchelor (1967) or Pedlovsky (1971).

We can see at once that the stream function $\psi(x, y, t)$ of the motion is

$$\psi = p/\rho f. \tag{8}$$

This is so because, from Equation (6) we have

$$v = \partial\psi/\partial x, \qquad u = -\partial\psi/\partial y$$

due to the fact that ρ is a constant and f will not vary within the limited extent of colatitude. Because of the same reasons, we can write

$$\psi_t = \frac{\partial\psi}{\partial t} = \frac{1}{\rho f}\frac{\partial p}{\partial z}\frac{dz}{dt} = -\frac{g}{f}\frac{d}{dt}(h + \zeta)$$

$$= -\frac{g}{f}\zeta_t, \tag{9}$$

where subscripts denote, as usual, partial derivatives. The symbol ζ stands for the tidal (or long wave) variation of the height of the liquid column above the average depth h of the fluid shell.

Having defined a model for our fluid ocean, let us now consider a vortex tube and note that, since in our approximation the horizontal velocities are virtually independent of the z-coordinate and much larger than the vertical velocity, the vortex lines can be expected to be nearly vertical and the vorticity $\nabla \times \mathbf{q} \equiv (X, Y, Z)$ of the tube shall consist primarily of its vertical component Z.

The strength of a vortex tube is the product of its vorticity and its cross-section area; mass conservation requires that the cross-section be inversely proportional to the length $h + \zeta$ of the tube. The strength of the tube is then proportional to the potential vorticity $Z/(h + \zeta)$. The absolute potential vorticity is

$$F \equiv \frac{f + Z}{h + \zeta}, \tag{10}$$

and consists of the vorticity f due to the planet rotation and the vorticity Z of its relative motion. By specifying that we deal with a rapidly-rotating planet, we imply that $f > Z$, provided that we are not too close to the equatorial belt, where $\theta_0 \sim \pi/2$. One can prove that this quantity F is conserved, which means that

$$\frac{D}{Dt}\left(\frac{f + Z}{h + \zeta}\right) = 0\,; \tag{11}$$

see, e.g., Milne–Thomson (1960), Batchelor (1967), and Platzman (1971).

Performing the operation implied by Equation (11), we find that

$$\frac{\mathrm{D}}{\mathrm{D}t}(f+Z)=F\frac{\mathrm{D}}{\mathrm{D}t}(h+\zeta)\,.$$

Now, we can expand the various terms as follows:

$$\frac{\mathrm{D}Z}{\mathrm{D}t}=Z_t+uZ_x+vZ_y=Z_t+\psi_xZ_y-\psi_yZ_x\,, \tag{A}$$

where we have made use of the stream function; next we get

$$\frac{\mathrm{D}f}{\mathrm{D}t}=\frac{v}{R+h}\frac{\partial f}{\partial\theta}=-\frac{2\omega\sin\theta}{R+h}\psi_x\,, \tag{B}$$

because f does not vary with time and depends only on θ; also, because of our assumption of limited variation in latitude, we can assign a mean value θ_0 for the colatitude, whereby we can consider $f=2\omega\cos\theta_0$ to be a constant of motion and take

$$\frac{\mathrm{D}f}{\mathrm{D}t}=-\frac{2\omega\sin\theta_0}{R+h}\psi_x=\beta\psi_x$$

with β a constant, (the so-called β-plane approximation); (C) h does not vary either geographically or with time, we have then $\mathrm{D}h/\mathrm{D}t\equiv 0$; and finally,

$$F\frac{\mathrm{D}\zeta}{\mathrm{D}t}=F\zeta_t+F(u\zeta_x+v\zeta_y)\,; \tag{D}$$

we can ignore the last two quadratic terms, approximate F by f/h, and use Equation (9) to write

$$F\frac{\mathrm{D}\zeta}{\mathrm{D}t}\sim\frac{f}{h}\zeta_t=-\frac{f^2}{gh}\psi_t=-\frac{1}{A^2}\psi_t\,,$$

where A is a positive quantity having the dimension of a length; the approximation used here for F is based upon the fact that $\zeta\ll h$ and that $Z<f=2\omega\cos\theta_0$ because of our assumption of a rapidly rotating planet, provided we are not considering any motion in the equatorial belt.

Upon collecting the above partial results, we can write the equation that governs the motion of a vortex tube as

$$Z_t-\frac{1}{A^2}\psi_t+\beta\psi_x+\psi_xZ_y-\psi_yZ_x=0\,, \tag{12}$$

where A and β are to be considered constants; the vertical component Z of the vorticity can be expressed in terms of the stream function as

$$Z=v_x-u_y=\psi_{xx}+\psi_{yy}\,. \tag{13}$$

3. Existence of Waves of Permanent Form

The vorticity equation, Equations (12)–(13), is a nonlinear partial differential equation (p.d.e.) of the third order for the stream function $\psi(x, y, t)$. We plan to investigate the nature of its solutions, in particular to ascertain whether any of them can represent travelling waves of permanent form. In other words, we wish to find a function $\Lambda(x - ct, y)$, where c is the constant velocity of propagation that will coincide with the stream function

$$\psi(x, y, t) \equiv \Lambda(x - ct, y) \tag{14}$$

and necessarily will satisfy the vorticity equation. In terms of this function, the vorticity Z can be represented as

$$Z(x - ct, y) = \Lambda_{xx} + \Lambda_{yy} . \tag{15}$$

We can now eliminate the time variable and get a new p.d.e. in the x, y variables:

$$-cZ_x + \frac{c}{A^2}\Lambda_x + \beta\Lambda_x + \Lambda_x Z_y - \Lambda_y Z_x = 0 . \tag{16}$$

Upon close examination of the above expression, we realize that by adding and subtracting the quantity $\Lambda_x\Lambda_y/A^2$ to the left-hand side of Equation (16), this expression becomes the expansion of the Jacobian of the functions $Z - (\Lambda/A^2) + \beta y$ and $\Lambda + cy$ with respect to the x, y variables. Thus, Equation (16) can be rewritten as

$$\partial J\left(Z - \frac{\Lambda}{A^2} + \beta y; \Lambda + cy\right)\Big/\partial(x, y) = 0 . \tag{17}$$

This situation is indicative of the existence of a functional relationship between these two quantities, which we shall represent as

$$Z - \frac{\Lambda}{A^2} + \beta y = H(\Lambda + cy) , \tag{18}$$

where H is any arbitrary differentiable function whose dimension must be $[H] =$ cm^{-2}; see also Stoker (1957) and Whitham (1974).

It is convenient, at this stage, to use the polar coordinates (r, α) in the tangent plane to the sphere, where α is the azimuth angle reckoned from the meridian. We find then that

$$Z(x - ct, y) = \Lambda_{xx} + \Lambda_{yy} = \frac{1}{r}\Lambda_r + \Lambda_{rr} + \frac{1}{r^2}\Lambda_{\alpha\alpha} ,$$

and Equation (18) evolves into

$$r^2\Lambda_{rr} + r\Lambda_r + \Lambda_{\alpha\alpha} - \frac{r^2}{A^2}\Lambda + \beta r^2 y = r^2 H(\Lambda + cy)\,, \tag{19}$$

where $y = r\cos\alpha$.

We must solve Equation (19) where H is still an arbitrary function. Although we desire to get as general a solution as possible, we must, nevertheless, be interested in producing solutions that can be represented by means of convergent power series and/or known special functions; these solutions must also vanish at infinity.

For this purpose, we have found it expedient to proceed according to the following steps:

(1) separation of the two variables

$$\Lambda(r, \alpha) \equiv \Lambda_1(r)\Lambda_2(\alpha)\,; \tag{20}$$

(2) choose $\Lambda_{\alpha\alpha} \equiv \Lambda_1\Lambda_2''$ in a most convenient way: e.g.,

$$\Lambda_2'' = -\nu^2\Lambda_2 \tag{21}$$

with ν^2 a positive constant; primes here denote derivatives with respect to the only variable in the given function; this gives rise to $\Lambda_2(\alpha) \equiv \cos(\nu\alpha)$; and

(3) choose a simple functional expression for H, e.g., $H(x) \equiv kx$, where k is a proportionality factor, whose dimension is $[k] = \mathrm{cm}^{-2}$.

Equation (19) now becomes

$$r^2\Lambda_1'' + r\Lambda_1' - \left[\nu^2 + \left(k + \frac{1}{A^2}\right)r^2\right]\Lambda_1 = r^3(kc - \beta)\frac{\cos\alpha}{\Lambda_2}\,. \tag{22}$$

We now have two choices for k at our disposal. By choosing $k_1 = \beta/c$, we cause the right-hand side of the equation to vanish; the equation reduces to

$$r^2\Lambda_1'' + r\Lambda_1' - \left(\nu^2 + \frac{\mu^2 r^2}{a^2}\right)\Lambda_1 = 0$$

with

$$\frac{\mu^2}{a^2} = \frac{\beta}{c} + \frac{1}{A^2}\,,$$

where a has the dimension of a length and μ is a pure number. This is a differential equation of the Bessel family. Thus, the choice $k_1 = \beta/c$ yields

$$\begin{aligned} \Lambda_1(r) &\equiv c_1 I_\nu\left(\frac{\mu r}{a}\right) + c_2 K_\nu\left(\frac{\mu r}{a}\right), \\ \Lambda_2(\alpha) &\equiv \cos(\nu\alpha)\,; \end{aligned} \tag{23}$$

where I_ν and K_ν are the modified Bessel functions of order ν, and c_1, c_2 are arbitrary constants. We use here the Watson notation for the Bessel functions; see Watson (1966) or Abramowitz and Stegun (1965).

The other possibility consists of choosing $k_2 + (1/A^2) = \sigma^2/a^2$, σ being a pure number. This, however, entails the selection of $\Lambda_2 \equiv \cos\alpha$, i.e., $\nu = 1$, if we want the separation of variables to occur. Equation (22) becomes

$$r^2\Lambda_1'' + r\Lambda_1' - \left(1 + \frac{\sigma^2 r^2}{a^2}\right)\Lambda_1 = r^3 c\left(k_2 - \frac{\beta}{c}\right),$$

which is again a differential equation of the Bessel family. Thus, the selection $k_2 = (\sigma^2/a^2) - (1/A^2)$ yields

$$\begin{aligned} \Lambda_1(r) &\equiv c_1 I_1\left(\frac{\sigma r}{a}\right) + c_2 K_1\left(\frac{\sigma r}{a}\right) + B\frac{r}{a}, \\ \Lambda_2(\alpha) &\equiv \cos\alpha; \qquad B = -ac\left(1 - \frac{\mu^2}{\sigma^2}\right). \end{aligned} \tag{24}$$

We know that K_ν becomes infinite at $r = 0$ and vanishes at infinity, whereas I_ν is bounded at $r = 0$. We must, therefore, combine these two functions in order to obtain a composite solution that remains finite everywhere and vanishes at infinity. We must, however, impose continuity for Λ_1, Λ_1' and the vorticity Z at the patch-point, which we choose to be $r = a$. To this end, we must have at our disposal a certain number of free parameters within the functions we employ in order to satisfy the conditions imposed by continuity. Because of these reasons, we will choose $\Lambda_1(r) \equiv c_2 K_1(\mu r/a)$ corresponding to $k_1 = \beta/c$ with $\nu = 1$ for $r > a$, and $\Lambda_1(r) \equiv c_1 I_1(\sigma r/a) + Br/a$ corresponding to $k_2 = (\sigma^2/a^2) - (1/A^2)$ for $r < a$.

Continuity for $\Lambda_1(r)$ at $r = a$ requires a choice of the parameters so as to render the complete solution to be

$$\Lambda(r, \alpha) \equiv -ac(\cos\alpha)\begin{cases} \dfrac{\mu^2}{\sigma^2}\dfrac{1}{I_1(\sigma)} I_1\left(\dfrac{\sigma r}{a}\right) + \left(1 - \dfrac{\mu^2}{\sigma^2}\right)\dfrac{r}{a}, & r < a \\[2ex] \dfrac{1}{K_1(\mu)} K_1\left(\dfrac{\mu r}{a}\right), & r > a\,. \end{cases} \tag{25}$$

Continuity for Λ_1' at $r = a$ imposes the condition that the parameters μ and σ satisfy the following equation

$$\frac{1}{\sigma}\frac{I_2(\sigma)}{I_1(\sigma)} = -\frac{1}{\mu}\frac{K_2(\mu)}{K_1(\mu)}, \tag{26}$$

which has an infinity of solutions; see Watson (1966).

Finally, evaluation of the vorticity from Equation (25) yields

$$Z(r, \alpha) \equiv -\mu^2 \frac{c}{a} \cos\alpha \begin{cases} I_1(\sigma r/a)/I_1(\sigma), & r < a \\ K_1(\mu r/a)/K_1(\mu), & r > a \end{cases} \tag{27}$$

which is continuous at $r = a$.

4. The Soliton Connection

In conclusion, we have (1) determined the equation of motion for vortices within a rotating ocean, and this turns out to be a nonlinear p.d.e. for the stream function; and (2) obtained solutions for such an equation that represent waves of permanent form. The next question that we should ask is how these waves interact with each other at collision.

Solitons are solutions of nonlinear p.d.e.(s), that behave like solitary waves of unchanged form that decay at infinity; however, they do not interact with each other, whereby at separation after collision they take back their original profile. The theory of solitons as solutions of the Korteweg–de Vries equation and their connection to the Inverse Scattering problem can be gathered from early publications such as Zabusky and Kruskal (1965), Miura *et al.* (1968), and Ablowitz *et al.* (1979).

We have indeed found solutions to a nonlinear p.d.e. (vorticity equation) in the form of permanent waves decaying at infinity; the question whether or not our solutions behave like solitons when they interact with each other must, however, be settled from a numerical point of view and, to the best of our efforts, still remains unresolved.

Other localized vortices of permanent form have been obtained in the past and studied by McWilliams and Zabusky (1982); these authors were able to prove that their solutions did not behave like solitons.

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